



HAL
open science

A new family of periodic functions as explicit roots of a class of polynomial equations

Marc Artzrouni

► **To cite this version:**

Marc Artzrouni. A new family of periodic functions as explicit roots of a class of polynomial equations. Australian Journal of Mathematical Analysis and Applications, 2006, 3 (2), pp.4; 1-16. hal-00003146

HAL Id: hal-00003146

<https://hal.science/hal-00003146>

Submitted on 25 Oct 2004

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A new family of periodic functions as explicit roots of a class of polynomial equations

Marc Artzrouni
Department of Mathematics
University of Pau
64000 Pau
FRANCE

Marc.Artzrouni@univ-pau.fr

October 22, 2004

Abstract

For any positive integer n , a new family of periodic functions in power series form and of period n is used to solve in closed form a class of polynomial equations of order n . The n roots are the values of the appropriate function from that family taken at $0, 1, \dots, n - 1$.

1 Background

For centuries mathematicians have sought closed-form expressions for the roots of polynomial equations of arbitrary order n . It is now well known that polynomials of order four or less can be solved explicitly using rational operations and finite root extractions. The Abel-Ruffini theorem shows that this cannot be done for orders five and above [1].

Several authors have proposed series solutions of algebraic (and polynomial) equations ([2], [3], [4], [5], [6]). These solutions, which rely on hypergeometric functions, are cumbersome to implement and have not provided feasible alternatives to standard numerical methods.

For a positive integer n , we will describe here in elementary fashion a family of functions

$$x(t, u) = \sum_{m=1}^{\infty} \beta_m e^{2u\pi m \times i/n} \times t^m \quad (1.1)$$

of two real variables t and u that are power series in t and are periodic of period n in the variable u . (These functions are analogous to complex Fourier series except that the summation extends over positive indices only).

For a limited class of polynomial equations of order n (parameterized in some way by t) the β'_m 's will be found explicitly so that the series $\sum |\beta_m t^m|$ converges and the values of $x(t, u)$ at t and $u = 0, 1, \dots, n - 1$ will be the n roots of the equation. The function $x(t, u)$, which could be thought as "elementary" in the same way

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} / (2n)! \quad (1.2)$$

is an elementary function, thus provides explicit solutions simply through its values at t and $u = 0, 1, \dots, n - 1$.

This will be only a first step as the class of polynomial equations solved explicitly with these power series is limited. The ultimate goal is to generalize the approach proposed here to any polynomial equation.

2 Preliminaries

We start off with a polynomial equation in the form

$$x^n = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1x + a_0 \quad (2.1)$$

where the a'_k s are real or complex coefficients and $a_0 = \rho e^{i\theta}$ ($\rho > 0$, $-\pi < \theta \leq \pi$) is assumed throughout to be non-zero. (Otherwise (2.1) can trivially be reduced to an equation of degree $n - 1$).

Equation (2.1) is transformed by multiplying the right-hand side by t^n where t is a real variable that we may initially think of as small but is destined to take on any real value including 1:

$$x^n = (a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1x + a_0)t^n. \quad (2.2)$$

We will find a solution in the form of a function $x(t, u)$ given in Eq. (1.1). We will show that $\sum |\beta_m t^m|$ converges (and $x(t, u)$ is therefore defined) when t is small enough or $|a_0|$ large enough, or the moduli $|a_k|$ other than $|a_0|$ small enough.

When the series converges the values of $x(t, u)$ at t and $u = 0, 1, \dots, n - 1$ will provide the n roots of Equation (2.2). Indeed, we will show that if we define the partial sums

$$x(t, u)_q \stackrel{\text{def.}}{=} \sum_{m=1}^q \beta_m e^{2u\pi m \times i/n} \times t^m, \quad (2.3)$$

then for $u = 0, 1, \dots, n - 1$ the polynomial in t

$$E\left(x(t, u)_q\right) \stackrel{\text{def.}}{=} \left[x(t, u)_q \right]^n - \left(a_{n-1} \left[x(t, u)_q \right]^{n-1} + a_{n-2} \left[x(t, u)_q \right]^{n-2} + \dots + a_1 x(t, u)_q + a_0 \right) t^n \quad (2.4)$$

will approach 0 for $q \rightarrow \infty$.

We begin by seeking the roots expressed as the infinite series

$$x(t) = b_1 t + b_2 t^2 + b_3 t^3 + \dots \quad (2.5)$$

where *a priori* we will need n different sequences $\{b_m\}$ to generate the n roots. We will find these b'_m s and show that in fact they are each of the form

$$b_m = \beta_m e^{2k\pi m \times i/n}, k = 0, 1, \dots, n - 1 \quad (2.6)$$

required in Eq. (1.1) with a single sequence $\{\beta_m\}$ that will be obtained explicitly through a simple discrete dynamical system.

The equation to solve is now

$$x(t)^n = (a_{n-1}x(t)^{n-1} + a_{n-2}x(t)^{n-2} + \dots + a_1x(t) + a_0)t^n. \quad (2.7)$$

Before proceeding we need some notations and preliminary results.

We define the powers B_q^r of the partial sums of $x(t)$:

$$B_q^r \stackrel{def.}{=} (b_1t + b_2t^2 + \dots + b_qt^q)^r, \quad q, r \in \mathbb{N}, \quad (2.8)$$

where for ease of notation the functional dependence of B_q^r on t is omitted. We let $K(d, B_q^r)$ denote the coefficient of t^d in B_q^r . ($K(d, B_q^r)$ does not depend on t and $K(d, B_q^r) = 0$ if $d < r$ or $d > r \cdot q$).

Proposition 2.1. *With $d \geq r$, the coefficients $K(d, B_q^r)$ satisfy*

$$K(d, B_q^r) = K(d, B_{d-r+1}^r) \quad \forall q \geq d - r + 1, \quad (2.9)$$

and

$$K(d, B_q^r) = \sum_{m=1}^{\min(d-r+1, q)} b_m K(d-m, B_q^{r-1}), \quad r \geq 2, \quad d \geq r, \quad (2.10)$$

$$K(q+s, B_{q+s}^{s+1}) = b_1 K(q+s-1, B_{q+s-1}^s) + \sum_{m=2}^{q-1} b_m K(q+s-m, B_{q+s-m}^s) + b_q b_1^s. \quad (2.11)$$

Proof. Equation (2.9) expresses the fact that when $q \geq d - r + 1$ then only the first $d - r + 1$ b_i 's enter into $K(d, B_q^r)$. Equation (2.10) is the convolution rule used to express the coefficient of t^d in B_q^r considered as the product $B_q^{r-1} \times B_q^1$. Equation (2.11) uses (2.9) to express a form of Eq. (2.10) in which all the $K(a, B_u^v)$'s have identical values for a and u . □

With $x(t)$ given in Eq. (2.5) both sides of Eq. (2.7) are polynomials in t with powers $\geq n$. The goal is to find recursively the b'_m 's so that for each $p \geq n$ the coefficients of t^p are equal on both sides of (2.7). This will be done by considering the roots $x(t)$ in the form of the gradually expanding partial sums B_q^1 . The coefficient b_1 will be found by seeking a solution of the form B_1^1 for which the coefficients of t^n on both sides of (2.7) are equal. With b_1 thus determined, b_2 is found by seeking a solution of the form B_2^1 for which the coefficients of t^{n+1} on both sides of (2.7) are equal, etc.

We begin with a candidate solution $B_1^1 = b_1 t$ by setting equal the coefficients of t^n on both side of Eq. (2.7), i.e.

$$B(n, B_1^n) = b_1^n = a_0. \quad (2.12)$$

Therefore the n possible values of b_1 are

$$b_1 = \rho^{1/n} e^{i(\theta+2k\pi)/n} \quad k = 0, 1, \dots, n-1 \quad (2.13)$$

which is Eq. (2.6) with $m = 1$ and

$$\beta_1 \stackrel{\text{def.}}{=} \rho^{1/n} e^{i\theta/n}. \quad (2.14)$$

We note that the n values $x(t) = b_1 t$ with b_1 of Eq. (2.13) are the exact trivial solutions of Eq. (2.7) when all coefficients other than a_0 are 0; otherwise they provide crude first-approximation solutions when t is small.

In order to find b_2 with a candidate solution $B_2^1 = b_1 t + b_2 t^2$, we note that the coefficient of t^{n+1} on the left side of Eq. (2.7) is $K(n+1, B_2^n)$ and on the right is $a_1 K(1, B_2^1)$. Therefore we want

$$K(n+1, B_2^n) = a_1 K(1, B_2^1). \quad (2.15)$$

Similarly, equating the coefficients of t^{n+2} with a solution B_3^1 yields

$$K(n+2, B_3^n) = a_2 K(2, B_3^2) + a_1 K(2, B_3^1). \quad (2.16)$$

When we equate the coefficients of t^{n+3} a similar expression arises with a third term involving the coefficient a_3 . In general, to equate the coefficients of t^{n+q} on both sides of Eq. (2.7) (with a candidate solution B_{q+1}^1) one needs:

$$K(n+q, B_{q+1}^n) = \sum_{m=1}^{\min(n-1, q)} a_m K(q, B_{q+1}^m) \quad q = 1, 2, \dots \quad (2.17)$$

Equation (2.9) shows that for any $s \geq 1$

$$K(n+q, B_{q+1}^n) = K(n+q, B_{q+s}^n) \quad (2.18)$$

$$K(q, B_{q+1}^m) = K(q, B_{q-m+1}^m) = K(q, B_{q+s}^m). \quad (2.19)$$

These equations show that (2.17) implies for any s

$$K(n+q, B_{q+s}^n) = \sum_{m=1}^{\min(n-1, q)} a_m K(q, B_{q+s}^m). \quad (2.20)$$

Therefore if (2.17) is satisfied, the coefficients of t^{n+q} on both sides of Eq. (2.7) are also equal when any number of terms $b_k t^k$ are added to the partial sum B_{q+1}^1 .

We next write the $K(a, B_u^v)$'s appearing in Eq. (2.17) in such a way that the indices a and u are equal. Equation (2.9) shows that Eq. (2.17) is equivalent to

$$K(n+q, B_{n+q}^n) = \sum_{m=1}^{\min(n-1, q)} a_m K(q, B_q^m) \quad q = 1, 2, \dots \quad (2.21)$$

Given that $b_q = K(q, B_q^1)$, the task will be to find n sequences of b'_m 's for which the infinite system (2.21) is satisfied for all q . We will achieve this through a simple discrete dynamical system that generates the b'_q 's. A change of variable on the dynamical system will show that the n sequences $\{b_m\}$ are of the form $\{\beta_m e^{2k\pi m \times i/n}\}$ required in Eq. (1.1) (with $k = 0, 1, \dots, n-1$). Finally we will give the convergence conditions for the series $\sum |\beta_m t^m|$ and show that the $x(t, k)$'s are the n roots in the sense of (2.4).

3 Matrix formulation

We first define the sequence of $n-1$ dimensional vectors

$$V(q) \stackrel{def.}{=} (K(q, B_q^1) \quad K(q+1, B_{q+1}^2) \quad \dots \quad K(q+n-2, B_{q+n-2}^{n-1}))' \quad q = 1, 2, \dots \quad (3.1)$$

where the apostrophe means "transpose" so that $V(q)$ is a column vector. We note that

$$V(1) = (b_1 \quad b_1^2 \quad b_1^3 \quad \dots \quad b_1^{n-1})'. \quad (3.2)$$

Also the s -th component $V(2)_s$ of $V(2)$ is $K(s+1, B_{s+1}^s) = s.b_2 b_1^{s-1}$ with b_2 to be determined from Eq. (2.21) for $q = 1$:

$$n.b_2 b_1^{n-1} = K(n+1, B_{n+1}^n) = a_1 b_1 \quad (3.3)$$

from which $b_2 = a_1 / (n b_1^{n-2})$ and therefore

$$V(2) = (a_1/n b_1^{n-2} \quad 2a_1/n b_1^{n-3} \quad \dots \quad \dots \quad (n-1) a_1/n b_1^0)'. \quad (3.4)$$

We next define the sequence of $n-1$ dimensional vectors $X(q)$ obtained as the convolution of the b'_i 's with the vectors $V(q)$:

$$X(q-1) \stackrel{def.}{=} \sum_{m=2}^{q-1} b_m V(q-m+1) \quad q = 3, 4, \dots \quad (3.5)$$

We let $V(q)_r$ and $X(q)_r$ denote the r -th components of $V(q)$ and $X(q)$. The left-hand side of Eq. (2.11) is $V(q)_{s+1}$ and Eq. (2.11) with $s = 1$ is then

$$V(q)_2 = b_1 V(q)_1 + X(q-1)_1 + V(q)_1 b_1 = 2b_1 V(q)_1 + X(q-1)_1. \quad (3.6)$$

With $s = 2$ Eq. (2.11) is

$$V(q)_3 = b_1 V(q)_2 + X(q-1)_2 + V(q)_1 b_1^2 \quad (3.7)$$

and more generally

$$V(q)_{s+1} = b_1 V(q)_s + X(q-1)_s + V(q)_1 b_1^s, \quad s = 1, 2, \dots, n-2. \quad (3.8)$$

Equation (2.11) with $s = n-1$ is

$$K(q+n-1, B_{q+n-1}^n) = b_1 V(q)_{n-1} + X(q-1)_{n-1} + V(q)_1 b_1^{n-1}. \quad (3.9)$$

If we define the $n-1$ dimensional square matrix

$$M_1 = \begin{pmatrix} 2b_1 & -1 & 0 & 0 & \dots & 0 \\ b_1^2 & b_1 & -1 & 0 & \dots & 0 \\ b_1^3 & 0 & b_1 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_1^{n-2} & 0 & \dots & 0 & b_1 & -1 \\ b_1^{n-1} & 0 & \dots & \dots & \dots & b_1 \end{pmatrix} \quad (3.10)$$

then Eqs. (3.6)- (3.9) (for $s = 1, 2, \dots, n-1$) can be written compactly as

$$-X(q-1) = M_1 V(q) - \begin{pmatrix} 0 \\ 0 \\ \dots \\ \dots \\ K(q+n-1, B_{q+n-1}^n) \end{pmatrix} \quad (3.11)$$

The goal is to express $V(q)$ as a function of past vectors $V(q-1), V(q-2), \dots, V(1)$. The inverse of M_1 is

$$M_1^{-1} = \frac{1}{nb_1^{n-1}} \begin{pmatrix} b_1^{n-2} & b_1^{n-1} & \dots & b_1^2 & b_1 & 1 \\ -(n-2)b_1^{n-1} & 2b_1^{n-2} & 2b_1^{n-3} & \dots & 2b_1^2 & 2b_1 \\ -(n-3)b_1^n & -(n-3)b_1^{n-1} & 3b_1^{n-2} & \dots & 3b_1^3 & 3b_1^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -2b_1^{2n-5} & -2b_1^{2n-6} & \dots & -2b_1^{n-1} & (n-2)b_1^{n-2} & (n-2)b_1^{n-3} \\ -b_1^{2n-4} & -b_1^{2n-5} & -b_1^{2n-6} & \dots & -b_1^{n-1} & (n-1)b_1^{n-2} \end{pmatrix} \quad (3.12)$$

If $(M_1^{-1})_{i,j}$ is the entry in the i -th row, j -th entry of this inverse, Eq. (3.12) can be written as

$$(M_1^{-1})_{i,j} = \frac{1}{nb_1^{n-1}} \begin{cases} -(n-i)b_1^{n+i-j-2} & \text{if } i > j \\ ib_1^{n+i-j-2} & \text{if } i \leq j \end{cases} \quad (3.13)$$

We multiply both sides of Eq. (3.11) by M_1^{-1} to obtain

$$V(q) = -M_1^{-1} X(q-1) + M_1^{-1} \begin{pmatrix} 0 \\ 0 \\ \dots \\ \dots \\ K(q+n-1, B_{q+n-1}^n) \end{pmatrix}. \quad (3.14)$$

From Eq. (2.21) we note that

$$K(q+n-1, B_{q+n-1}^n) = \sum_{m=1}^{\min(n-1, q-1)} a_m K(q-1, B_{q-1}^m) \quad (3.15)$$

$$= \sum_{m=1}^{\min(n-1, q-1)} a_m V(q-m)_m = \sum_{m=1}^{\min(n-1, q-1)} [A_m \cdot V(q-m)] \quad (3.16)$$

where each A_p ($p = 1, 2, \dots, n-1$) is an $(n-1)$ -dimensional row vector with a_p in p -th position and zeros elsewhere. (Here and elsewhere an expression in square brackets will generally represent a scalar product).

The vector $V(q)$ of Eq. (3.14) is now

$$V(q) = -M_1^{-1}X(q-1) + \frac{\sum_{m=1}^{\min(n-1, q-1)} [A_m \cdot V(q-m)]}{nb_1^{n-1}} \begin{pmatrix} 1 \\ 2b_1 \\ 3b_1^2 \\ \dots \\ (n-1)b_1^{n-2} \end{pmatrix}. \quad (3.17)$$

For each one of the n values $\rho^{1/n} e^{i(\theta+2k\pi)/n}$ ($k = 0, 1, \dots, n-1$) of b_1 the sequence $V(q)$ yields all the subsequent b_i 's since b_i is the first component of $V(i)$. We thus have the n sequences $\{b_k\}$ that will provide the n roots (under the right as-yet-unproved convergence conditions).

A change of variable on the sequence $V(m)$ which defines a new sequence $W(m)$ will now show that the n values of each b_m can be expressed in the form $\beta_m e^{2k\pi m \times i/n}$ ($k = 0, 1, \dots, n-1$) where a single sequence $\{\beta_m\}$ is defined recursively as a function of the a_k 's .

We will first need the diagonal matrix

$$B \stackrel{def.}{=} \begin{pmatrix} b_1^{n-1} & 0 & 0 & 0 & 0 \\ 0 & b_1^{n-2} & 0 & 0 & 0 \\ 0 & 0 & b_1^{n-3} & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & 0 & b_1^1 \end{pmatrix} \quad (3.18)$$

and

$$M \stackrel{def.}{=} \frac{1}{n} \begin{pmatrix} 1 & 1 & \dots & \dots & 1 \\ -(n-2) & 2 & 2 & \dots & 2 \\ -(n-3) & -(n-3) & 3 & \dots & 3 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -1 & \dots & -1 & n-1 \end{pmatrix} \quad (3.19)$$

The matrix M is the inverse M_1^{-1} of (3.12) in which b_1 would be set to 1. With these notations

$$M_1^{-1} = \frac{1}{b_1} B^{-1} M B \quad (3.20)$$

The sequence $W(q)$ derived from $V(q)$ is defined as

$$W(q) \stackrel{def.}{=} b_1^{1-q} B.V(q), \quad q = 1, 2, \dots \quad (3.21)$$

We will see that the sequence $W(q)$ is independent of the particular b_1 chosen among its n possible values $\rho^{1/n} e^{i(\theta+2k\pi)/n}$ ($k = 0, 1, \dots, n-1$). The fact that each one of the b'_m s is of the form $\beta_m e^{2k\pi m \times i/n}$ ($k = 0, 1, \dots, n-1$) follows then from the fact that the first components $V(q)_1$ and $W(q)_1$ satisfy

$$V(q)_1 = W(q)_1 b_1^q / a_0. \quad (3.22)$$

In the theorem below we give for the sequence $W(q)$ an expression analogous to (3.17) for $V(q)$ and provide the expressions for the β'_m s.

Theorem 3.1. *The sequence $W(q)$ is defined as*

$$W(1) = a_0 (1 \ 1 \ \dots \ \dots \ 1)' \quad (3.23)$$

$$W(2) = \frac{a_1}{n} (1 \ 2 \ \dots \ \dots \ n-1)' \quad (3.24)$$

$$W(q) = \frac{1}{a_0} \left[-M \sum_{p=1}^{q-2} [u.W(p+1)] W(q-p) + U \sum_{p=1}^{\min(n-1, q-1)} [A_p.W(q-p)] \right], \quad q = 3, 4, \dots \quad (3.25)$$

where:

1. u is the $(n-1)$ -dimensional row vector having 1 in first position and zeros elsewhere.
2. U is the $(n-1)$ -dimensional column vector $\frac{1}{n} (1 \ 2 \ \dots \ \dots \ n-1)'$.

The β'_m s that define the n sequences

$$b_m = \beta_m e^{2k\pi m \times i/n}, \quad k = 0, 1, \dots, n-1; \quad m = 1, 2, \dots \quad (3.26)$$

are

$$\beta_m = \rho^{m/n} \times e^{i(m\theta)/n} \times W(m)_1 / a_0, \quad m = 1, 2, \dots \quad (3.27)$$

Proof. The expressions of (3.23)- (3.24) follow from the definition in Eq. (3.21) and Eqs. (3.2) and (3.4) which define $V(1)$ and $V(2)$.

Using Eq. (3.5) to express $X(q-1)$ appearing in $V(q)$ of Eq. (3.17) yields

$$V(q) = b_1^{q-1} B^{-1} W(q) = -M_1^{-1} \left(\sum_{m=2}^{q-1} b_m b_1^{q-m} B^{-1} W(q-m+1) \right) + \frac{\sum_{p=1}^{\min(n-1, q-1)} \left[A_p \cdot \left(b_1^{q-p-1} B^{-1} W(q-p) \right) \right]}{n b_1^{n-1}} \begin{pmatrix} 1 \\ 2b_1 \\ 3b_1^2 \\ \dots \\ (n-1) b_1^{n-2} \end{pmatrix}, \quad q = 3, 4, \dots \quad (3.28)$$

Bearing in mind that:

- * $B \times M_1^{-1} \times B^{-1} = M/b_1$; $b_1^n = a_0$
- * each b_m is b_1^{m-n} multiplied by the first component $[u \cdot W(m)]$ of $W(m)$
- * the p -th component of $B^{-1} W(q-p)$ is $b_1^{p-n} W(q-p)_p = b_1^p W(q-p)_p / a_0$, we multiply both sides of Eq. (3.28) by $b_1^{1-q} B$ to obtain

$$W(q) = \frac{1}{a_0} \left[-M \sum_{m=2}^{q-1} [u \cdot W(m)] W(q-m+1) + U \sum_{p=1}^{\min(n-1, q-1)} [A_p \cdot W(q-p)] \right], \quad q = 3, 4, \dots \quad (3.29)$$

which is Eq. (3.25) once the index m in the first sum is set equal to $p+1$ with p going from 1 to $q-2$.

In view of Eq. (3.22) the first component b_m of $V(m)$ is equal to

$$b_m = b_1^m W(m)_1 / a_0 = \rho^{m/n} \times e^{m(\theta+2k\pi)i/n} W(m)_1 / a_0 \quad (3.30)$$

which is (3.26) with β_m given in Eq. (3.27). □

4 Convergence results

In order to establish convergence conditions for the series of (2.5) we need to assess the growth of the first components $W(m)_1$ of the vectors $W(m)$. We will use the ℓ_∞ norm $|V|_\infty = \max_i |V_i|$ of a complex vector $V = (v_i)$. We also define the row-sum norm

$$\|A\| = \max_i \sum_{j=1}^m |a_{ij}| \quad (4.1)$$

of an m -dimensional square matrix $A = (a_{ij})$ and recall that $|AV|_\infty \leq \|A\| |V|_\infty$.

If we define the function of two variables

$$z(u, n) = -2u^2 + u(2n+1) - n \quad (4.2)$$

then the norm of the $(n-1)$ -dimensional matrix M of Eq. (3.19) is

$$\|M\| = \frac{1}{n} \max(z[\text{floor}(n/2 + 1/4), n], z[\text{ceil}(n/2 + 1/4), n]). \quad (4.3)$$

where the $\text{floor}(x)$ and $\text{ceil}(x)$ functions are the largest integer smaller than x and the smallest integer less than x .

We next define the sequence of moduli

$$w(q) \stackrel{\text{def.}}{=} |W(q)|_\infty, \quad q = 1, 2, \dots \quad (4.4)$$

which are upper bounds for the moduli of the $W(q)_1$'s of interest. We also define the maximum modulus of the coefficients a_k other than a_0 :

$$\alpha \stackrel{\text{def.}}{=} \max_{k=1,2,\dots,n-1} |a_k|. \quad (4.5)$$

We note that $w(1) = |a_0|$ and $w(2) \leq \alpha$. From Eq. (3.25) we now have

$$w(q) \leq \frac{1}{|a_0|} \left(\sum_{p=1}^{q-2} w(p+1) \|M\| \times w(q-p) + \alpha \sum_{p=1}^{\min(n-1, q-1)} w(q-p) \right), \quad q = 3, 4, \dots \quad (4.6)$$

with an important distinction to be made between the cases $q \leq n$ and $q > n$. In the former case the p in the second sum on the right-hand side of (4.6) goes to $q-1$ with a last term $w(1) = |a_0|$. The inequality of (4.6) is then

$$w(q) \leq \frac{1}{|a_0|} \left(\sum_{p=1}^{q-2} [\|M\|w(p+1) + \alpha]w(q-p) \right) + \alpha, \quad q = 3, 4, \dots, n \quad (4.7)$$

We next define

$$\mu = \|M\|/|a_0| \quad (4.8)$$

and the sequence of functions

$$\sigma_p(\alpha) = \begin{cases} \alpha/\|M\|, & \text{if } p \leq n \\ 0, & \text{if } p > n \end{cases}, \quad p = 2, 3, \dots \quad (4.9)$$

In the case $q > n$ the index p in the second sum on the right-hand side of (4.6) goes to $n-1$ and (4.6) can now be written

$$w(q) \leq \mu \left(\sum_{p=1}^{q-2} [w(p+1) + \sigma_{p+1}(\alpha)]w(q-p) \right) \quad (4.10)$$

$$\leq \mu \left(\sum_{p=1}^{q-2} [w(p+1) + \sigma_{p+1}(\alpha)][w(q-p) + \sigma_{q-p}(\alpha)] \right), \quad q = n+1, n+2, \dots \quad (4.11)$$

where the purpose of this last inequality is to bound the sequence $\{w(q)\}$ by a particular type of discrete convolution which will be considered below.

We first define recursively a sequence of nonnegative functions $S_q(\alpha, |a_0|)$:

$$S_2(\alpha, |a_0|) \stackrel{\text{def.}}{=} \alpha + \sigma_2(\alpha) \quad (4.12)$$

$$S_q(\alpha, |a_0|) \stackrel{def.}{=} \frac{1}{|a_0|} \left(\sum_{p=1}^{q-2} [\|M\| S_{p+1}(\alpha, |a_0|) + \alpha] S_{q-p}(\alpha, |a_0|) \right) + \alpha + \sigma_q(\alpha), q = 3, 4, \dots, n \quad (4.13)$$

where the functional notation emphasizes for future reference a dependence of S_q on α and $|a_0|$ (even though $S_2(\alpha, |a_0|)$ of (4.12) does not depend on $|a_0|$).

We next continue the sequence of S'_q s with

$$S_q(\alpha, |a_0|) \stackrel{def.}{=} \mu \left(\sum_{p=1}^{q-2} S_{p+1}(\alpha, |a_0|) S_{q-p}(\alpha, |a_0|) \right), q = n + 1, n + 2, \dots \quad (4.14)$$

Lemma 4.1. *With the notations given above,*

$$w(q) \leq w(q) + \sigma_q(\alpha) \leq S_q(\alpha, |a_0|), \quad q = 2, 3, \dots \quad (4.15)$$

Proof. The inequality of (4.15) is true for $q = 2$ because $w(2) \leq \alpha$. In view of (4.7) and (4.13) it is then also true for $q = 3, 4, \dots, n$. For $q = n + 1$, (4.11) yields

$$\begin{aligned} w(n + 1) &= w(n + 1) + \sigma_{n+1}(\alpha) \leq \mu \left(\sum_{p=1}^{n-1} [w(p + 1) + \sigma_{p+1}(\alpha)] [w(n + 1 - p) + \sigma_{n+1-p}(\alpha)] \right) \\ &\leq \mu \left(\sum_{p=1}^{n-1} S_{p+1}(\alpha, |a_0|) \times S_{n+1-p}(\alpha, |a_0|) \right) = S_{n+1}(\alpha, |a_0|). \end{aligned} \quad (4.16)$$

To prove the result by induction we assume (4.15) is true up to order $n + r$. Then at order $n + r + 1$ we have

$$w(n + r + 1) = w(n + r + 1) + \sigma_{n+r+1}(\alpha) \quad (4.17)$$

$$\leq \mu \left(\sum_{p=1}^{n+r-1} [w(p + 1) + \sigma_{p+1}(\alpha)] [w(n + r + 1 - p) + \sigma_{n+r+1-p}(\alpha)] \right) \quad (4.18)$$

$$\leq \mu \left(\sum_{p=1}^{n+r-1} S_{p+1}(\alpha, |a_0|) \times S_{n+r+1-p}(\alpha, |a_0|) \right) = S_{n+r+1}(\alpha, |a_0|) \quad (4.19)$$

which completes the proof. □

We now provide some definitions and results pertaining to convolution-type sequences such as the S'_q s of (4.14).

Proposition 4.2. For any scalar μ , a μ -convolution of order m is an infinite sequence $\{u_k\}$ consisting of m initial scalars u_1, u_2, \dots, u_m with subsequent terms defined as

$$u_q = \mu \left(\sum_{p=1}^{q-1} u_p u_{q-p} \right), \quad q = m+1, m+2, \dots \quad (4.20)$$

For a μ -convolution $\{v_q\}$ of order $m=1$, we have

$$v_{q+1} = v_1 C_q (v_1 \mu)^q, \quad q = 0, 1, \dots \quad (4.21)$$

where the C_r 's are the Catalan numbers

$$C_r = \frac{(2r)!}{(r+1)!r!}, \quad r = 0, 1, 2, \dots; \quad C_r \sim \frac{4^r}{\sqrt{\pi} r^{3/2}} \quad \text{for } r \rightarrow \infty. \quad (4.22)$$

For a μ -convolution of order m with $\mu > 0$ consisting of m initial nonnegative terms u_1, u_2, \dots, u_m we define

$$v_1 \stackrel{\text{def.}}{=} \max_{k=1,2,\dots,m} \left(\frac{u_k}{C_{k-1} \mu^{k-1}} \right)^{1/k}. \quad (4.23)$$

The sequence u_k is then bounded by the μ -convolution $\{v_k\}$ of order 1 and initial term v_1 , i.e. for any q :

$$u_{q+1} \leq v_{q+1} = v_1^{q+1} \mu^q C_q \sim v_1^{q+1} \mu^q \frac{4^q}{\sqrt{\pi} q^{3/2}} \quad \text{for } q \rightarrow \infty. \quad (4.24)$$

Proof. One of the defining relations for the Catalan numbers is

$$C_s = C_0 C_{s-1} + C_1 C_{s-2} + \dots + C_{s-1} C_0, \quad s = 1, 2, \dots \quad (4.25)$$

Equation (4.21) follows then from a straightforward proof by induction and the asymptotic form of (4.22) results from Stirling's formula [7]. The result of (4.24) is a direct consequence of (4.23) which states that $u_k \leq v_k$ for $k = 1, 2, \dots, m$.

□

We will now use these results to assess the radius of convergence of the power series $\sum \beta_m t^m$.

Theorem 4.3. A lower bound for the radius of convergence of $\sum \beta_m t^m$ is

$$LBRC(\alpha, |a_0|) \stackrel{\text{def.}}{=} \frac{1}{4} \left(\max_{k=1,2,\dots,n-1} \left(\frac{S_{k+1}(\alpha, |a_0|) \|M\|}{C_{k-1} (|a_0|)^{1-k/n}} \right)^{1/k} \right)^{-1}. \quad (4.26)$$

Proof. We first define

$$v_1(\alpha, |a_0|) \stackrel{def.}{=} \max_{k=1,2,\dots,n-1} \left(\frac{S_{k+1}(\alpha, |a_0|)}{C_{k-1}\mu^{k-1}} \right)^{1/k}, \quad (4.27)$$

and bear in mind that the μ -convolution $S_q(\alpha, |a_0|)$ of order $n-1$ starts at $q=2$. Therefore $S_2(\alpha, |a_0|)$ is u_1 of Proposition 4.2, and more generally $S_m(\alpha, |a_0|)$ is u_{m-1} . The results of (4.15) and (4.24) then yield for any m

$$w(m) \leq S_m(\alpha, |a_0|) \leq v_1(\alpha, |a_0|)^{m-1} \times \mu^{m-2} \times C_{m-2} \quad (4.28)$$

$$\sim v_1(\alpha, |a_0|)^{m-1} \mu^{m-2} \frac{4^{m-2}}{\sqrt{\pi}(m-2)^{3/2}} \text{ for } m \rightarrow \infty. \quad (4.29)$$

The series $\sum |\beta_m t^m|$ converges if the sequence $\left\{ |\beta_m|^{1/m} |t| \right\}_{m=1,2,\dots}$ can be bounded by a sequence converging to a limit < 1 . Given Eq. (3.27) and the fact that $\rho = |a_0|$, we have

$$|\beta_m|^{1/m} |t| = \rho^{1/n-1/m} |W(m)_1|^{1/m} |t| \quad (4.30)$$

$$\leq \rho^{1/n-1/m} w(m)^{1/m} |t| \leq |t| \rho^{1/n-1/m} (v_1(\alpha, |a_0|)^{m-1} \mu^{m-2} C_{m-2})^{1/m} \quad (4.31)$$

$$\sim \frac{4|t| \times \|M\| \times v_1(\alpha, |a_0|)}{(|a_0|)^{1-1/n}} = 4|t| \max_{k=1,2,\dots,n-1} \left(\frac{S_{k+1}(\alpha, |a_0|) \|M\|}{C_{k-1} (|a_0|)^{1-k/n}} \right)^{1/k} \text{ for } m \rightarrow \infty \quad (4.32)$$

which yields the desired result of (4.26). □

We now know that $\sum \beta_m t^m$ has a positive radius of convergence which we call $RC(a)$ to emphasize its dependence on the vector $a = (a_k)$ of coefficients. We do not have an analytical expression for $RC(a)$ but we do have the lower bound $LBRC(\alpha, |a_0|)$.

For any $|t| < RC(a)$ we may then define

$$c \stackrel{def.}{=} \frac{t/RC(a) + 1}{2} < 1 \quad (4.33)$$

in which case there exists $A > 0$ such that

$$|b_m t^m| = |K(m, B_m^1) t^m| = |\beta_m| |t|^m \leq A c^m, \quad m = 1, 2, \dots \quad (4.34)$$

In order to show that when the series $\sum x(t, k) = \sum \beta_m e^{2k\pi m \times i/n} t^m$ converge they provide the n roots of Equation (2.2), we need to prove a result on the growth of the $K(d, B_{q+1}^p)$'s.

Lemma 4.4. *If $|t| < RC(a)$ then there exists $A > 0$ and c ($0 < c < 1$) such that*

$$|K(d, B_{q+1}^p)| \leq A^p(q+1)^{p-1}(c/|t|)^d, \quad \forall d \geq p \geq 1, \quad q \geq 0. \quad (4.35)$$

Proof. We prove the result by induction on p . Equation (4.34) shows that if $d \leq q+1$ then

$$|K(d, B_{q+1}^1)| = |\beta_d| \leq A(c/|t|)^d \quad (4.36)$$

which proves (4.35) for $p = 1$; (4.35) is trivially true if $d > q+1$ since then $K(d, B_{q+1}^1) = 0$. For $p = 2$, Eq. (2.10) yields

$$|K(d, B_{q+1}^2)| = \left| \sum_{m=1}^{\min(d-1, q+1)} b_m K(d-m, B_{q+1}^1) \right| \leq \quad (4.37)$$

$$\sum_{m=1}^{q+1} |b_m| |K(d-m, B_{q+1}^1)| \leq \sum_{m=1}^{q+1} A(c/|t|)^m A(c/|t|)^{d-m} = A^2(q+1)(c/|t|)^d. \quad (4.38)$$

We now assume that (4.35) is true up to order $p < d$ and calculate $|K(d, B_{q+1}^{p+1})|$. Equation (2.10) yields

$$|K(d, B_{q+1}^{p+1})| = \left| \sum_{m=1}^{\min(d-p, q+1)} b_m K(d-m, B_{q+1}^p) \right| \leq \sum_{m=1}^{q+1} |b_m| |K(d-m, B_{q+1}^p)| \quad (4.39)$$

$$\leq \sum_{m=1}^{q+1} A(c/|t|)^m A^p(q+1)^{p-1}(c/|t|)^{d-m} = A^{p+1}(q+1)^p(c/|t|)^d \quad (4.40)$$

which is the desired result. □

In the next section we bring together previous results and prove that when it converges, the power series $x(t, k)$ provides the roots of Equation (2.2).

5 Main result

Theorem 5.1. *We consider the following polynomial equation, parameterized by $t > 0$ and with $a_0 = \rho e^{i\theta} \neq 0$:*

$$x^n = (a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1x + a_0)t^n. \quad (5.1)$$

We define $\alpha \stackrel{\text{def.}}{=} \max_{k=1,2,\dots,n-1} |a_k|$ and recall the definition of the matrix M of (3.19). We then define $n-1$ numbers $S_k(\alpha, |a_0|)$ recursively as follows:

$$S_2(\alpha, |a_0|) \stackrel{\text{def.}}{=} \alpha + \alpha/\|M\|, \quad (5.2)$$

$$S_q(\alpha, |a_0|) \stackrel{\text{def.}}{=} \frac{1}{|a_0|} \left(\sum_{p=1}^{q-2} [\|M\| S_{p+1}(\alpha, |a_0|) + \alpha] S_{q-p}(\alpha, |a_0|) \right) + \alpha + \alpha/\|M\|, q = 3, 4, \dots, n \quad (5.3)$$

We also define recursively the infinite sequence of $(n-1)$ dimensional vectors $W(q)$:

$$W(1) = a_0(1 \ 1 \ \dots \ 1)' ; W(2) = \frac{a_1}{n}(1 \ 2 \ \dots \ n-1)' \quad (5.4)$$

$$W(q) = \frac{1}{a_0} \left[-M \sum_{p=1}^{q-2} [u \cdot W(p+1)] W(q-p) + U \sum_{p=1}^{\min(n-1, q-1)} [A_p \cdot W(q-p)] \right], q = 3, 4, \dots \quad (5.5)$$

where:

1. u is the $(n-1)$ -dimensional row vector having 1 in first position and zeros elsewhere.
2. U is the $(n-1)$ -dimensional column vector $\frac{1}{n}(1 \ 2 \ \dots \ n-1)'$.
3. Each A_p ($p=1, 2, \dots, n-1$) is the $(n-1)$ -dimensional row vector with a_p in p -th position and zeros elsewhere.

If we define the sequence

$$\beta_m = \rho^{m/n} \times e^{i(m\theta)/n} \times W(m)_1/a_0, \quad m = 1, 2, \dots \quad (5.6)$$

then $\sum |\beta_m t^m|$ converges for $|t| < RC(a)$ and in particular for

$$|t| < LBRC(\alpha, |a_0|) \stackrel{\text{def.}}{=} \frac{1}{4} \left(\max_{k=1, 2, \dots, n-1} \left(\frac{S_{k+1}(\alpha, |a_0|) \|M\|}{C_{k-1}(|a_0|)^{1-k/n}} \right)^{1/k} \right)^{-1}. \quad (5.7)$$

If $|t| < RC(a)$ the n convergent series

$$x(t, k) = \sum_{m=1}^{\infty} \beta_m e^{2k\pi m \times i/n} \times t^m, \quad k = 0, 1, \dots, n-1 \quad (5.8)$$

are the n roots of Eq. (5.1) in the sense that for each $k = 0, 1, \dots, n-1$ the expression

$$E(x(t, k)_q) \stackrel{\text{def.}}{=} [x(t, k)_q]^n - \left(a_{n-1} [x(t, k)_q]^{n-1} + a_{n-2} [x(t, k)_q]^{n-2} + \dots + a_1 x(t, k)_q + a_0 \right) t^n \quad (5.9)$$

approaches 0 for $q \rightarrow \infty$. (The $x(t, k)_q$'s are the partial sums up to order q (Eq. (2.3)).

Proof. Given Eq. (2.17) the polynomial in t on the right-hand side of (5.9) only has terms of the form t^{n+q+s} with $s \geq 0$. We will first show that $[x(t, k)_q]^n$ contributes a series of such terms that approaches 0 when $q \rightarrow \infty$. Indeed, (4.35) shows that the modulus of

the contribution of $\left[x(t, k)_q\right]^n$ to the terms t^{n+q+s} is

$$\left| \sum_{s=0}^{q(n-1)-n} t^{n+q+s} K(n+q+s, B_q^n) \right| \leq \sum_{s=0}^{\infty} |t|^{n+q+s} A^n q^{n-1} (c/|t|)^{n+q+s} = \quad (5.10)$$

$$A^n q^{n-1} c^{n+q} \sum_{s=0}^{\infty} c^s \quad (5.11)$$

which approaches 0 when $q \rightarrow \infty$.

We similarly consider the modulus of the contribution of each $\left[x(t, k)_q\right]^p$ of (5.9) (for $1 \leq p \leq n-1$) to the series of terms t^{n+q+s} . Given the term t^n appearing on the right-hand side of (5.9), this modulus is

$$\left| \sum_{s=0}^{q(p-1)} t^{q+s} K(q+s, B_q^p) \right| \leq \sum_{s=0}^{\infty} |t|^{q+s} A^p q^{p-1} (c/|t|)^{q+s} = \quad (5.12)$$

$$A^p q^{p-1} c^q \sum_{s=0}^{\infty} c^s \quad (5.13)$$

which also approaches 0 when $q \rightarrow \infty$. (Because $a_0 = b_1^n$ the term $a_0 t^n$ on the right-hand side of (5.9) cancels out with the term $b_1^n t^n$ appearing in the expansion of $\left[x(t, k)_q\right]^n$.) This completes the proof since each one of the $\left[x(t, k)_q\right]^p$ terms on the right-hand side of $E\left(x(t, k)_q\right)$ approaches 0 when $q \rightarrow \infty$.

□

Remarks:

1. The functions $x(t, k)$ of Eq. (5.8) can be viewed as Taylor expansions in the variable t of the roots of Equation (5.1): when t is small the first few terms of the series provide approximate values for the roots (a numerical example is given below).

2. In view of (5.6) the radius of convergence $RC(a)$ of $x(t, u)$ as a power series in t can be calculated numerically as

$$RC(a) = \lim_{m \rightarrow \infty} \inf Q(m) \quad (5.14)$$

where

$$Q(m) \stackrel{def.}{=} \frac{1}{(|a_0|)^{1/n} (|W(m)_1|)^{1/m}} \quad (5.15)$$

3. Each term $S_q(\alpha, |a_0|)$ is a polynomial in α with no constant term and positive coefficients. This insures that $LBRC(\alpha, |a_0|)$ tends to infinity for $\alpha \rightarrow 0$. Therefore

(5.7) is satisfied for α small enough which shows that the functions $x(t, u)$ provide the roots when the coefficients a_k other than a_0 are small enough.

4. Each $S_q(\alpha, |a_0|)$ is a decreasing function of $|a_0|$ that approaches $\alpha + \alpha/\|M\|$ when $|a_0| \rightarrow \infty$. With $(|a_0|)^{1-k/n}$ in the denominator of (5.7), the function $LBRC(\alpha, |a_0|)$ tends to infinity for $|a_0| \rightarrow \infty$. Therefore the functions $x(t, u)$ provide the roots when $|a_0|$ is large enough.

As a numerical illustration we consider the equation

$$x^6 = (-x^5 + x^4 - 2x^3 - 3x^2 + 2x + 8)t^6 \quad (5.16)$$

with t set equal to 1. The lower bound $LBRC(\alpha, |a_0|)$ for the radius of convergence of the power series $x(t, u)$ is 0.094. Therefore we do not know whether $x(t, u)$ converges with $t = 1$. However $LBRC(\alpha, |a_0|)$ is an extremely conservative bound calculated only with $\alpha, |a_0|$ and $\|M\|$; $LBRC(\alpha, |a_0|)$ thus represents a worst-case scenario on the coefficients a_k . For these reasons the condition $|t| < LBRC(\alpha, |a_0|)$ of (5.7) is probably of limited practical use. Its usefulness lies in the qualitative Remarks 3 and 4 above, which state that the function $x(t, u)$ does provide the roots for α small enough or $|a_0|$ large enough.

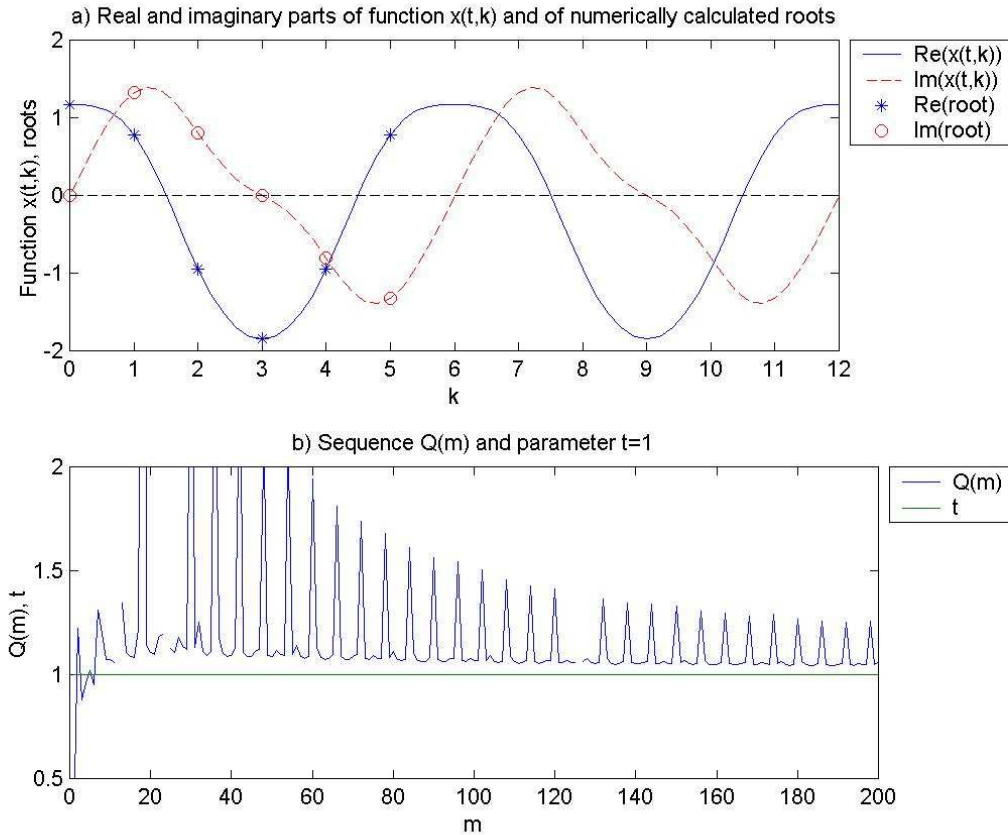


Figure 5.1: Function $x(t,k)$ (with $t=1$) and sequence $Q(m)$

From Figure 5.1b the radius of convergence $RC(a)$ assessed numerically as the limit of $Q(m)$ appears to be about 1.05. Therefore the power series $x(t, u)$ does converge with $t = 1$. (A few missing values in the $Q(m)$ sequence arise when $W(m)_1 = 0$, i.e. $Q(m) = \infty$.)

The real and imaginary parts of $x(t, k)$ are plotted in Figure 5.1a over two periods of length $n = 6$ (200 terms are used to calculate the series). These two parts taken at $t = 1$ and $k = 0, 1, 2, 3, 4, 5$ coincide with those obtained using Matlab's built-in polynomial equation routine to solve Eq. (5.16) with $t = 1$ (stars and circles in Figure 5.1a).

With real coefficients for the polynomial equation the complex roots come in conjugate pairs. There are two such pairs. One for $k = 1, k = 5$ and the other for $k = 2, k = 4$. In addition there are two real roots at $k = 0$ and at $k = 3$. In this and some other cases these real roots appear to be at local minima or maxima of the $Re(x(t, k))$ function. These special behaviors of and relationships between the real and imaginary parts no doubt arise from particular patterns in the sequence $\{W(q)_1\}$ which have yet to be explored. These structures disappear with complex coefficients since in this case there are no more complex conjugate roots.

The first five terms of the series $x(t, u)$ are

$$x(t, u) \approx \sqrt{2}e^{2u\pi \times i/6}t + 0.08333e^{4u\pi \times i/6}t^2 - 0.18414e^{6u\pi \times i/6}t^3 - 0.14506e^{8u\pi \times i/6}t^4 + 0.11441e^{10u\pi \times i/6}t^5. \quad (5.17)$$

There are not enough terms to calculate the roots when $t = 1$. With $t = 0.4$ however, the values provided by (5.17) are very close to those calculated numerically (Table 5.1).

Table 5.1: $x(t, k)$ and numerically calculated roots of Equation (5.16) with $t=0.4$

	k=0	k=1	k=2	k=3	k=4	k=5
$x(0.4, k)$	0.565	0.290+0.504i	-0.300+0.474i	-0.545	-0.300-0.474i	0.290-0.504i
num. root	0.564	0.290+0.504i	-0.301+0.474i	-0.546	-0.301-0.474i	0.290-0.504i

If the constant term $a_0 = 8$ in Eq. (5.16) is less than approximately 4, the radius of convergence for the series $x(t, u)$ drops below 1. The method can no longer be used to solve Eq. (5.16) with $t = 1$. Several approaches have been tried in order to extend the method to the case a_0 small in which at least one root becomes close to 0. One possibility would be to transform the unknown in such a way that small roots are moved away from 0. For example one could write the polynomial equation in terms of a changed unknown $y = 1/x$: if a root x is small then the corresponding y is large. However this approach did not change the problem. Another more promising possibility would be to inject the parameter t differently into the equation. For example one could use a similar approach after multiplying the left side of Eq. (2.1) by t^n instead of the right side. Or one could

multiply each term of the equation by a power t^p with a different and well-chosen p for each term. To date such attempts have proved largely inconclusive.

Ours is only a first step, which shows that the roots of a particular class of polynomial equations can be expressed explicitly with an infinite number of rational operations and root extractions. It is to be hoped that some variant of the family of periodic functions $x(t, u)$ will eventually emerge to provide closed-form expressions for the roots of arbitrary polynomial equations.

REFERENCES

- [1] N.H. Abel. "Beweis der Unmöglichkeit, algebraische Gleichungen von höheren Graden als dem vierten allgemein aufzulösen." *J. reine angew. Math.* 1, 65, 1826. Reprinted in Abel, N. H. *Oeuvres Completes* (Ed. L. Sylow and S. Lie). New York: Johnson Reprint Corp., pp. 66-87, 1988.
- [2] G. Belardinelli, Fonctions hypergéométriques de plusieurs variables et résolution analytique des équations algébrique générales. *Mémoire des Sci. Math.* 145, 1960.
- [3] R. Birkeland. Résolution de l'équation algébrique générale par des fonctions hypergéométriques supérieures, *Comp. Rend.* 171, 778 ,1921.
- [4] Eisenstein, G. Allgemeine Auflösung der Gleichungen von den ersten vier Graden, *J. reine angew. Math.* 27, 81, 1844.
- [5] P. L. Chebychev, Calculations of the roots of an equation (in Russian), in P. L. Chebychev: *Collected Works*, v. V, Akademii Nauk SSSR, Moscow, 1951.
- [6] P. A. Lambert, On the solution of algebraic equations in infinite series, *Bull. Am. Math. Soc.* XIV, 467, 1908.
- [7] Weisstein E.W. *et al.* "Catalan Number." From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/CatalanNumber.html>.