HURWITZ ACTION ON TUPLES OF EUCLIDEAN REFLECTIONS

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This note was prompted by the reading of [4], which purports to show that if an \( n \)-tuple of Euclidean reflections has a finite orbit under the Hurwitz action of the braid group, then the generated group is finite. I noticed that the proof given is fatally flawed \(^1\); however, using the argument of Vinberg given in [3], I found a short (hopefully correct) proof which at the same time considerably simplifies the computational argument given in [3]. This is what I expound below. I first recall all the necessary notation and assumptions, expounding some facts in slightly more generality than necessary.

0.1. Hurwitz action.

Definition. Given a group \( G \), we call Hurwitz action the action of the \( n \)-strand braid group \( \mathcal{B}_n \) on \( G^n \) given by

\[
\sigma_i(s_1, \ldots, s_n) = (s_1, \ldots, s_{i-1}, s_{i+1}, s_i s_{i+1}^{-1}, s_{i+2}, \ldots, s_n).
\]

The inverse is given by \( \sigma_i^{-1}(s_1, \ldots, s_n) = (s_1, \ldots, s_{i-1}, s_i s_{i+1}, s_{i+2}, \ldots, s_n) \).

Here \( a^b = b^{-1}ab \) and \( b^a = bab^{-1} \).

This action preserves the product of the \( n \)-tuple. We need to repeat some remarks in [4]. By decreasing induction on \( i \) one sees that \( \sigma_1 \ldots \sigma_n(s_1, \ldots, s_n) = (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \). In particular if \( \gamma = \sigma_1 \ldots \sigma_{n-1} \) we get \( \gamma(s_1, \ldots, s_n) = (s_i, s_{i+1}, \ldots, s_n) \) whence, if \( c = s_1 \ldots s_n \), we get that \( \gamma^n(s_1, \ldots, s_n) = (s_1, \ldots, s_n)^c \).

We also deduce that given any subsequence \((i_1, \ldots, i_k)\) of \((1, \ldots, n)\), there exists an element of the Hurwitz orbit of \((s_1, \ldots, s_n)\) which begins by \((s_{i_1}, \ldots, s_{i_k})\).

Assume now that the Hurwitz orbit of \((s_1, \ldots, s_n)\) is finite. Then some power of \( \gamma \) fixes \((s_1, \ldots, s_n)\), thus some power of \( c \) is central in the subgroup generated by the \( s_i \). Similarly, by looking at the action of \( \sigma_1 \ldots \sigma_{n-1} \) on an element of the orbit beginning by \((s_{i_1}, \ldots, s_{i_k})\) we get that for any subsequence \((i_1, \ldots, i_k)\) of \((1, \ldots, n)\) there exists a power of \( s_{i_1} \ldots s_{i_k} \) central in the subgroup generated by \((s_{i_1}, \ldots, s_{i_k})\).

0.2. Reflections. Let \( V \) be a vector space on some subfield \( K \) of \( \mathbb{C} \). We call complex reflection a finite order element \( s \in \text{GL}(V) \) whose fixed points are a hyperplane. If \( \zeta \) (a root of unity) is the unique non-trivial eigenvalue of \( s \), the action of \( s \) can be written \( s(x) = x - \bar{r}(x)r \) where \( r \in V \) and \( \bar{r} \) is an element of the dual of \( V \) satisfying

\[ x = \bar{x} = -x \text{ for any } x \in V. \]

\(^{1}\)The problem is in proposition 2.3, which is essential to the main theorem (1.1) of the paper. The argument given there is basically that if a Coxeter group has a reflection representation where the image of the Coxeter element is of finite order, then the image of that representation is finite. However this is false: the Cartan matrix \( \left( \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right) \) where \( l^2 + l = \sqrt{2} \) defines Euclidean reflections which give a representation of an infinite rank 3 Coxeter group, such that the image of the Coxeter group is infinite but the image of the Coxeter element is of order 8 (personal communication of F.Zara).
These elements are unique up to multiplying \( r \) by a scalar and \( \bar{r} \) by the inverse scalar. We say that \( r \) (resp. \( \bar{r} \)) is a root (resp. coroot) associated to \( s_i \).

### 0.3. Cartan Matrix

If \((s_1, \ldots, s_n)\) is a tuple of complex reflections and if \(r_i, \bar{r}_i\) are corresponding roots and coroots, we call Cartan matrix the matrix \(C = \{\bar{r}_i(r_j)\}\). This matrix is unique up to conjugating by a diagonal matrix. Conversely, a class modulo the action of diagonal matrices of Cartan matrices is an invariant of the \(\text{GL}(V)\)-conjugacy class of the tuple. It determines this class if it is invertible and \(n = \dim V\). Indeed, this implies that the \(r_i\) form a basis of \(V\); and in this basis the matrix \(s_i\) differs from the identity matrix only on the \(i\)-th line, where the opposed of the \(i\)-th line of \(C\) has been added; thus \(C\) determines the \(s_i\).

If \(C\) can be chosen Hermitian (resp. symmetric), such a choice is then unique up to conjugating by a diagonal matrix of norm 1 elements of \(K\) (resp. of signs).

If \(C\) is Hermitian (which implies that the \(s_i\) are of order 2), then the sesquilinear form given by \(t^* C\) is invariant by the \(s_i\) (if the \(s_i\) are not of order 2, but the matrix obtained by replacing all elements on the diagonal of \(t^* C\) by 2's is Hermitian, then the latter matrix defines a sesquilinear form invariant by the \(s_i\)).

### 0.4. Coxeter element

We keep the notation as above and we assume that the \(r_i\) form a basis of \(V\). We recall a result of \([2]\) on the “Coxeter” element \(c = s_1 \ldots s_k\). If we write \(C = U + V\) where \(U\) is upper triangular unipotent and where \(V\) is lower triangular (with diagonal terms \(-\zeta_i\), thus \(V\) is also unipotent when \(s_i\) are of order 2), then the matrix of \(c\) in the \(r_i\) basis is \(-U^{-1}V\) (to see this write it as \(U s_1 \ldots s_k = -V\) and look at partial products in the left-hand side starting from the left). As \(U\) is of determinant 1, we deduce that \(\chi(c) = \det(xI + U^{-1}V) = \det(xU + V)\) where \(\chi(c)\) denotes the characteristic polynomial: in particular \(\det(C) = \chi(c)|_{x=1}\); one also gets that the fix-point set of \(c\) is the kernel of \(C\), equal to the intersection of the reflecting hyperplanes.

### 0.5. The main theorem

The next theorem implies the statement given in \([2]\) \(([4], 1.1]\) considers Euclidean reflections with the \(r_i\) linearly independent; if the \(r_i\) are chosen of the same length this implies that \(C\) is symmetric, and as \(C\) is then the Gram matrix of the \(r_i\) it is invertible:

**Theorem.** Let \((s_1, \ldots, s_n)\) be a tuple of reflections in \(\text{GL}(\mathbb{R}^n)\) which have an associated Cartan matrix symmetric and invertible. Assume in addition that the Hurwitz orbit of the tuple is finite. Then the group generated by the \(s_i\) is finite.

**Proof.** In the next paragraph, we just need that \((s_1, \ldots, s_n)\) is a tuple of complex reflections with a finite Hurwitz orbit and with the \(r_i\) a basis of \(V\).

A straightforward computation shows that an element of \(\text{GL}(V)\) commutes to the \(s_i\), if and only if it acts as a scalar on the subspaces generated by \(\{r_i\}_{i \in I}\) where \(I\) is a block of \(C\) (i.e., a connected component of the graph with vertices \(\{1, \ldots, n\}\) and edges \((i, j)\) for each pair such that either \(C_{i,j}\) or \(C_{j,i}\) is not zero). The finiteness of the Hurwitz orbit implies that for any subsequence \((i_1, \ldots, i_k)\) of \(\{1, \ldots, n\}\), there exists a power of \(s_{i_1} \ldots s_{i_k}\) which commutes to \(s_{i_1}, \ldots, s_{i_k}\). This power acts thus as a scalar on each subspace generated by the \(r_{i_1}\) in a block of the submatrix of \(C\) determined by \((i_1, \ldots, i_k)\). As the determinant of each \(s_i\) on this subspace is a root of unity, the scalar must be a root of unity. Thus, the restriction of each \(s_{i_1} \ldots s_{i_k}\) to the subspace \(\langle r_{i_1}, \ldots, r_{i_k} \rangle\) generated by the \(r_{i_j}\) is of finite order.
We use from now on all the assumptions of the theorem. Thus the $s_i$ are order 2 elements of $O(C)$, the orthogonal group of the quadratic form defined by $C$.

Also, $\chi(c)$ is a polynomial with real coefficients. As $c$ is of finite order, any real root of $\chi(c)$ is 1 or $-1$. This implies that $\chi(c)|_{x=1}$ is a nonnegative real number, and thus $\det C$ also. The same holds for any principal minor of $C$, since such a minor is $\chi(c')|_{x=1}$ where $c'$ is the restriction of some $s_{i_1} \ldots s_{i_k}$ to $<r_{i_1}, \ldots, r_{i_k}>$. The quadratic form defined by $C$ is thus positive, and as $\det C \neq 0$ it is positive definite (cf. §7, exercice 2).

We now digress about the Cartan matrix of two reflections $s_1$ et $s_2$. Such a matrix is of the form $\begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix}$. If $a = 0$ and $b \neq 0$ or $a \neq 0$ and $b = 0$ then $s_1 s_2$ is of infinite order. Otherwise, the number $ab$ is a complete invariant of the conjugacy class of $(s_1, s_2)$ restricted to $<r_1, r_2>$, and $s_1 s_2$ restricted to this subspace is of finite order $m$ if and only if there exists $k$ prime to $m$ such that $ab = 4 \cos^2 k\pi/m$.

Since $C$ is symmetric and since the restriction of $s_i s_j$ to $<r_i, r_j>$ is of finite order, there exists prime integer pairs $(k_{i,j}, m_{i,j})$ such that $C_{i,j} = \pm 2 \cos k_{i,j} \pi/m_{i,j}$. If $K$ is the cyclotomic subfield containing the lcm$(2m_{i,j})$-th roots of unity, and if $O$ is the ring of integers of $K$, we get that all coefficients of $C$ lie in $O$. It follows, if $G$ is the group generated by the $s_i$, that in the $r_i$ basis we have $G \subset GL(O^n)$.

We now apply Vinberg’s argument as in [3, 1.4.2]. Let $\sigma \in \text{Gal}(K/\mathbb{Q})$. Then $\sigma(C)$ is again positive definite: all arguments used to prove that $C$ is positive definite still apply for $\sigma(C)$: it is real, symmetric, invertible and the Hurwitz orbit of $(\sigma(s_1), \ldots, \sigma(s_n))$ is still finite. Since $G \subset O(C)$, which is compact, the entries of the elements of $G$ in the $r_i$ basis are of bounded norm. Since $O(\sigma(C))$ is also compact for any $\sigma \in \text{Gal}(K/\mathbb{Q})$, we get that entries of elements of $G$ are elements of $O$ all of whose complex conjugates have a bounded norm. There is a finite number of such elements, so $G$ is finite.

\[ \square \]

References