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QUASI-POISSON ACTIONS AND MASSIVE NON-ROTATING BTZ BLACK HOLES

SÉBASTIEN RACANIÈRE

Abstract. Using ideas from an article of P. Bieliavsky, M. Rooman and Ph. Spindel on BTZ black holes, I construct a family of interesting examples of quasi-Poisson actions as defined by A. Alekseev and Y. Kosmann-Schwarzbach. As an application, I obtain a genuine Poisson structure on SL(2, \mathbb{R}) which induces a Poisson structure on a BTZ black hole.

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I would like to thank Pierre Bieliavsky for advising me to look at the article [4] and then answering my questions about it. Thank you to Stéphane Detournay for talking to me about physics in a way I could understand and for answering many questions I had about [3]. And finally, a big thanks to David Iglesias-Ponte for interesting conversations about [1].

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1. Introduction

In [4], P. Bieliavsky, M. Rooman and Ph. Spindel construct a Poisson structure on massive non-rotating BTZ black holes; in [3], P. Bieliavsky, S. Detournay, Ph. Spindel and M. Rooman construct a star product on the same black hole. The direction of this deformation is a Poisson bivector field which has the same symplectic leaves as the Poisson bivector field of [4]: roughly speaking, they correspond to orbits under a certain twisted action by conjugation.

In the present paper, I wish to show how techniques used in [4] in conjunction with techniques of the theory of quasi-Poisson manifolds (see [1] and [2]) can be used to construct an interesting family of manifolds with a quasi-Poisson action and how a particular case of this family leads to a genuine Poisson structure on a massive non-rotating BTZ black hole with similar symplectic leaves as in [4] and [3].

2. Main results

I will not recall here the basic definitions in the theory of quasi-Poisson manifolds and quasi-Poisson actions. The reader will find these definitions in A. Alekseev and Y. Kosmann-Schwarzbach [1], and in A. Alekseev, Y. Kosmann-Schwarzbach and E. Meinrenken [2].

Let \( G \) be a connected Lie group of dimension \( n \) and \( \mathfrak{g} \) its Lie algebra, on which \( G \) acts by the adjoint action \( \text{Ad} \). Assume we are given an \( \text{Ad} \)-invariant non-degenerate bilinear form \( K \) on \( \mathfrak{g} \). For example, if \( G \) is semi-simple, then \( K \) could be the Killing form. In the following, I will denote by \( K \) again the linear isomorphism

\[
\mathfrak{g} \longrightarrow \mathfrak{g}^* \quad \quad \quad \quad \quad \mathfrak{g} \quad \longrightarrow \quad K(\cdot, \cdot).
\]
Let \( D = G \times G \) and \( \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g} \) its Lie algebra. Define an \( \text{Ad} \)-invariant non-degenerate bilinear form \( \langle \cdot , \cdot \rangle \) of signature \( (n,n) \) by
\[
\mathfrak{d} \times \mathfrak{d} = (\mathfrak{g} \oplus \mathfrak{g}) \times (\mathfrak{g} \oplus \mathfrak{g}) \rightarrow \mathbb{R}
\]
\[
\langle (x,y), (x',y') \rangle \mapsto K(x,x') - K(y,y').
\]
Assume there is an involution \( \sigma \) on \( G \) which induces an orthogonal involutive morphism, again denoted by \( \sigma \), on \( \mathfrak{g} \). Let \( \Delta_+ : G \rightarrow D \) and \( \Delta_+^\sigma : G \rightarrow D \) be given by
\[
\Delta_+ (g) = (g,g)
\]
and
\[
\Delta_+^\sigma (g) = (g, \sigma(g)).
\]
Denote by \( G_+ \) and \( G_+^\sigma \) their respective images in \( D \). Let \( S = D/G_+ \) and \( S^\sigma = D/G_+^\sigma \). Then both \( S \) and \( S^\sigma \) are isomorphic to \( G \). The isomorphism between \( S \) and \( G \) is induced by the map
\[
D \rightarrow G
\]
\[
(g,h) \mapsto gh^{-1},
\]
whereas the isomorphism between \( S^\sigma \) and \( G \) is induced by
\[
D \rightarrow G
\]
\[
(g,h) \mapsto g\sigma(h)^{-1}.
\]
I will use these two isomorphisms to identify \( S \) and \( G \), and \( S^\sigma \) and \( G \). Denote again by \( \Delta_+ : \mathfrak{g} \rightarrow \mathfrak{d} \) and \( \Delta_+^\sigma : \mathfrak{g} \rightarrow \mathfrak{d} \) the morphisms induced by \( \Delta_+ : G \rightarrow D \) and \( \Delta_+^\sigma : G \rightarrow D \) respectively. Let \( \Delta_- : \mathfrak{g} \rightarrow \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g} \) and \( \Delta_-^\sigma : \mathfrak{g} \rightarrow \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g} \) be defined by
\[
\Delta_-(x) = (x,-x),
\]
and
\[
\Delta_-^\sigma (x) = (x,-\sigma(x)).
\]
Let \( \mathfrak{g}_- = \text{Im}(\Delta_-) \) and \( \mathfrak{g}_-^\sigma = \text{Im}(\Delta_-^\sigma) \). We have two quasi-triples \( (D, G_+, \mathfrak{g}_-) \) and \( (D, G_+^\sigma, \mathfrak{g}_-^\sigma) \). They induce two structures of quasi-Poisson Lie group on \( D \), of respective bivector fields \( P_D \) and \( P_D^\sigma \), and two structures of quasi-Poisson Lie group on \( G_+ \) and \( G_+^\sigma \) of respective bivector fields \( P_{G_+} \) and \( P_{G_+}^\sigma \). I will simply write \( G_+ \), respectively \( G_+^\sigma \), to denote the group together with its quasi-Poisson structure. Of course, these quasi-Poisson structures are pairwise isomorphic. More precisely, the isomorphism \( \text{Id} \times \sigma : (g,h) \mapsto (g,\sigma(h)) \) of \( D \) sends \( P_D \) on \( P_D^\sigma \) and vice-versa. This isomorphism can be used to deduce some of the results given at the beginning of the present article from the results of Alekseev and Kosmann-Schwarzbach [1]; but it takes just as long to redo the computations, and that is what I do here.

According to [1], the bivector field \( P_D \), respectively \( P_D^\sigma \), is projectable onto \( S \), respectively \( S^\sigma \). Let \( P_S \) and \( P_{S^\sigma} \) be their respective projections. Using the identifications between \( S \) and \( G \), and \( S^\sigma \) and \( G \), one can check that \( P_S \) and \( P_{S^\sigma} \) are the same bivector fields on \( G \). What is more interesting, and what I will prove, is the following Theorem.

**Theorem 2.1.** The bivector field \( P_D^\sigma \) is projectable onto \( S \). Let \( P_S^\sigma \) be its projection. Identify \( S \) with \( G \) and trivialise their tangent space using right translations, then for \( s \) in \( S \) and \( \xi \) in \( \mathfrak{g}^* \simeq T^*_xS \) there is the following explicit formula
\[
P_S^\sigma(s)(\xi) = \frac{1}{2}(\text{Ad}_{\sigma(s)})^{-1} - \text{Ad}_s \circ \sigma \circ K^{-1}(\xi).
\]
Moreover, the action
\[
G_+^\sigma \times S \rightarrow S
\]
\[
(g, s) \mapsto g s \sigma(g)^{-1}
\]
of $G^\sigma_+ \times (S, P^\sigma_+)$ is quasi-Poisson in the sense of Alekseev and Kosmann-Schwarzbach \cite{1}. The image of $P^\sigma_+(s)$, seen as a map $T^*_s S \rightarrow T_s S$, is tangent to the orbit through $s$ of the action of $G^\sigma$ on $S$.

In the setting of the above Theorem, the bivector field $P^\sigma_+$ is $G^\sigma$ invariant; hence if $F$ is a subgroup of $G^\sigma$ and $I$ is an $F$-invariant open subset of $S$ such that the action of $F$ on $I$ is principal then $F \setminus I$ is a smooth manifold and $P^\sigma_+$ descends to a bivector field on it. An application of this remark is the following Theorem.

**Theorem 2.2.** Let $G = \text{SL}(2, \mathbb{R})$. Let

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and choose $\sigma = \text{Ad}_H$. Let

$$I = \{ \begin{bmatrix} u + x & y + t \\ y - t & u - x \end{bmatrix} \mid u^2 - x^2 - y^2 + t^2 = 1, t^2 - y^2 > 0 \}$$

be an open subset of $S$. Let $F$ be the following subgroup of $G$

$$F = \{ \exp(n \pi H), n \in \mathbb{N} \}.$$ 

The quotient $F \setminus I$ (together with an appropriate metric) is a model of massive non-rotating BTZ black hole (see \cite{4}). The bivector field it inherits following the above remark, is Poisson. Its symplectic leaves consist of the projection to $F \setminus I$ of the orbits of the action of $G^\sigma$ on $S$ except along the projection of the orbit of the identity. Along this orbit, the bivector field vanishes and each point forms a symplectic leaf.

In the coordinates (46) of \cite{4} (or (3) of the present article), the Poisson bivector field is

$$2 \cosh^2 \left( \frac{\rho}{2} \right) \sin(\tau) \sinh(\rho) \partial_\tau \wedge \partial_\theta.$$

The above Poisson bivector field should be compared with the one defined in \cite{4} and given by

$$\frac{1}{\cosh^2 \left( \frac{\rho}{2} \right) \sin(\tau)} \partial_\tau \wedge \partial_\theta.$$

The symplectic leaves of this Poisson structure are the images under the projection $I \rightarrow F \setminus I$ of the action of $G^\sigma_+$ on $S$.

3. LET THE COMPUTATIONS BEGIN

Throughout the present article, I will use the notations introduced in the previous Section. To begin with, I will prove that $(D, G^\sigma_+, g^\sigma_+)$ does indeed form a quasi-triple.

Because $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$, one also has a decomposition $\mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}^*$. One also has $\mathfrak{d} = \mathfrak{g}^+_+ \oplus \mathfrak{g}^-_-$ and accordingly $\mathfrak{d}^* = \mathfrak{g}^*_+ \oplus \mathfrak{g}^*_-$.

Denote $p_{\mathfrak{g}^+_+}$ and $p_{\mathfrak{g}^-_-}$ the projections on respectively $\mathfrak{g}^*_+$ and $\mathfrak{g}^*_-$. So that $1_\mathfrak{g} = p_{\mathfrak{g}^+_+} + p_{\mathfrak{g}^-_-}$.

In this article, I express results using mostly the decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$. Using it, we have

$$\mathfrak{g}^*_+ = \{ (\xi, \xi \circ \sigma) \mid \xi \in \mathfrak{g}^* \}$$

and

$$\mathfrak{g}^*_- = \{ (\xi, -\xi \circ \sigma) \mid \xi \in \mathfrak{g}^* \}.$$
Finally, the r-matrix is defined as

\[ \xi, \eta \in \mathfrak{g}^* \]

and finally the r-matrix

\[ r^\sigma : \mathfrak{g}^* \oplus \mathfrak{g}^* \rightarrow \mathfrak{g} \oplus \mathfrak{g} \]

\( (\xi, \eta) \mapsto \frac{1}{2} \Delta^\sigma \circ K^{-1}(\xi + \eta \circ \sigma) \).

\textbf{Proof.} It is straightforward to prove that \( \mathfrak{d} = \mathfrak{g}^+_\sigma \oplus \mathfrak{g}^- \) and that both \( \mathfrak{g}^+ \) and \( \mathfrak{g}^- \) are isotropic in \( (\mathfrak{d}, \langle \cdot, \cdot \rangle) \). This proves that \( (D, G^\sigma, \mathfrak{g}^\sigma) \) is a quasi-triple.

For \( (\xi, \eta) \) in \( \mathfrak{g}^+ \) and \( (x, \sigma(x)) \) in \( \mathfrak{g}^\sigma \)

\[ \langle j(\xi, \eta), (x, \sigma(x)) \rangle = \langle \xi, \eta \rangle \circ (x, \sigma(x)) \].

The map \( j \) is actually characterised by this last property. The equality

\[ (\Delta^\sigma \circ K^{-1} \circ \Delta^\sigma_+)(\xi, \eta, \sigma) = (\xi, \xi \circ \rho_{\xi}(\eta)) \]

proves that

\[ j(\xi, \xi \circ \rho_{\xi}(\eta)) = \Delta^\sigma_+ \circ K^{-1} \circ \Delta^\sigma_+(\xi, \xi \circ \rho_{\xi}(\eta)) \]

Since \( \sigma \) is a Lie algebra morphism, we have \([\mathfrak{g}^-, \mathfrak{g}^\sigma] \subset \mathfrak{g}_+^\sigma \). This proves that \( F^\sigma : \bigwedge^2 \mathfrak{g}^{\sigma +} \rightarrow \mathfrak{g}^- \), given by

\[ F^\sigma(\xi, \eta) = p_{\mathfrak{g}^-}[j(\xi), j(\eta)] \]

vanishes.

I will now compute \( \varphi^\sigma \). It is defined as

\[ \varphi^\sigma((\xi, \sigma \circ \xi), (\eta, \sigma \circ \eta), (\nu, \sigma \circ \nu)) = (\nu, \sigma \circ \nu) \circ p_{\mathfrak{g}^+}(j(\xi, \sigma \circ \xi), j(\eta, \sigma \circ \eta)) \]

\[ = \langle j(\nu, \sigma \circ \nu), j(\xi, \sigma \circ \xi), j(\eta, \sigma \circ \eta) \rangle \]

\[ = \langle \Delta^\sigma_+ \circ K^{-1}(\nu), \Delta^\sigma_- \circ K^{-1}(\xi), \Delta^\sigma_- \circ K^{-1}(\eta) \rangle \]

\[ = 2K(K^{-1}(\nu), K^{-1}(\xi), K^{-1}(\eta)) \]

Finally, the r-matrix is defined as

\[ r^\sigma : \mathfrak{g}^+ \oplus \mathfrak{g}^+ \rightarrow \mathfrak{g} \oplus \mathfrak{g} \]

\( ((\xi, \sigma \circ \xi), (\eta, \sigma \circ \eta)) \mapsto (0, j(\xi, \sigma \circ \eta)) \).

If \( (\xi, \eta) \) is in \( \mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}^* \) then its decomposition in \( \mathfrak{g}^+_\sigma \oplus \mathfrak{g}^- \) is \( \frac{1}{2}((\xi + \eta \circ \sigma, \xi \circ \sigma + \eta), (\xi - \eta \circ \sigma, -\xi \circ \sigma + \eta)) \). The result follows.

I now wish to compute the bivector \( P^\sigma_D \) on \( D \). By definition, it is equal to \( (r^\sigma)_{\lambda} - (r^\sigma)_{\rho} \), where the upper script \( \lambda \) means the left invariant section of \( \Gamma(TD \otimes TD) \) generated by \( r^\sigma \), while the upper script \( \rho \) means the right invariant section of \( \Gamma(TD \otimes TD) \) generated by \( r^\sigma \).

\textbf{Proposition 3.2.} Identify \( T_dD \) to \( \mathfrak{d} \) by right translations. The value of \( P^\sigma_D \) at \( d = (a, b) \) is

\[ \mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}^* \]

\( (\xi, \eta) \rightarrow \frac{1}{2}(K^{-1}(\eta \circ \sigma \circ (\text{Ad}_{\sigma(b)} - 1)), -K^{-1}(\xi \circ \sigma \circ (\text{Ad}_{\sigma(a)} - 1))) \).
Proof. Fix \(d = (a, b)\) in \(D\). I choose to trivialise the tangent bundle, and its dual, of \(D\) by using right translations. See \((r^*_p)^\alpha\) as a map from \(T^*D\) to \(TD\). If \(\alpha\) is in \(\mathfrak{d}^*\), then
\[
(r^*_p)^\alpha(d)(\alpha^\rho) = (r^*_p(\alpha))^\rho(d),
\]
whereas
\[
(r^*_p)^\lambda(d)(\alpha^\rho) = (\text{Ad}_d \circ r^*_p(\alpha \circ \text{Ad}_d))^\rho(d).
\]
Thus \(P^*_D\) at the point \(d = (a, b)\) is
\[
\mathfrak{d}^* = g^* \oplus g^* \quad \longrightarrow \quad \mathfrak{d} = g \oplus g
\]
\[
(\xi, \eta) \quad \longrightarrow \quad -\frac{1}{2} \Delta_\pm \circ K^{-1}(\xi + \eta \circ \sigma) + \frac{1}{2} \text{Ad}_d \circ \Delta_\pm \circ K^{-1}(\xi \circ \text{Ad}_a + \eta \circ \text{Ad}_b \circ \sigma).
\]
The above description of \(P^*_D\) can be simplified:
\[
P^*_D(d)(\xi, \eta) = -\frac{1}{2} \Delta_\pm \circ K^{-1}(\xi + \eta \circ \sigma) + \frac{1}{2} \text{Ad}_d \circ K^{-1}(\xi \circ \text{Ad}_a + \eta \circ \text{Ad}_b \circ \sigma),
\]
\[
= \frac{1}{2} \text{Ad}_d \circ K^{-1}(\xi \circ \text{Ad}_a + \eta \circ \text{Ad}_b \circ \sigma) - \sigma \circ K^{-1}(\xi + \eta \circ \sigma) +
\]
\[
\frac{1}{2} (K^{-1}(\xi + \eta \circ \sigma) - \sigma \circ K^{-1}(\xi + \eta \circ \sigma)) +
\]
\[
\frac{1}{2} (K^{-1}(\xi + \eta \circ \sigma \circ \text{Ad}_b(-1)), -\sigma \circ K^{-1}(\xi \circ \text{Ad}_a(-1) - 1)), -\sigma \circ K^{-1}(\xi \circ \sigma \circ (\text{Ad}_b(-1) - 1))\].
\]
It follows from \([\text{II}]\) that \(P^*_D\) is projectable on \(S^\sigma = D/G^\sigma_+\). Actually, the following is also true

**Proposition 3.3.** The bivector \(P^*_D\) is projectable to a bivector \(P^*_S\) on \(S = D/G_+\). Identify \(S\) with \(G\) through the map
\[
D \quad \longrightarrow \quad G
\]
\[
(a, b) \quad \longmapsto \quad ab^{-1}.
\]
Trivialise the tangent space to \(G\), and hence to \(S\), by right translations. If \(s\) is in \(S\), then using the above identification, \(P^*_S\) at the point \(s\) is
\[
P^*_S(s)(\xi) = \frac{1}{2} (\text{Ad}_{\sigma(s)^{-1}} - \text{Ad}_a) \circ \sigma \circ K^{-1}(\xi).
\]

Proof. Assume \(s\) in \(S\) is the image of \((a, b)\) in \(D\), that is \(s = ab^{-1}\). The tangent map of
\[
D \quad \longrightarrow \quad G
\]
\[
(a, b) \quad \longmapsto \quad ab^{-1}
\]
at \((a, b)\) is
\[
p : \mathfrak{d} \quad \longrightarrow \quad \mathfrak{g}
\]
\[
(x, y) \quad \longmapsto \quad x - \text{Ad}_{ab^{-1}} y.
\]
The dual map of \(p\) is
\[
p^* : \mathfrak{g}^* \quad \longrightarrow \quad \mathfrak{d}^*
\]
\[
\xi \quad \longmapsto \quad (\xi, -\xi \circ \text{Ad}_{ab^{-1}})
\]
The bivector \(P^*_D\) is projectable onto \(S\) if and only if for all \((a, b)\) in \(D\) and \(\xi\) in \(\mathfrak{g}^*\), the expression
\[
p(P^*_D(a, b)(p^*\xi))
\]
depends only on \( s = ab^{-1} \). It will then be equal to \( P_S^g(s)(\xi) \). This expression is equal to

\[
P(P_S^g(a, b)(\xi, -\xi \circ \text{Ad}_{ab^{-1}}))
\]

\[
= \frac{1}{2} p((\text{Ad}_{ab^{-1}} - 1) \circ \sigma \circ K^{-1}(-\xi \circ \text{Ad}_{ab^{-1}}, (1 - \text{Ad}_{ab^{-1}}) \circ \sigma \circ K^{-1}(\xi))
\]

\[
= \frac{1}{2} p((\text{Ad}_{ab^{-1}} - \text{Ad}_{ab^{-1}}) \circ \sigma \circ K^{-1}(\xi), (1 - \text{Ad}_{ab^{-1}}) \circ \sigma \circ K^{-1}(\xi))
\]

\[
= \frac{1}{2} (\text{Ad}_{ab^{-1}} - \text{Ad}_{ab^{-1}} - \text{Ad}_{ab^{-1}} + \text{Ad}_{ab^{-1}}) \circ \sigma \circ K^{-1}(\xi)
\]

This both proves that \( P_D^g \) is projectable on \( S \) and gives a formula for the projected bivector.

\[\square\]

To prove that there exists a quasi-Poisson action of \( G^\sigma \) on \((S, P_D^g)\), I must compute \([P_S^g, P_S^g]\), where \([,]\) is the Schouten-Nijenhuis bracket on multi-vector fields.

**Lemma 3.4.** For \( x, y \) and \( z \) in \( g \), let \( \xi = K(x) \), \( \eta = K(y) \) and \( \nu = K(z) \). We have

\[
\frac{1}{2}[P_S^g(s), P_S^g(s)](\xi, \eta, \nu) = \frac{1}{4} K(x, [y, \tau_s(z)] + [\tau_s(y), z] - \tau_s([y, z])),
\]

where \( \tau_s = \text{Ad}_s \circ \sigma - \sigma \circ \text{Ad}_{s^{-1}} \).

**Proof.** Let \((a, b)\) in \( D \) be such that \( s = ab^{-1} \). Let \( p \) be as in the proof of Proposition 3.3.

The bivector \( P_D^g(s) \) is \( p(P_D^g(a, b)) \). Hence,

\[
[P_S^g(s), P_S^g(s)] = p([P_D^g(a, b), P_D^g(a, b)]).
\]

But it is proved in \[\square\] that

\[
[P_D^g(a, b), P_D^g(a, b)] = (\varphi^\sigma)^p(a, b) - (\varphi^\sigma)^h(a, b).
\]

Hence

\[
\frac{1}{2}[P_S^g(s), P_S^g(s)] = p((\varphi^\sigma)^p(a, b)) - p((\varphi^\sigma)^h(a, b)).
\]

Now, it is tedious but straightforward and very similar to the above computations to check that

\[
p((\varphi^\sigma)^p(a, b))(\xi, \eta, \nu) = \frac{1}{4} K(x, [y, \tau_s(z)] + [\tau_s(y), z] - \tau_s([y, z])),
\]

and

\[
p((\varphi^\sigma)^h(a, b))(\xi, \eta, \nu) = 0.
\]

\[\square\]

The group \( D \) acts on \( S = D/G_+ \) by multiplication on the left. This action restricts to an action of \( G_+^\sigma \) on \( S \). Identifying \( G \) and \( G_+^\sigma \) via \( \Delta_+^\sigma \), this action is

\[
G \times S \rightarrow S
\]

\[
(g, s) \mapsto g s \sigma(g)^{-1}.
\]

The infinitesimal action of \( g \) at the point \( s \) in \( S \) reads

\[
\begin{align*}
g & \quad \mapsto \quad T_s S \simeq g \\
x & \quad \mapsto \quad x - \text{Ad}_s \circ \sigma(x),
\end{align*}
\]

with dual map

\[
T_s^* S \simeq g^* \quad \mapsto \quad g^*
\]

\[
\xi \quad \mapsto \quad \xi - \xi \circ \text{Ad}_s \circ \sigma.
\]

Denote by \((\varphi^\sigma)_S\) the induced trivector field on \( S \). If \( \xi, \eta \) and \( \nu \) are in \( g^* \) then

\[
(\varphi^\sigma)_S(s)(\xi, \eta, \nu) = \varphi^\sigma(\xi - \xi \circ \text{Ad}_s \circ \sigma, \eta - \eta \circ \text{Ad}_s \circ \sigma, \nu - \nu \circ \text{Ad}_s \circ \sigma).
\]

Computing the right hand side in the above equality is a simple calculation which proves the following Lemma.
Lemma 3.5. The bivector field $P^c_S$ and the trivector field $(\varphi^c)_S$ satisfy
\[ \frac{1}{2}[P^c_S, P^c_S] = (\varphi^c)_S. \]

To prove that the action of $G^c_+$ on $(S, P^c_S)$ is indeed quasi-Poisson, there only remains to prove that $P^c_S$ is $G^c_+$-invariant.

Lemma 3.6. The bivector field $P^c_S$ is $G^c_+$-invariant.

Proof. Fix $g$ in $G \simeq G^c_+$. Denote $\Sigma_g$ the action of $g$ on $S$. The tangent map of $\Sigma_g$ at $s \in S$ is
\[ T_s S \simeq g \quad \mapsto \quad T_{g\sigma(g)^{-1}} s \simeq g. \]

Also, if $\xi$ is in $\mathfrak{g}^*$
\[ P^c_S(g\sigma(g)^{-1})(\xi) = \frac{1}{2}(\text{Ad}_{\sigma(g)^{-1}} - \text{Ad}_{g\sigma(g)^{-1}}) \circ \sigma \circ K^{-1}(\xi) = \text{Ad}_g \circ (\text{Ad}_{\sigma(g)^{-1}} - \text{Ad}_s) \circ \text{Ad}_{\sigma(g)^{-1}} \circ K^{-1}(\xi) = \text{Ad}_g(P^c_S(s)(\xi \circ \text{Ad}_g)) = (\Sigma_g)_*(P^c_S)(\Sigma_g(s))(\xi). \]

Lemma 3.7. Let $s$ be in $S$. The image of $P^c_S(s)$ is
\[ \text{Im} P^c_S(s) = \{(1 - \text{Ad}_s \circ \sigma \circ (1 + \text{Ad}_s \circ \sigma)(y) \mid y \in \mathfrak{g}\}. \]
In particular, it is included in the tangent space to the orbit through $s$ of the action of $G^c$.

Proof. The image of $P^c_S(s)$ is by Proposition 3.6
\[ \text{Im} P^c_S(s) = \{(1 - \text{Ad}_s \circ \sigma)(y) \mid y \in \mathfrak{g}\}. \]

The Lemma follows by setting $x = Ad_s \circ \sigma(y) = \sigma \circ Ad_s \circ \sigma(y)$ and noticing that $(1 - (\text{Ad}_s \circ \sigma)^2 = (1 - \text{Ad}_s \circ \sigma) \circ (1 + \text{Ad}_s \circ \sigma). \]

This finishes the proof of Theorem 2.1.

Choose $G$ and $\sigma$ as in Theorem 2.2. The trivector field $[P^c_S, P^c_S]$ is tangent to the orbit of the action of $G^c_+$ on $S$. These orbits are of dimension at most 2, therefore the trivector field $[P^c_S, P^c_S]$ vanishes and $P^c_S$ defines a Poisson structure on $\text{SL}(2, \mathbb{R})$ which is invariant under the action
\[ \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \longrightarrow \text{SL}(2, \mathbb{R}) \]
\[ (g, s) \quad \mapsto \quad g\sigma(g)^{-1}. \]

Lemma 3.7 and a simple computation prove that along the orbit of the identity, the bivector field $P^c_S$ vanishes; and that elsewhere, its image coincides with the tangent space to the orbits of the above action. Recall that in $I$, the domain $I$ is given by
\[ z(\tau, \theta, \rho) = \begin{bmatrix} \sinh(\frac{\xi}{2}) + \cosh(\frac{\xi}{2}) \cos(\tau) & \exp(\theta) \cosh(\frac{\xi}{2}) \sin(\tau) \\ -\exp(-\theta) \cosh(\frac{\xi}{2}) \sin(\tau) & -\sinh(\frac{\xi}{2}) + \cosh(\frac{\xi}{2}) \cos(\tau) \end{bmatrix}. \]

This formula also defines coordinates on $I$. Using Formula 2 of Proposition 3.6 and a computer, it is easy to check that $P^c_S$ if indeed given by Formula 4. This ends the proof of Theorem 2.2.
4. A final remark

One might ask how different is the quasi-Poisson action of $G_{+}^{\sigma}$ on $(S, P_{S}^{\sigma})$ from the usual quasi-Poisson action of $G_{+}$ on $(S, P_{S})$. For example, if one takes $G = SU(2)$, $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\sigma = \text{Ad}_H$ then the multiplication on the right in SU(2) by $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ defines an isomorphism between the two quasi-Poisson actions.

Nevertheless, in the example of Theorem 2.2 the two structures are indeed different since for example the action of SL(2, $\mathbb{R}$) on itself by conjugation has two fixed points whereas the action of SL(2, $\mathbb{R}$) on itself used in Theorem 2.2 does not have any fixed point.

References


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