Quasi-Poisson actions and massive non-rotating BTZ black holes
Sébastien Racanière

To cite this version:
Sébastien Racanière. Quasi-Poisson actions and massive non-rotating BTZ black holes. 8 pages. Comments welcomed. 2004. <hal-00002975>

HAL Id: hal-00002975
https://hal.archives-ouvertes.fr/hal-00002975
Submitted on 29 Sep 2004

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
QUASI-POISSON ACTIONS AND MASSIVE NON-ROTATING BTZ BLACK HOLES

SÉBASTIEN RACANIÈRE

ABSTRACT. Using ideas from an article of P. Bieliavsky, M. Rooman and Ph. Spindel on BTZ black holes, I construct a family of interesting examples of quasi-Poisson actions as defined by A. Alekseev and Y. Kosmann-Schwarzbach. As an application, I obtain a genuine Poisson structure on SL(2, R) which induces a Poisson structure on a BTZ black hole.

ACKNOWLEDGEMENT

I would like to thank Pierre Bieliavsky for advising me to look at the article [4] and then answering my questions about it. Thank you to Stéphane Detournay for talking to me about physics in a way I could understand and for answering many questions I had about [3]. And finally, a big thanks to David Iglesias-Ponte for interesting conversations about [1].

The author was supported by a Marie Curie Fellowship, EC Contract Ref: HPMF-CT-2002-01850

1. INTRODUCTION

In [4], P. Bieliavsky, M. Rooman and Ph. Spindel construct a Poisson structure on massive non-rotating BTZ black holes; in [3], P. Bieliavsky, S. Detournay, Ph. Spindel and M. Rooman construct a star product on the same black hole. The direction of this deformation is a Poisson bivector field which has the same symplectic leaves as the Poisson bivector field of [4]: roughly speaking, they correspond to orbits under a certain twisted action by conjugation.

In the present paper, I wish to show how techniques used in [4] in conjunction with techniques of the theory of quasi-Poisson manifolds (see [1] and [2]) can be used to construct an interesting family of manifolds with a quasi-Poisson action and how a particular case of this family leads to a genuine Poisson structure on a massive non-rotating BTZ black hole with similar symplectic leaves as in [4] and [3].

2. MAIN RESULTS

I will not recall here the basic definitions in the theory of quasi-Poisson manifolds and quasi-Poisson actions. The reader will find these definitions in A. Alekseev and Y. Kosmann-Schwarzbach [1], and in A. Alekseev, Y. Kosmann-Schwarzbach and E. Meinrenken [2].

Let $G$ be a connected Lie group of dimension $n$ and $\mathfrak{g}$ its Lie algebra, on which $G$ acts by the adjoint action $\text{Ad}$. Assume we are given an $\text{Ad}$-invariant non-degenerate bilinear form $K$ on $\mathfrak{g}$. For example, if $G$ is semi-simple, then $K$ could be the Killing form. In the following, I will denote by $K$ again the linear isomorphism

\[
\begin{align*}
\mathfrak{g} & \longrightarrow \mathfrak{g}^* \\
x & \longmapsto K(x, \cdot).
\end{align*}
\]
Let $D = G \times G$ and $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ its Lie algebra. Define an Ad-invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ of signature $(n, n)$ by

\[
\mathfrak{d} \times \mathfrak{d} = (\mathfrak{g} \oplus \mathfrak{g}) \times (\mathfrak{g} \oplus \mathfrak{g}) \longrightarrow \mathbb{R}
\]

\[
\langle (x, y), (x', y') \rangle \mapsto K(x, x') - K(y, y').
\]

Assume there is an involution $\sigma$ on $G$ which induces an orthogonal involutive morphism, again denoted by $\sigma$, on $\mathfrak{g}$. Let $\Delta_+: G \to D$ and $\Delta^+_\sigma: G \to D$ be given by

\[
\Delta_+(g) = (g, g)
\]

and

\[
\Delta^+_\sigma(g) = (g, \sigma(g)).
\]

Denote by $G_+$ and $G^\sigma_+$ their respective images in $D$. Let $S = D/G_+$ and $S^\sigma = D/G^\sigma_+$. Then both $S$ and $S^\sigma$ are isomorphic to $G$. The isomorphism between $S$ and $G$ is induced by the map

\[
D \longrightarrow G,
\]

\[
(g, h) \longmapsto gh^{-1},
\]

whereas the isomorphism between $S^\sigma$ and $G$ is induced by

\[
D \longrightarrow G,
\]

\[
(g, h) \longmapsto g\sigma(h)^{-1}.
\]

I will use these two isomorphisms to identify $S$ and $G$, and $S^\sigma$ and $G$. Denote again by $\Delta_+: \mathfrak{g} \to \mathfrak{d}$ and $\Delta^+_\sigma: \mathfrak{g} \to \mathfrak{d}$ the morphisms induced by $\Delta_+: G \to D$ and $\Delta^+_\sigma: G \to D$ respectively. Let $\Delta_+: \mathfrak{g} \to \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ and $\Delta^+_\sigma: \mathfrak{g} \to \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ be defined by

\[
\Delta_-(x) = (x, -x),
\]

and

\[
\Delta^+\sigma(x) = (x, -\sigma(x)).
\]

Let $\mathfrak{g}_- = \text{Im}(\Delta_-)$ and $\mathfrak{g}^\sigma_- = \text{Im}(\Delta^+_\sigma)$. We have two quasi-triples $(D, G_+, \mathfrak{g}_-)$ and $(D, G^\sigma_+, \mathfrak{g}^\sigma_-)$. They induce two structures of quasi-Poisson Lie group on $D$, of respective bivector fields $P_D$ and $P^\sigma_D$, and two structures of quasi-Poisson Lie group on $G_+$ and $G^\sigma_+$ of respective bivector fields $P_{G_+}$ and $P^\sigma_{G_+}$. I will simply write $G_+$, respectively $G^\sigma_+$, to denote the group together with its quasi-Poisson structure. Of course, these quasi-Poisson structures are pairwise isomorphic. More precisely, the isomorphism $\text{Id} \times \sigma: (g, h) \longmapsto (g, \sigma(h))$ of $D$ sends $P_D$ on $P^\sigma_D$ and vice-versa. This isomorphism can be used to deduce some of the results given at the beginning of the present article from the results of Alekseev and Kosmann-Schwarzbach [1]; but it takes just as long to redo the computations, and that is what I do here.

According to [1], the bivector field $P_D$, respectively $P^\sigma_D$, is projectable onto $S$, respectively $S^\sigma$. Let $P_S$ and $P^\sigma_S$ be their respective projections. Using the identifications between $S$ and $G$, and $S^\sigma$ and $G$, one can check that $P_S$ and $P^\sigma_S$ are the same bivector fields on $G$.

What is more interesting, and what I will prove, is the following Theorem.

**Theorem 2.1.** The bivector field $P^\sigma_S$ is projectable onto $S$. Let $P^\sigma_S$ be its projection. Identify $S$ with $G$ and trivialise their tangent space using right translations, then for $s$ in $S$ and $\xi$ in $\mathfrak{g}^* \simeq T^*_s S$ there is the following explicit formula

\[
P^\sigma_S(s)(\xi) = \frac{1}{2}(\text{Ad}_{\sigma(s)} - \text{Ad}_s) \circ \sigma \circ K^{-1}(\xi).
\]

Moreover, the action

\[
G^\sigma \times S \longrightarrow S
\]

\[
(g, s) \longmapsto g \sigma(g)^{-1}
\]
of $G^\sigma_+$ on $(S, P^\sigma_S)$ is quasi-Poisson in the sense of Alekseev and Kosmann-Schwarzbach [1].

The image of $P^\sigma_S(s)$, seen as a map $T_s^*S \to T_sS$, is tangent to the orbit through $s$ of the action of $G^\sigma$ on $S$.

In the setting of the above Theorem, the bivector field $P^\sigma_S$ is $G^\sigma$ invariant; hence if $F$ is a subgroup of $G^\sigma$ and $I$ is an $F$-invariant open subset of $S$ such that the action of $F$ on $I$ is principal then $F\backslash I$ is a smooth manifold and $P^\sigma_S$ descends to a bivector field on it. An application of this remark is the following Theorem.

**Theorem 2.2.** Let $G = \text{SL}(2, \mathbb{R})$. Let

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and choose $\sigma = \text{Ad}_H$. Let

$$I = \{ \begin{bmatrix} u+x & y+t \\ y-t & u-x \end{bmatrix} \mid u^2 - x^2 - y^2 + t^2 = 1, t^2 - y^2 > 0 \}$$

be an open subset of $S$. Let $F$ be the following subgroup of $G$,

$$F = \{ \exp(n\pi H), n \in \mathbb{N} \}.$$

The quotient $F\backslash I$ (together with an appropriate metric) is a model of massive non-rotating BTZ black hole (see [3]). The bivector field it inherits following the above remark, is Poisson. Its symplectic leaves consist of the projection to $F\backslash I$ of the orbits of the action of $G^\sigma$ on $S$ except along the projection of the orbit of the identity. Along this orbit, the bivector field vanishes and each point forms a symplectic leaf.

In the coordinates (46) of [3] (or (3) of the present article), the Poisson bivector field is

$$(1) \quad 2\cosh^2\left(\frac{\rho}{2}\right)\sin(\tau) \sinh(\rho) \partial_{\tau} \wedge \partial_{\theta}.$$

The above Poisson bivector field should be compared with the one defined in [3] and given by

$$\frac{1}{\cosh^2\left(\frac{\rho}{2}\right)\sin(\tau)} \partial_{\tau} \wedge \partial_{\theta}.$$

The symplectic leaves of this Poisson structure are the images under the projection $I \to F\backslash I$ of the action of $G^\sigma_+$ on $S$.

3. **LET THE COMPUTATIONS BEGIN**

Throughout the present article, I will use the notations introduced in the previous Section. To begin with, I will prove that $(D, G^\sigma_+, g^\sigma_+)$ does indeed form a quasi-triple.

Because $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$, one also has a decomposition $\mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}^*$. One also has $\mathfrak{d} = g^\sigma_+ \oplus g^\sigma_-$ and accordingly $\mathfrak{d}^* = g^\sigma_+^* \oplus g^\sigma_-^*$. Denote $p_{g^\sigma_+}$ and $p_{g^\sigma_-}$ the projections on respectively $g^\sigma_+^*$ and $g^\sigma_-^*$ induced by the decomposition $\mathfrak{d} = g^\sigma_+ \oplus g^\sigma_-$.

In this article, I express results using mostly the decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$. Using it, we have

$$g^\sigma_+^* = \{ (\xi, \xi \circ \sigma) \mid \xi \in g^\sigma \}$$

and

$$g^\sigma_-^* = \{ (\xi, -\xi \circ \sigma) \mid \xi \in g^\sigma \}.$$
Finally, the $r$-matrix is defined as

$$\varepsilon : \mathfrak{g}_+^* \otimes \mathfrak{g}_+^* \longrightarrow \mathfrak{g}_+^* \otimes \mathfrak{g}_+^*$$

and finally the $r$-matrix $r_3^\sigma$ are

$$j : \mathfrak{g}_+^* \longrightarrow \mathfrak{g}_-^*$$

$$(\xi, \xi \circ \sigma) \longrightarrow \Delta_\sigma \circ K^{-1}(\xi),$$

and

$$F^\sigma = 0,$$

and

$$\varphi^\sigma : \wedge^3 \mathfrak{g}_+^*, \mathfrak{g}_-^* \longrightarrow \mathbb{R}$$

$$((\xi, \sigma \circ \xi), (\eta, \sigma \circ \eta), (\nu, \sigma \circ \eta)) \longrightarrow 2K(K^{-1}(\nu), [K^{-1}(\xi), K^{-1}(\eta)],$$

and finally the $r$-matrix

$$r_3^\sigma : \mathfrak{g}_+^* \otimes \mathfrak{g}_+^* \longrightarrow \mathfrak{g}_-^* \otimes \mathfrak{g}_-^*$$

$$(\xi, \eta) \longrightarrow \frac{1}{2} \Delta_\sigma \circ K^{-1}(\xi + \eta \circ \sigma).$$

Proof. It is straightforward to prove that $\mathfrak{d} = \mathfrak{g}_+^* \oplus \mathfrak{g}_-^*$ and that both $\mathfrak{g}_+^*$ and $\mathfrak{g}_-^*$ are isotropic in $(\mathfrak{d}, \langle \rangle)$. This proves that $(D, G^\sigma, \mathfrak{g}^\sigma)$ is a quasi-triple.

For $(\xi, \xi \circ \sigma)$ in $\mathfrak{g}_+^*$ and $(x, \sigma(x))$ in $\mathfrak{g}_+^*$

$$\langle j(\xi, \xi \circ \sigma), (x, \sigma(x)) \rangle = (\xi, \xi \circ \sigma)(x, \sigma(x)).$$

The map $j$ is actually characterised by this last property. The equality

$$(\Delta_\sigma \circ K^{-1} \circ \Delta_\sigma^*(\xi, \xi \circ \sigma), (x, \sigma(x)) = (\xi, \xi \circ \sigma)(x, \sigma(x)),$$

proves that

$$j(\xi, \xi \circ \sigma) = \Delta_\sigma \circ K^{-1} \circ \Delta_\sigma^*(\xi, \xi \circ \sigma) = \Delta_\sigma \circ K^{-1}(\xi).$$

Since $\sigma$ is a Lie algebra morphism, we have $[\mathfrak{g}_-^*, \mathfrak{g}_-^*] \subset \mathfrak{g}_+^*$. This proves that $F^\sigma : \wedge^2 \mathfrak{g}_+^* \longrightarrow \mathfrak{g}_-^*$, given by

$$F^\sigma(\xi, \eta) = p_{\mathfrak{g}_-^*}[j(\xi), j(\eta)],$$

vanishes.

I will now compute $\varphi^\sigma$. It is defined as

$$\varphi^\sigma((\xi, \sigma \circ \xi), (\eta, \sigma \circ \eta), (\nu, \sigma \circ \nu)) = (\nu, \sigma \circ \nu) \circ p_{\mathfrak{g}_+^*}([j(\xi, \sigma \circ \xi), j(\eta, \sigma \circ \eta)])$$

$$= \langle j(\nu, \sigma \circ \nu), j(\xi, \sigma \circ \xi), j(\eta, \sigma \circ \eta) \rangle$$

$$= \langle \Delta_\sigma \circ K^{-1}(\nu), \Delta_\sigma \circ K^{-1}(\xi), \Delta_\sigma \circ K^{-1}(\eta) \rangle$$

$$= 2K(K^{-1}(\nu), [K^{-1}(\xi), K^{-1}(\eta)])$$

Finally, the $r$-matrix is defined as

$$r_3^\sigma : \mathfrak{g}_+^* \otimes \mathfrak{g}_+^* \longrightarrow \mathfrak{g}_-^* \otimes \mathfrak{g}_-^*$$

$$(\xi, \xi \circ \sigma, (\eta, \eta \circ \sigma)) \longrightarrow (0, j(\xi, \xi \circ \sigma)).$$

If $(\xi, \eta)$ is in $\mathfrak{d}^\ast = \mathfrak{g}^* \oplus \mathfrak{g}^*$ then its decomposition in $\mathfrak{g}_+^* \otimes \mathfrak{g}_+^*$ is $\frac{1}{2}((\xi + \eta \circ \sigma, \xi \circ \sigma + \eta), (\xi - \eta \circ \sigma, -\xi \circ \sigma + \eta))$. The result follows.

I now wish to compute the bivector $P_3^\sigma$ on $D$. By definition, it is equal to $(r_3^\sigma)^\lambda - (r_3^\sigma)^\rho$, where the upper script $\lambda$ means the left invariant section of $\Gamma(TD \otimes TD)$ generated by $r_3^\sigma$, while the upper script $\rho$ means the right invariant section of $\Gamma(TD \otimes TD)$ generated by $r_3^\sigma$.

Proposition 3.2. Identify $T_aD$ to $\mathfrak{d}$ by right translations. The value of $P_3^\sigma$ at $d = (a, b)$ is

$$\mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}^*$$

$$(\xi, \eta) \longrightarrow \frac{1}{2}(K^{-1}(\eta \circ \sigma \circ (\text{Ad}_{\sigma(b)}a^{-1})), -K^{-1}(\xi \circ \sigma \circ (\text{Ad}_{\sigma(b)}a^{-1}))).$$
Proof. Fix $d = (a, b)$ in $D$. I choose to trivialise the tangent bundle, and its dual, of $D$ by using right translations. See $(r^*_g)^\alpha$ as a map from $T^*D$ to $TD$. If $\alpha$ is in $\mathfrak{d}^*$, then

$$(r^*_g)^\alpha(d)(\alpha^\rho) = (r^*_g(\alpha))(\alpha^\rho)(d),$$

whereas

$$(r^*_g)^\lambda(d)(\alpha^\rho) = (\operatorname{Ad}_d \circ r^*_g(\alpha \circ \operatorname{Ad}_d))(\alpha^\rho)(d).$$

Thus $P_D^\sigma$ at the point $d = (a, b)$ is

$$\mathfrak{d} = \mathfrak{g}^* \oplus \mathfrak{g}^* \quad \mapsto \quad \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$$

$$(\xi, \eta) \quad \mapsto \quad -\frac{1}{2}\Delta_\eta \circ K^{-1}(\xi + \eta \circ \sigma) + \frac{1}{2}\operatorname{Ad}_d \circ \Delta_\eta \circ K^{-1}(\xi \circ \operatorname{Ad}_a + \eta \circ \operatorname{Ad}_b \circ \sigma).$$

The above description of $P_D^\sigma$ can be simplified:

$$P_D^\sigma(d)(\xi, \eta) = -\frac{1}{2}\Delta_\eta \circ K^{-1}(\xi + \eta \circ \sigma) + \frac{1}{2}(\operatorname{Ad}_a \circ K^{-1}(\xi \circ \operatorname{Ad}_a + \eta \circ \operatorname{Ad}_b \circ \sigma) - \sigma \circ \operatorname{Ad}_{\sigma(b)} \circ K^{-1}(\xi \circ \operatorname{Ad}_{a(b)} + \eta \circ \operatorname{Ad}_b \circ \sigma \circ \operatorname{Ad}_{\sigma(b)}(\eta))$$

$$= -\frac{1}{2}(K^{-1}(\xi + \eta \circ \sigma), -\sigma \circ K^{-1}(\xi + \eta \circ \sigma)) + \frac{1}{2}(K^{-1}(\xi \circ \sigma \circ \operatorname{Ad}_{\sigma(b)}(\eta - 1), -\sigma \circ K^{-1}(\xi \circ \sigma \circ \operatorname{Ad}_{\sigma(b)}(\eta - 1)))$$

$$= -\frac{1}{2}(K^{-1}(\eta \circ \sigma \circ \operatorname{Ad}_{\sigma(b)}(\eta - 1), -\sigma \circ K^{-1}(\xi \circ \sigma \circ \operatorname{Ad}_{\sigma(b)}(\eta - 1))).$$

□

It follows from [1] that $P_D^\sigma$ is projectable on $S^\sigma = D/G^\sigma_+$. Actually, the following is also true

**Proposition 3.3.** The bivector $P_D^\sigma$ is projectable to a bivector $P_S^\sigma$ on $S = D/G_+$. Identify $S$ with $G$ through the map

$$\begin{array}{ccc}
D & \rightarrow & G \\
(a, b) & \mapsto & ab^{-1}.
\end{array}$$

Trivialise the tangent space to $G$, and hence to $S$, by right translations. If $s$ is in $S$, then using the above identification, $P_S^\sigma$ at the point $s$ is

$$(2) \quad P_S^\sigma(s)(\xi) = \frac{1}{2}(\operatorname{Ad}_{\sigma^{-1}}(b) - \operatorname{Ad}_{a}) \circ \sigma \circ K^{-1}(\xi).$$

Proof. Assume $s$ in $S$ is the image of $(a, b)$ in $D$, that is $s = ab^{-1}$. The tangent map of

$$\begin{array}{ccc}
D & \rightarrow & G \\
(a, b) & \mapsto & ab^{-1}
\end{array}$$

at $(a, b)$ is

$$p : \mathfrak{d} \quad \mapsto \quad \mathfrak{g}$$

$$(x, y) \quad \mapsto \quad x - \operatorname{Ad}_{ab^{-1}}y.$$

The dual map of $p$ is

$$p^* : \mathfrak{g}^* \quad \mapsto \quad \mathfrak{d}^*$$

$$\xi \quad \mapsto \quad (\xi, -\xi \circ \operatorname{Ad}_{ab^{-1}})$$

The bivector $P_D^\sigma$ is projectable onto $S$ if and only if for all $(a, b)$ in $D$ and $\xi$ in $\mathfrak{g}^*$, the expression

$$p(P_D^\sigma(a, b)(p^*\xi))$$
depends only on $s = ab^{-1}$. It will then be equal to $P_S^s(s)(\xi)$. This expression is equal to
\[
p(P_S^s(a, b)(\xi, -\xi \circ \text{Ad}_{ab^{-1}})) = \frac{1}{2} p((\text{Ad}_{ab^{-1}} - 1) \circ \sigma \circ K^{-1}(-\xi \circ \text{Ad}_{ab^{-1}}), (1 - \text{Ad}_{ab^{-1}}) \circ \sigma \circ K^{-1}(\xi))
\]
\[
= \frac{1}{2} p((\text{Ad}_{b^{-1}} - \text{Ad}_{ab^{-1}}) \circ \sigma \circ K^{-1}(\xi), (1 - \text{Ad}_{ab^{-1}}) \circ \sigma \circ K^{-1}(\xi))
\]
\[
= \frac{1}{2}(\text{Ad}_{b^{-1}} - \text{Ad}_{ab^{-1}} - \text{Ad}_{ab^{-1}} + \text{Ad}_{ab^{-1}}) \circ \sigma \circ K^{-1}(\xi)
\]
This both proves that $P_D^s$ is projectable on $S$ and gives a formula for the projected bivector.

To prove that there exists a quasi-Poisson action of $G^\sigma$ on $(S, P_S^s)$, I must compute $[P_S^s, P_S^s]$, where $[,]$ is the Schouten-Nijenhuis bracket on multi-vector fields.

**Lemma 3.4.** For $x$, $y$ and $z$ in $g$, let $\xi = K(x)$, $\eta = K(y)$ and $\nu = K(z)$. We have
\[
\frac{1}{2}[P_S^s(s), P_S^s(s)](\xi, \eta, \nu) = \frac{1}{4} K(x, [y, \tau_s(z)] + [\tau_s(y), z] - \tau_s([y, z])),
\]
where $\tau_s = \text{Ad}_a \circ \sigma - \sigma \circ \text{Ad}_s^{-1}$.

**Proof.** Let $(a, b)$ in $D$ be such that $s = ab^{-1}$. Let $p$ be as in the proof of Proposition 3.3. The bivector $P_S^s(s)$ is $p(P_D^s(a, b))$. Hence,
\[
[P_S^s(s), P_S^s(s)] = p([P_D^s(a, b), P_D^s(a, b)]).
\]
But it is proved in [11] that
\[
[P_D^s(a, b), P_D^s(a, b)] = (\varphi^\sigma)^p(a, b) - (\varphi^\sigma)^h(a, b).
\]
Hence
\[
\frac{1}{2}[P_S^s(s), P_S^s(s)] = p((\varphi^\sigma)^p(a, b)) - p((\varphi^\sigma)^h(a, b)).
\]
Now, it is tedious but straightforward and very similar to the above computations to check that
\[
p((\varphi^\sigma)^p(a, b))(\xi, \eta, \nu) = \frac{1}{4} K(x, [y, \tau_s(z)] + [\tau_s(y), z] - \tau_s([y, z])),
\]
and
\[
p((\varphi^\sigma)^h(a, b))(\xi, \eta, \nu) = 0.
\]

The group $D$ acts on $S = D/G_+$ by multiplication on the left. This action restricts to an action of $G_+^2$ on $S$. Identifying $G$ and $G_+^2$ via $\Delta_+^2$, this action is
\[
G \times S \longrightarrow S
\]
\[
(g, s) \longmapsto gs\sigma(g)^{-1}.
\]
The infinitesimal action of $g$ at the point $s$ in $S$ reads
\[
g \longrightarrow T_sS \simeq g
\]
\[
x \longmapsto x - \text{Ad}_x \circ \sigma(x),
\]
with dual map
\[
T_s^*S \simeq g^* \longrightarrow g^*
\]
\[
\xi \longmapsto \xi - \xi \circ \text{Ad}_s \circ \sigma.
\]
Denote by $(\varphi^\sigma)_s$ the induced trivector field on $S$. If $\xi$, $\eta$ and $\nu$ are in $g^*$ then
\[
(\varphi^\sigma)_s(s)(\xi, \eta, \nu) = \varphi^\sigma(\xi - \xi \circ \text{Ad}_x \circ \sigma, \eta - \eta \circ \text{Ad}_x \circ \sigma, \nu - \nu \circ \text{Ad}_x \circ \sigma).
\]
Computing the right hand side in the above equality is a simple calculation which proves the following Lemma.
**Lemma 3.5.** The bivector field $P_S^\sigma$ and the trivector field $(\varphi^\sigma)_S$ satisfy
\[
\frac{1}{2}[P_S^\sigma, P_S^\sigma] = (\varphi^\sigma)_S.
\]

To prove that the action of $G^\sigma_+$ on $(S, P_S^\sigma)$ is indeed quasi-Poisson, there only remains to prove that $P_S^\sigma$ is $G^\sigma_+$-invariant.

**Lemma 3.6.** The bivector field $P_S^\sigma$ is $G^\sigma_+$-invariant.

**Proof.** Fix $g$ in $G \simeq G^\sigma_+$. Denote $\Sigma_g$ the action of $g$ on $S$. The tangent map of $\Sigma_g$ at $s \in S$ is
\[
T_s S \simeq g \quad \longrightarrow \quad T_{g \sigma(g)^{-1}}S \simeq g \
\]

Also, if $\xi$ is in $g^*$
\[
P_S^\sigma(g \sigma(g)^{-1})(\xi) = \frac{1}{2}(\text{Ad}_{\sigma(g)^{-1}} - \text{Ad}_{g \sigma(g)^{-1}}) \circ \sigma \circ K^{-1}(\xi)
\]
\[
= \frac{1}{2}\text{Ad}_g \circ (\text{Ad}_{\sigma(g)^{-1}} - \text{Ad}_\sigma) \circ \text{Ad}_{g \sigma(g)^{-1}} \circ \sigma \circ K^{-1}(\xi)
\]
\[
= \text{Ad}_g(P_S^\sigma(s)(\xi \circ \text{Ad}_g))
\]
\[
= (\Sigma_g)_*(P_S^\sigma)(\Sigma_g(s))(\xi).
\]

□

**Lemma 3.7.** Let $s$ be in $S$. The image of $P_S^\sigma(s)$ is
\[
\text{Im}P_S^\sigma(s) = \{(1 - \text{Ad}_s \circ \sigma) \circ (1 + \text{Ad}_s \circ \sigma)(y) \mid y \in g\}.
\]

In particular, it is included in the tangent space to the orbit through $s$ of the action of $G^\sigma$.

**Proof.** The image of $P_S^\sigma(s)$ is by Proposition 3.6
\[
\text{Im}P_S^\sigma(s) = \{(\text{Ad}_{\sigma(s)^{-1}} - \text{Ad}_s)\sigma(x) \mid x \in g\}
\]

The Lemma follows by setting $x = \text{Ad}_s \circ \sigma(y) = \sigma \circ \text{Ad}_{\sigma(s)}(y)$ and noticing that $(1 - (\text{Ad}_s \circ \sigma)^2) = (1 - (\text{Ad}_s \circ \sigma) \circ (1 + \text{Ad}_s \circ \sigma))$. □

This finishes the proof of Theorem 2.1.

Choose $G$ and $\sigma$ as in Theorem 2.2. The trivector field $[P_S^\sigma, P_S^\sigma]$ is tangent to the orbit of the action of $G^\sigma_+$ on $S$. These orbits are of dimension at most 2, therefore the trivector field $[P_S^\sigma, P_S^\sigma]$ vanishes and $P_S^\sigma$ defines a Poisson structure on $\text{SL}(2, \mathbb{R})$ which is invariant under the action
\[
\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \longrightarrow \text{SL}(2, \mathbb{R})
\]
\[
(g, s) \quad \mapsto \quad g \sigma(g)^{-1}.
\]

**Lemma 3.7** and a simple computation prove that along the orbit of the identity, the bivector field $P_S^\sigma$ vanishes; and that elsewhere, its image coincides with the tangent space to the orbits of the above action. Recall that in [1], the domain $I$ is given by
\[
\begin{pmatrix}
\sinh(\frac{\tau}{2}) + \cosh(\frac{\tau}{2})\cos(\theta) & \exp(\theta)\cosh(\frac{\tau}{2})\sin(\theta) \\
-\exp(-\theta)\cosh(\frac{\tau}{2})\sin(\theta) & -\sinh(\frac{\tau}{2}) + \cosh(\frac{\tau}{2})\cos(\theta)
\end{pmatrix}.
\]

This formula also defines coordinates on $I$. Using Formula 2 of Proposition 3.6 and a computer, it is easy to check that $P_S^\sigma$ if indeed given by Formula (1). This ends the proof of Theorem 2.2.
4. A final remark

One might ask how different is the quasi-Poisson action of $G_+^g$ on $(S, P_S^g)$ from the usual quasi-Poisson action of $G_+$ on $(S, P_S)$. For example, if one takes $G = \text{SU}(2)$, $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\sigma = \text{Ad}_H$ then the multiplication on the right in SU(2) by $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ defines an isomorphism between the two quasi-Poisson actions.

Nevertheless, in the example of Theorem 2.2, the two structures are indeed different since for example the action of $\text{SL}(2, \mathbb{R})$ on itself by conjugation has two fixed points whereas the action of $\text{SL}(2, \mathbb{R})$ on itself used in Theorem 2.2 does not have any fixed point.

References


Department of Mathematics, South Kensington Campus, Imperial College London, SW7 2AZ, UK

E-mail address: s.racaniere@ic.ac.uk
URL: www.ma.ic.ac.uk/~racani