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Abstract

Damage theory is, by its very essence, a mean-field theory. In this note, we argue that considering the effective interaction kernel between an additional micro-crack, and the effective equivalent damaged matrix, the power-law decay of the influence function (or Green's function) becomes more and more long-ranged as the tangent modulus vanishes. Moreover, the reloaded region becomes a narrower and narrower "cone", so that the damage in this cone becomes closer and closer to the so-called global load sharing rule used, for instance, to study a fiber bundle. This constitutes a formal justification of the relevance of such a mean-field approach as the peak stress is approached.

Keywords: Continuum Damage Mechanics, Mode III crack, local load sharing, global load sharing, mean field regime, far-field stress.

I. INTRODUCTION

Continuum Damage Mechanics (CDM) consists in modeling the gradual degradation occurring in materials (e.g., cavity nucleation and growth, multiple cracking) within the framework of Continuum Mechanics[1] and Continuum Thermodynamics[2]. Internal variables are introduced and their mathematical nature depends on the degradation phenomenon one wants to describe and the physical basis one is ready to consider. In many years, an isotropic damage description was used[3] even though anisotropic descriptions are more realistic when considering multiple (micro)cracking of quasi-brittle materials[4–8]. Discrete approaches can also be used[9] to study a set of cracks or voids. They explicitly account for defect interactions, viz. shielding or amplification. This approach is more detailed than the previous one but is also more limited since the number of discrete defects is still relatively low compared to situations where the CDM framework is applicable. In the latter case, the stress redistributions are dealt with through the constitutive equation itself. It can be noted that if local stress amplifications were the controlling feature then the material would essentially behave as an elastic-brittle medium.

What CDM usually describes is a progressive reduction in stiffness of a damaged material[10–13]. As a volume element is unloaded, its behavior is purely elastic with a lower Young's modulus or a higher compliance. In contrast, for an element whose stress state leads to a damage change, a further increase in load will induce an incremental stress/strain relation which will account for both the elastic loading and the incremental damage, so that the tangent loading and unloading stiffnesses differ. As a result of this difference, we argue in this note, that the influence function of given volume element will give rise to a stress redistribution kernel whose decay with distance is different from (i.e., slower than) the purely elastic case. The exact form of this kernel will be derived in the simple case of a mode III loading, although a similar computation can be performed in other cases. Concerning the consequences of this simple observation, we believe that it sheds some light on the relevance of a mean-field approach to damage. The load redistribution from one volume element to other ones is a basic process at play in the kinetics of damage. A transverse cut through a specimen loaded in tension may then be modeled with the help of a line spring model for a homogeneous material[14–16] or by closely related descriptions for an interface between two dissimilar media[17,18]. For this modelling to be meaningful, the boundary conditions specifying the load or the displacement has to be modified as compared to the two well studied limits, viz. the Mean Field (MF) or Global Load Sharing (GLS) hypothesis for which each volume element will interact globally as a result of a homogenized effect of the surrounding medium and the Local Load Sharing (LLS) hypothesis for which the immediate neighborhood is relevant. If the medium were purely elastic then the description of the load redistribution would make use of a standard Green's function, which already is an intermediate case between MF and LLS. For illustration purposes and to be more precise, we refer here to simple fiber-bundle studies that give rise naturally to a damage behavior and illustrate what we refer to as MF.

A fiber bundle model consists in studying the mechanical behavior of a one dimensional array of parallel fibers loaded in tension[19–21]. The stiffness of the fibers can be considered uniform, but their strength is randomly distributed. Depending on the precise rule of load transfer upon fiber breakage, we can observe different responses. Two major limits have been considered[22]: the GLS rule corresponds to the case of clamping the fibers between two rigid crossheads. It follows that each unbroken fiber supports the same load and the load carried by a breaking fiber is evenly redistributed on the surviving fibers. In our terminology, this is an MF regime. The resulting behavior is simple damage[23,24]. Consequently, the perturbations are insensitive to the spatial correlations. A fiber will be subjected to a load which is the total force exerted on one crosshead, divided by the number of fibers, irrespective of their location. Exchanging two fibers (broken or intact) has no impact. This is in contrast with the LLS rule where the load on a fiber is a function of its local environment. LLS consists in transferring the load supported by a fiber that fails, to its nearest (right and left) surviving fibers[25] or fiber bundles[26]. This is physically the limit of a very compliant crosshead. The resulting mechanical behavior of the bundle consists in two stages. The first one is a damage behavior where the weakest fibers fail. Then when a few neighboring fibers have broken, so that the extension of this defect reaches a critical size, the load transferred to the ends of this "crack" is so large that this crack propagates in an unstable manner throughout the entire bundle. The behavior thus appears to be brittle.

Both of these limits are over-simplifications, and can hardly be interpreted as representative of the bulk of a two- or three-dimensional system. It can be noted that the ultimate strength of some unidirectional fiber-reinforced composites can be modeled within this framework[27–31]. To progress toward a more realistic description of the mechanical coupling of the fibers within the bundle, so that the correspondence with a two-dimensional system appears more convincing, an intermediate case has been introduced[32,33] replacing the two connecting crossheads by two semi-infinite elastic domains. The load transfer between the surviving fibers is then computed by solving for the stress distribution within those domains. Now depending on the relative stiffness of the fibers and the elastic properties of the semi-infinite domains, we can observe either a non-linear (damage) behavior (compliant fibers) or a damage-brittle transition (stiff fibers). In the first case, past a bifurcation point reached at the peak stress for an infinite system, we can study the progressive localization of damage on a smaller and smaller damaged region while the rest of the bundle is progressively unloaded. However, damage is confined at the interface between the elastic domains, and thus this model still cannot be argued to be representative of a homogeneously damaged medium.

Nevertheless we conclude that the precise way in which the load is transferred from one fiber on to its surrounding has a drastic influence on the type of behavior. This load transfer can be formulated by using a "Green's function" type of formalism, i.e., by studying how the breakage of a fiber supporting a unit force will load another fiber at a distance r by a force $\phi(r)$. The GLS rule corresponds to $\phi(r) = \mathit{constant}$ for $r \neq 0$, and $\phi(0) = -1$ whereas the LLS rule is a very rapid decay $\phi(r = 0) = -1$, $\phi(r = \pm 1) = 1/2$, and $\phi(r) = 0$ for $|r| > 1$. The coupling with an elastic block is characterized by a power-law decay $\phi(r) \propto r^{-1}$. The Green's function ϕ can be characterized by the exponent γ of the decay of ϕ with r, $\phi(r) \propto r^{-\gamma}$. The three cases considered above are $\gamma = 0$ (GLS), ∞ (LLS) and 1 (Elastic block). Other functions ϕ have been considered by Wu and Leath[34].

The aim of this note is to show that when considering a two-dimensional system, as damage progresses in a homogeneous manner, the effective "Green's function" has a continuously evolving exponent γ, varying from 1 in the initial stage (elastic regime) to 0 as the peak stress is reached. Moreover the shape of the reloaded region is a "cone" (far from the perturbation region) whose aperture angle vanishes as the peak stress is approached. This gives some credit to the modelling of the final stage of damage using a one dimensional description based on the fiber bundle model with the Global Load Sharing rule. As a result, we conclude that the standard treatment which consists in focusing only on the far-field stress induced by micro-cracking, is legitimate.

II. HOMOGENEOUS MODE III LOADING

We consider an infinite medium subjected to a uniform remote stress σ_{yz}. The problem will be focused essentially on an anti-plane geometry, so that every field will be invariant along the z-axis. We will consider below the perturbation in the stress field induced by the presence of a crack (or a slightly more damaged zone). The crack will thus be assumed to be invariant along the z-axis, and aligned in the (x, z)-plane. In this case, a pure mode III condition is met.

In the absence of crack, the medium is subjected to a uniform stress and strain. By using the anti-plane property, the problem reduces to the determination of a displacement field aligned with the z-axis and independent of z, $U_z = w(x, y)$. The strains are characterized by the displacement gradients $\gamma_{xz} = w_{,x}$, $\gamma_{yz} = w_{,y}$. If the material behavior were elastic, the following stress $\Sigma = (\sigma_{xz} \;\; \sigma_{yz})^T$ / displacement gradient $\Gamma = (\gamma_{xz} \;\; \gamma_{yz})^T$ relationship would apply

$$\Sigma = \underline{K}\Gamma \tag{1}$$

where $\underline{K}$ is the reduced stiffness tensor

$$\underline{K} = \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

and μ the shear modulus.

We consider now a damageable material. For any constitutive law, (provided the prescribed stress can be sustained by the material) the homogeneous solution remains valid. Because of the simplicity of the anti-plane geometry, we can easily deal with anisotropic damage. Assuming now that the stress has been applied through a simple proportional loading, the transverse stiffness K_{xx} remains unaffected by the loading, i.e. $K_{xx} = K_0 = \mu$, whereas the longitudinal one, K_{yy}, is reduced to $K_u = (1 - D)\mu$ upon unloading (i.e., if $\dot{\sigma}_{yz} < 0$) and to K_l upon further loading (i.e., if $\dot{\sigma}_{yz} > 0$). The expression of the tangent stiffness K_l can be extracted from the constitutive law if $D(\gamma_{yz})$ is specified, through

$$K_l = \mu \left[(1 - D) - \frac{\partial D}{\partial \gamma_{yz}} \gamma_{yz} \right] . \tag{3}$$

This defines the incremental stiffness from a uniform stress state, $\Sigma = \Sigma^{(0)}$.

We assume the possible occurrence of a crack. As shown schematically in Fig. 1, we imagine a crack centered at the origin, lying in the (x, z) plane, and which fulfills the anti-plane conditions. Therefore the crack is assumed to be infinite along the z-axis and of size ℓ along the x direction. The presence of the crack induces a modification of the stress field Σ which is now written

$$\Sigma(x,y) = \Sigma^{(0)} + \Sigma^{(1)}(x,y) \tag{4}$$

In the immediate vicinity of the crack, the singular mode III stress field can only be determined if the entire constitutive law is specified. For example, HRR solutions can be exhibited for a power-law hardening material. For an elastic-damaged material, solutions can also be constructed in the vicinity of a crack tip[35–37]. However, the far-field influence of the crack assumes a much more general character since the additional contribution $\Sigma^{(1)}$ is much smaller than the background $\Sigma^{(0)}$. In this case, only the tangent stiffness is required provided one focuses on distances to the origin, r, much larger than the extent ℓ of the crack itself, $r \gg \ell$. It can be noted that the considered defect may be of any nature provided it *softens* a volume element lying at the origin. It could be a crack but also a more damaged region. The extension ℓ is assumed to be small compared to the specimen size L. Close to this defect, the stress field modification (from the uniformly damaged medium) is certainly very sensitive to the nature of this defect (e.g., stress concentration or even stress singularity). However past distances of the order of a few times ℓ (depending on the precision required) all such defects will induce a similar perturbation in an infinite medium as described below. In all cases, the stress modification will weaken in magnitude as the distance to the defect increases, and it will spread over wider regions. If one were to consider the effect of a boundary, although a complete computation should be performed, the small amplitude of the stress modification and its wide spreading should not alter drastically the solution proposed herein. Consequently, the solution is applicable only provided there is a clear separation of length scales between the defect size and the system size L, $\ell \ll L$.

In order to address this problem, without having to tackle the immediate vicinity of the crack, we expand the far-field in contributions of multipoles of an equivalent force distribution within a domain which contains the crack. Since no external force is exerted on the crack itself, the algebraic sum of all equivalent forces is zero (the monopole term vanishes). Thus the leading non-vanishing contribution comes from a dipolar term. It can be expressed physically as the effect of two opposite forces, of magnitude $\sigma_{yz}^{(0)}\ell$, applied on the two mouths of the crack. In the following, we only consider this dominant contribution, since all higher order terms (quadrupolar, ...) vanish with distance r more quickly than the dipole term.

In the case of a virgin elastic material, the dipolar field can be conveniently expressed in the complex plane as

$$\mu w^{(1)} = \Re\left(\frac{A}{\zeta}\right) \tag{5}$$

where $w^{(1)}$ is the displacement field due to the presence of the crack and $\zeta = x + iy = re^{i\theta}$. Thus the stress field $\Sigma^{(1)}$ vanishes as $1/r^2$.

Coming back to the original damage case, we have to distinguish between two domains, $\mathcal{D}_u$ and $\mathcal{D}_l$ depending on whether the stress component $\sigma_{yz}^{(1)}$ is positive (subscript l for "loading") or negative (subscript u for "unloading"). At a large distance from the crack, where the dipolar approximation is appropriate, there is no longer any length scale in the problem, and thus the domains have to be invariant under a dilation with respect to the origin. Thus they are cones (or sectors) in the plane as sketched in Fig. 2

In the sequel, we will mostly focus on the additional stress field $\Sigma^{(1)}$, and its corresponding displacement field $w^{(1)}$. Thus for simplicity of notations, we will drop the superscript (1).

The difficulty of the problem is that the boundaries of the domains are unknown, and should be obtained self-consistently with the solution. In each domain, we have to solve the following equilibrium equation

$$K_0 \frac{\partial^2 w_i}{\partial x^2} + K_i \frac{\partial^2 w_i}{\partial y^2} = 0 \tag{6}$$

where the index i is either u or l. The boundary between $\mathcal{D}_u$ and $\mathcal{D}_l$ is such that the stress σ_{yz} is vanishing and the stress vector is continuous

$$\frac{\partial w_i}{\partial y} = 0 \tag{7}$$

and

$$\left[K_u \frac{\partial w_u}{\partial x} - K_l \frac{\partial w_l}{\partial x}\right] n_x = 0 \tag{8}$$

where n_x is the x-component of the normal to the boundary between the two domains. Finally concerning the boundary conditions, the displacement field w_i should vanish at infinity and be singular at the origin. A dipolar field will naturally be selected if the proper symmetries are imposed: w_i symmetric under $x \to -x$ reversal, and antisymmetric with respect to $y \to -y$.

The absence of characteristic length scale (in the limit $r/\ell \gg 1$) in this problem imposes that the shape of the domains is a "cone". The boundary has thus to be $x = \pm ay$ where

a has to be determined (Fig. 2). Moreover, the displacement field admits a decoupled form $V = r^{-\alpha}\varphi(\theta)$ in polar coordinates, for the same reason of absence of characteristic scale. This singular form is classically encountered when analyzing scale-free problems in elasticity[38].

In each domain, $i = u$ or l, one may define a rescaled coordinate system[39,40], $y'_i = y/\sqrt{K_i/K_0}$, and $x'_i = x$, such that the problem to solve is simply

$$\nabla'^2 w_i = 0 \tag{9}$$

where ∇'^2 is the Laplacian operator in the corresponding (x', y') space.

The above remarks together with symmetry considerations indicate that the potential admits the simple form

$$\begin{aligned} \mu w_u &= \Re(A_u {z'_u}^{-\alpha}) \\ \mu w_l &= \Re(A_l (z'_l e^{-i\pi/2})^{-\alpha}) \end{aligned} \tag{10}$$

where A_i are reals.

Returning to the original coordinates, and introducing the notations $k_i = K_i/K_0$ for $i = u, l$ we have in domain $\mathcal{D}_u$

$$\mu w_u = A_u (y^2/k_u + x^2)^{-\alpha/2} \cos(\alpha \mathrm{A}tan(x\sqrt{k_u}/y)) \tag{11}$$

and similarly in domain $\mathcal{D}_l$

$$\mu w_l = A_l (y^2/k_l + x^2)^{-\alpha/2} \sin(\alpha(\mathrm{A}tan(x\sqrt{k_l}/y) - \pi/2)) \tag{12}$$

We now simply need to determine A_u/A_l, α and a.

The stress σ_{yz} in domain $\mathcal{D}_u$ is

$$\begin{aligned} \sigma_{yz} = A_u \alpha \sqrt{k_u} (y^2/k_u + x^2)^{-1-\alpha/2} \Big(&\frac{y}{\sqrt{k_u}} \cos(\alpha \mathrm{A}tan(x\sqrt{k_u}/y)) \\ &- x \sin(\alpha \mathrm{A}tan(x\sqrt{k_u}/y)) \Big) \end{aligned} \tag{13}$$

and similarly in domain $\mathcal{D}_l$

$$\begin{aligned} \sigma_{yz} = A_l \alpha \sqrt{k_l} (y^2/k_l + x^2)^{-1-\alpha/2} \Big(&\frac{y}{\sqrt{k_l}} \sin[\alpha(\mathrm{A}tan(x\sqrt{k_l}/y) - \pi/2)] \\ &+ x \cos[\alpha(\mathrm{A}tan(x\sqrt{k_l}/y) - \pi/2)] \Big) \end{aligned} \tag{14}$$

Setting these quantities equal to 0 along the boundary $x = ay$, we get

$$\alpha \mathrm{A}tan(a\sqrt{k_u}) = \mathrm{A}tan\left(\frac{1}{a\sqrt{k_u}}\right) \tag{15}$$

and for domain $\mathcal{D}_l$

$$\alpha(\mathrm{A}tan(a\sqrt{k_l}) - \pi/2) = -\mathrm{A}tan\left(a\sqrt{k_l}\right) \tag{16}$$

By introducing the notation $\psi_u = \mathrm{A}tan(a\sqrt{k_u})$, and similarly $\psi_l = \mathrm{A}tan(a\sqrt{k_u})$, we have to solve for

$$\begin{aligned} \psi_u &= \frac{\pi}{2(1+\alpha)} \\ \psi_l &= \frac{\alpha\pi}{2(1+\alpha)} \\ \frac{\tan(\pi/2(\alpha+1))}{\tan(\alpha\pi/2(\alpha+1))} &= \sqrt{\frac{k_u}{k_l}} \end{aligned} \tag{17}$$

The latter equation determines implicitly α from the ratio of secant to tangent stiffnesses, and the former gives the geometry of the interface between the two domains. Explicitly, a is obtained from the definition of either ψ_u or ψ_l, $a = \tan(\psi_u)/\sqrt{k_u}$.

Finally, the continuity of σ_{xz} along $x = ay$ imposes

$$A_u(k_u)^{(\alpha+3)/2}\cos^{\alpha+1}(\psi_u) = -A_l(k_l)^{(\alpha+3)/2}\cos^{\alpha+1}(\psi_l) \tag{18}$$

so that the ratio A_u/A_l can be determined. Any of the two amplitudes A_u or A_l is a simple scale factor that depends on the exact crack geometry and constitutive equations in the immediate vicinity of the crack. The order of magnitude of A_l can easily be obtained from the integral of the stress transferred across the $y = 0$ plane, which should balance $\sigma_{yz}^{(0)}\ell$. We deduce

$$A_l \simeq \frac{\sigma_{yz}^{(0)}\ell^{1+\alpha}}{\alpha\sqrt{k_l}} \tag{19}$$

With this last equation, all the quantities of interest are now determined. Figure 3 shows the change of the exponent α with the ratio k_l/k_u. There are two interesting limits to consider. When $k_l/k_u \approx 1$, the exponent $\alpha \approx 1$ and the elastic case is recovered. This case will be referred to as early damage regime. Conversely, when $k_l/k_u \approx 0$, the exponent α vanishes. This case will be referred to as peak stress regime.

A. Early damage regime

The first regime corresponds to the initial stage of damage. The contrast between k_u and k_l is small. By using the above equations, and introducing the notation $k_l/k_u = 1 - \epsilon$, we can obtain the following behavior for the solution

$$\begin{aligned}
\alpha &= 1 - \frac{\epsilon}{\pi} \\
\psi_u &= \frac{\pi}{4} + \frac{\epsilon}{8} \\
\psi_l &= \frac{\pi}{4} - \frac{\epsilon}{8} \\
a &= (k_u k_l)^{-1/4}
\end{aligned} \tag{20}$$

In particular, the exponent α is close to one as expected from the elastic case.

B. Peak stress regime

The other limit corresponds to the point where one approaches the peak of the stress-strain relationship. There, the tangent loading modulus vanishes (i.e., $k_l \to 0$), and we can expand the above relations to extract

$$\alpha = \frac{2}{\pi}\left(\frac{k_l}{k_u}\right)^{1/4} \tag{21}$$

and

$$\begin{aligned}
\psi_u &= \frac{\pi}{2} - \left(\frac{k_l}{k_u}\right)^{1/4} \\
\psi_l &= \left(\frac{k_l}{k_u}\right)^{1/4} \\
a &= (k_u k_l)^{-1/4}
\end{aligned} \tag{22}$$

The important feature is that the exponent α tends to 0. This means in practice that as the peak stress is approached, the effective interaction Green's function (decaying as the distance raised to the power $-\alpha$) becomes longer and longer ranged. It approaches the Global Load Sharing rule used in the fiber bundle model. It is also noteworthy that the loaded zone becomes a thinner and thinner "cone" along the x-axis, whose aperture angle is $(k_u k_l)^{1/4}$.

III. DISCUSSION AND CONCLUSION

The above discussion shows that the load transfer which arises from the opening of a new crack in mode III, has a long range part which evolves continuously as the tangent loading stiffness decreases. In particular at the onset of localization, the perturbation field due to the presence of the crack gives rise to a non-vanishing displacement in a direction aligned with the crack. Along this direction, we recover a condition similar to the global load sharing case.

However, we would like to point out that although the previous calculation appears as similar to a Green function, it is not the case, in the sense that the loading/unloading conditions prevent the construction of admissible solutions from simple superposition. For instance the case of two cracks in interaction is outside the scope of the present analysis. This strong non-linearity is certainly inherent to the damage behavior. Nevertheless, the validity of the present result concerning the far field is not confined to the case of a crack. It is also applicable to any increase of damage in a limited region of space.

Furthermore, in the framework of homogenization, provided there exists a scale above which no spatial correlations are relevant, one may homogenize (and use a mean-field theory) the elastic properties. Below this scale, even when all moduli are positive definite, such a procedure is not appropriate. The key feature in a damage theory is the fact that one does not control directly the spatial correlations in cracks. Moreover, as they are produced through elastic interactions which are long-range fields, the emergence of long-range (infinite) correlations may be feared. This may be at play in particular at the transition to a bifurcation such as strain or damage localization, and hence, even just prior to localization, the applicability of a mean-field procedure valid above a diverging length scale may be a nice mathematical property, but it will have limited practical consequences.

Lastly, even though the computation has been performed in the framework of anti-plane geometry, similar results should hold quite generally. A proper localization criterion which would take into account this loading/unloading condition is however out of reach of the present analysis.

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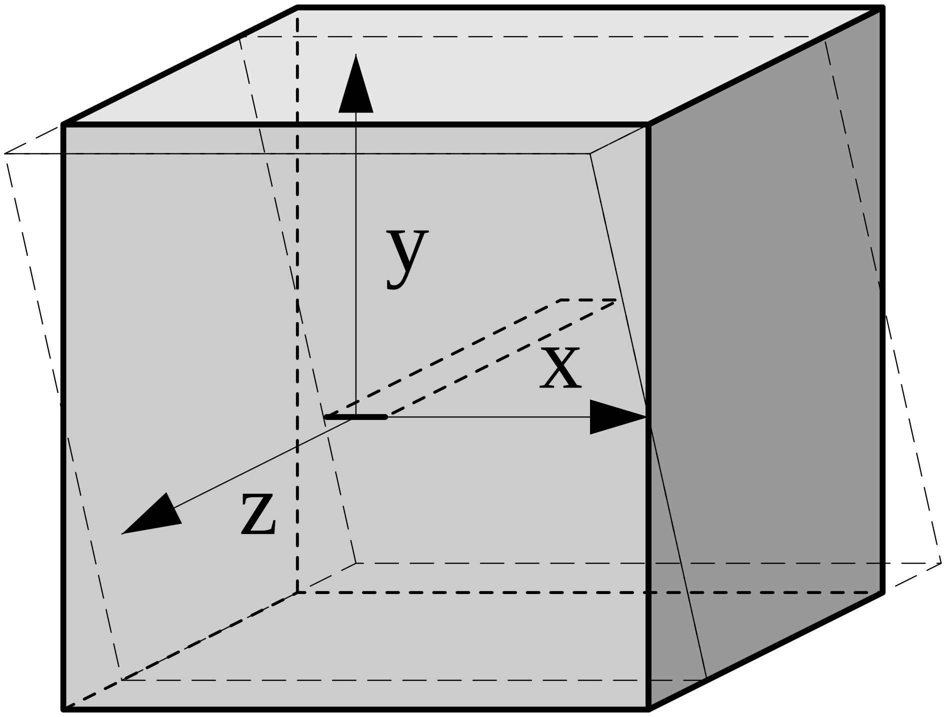

FIG. 1: An antiplane (mode III) loading condition is assumed (the displacement is everywhere along z, and independent of z). In addition to a homogeneous loading (leading to the sheared geometry shown with a dashed line), a crack centered along the z axis and parallel to the (x, z) plane is considered. The perturbed stress field due to this crack is computed in the far-field limit.

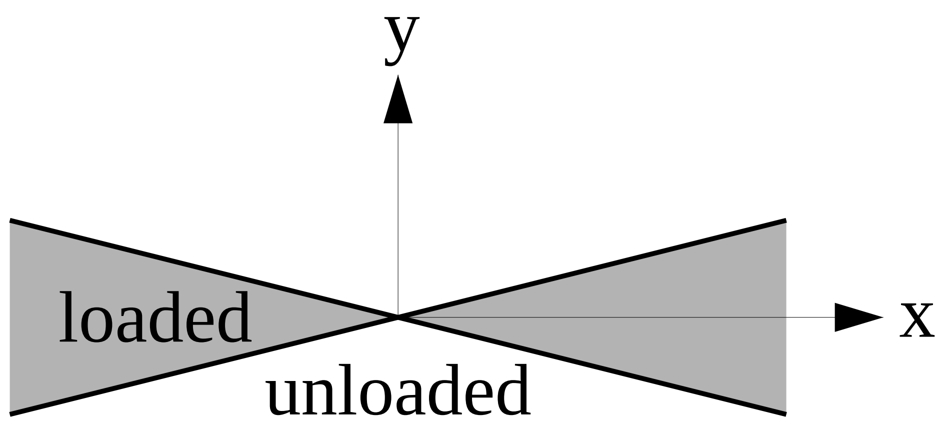

FIG. 2: Loading (shaded) and unloading domains in a far-field analysis. The crack is at the origin aligned along the x axis.

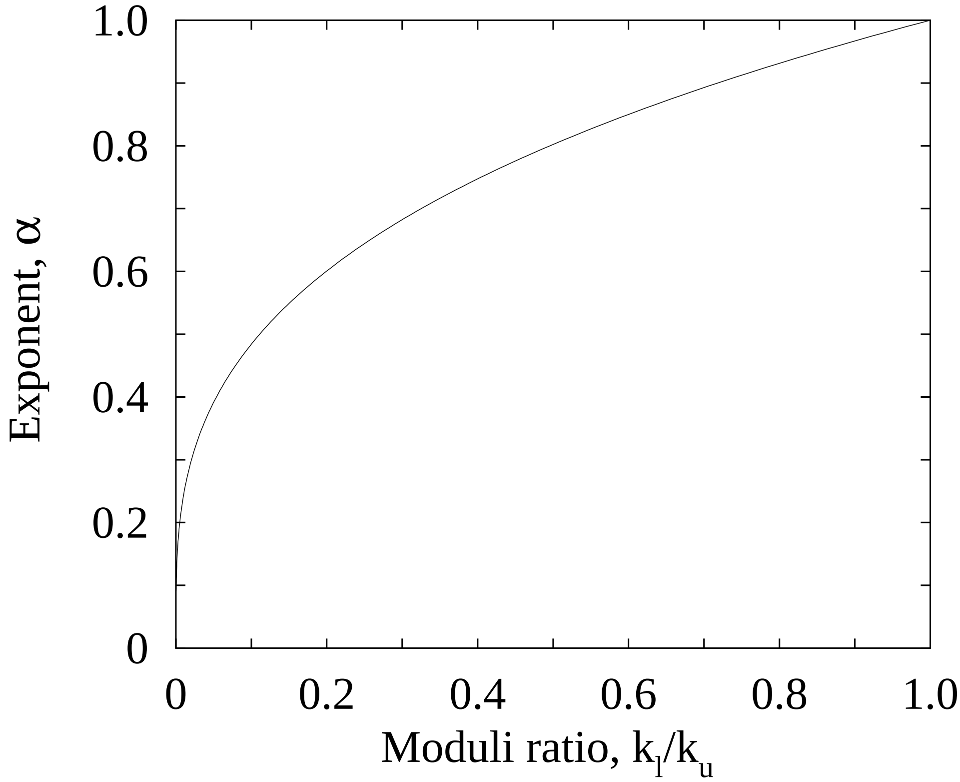

FIG. 3: Change of the exponent of the influence function with the ratio of tangent and secant moduli. Note that the exponent α vanishes as the 1/4th power of the ratio k_l/k_u.