Modular representations of cyclotomic Hecke algebras of type G(r,p,n)

Gwenaelle Genet, Nicolas Jacon

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Gwenaëlle Genet*and Nicolas Jacon†

Abstract

We give a classification of the simple modules for the cyclotomic Hecke algebras over $\mathbb{C}$ in the modular case. We use the unitriangular shape of the decomposition matrices of Ariki-Koike algebras and Clifford theory.

1 Introduction

Let r, p and n be integers such that p divides r. The complex reflection group of type $G(r, p, n)$ is defined to be the groups of $n \times n$ permutation matrices such that the entries are either 0 or r^{th} roots of unity and the product of the nonzero entries is a $\frac{r}{p}$-root of unity.

In [2] and [5], Ariki and Broué-Malle have defined a Hecke algebra associated to each complex reflection group $G(r, p, n)$. It can be seen as a deformation of the algebra of the group $G(r, p, n)$. According to a conjecture of Broué and Malle, such algebras, called cyclotomic Hecke algebras, should occur as endomorphism algebras of the Lusztig induced character. The representation theory of cyclotomic Hecke algebras of type $G(r, 1, n)$ (also known as Ariki-Koike algebras) is beginning to be well understood. They are cellular algebras, the simple modules have been classified in both semi-simple and modular cases and the decomposition matrices are known in characteristic 0 (see [21] for a survey of these results).

In [2], Ariki has shown that a Hecke algebra of type $G(r, p, n)$ can be considered as the 0-component of a graded system for a Hecke algebra of type $G(r, 1, n)$ with a special choice of parameters. As a consequence, in the semi-simple case, he has given a complete set of non isomorphic simple modules by using Clifford theory. In the modular case, partial results have been obtained by Hu in [17] (see also [16] and [18] where the case $r = p = 2$ which corresponds to Hecke algebras of type D_n is studied). The main problem is that the restrictions of the simple modules for non semi-simple Ariki-Koike algebras are much more complicated to describe than for semi-simple ones.

The purpose of this paper is precisely to give a parametrization of the simple modules for the non semi-simple cyclotomic Hecke algebras of type $G(r, p, n)$

*Address: UFR Mathématiques - Case 7012 - Université Denis Diderot - Paris 7 - 75251 PARIS Cedex 05, France. E-mail address: genet@math.jussieu.fr

†Address: Université de Caen, UFR des Sciences Département de Mathématiques Laboratoire LMNO, Campus II 14032 Caen cedex, France. E-mail address: jacon@math.unicaen.fr

over $\mathbb{C}$. It comes from the "canonical basic set" introduced by Geck and Rouquier in [13] and extended by the second author in [20]. This set induces a parametrization of the simple modules of non semi-simple Ariki-Koike algebras by some FLOTW multipartitions (this kind of multipartitions has been defined by Foda, Leclerc, Okado, Thibon and Welsh in [9]). We proceed in an analogous way than in [14, Theorem 2.1], using the unitriangular shape of the decomposition matrices of Ariki-Koike algebras and Clifford theory.

2 Cyclotomic Hecke algebras, Clifford theory

2.A Cyclotomic Hecke algebras

Let r, p and n be integers such that p divides r. We denote $d := \frac{r}{p}$. Let R be an integral ring, q a unit of R and a sequence $x = (x_1, \ldots, x_d)$ of elements in R. Denote $\mathfrak{H}^{q,x}_{r,p,n}(R)$, the Hecke R-algebra associated to the complex reflection group $G(r,p,n)$, for the parameters q, x, defined by the following presentation:

$$\begin{array}{ll}
\textit{generators}: & a_0, \ldots, a_n, \\
\textit{relations}: & (a_0 - x_1)\ldots(a_0 - x_d) = 0 \\
& (a_i - q)(a_i + 1) = 0,\ 1 \leq i \leq n, \\
& a_1 a_3 a_1 = a_3 a_1 a_3, \\
& a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1},\ 2 \leq i \leq n-1, \\
& (a_1 a_2 a_3)^2 = (a_3 a_1 a_2)^2, \\
& a_1 a_i = a_i a_1,\ 4 \leq i \leq n, \\
& a_i a_j = a_j a_i,\ 2 \leq i < j \leq n,\ j \geq i+2, \\
& a_0 a_1 a_2 = (q^{-1} a_1 a_2)^{2-r} a_2 a_0 a_1 + (q-1) \displaystyle\sum_{i=1}^{r-2} (q^{-1} a_1 a_2)^{1-k} a_0 a_1, \\
& \qquad\quad = a_1 a_2 a_0, \\
& a_0 a_i = a_i a_0,\ 3 \leq i \leq n.
\end{array}$$

We have the following special cases:

- if $r = p = 1$, $\mathfrak{H}^{q,x}_{r,p,n}(R)$ is the Hecke algebra of type A_{n-1},
- if $r = 2$ and $p = 1$, $\mathfrak{H}^{q,x}_{r,p,n}(R)$ is the Hecke algebra of type B_n,
- if $r = p = 2$, $\mathfrak{H}^{q,x}_{r,p,n}(R)$ is the Hecke algebra of type D_n,
- if $p = 1$, $\mathfrak{H}^{q,x}_{r,p,n}(R)$ is the Ariki-Koike algebra defined in [4].

Suppose the ring R contains a p^{th} root of unity η_p and a p^{th} root y_i of x_i, for each $i \in [1,d]$. We define a sequence $Q = (Q_1, \ldots, Q_r)$ of elements in R from the sequence x: for $j \in [1,r]$, $j = sp + t$ with $s \in [0, d-1]$ and $t \in [1,p]$, let $Q_j := y_{s+1} \eta_p^{t-1}$. Then, $\mathfrak{H}^{q,Q}_{r,1,n}(R)$ is the R-algebra defined by

$$\begin{array}{ll}
\textit{generators}: & T_1, \ldots, T_n, \\
\textit{relations}: & (T_1^p - x_1)\ldots(T_1^p - x_d) = 0, \\
& (T_i - q)(T_i + 1) = 0,\ 2 \leq i \leq n, \\
& T_1 T_2 T_1 T_2 = T_2 T_1 T_2 T_1, \\
& T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},\ 2 \leq i \leq n-1, \\
& T_i T_j = T_j T_i,\ 1 \leq i < j \leq n,\ j \geq i+2.
\end{array}$$

We use [2, Proposition 1.6] and identify a_0 with T_1^p, a_1 with $T_1^{-1}T_2T_1$ and a_i with T_i, for $i \in [2,n]$, such that $\mathfrak{H}^{q,x}_{r,p,n}(R)$ is considered as a subalgebra of $\mathfrak{H}^{q,Q}_{r,1,n}(R)$. Then by [15, §4.1], the Ariki-Koike algebra $\mathfrak{H}^{q,Q}_{r,1,n}(R)$ is graded over the cyclotomic Hecke algebra $\mathfrak{H}^{q,x}_{r,p,n}(R)$ by a cyclic group of order p:

$$\mathfrak{H}^{q,Q}_{r,1,n}(R) = \bigoplus_{i=0}^{p-1} T_1^i\, \mathfrak{H}^{q,x}_{r,p,n}(R).$$

Put $\mathfrak{H} := \mathfrak{H}^{q,Q}_{r,1,n}(R)$ and $\mathfrak{H}' := \mathfrak{H}^{q,x}_{r,p,n}(R)$. We can define an automorphism $g^{\mathfrak{H}}_{\mathfrak{H}'}$ of $\mathfrak{H}'$ by

$$g^{\mathfrak{H}}_{\mathfrak{H}'}(h) := T_1^{-1}hT_1, \text{ for all } h \in \mathfrak{H}'.$$

We also have an automorphism $f^{\mathfrak{H}}_{\mathfrak{H}'}$ of $\mathfrak{H}$ which is defined on the grading by

$$f^{\mathfrak{H}}_{\mathfrak{H}'}({T_1}^j h) = \eta_p^j\, T_1^j h, \text{ for } j \in [0, p-1], \ h \in \mathfrak{H}'.$$

The next theorem of Dipper and Mathas will be very useful in the following of this paper.

Theorem 2.1 ([8]) *With the previous notations, assume that we have a partition of Q:*

$$Q = Q^1 \coprod Q^2 \coprod \cdots \coprod Q^s$$

such that

$$f_\Gamma(q,Q) = \prod_{1 \le \alpha < \beta \le s} \prod_{\substack{Q_i \in Q^\alpha \\ Q_j \in Q^\beta}} \prod_{-n < a < n} (q^a Q_i - Q_j)$$

is a unit of R. Then, $\mathfrak{H}^{q,Q}_{r,1,n}(R)$ is Morita-equivalent to the following algebra:

$$\bigoplus_{\substack{n_1,\ldots,n_s > 0 \\ n_1 + \ldots + n_s = n}} \mathfrak{H}^{q,Q^1}_{r_1,1,n_1}(R) \otimes \ldots \otimes \mathfrak{H}^{q,Q^s}_{r_s,1,n_s}(R)$$

where for $i \in [1,s]$, $r_i = |Q^i|$.

2.B Clifford theory

Let $\mathbf{y}_1$, $\mathbf{y}_2$,...,$\mathbf{y}_d$ and $\mathbf{v}$ be indeterminates over $\mathbb{C}$. For $i \in [1,d]$, put

$$\mathbf{x}_i := \mathbf{y}_i^p$$

and for $i \in [1,r]$, $i = sp + t$ with $s \in [0, d-1]$, $t \in [1,p]$, $\eta_p := \exp(\frac{2i\pi}{p})$,

$$\mathbf{Q}_i := \eta_p^{t-1}\mathbf{y}_{s+1}.$$

Form the sequences $\mathbf{x} := (\mathbf{x}_1, \ldots, \mathbf{x}_d)$ and $\mathbf{Q} := (\mathbf{Q}_1, \ldots, \mathbf{Q}_r)$.

Let $A := \mathbb{C}[\mathbf{x}_1, \mathbf{x}_1^{-1}, \ldots, \mathbf{x}_d, \mathbf{x}_d^{-1}, \mathbf{v}, \mathbf{v}^{-1}]$, $K := \mathbb{C}(\mathbf{x}_1, \ldots, \mathbf{x}_d, \mathbf{v})$ its field of fractions.

Let $\theta : A \to \mathbb{C}$ be a ring homomorphism such that $\mathbb{C}$ is the field of fractions of $\theta(A)$.
We put for $i \in [1, d]$,

$$x_i := \theta(\mathbf{x}_i),$$

for $i \in [1, r]$,

$$Q_i := \theta(\mathbf{Q}_i)$$

and

$$v := \theta(\mathbf{v}).$$

Note that for $i \in [1, r]$, $i = sp + t$ with $s \in [0, d-1]$, $t \in [1, p]$, then $Q_i := \eta_p^{t-1} y_{s+1}$, for a complex number y_{s+1} such that $y_{s+1}^p = x_{s+1}$.
We also form the sequences $x := (x_1, \ldots, x_d)$ and $Q := (Q_1, \ldots, Q_r)$.

We will note for short $\mathbf{H}$, resp. H, for the Ariki-Koike algebras $\mathfrak{H}_{r,1,n}^{\mathbf{v},\mathbf{Q}}(K)$, resp. $\mathfrak{H}_{r,1,n}^{v,Q}(\mathbb{C})$ and $\mathbf{H}'$, resp. H', for the cyclotomic Hecke algebras $\mathfrak{H}_{r,p,n}^{\mathbf{v},\mathbf{x}}(K)$, resp. $\mathfrak{H}_{r,p,n}^{v,x}(\mathbb{C})$.

Since $\mathbf{H}$, resp. H, is graded over $\mathbf{H}'$, resp. H', we will give some results which comes from Clifford theory.

Let $\mathfrak{H}$ be the Ariki-Koike algebra $\mathbf{H}$, resp. H, and $\mathfrak{H}'$ be the cyclotomic Hecke algebra $\mathbf{H}'$, resp. H'. In both cases, we will denote by Res the functor of restriction from the category of the $\mathfrak{H}$-modules to the category of the $\mathfrak{H}'$-modules and by Ind the functor $\mathfrak{H} \otimes_{\mathfrak{H}'} -$ from the category of the $\mathfrak{H}'$-modules to the category of the $\mathfrak{H}$-modules.

Let V be a $\mathfrak{H}$-simple module and U be a simple submodule of $\operatorname{Res} V$. Put

$$o_{f,V} := \min\{k \in \mathbb{N}_{>0} \mid {}^{f^k}V \simeq V\}$$

where $f := f_{\mathfrak{H}'}^{\mathfrak{H}}$ and for $i \in [0, o_{f,V} - 1]$, ${}^{f^i}V$ is the $\mathfrak{H}$-module with the same underlying space as V and the structure of module is given by composing the original action of $\mathfrak{H}$ with f^i.
In the same way, let

$$o_{g,U} := \min\{k \in \mathbb{N}_{>0} \mid {}^{g^k}U \simeq U\}$$

where $g := g_{\mathfrak{H}'}^{\mathfrak{H}}$ and for $i \in [0, o_{g,U} - 1]$, ${}^{g^i}U$ is the $\mathfrak{H}'$-module with the same underlying space as U with the action of $\mathfrak{H}'$ twisted by g^i.
Denote $[X : Y]$ the multiplicity of a simple module Y in the semi-simple module X.

Lemma 2.2 *Denote for short $g := g_{\mathfrak{H}'}^{\mathfrak{H}}$ and $f := f_{\mathfrak{H}'}^{\mathfrak{H}}$. Every simple $\mathfrak{H}'$-module U appears as a direct summand of the restriction* $\operatorname{Res} V$ *of a simple $\mathfrak{H}$-module V with multiplicity-free. With such modules U and V, a conjugate ${}^{g^i}U$ of U, $i \in [0, o_{g,U} - 1]$, occurs in the restriction of the conjugate ${}^{f^j}V$ of V, for all*

$j \in [0, o_{f,V}-1]$ and the conjugates of V are the only simple $\mathfrak{H}$-modules whose restrictions contain such a ${}^{g^i}U$. Besides

$$\begin{aligned} \operatorname{Res} V &\simeq \bigoplus_{i=0}^{o_{f,V}-1} {}^{f^i}V, & (1)\\ \operatorname{Ind} U &\simeq \bigoplus_{i=0}^{o_{g,U}-1} {}^{g^i}U, & (2)\\ p &= o_{g,U}o_{f,V}. & (3) \end{aligned}$$

Proof: It is easy to see that

$$\operatorname{Res} V \simeq [\operatorname{Res} V : U] \bigoplus_{i=0}^{o_{g,U}-1} {}^{g^i}U, \tag{4}$$

(see [6, Theorem 11.1]). Using [14, Proposition 2.2], we get

$$\operatorname{Ind} U \simeq [\operatorname{Ind} U : V] \bigoplus_{i=0}^{o_{f,V}-1} {}^{f^i}V$$

as well as

$$\frac{p}{o_{f,V}\,o_{g,U}} = [\operatorname{Res} V : U][\operatorname{Ind} U : V]. \tag{5}$$

As $\operatorname{Res} V$ is g-stable, [6, Proposition 11.14] implies that $\operatorname{End}_{\mathfrak{H}}(\operatorname{Ind}\operatorname{Res} V)$ is graded over $\operatorname{End}_{\mathfrak{H}'}(\operatorname{Res} V)$ by a cyclic group of order p. By (4) and (5), the dimension of $\operatorname{End}_{\mathfrak{H}}(\operatorname{Ind}\operatorname{Res} V)$ over $\mathbb{C}$ is $\frac{p^2}{o_{f,V}}$ and by the grading over $\operatorname{End}_{\mathfrak{H}'}(\operatorname{Res} V)$, this dimension is equal to $p[\operatorname{Res} V : U]^2 o_{g,U}$. By comparison with (5), it follows that

$$[\text{Res } V : U] = [\text{Ind } U : V]. \tag{6}$$

Now, by definition of $o_{f,V}$, for short $\mathfrak{o}$, there exists an $\mathfrak{H}$-automorphism

$$t : V \to {}^{f^{\mathfrak{o}}}V.$$

So, $t^{\frac{p}{\mathfrak{o}}}$ is an $\mathfrak{H}$-isomorphism. By Schur's lemma, $t^{\frac{p}{\mathfrak{o}}}$ is a scalar. There exists a complex number c such that $(\frac{t}{c})^{\frac{p}{\mathfrak{o}}} = \mathrm{Id}_V$. Then

$$\operatorname{Res} V \simeq \bigoplus_{i=0}^{\frac{p}{\mathfrak{o}}-1} \operatorname{Ker}(\frac{t}{c} - \eta_p^{\mathfrak{o}i}\mathrm{Id}_V).$$

It is easy to see that for each $i \in [0, \frac{p}{\mathfrak{o}}-1]$, $\operatorname{Ker}(\frac{t}{c} - \eta_p^{\mathfrak{o}i}\mathrm{Id}_V)$ is an $\mathfrak{H}'$-module isomorphic to ${}^{g^i}\operatorname{Ker}(\frac{t}{c} - \mathrm{Id}_V)$. So the semi-simple module $\operatorname{Res} V$ decomposes into more than $\frac{p}{\mathfrak{o}}$ simple modules and by (5) by less than $\frac{p}{\mathfrak{o}}$ simple modules. Therefore $\operatorname{Res} V$ decomposes into exactly $\frac{p}{\mathfrak{o}}$ simple modules and $[\operatorname{Ind} U : V] = 1$ so $[\operatorname{Res} V : U] = 1$ by (6).
As the module induced from a simple $\mathfrak{H}'$-module to $\mathfrak{H}$ is semi-simple, the assertions of the Lemma are proved. ∎

2.C The semi-simple case

We keep the notations of the previous subsection.

We study the representation theory of the algebras $\mathbf{H} := \mathfrak{H}^{\mathbf{v},\mathbf{Q}}_{r,1,n}(K)$ and $\mathbf{H}' := \mathfrak{H}^{\mathbf{v},\mathbf{x}}_{r,p,n}(K)$. By [1, Main Theorem], $\mathbf{H}$ is a split semi-simple algebra. This implies that $\mathbf{H}'$ is also split semi-simple.

A complete set of non isomorphic modules for $\mathbf{H}$ has been given in [4] and [7]. Let Π^r_n be the set of r-partitions of n. We say that $\lambda = \big(\lambda(i)\big)_{i\in[1,r]}$ is a r-partition of size n if for every i, $\lambda(i)$ is a partition and $\Sigma^r_{i=1}|\lambda(i)| = n$.
For each r-partition $\lambda \in \Pi^r_n$, we can associate an $\mathfrak{H}^{\mathbf{v},\mathbf{Q}}_{r,1,n}(A)$-module $S^{\mathbf{v},\mathbf{Q}}(\lambda)$ called a "Specht module" that is free over A. Here, we use the definition of "classical" Specht module contrary to [7] where the dual Specht modules are defined. We have the following theorem.

Theorem 2.3 ([4],[7]) *The following set is a complete set of non isomorphic absolutely irreducible $\mathfrak{H}^{\mathbf{v},\mathbf{Q}}_{r,1,n}(K)$-modules:*

$$\left\{S^{\mathbf{v},\mathbf{Q}}_K(\lambda) := K \otimes_A S^{\mathbf{v},\mathbf{Q}}(\lambda) \mid \lambda \in \Pi^r_n\right\}.$$

For each $\lambda \in \Pi^r_n$, the set of standard tableaux of shape λ is a basis of the underlying $\mathbb{C}$-vector space of $S^{\mathbf{v},\mathbf{Q}}_K(\lambda)$ (see [4]). Using [4, Propositions 3.16, 3.17] and some combinatorial properties, it is easy to get the

Proposition 2.4 *Let $\lambda \in \Pi^r_n$. The map*

$$h^{\mathbf{v},\mathbf{Q}}_\lambda : {}^{f^{\mathbf{H}}_{\mathbf{H}'}}S^{\mathbf{v},\mathbf{Q}}_K(\lambda) \to S^{\mathbf{v},\mathbf{Q}}_K\big(\varpi(\lambda)\big) \tag{7}$$

that sends a standard λ-tableau $\mathbb{T} = (\mathbb{T}_i)_{i\in[1,r]}$ to the standard $\varpi(\lambda)$-tableau $\varpi(\mathbb{T}) := (\mathbb{T}_{\varpi^{-1}(i)})_{i\in[1,r]}$ is an $\mathbf{H}$-isomorphism where ϖ is the permutation of $\mathfrak{S}_r$ defined by $\mathbf{Q}_{\varpi(i)} := \eta_p\mathbf{Q}_i$, for all $i \in [1,r]$ and $\varpi(\lambda) = (\lambda(\varpi^{-1}(1)),\ldots,\lambda(\varpi^{-1}(r)))$.

Define

$$o_\lambda := \min\{k \in \mathbb{N}_{>0} \mid \varpi^k(\lambda) = \lambda\}.$$

It is clear that $o_\lambda = o_{f,S^{\mathbf{v},\mathbf{Q}}_K(\lambda)}$ and

$$\operatorname{Res} S^{\mathbf{v},\mathbf{Q}}_K(\lambda) \simeq \bigoplus_{i=0}^{\frac{p}{o_\lambda}-1} \operatorname{Ker}(h^{o_\lambda}_\lambda - \eta_p^{o_\lambda i}\operatorname{Id}_{S^{\mathbf{v},\mathbf{Q}}_K(\lambda)}),$$

where $h^{o_\lambda}_\lambda : {}^{f^{o_\lambda}}S^{\mathbf{v},\mathbf{Q}}_K(\lambda) \to S^{\mathbf{v},\mathbf{Q}}_K\big(\lambda\big)$ sends a standard λ-tableau $\mathbb{T}$ to the standard λ-tableau $\varpi^{o_\lambda}(\mathbb{T})$. For $i \in [0, \frac{p}{o_\lambda} - 1]$, put

$$S^{\mathbf{v},\mathbf{Q}}_K(\lambda, i) := \operatorname{Ker}(h^{o_\lambda}_\lambda - \eta_p^{o_\lambda i}\operatorname{Id}_{S^{\mathbf{v},\mathbf{Q}}_K(\lambda)}). \tag{8}$$

Let $\mathcal{L}$ be a set of representatives of r-partitions for the action of the cyclic group generated by ϖ over Π_n^r. It follows from Lemma 2.2, the

Proposition 2.5 *The set*

$$\{S_K^{\mathbf{v},\mathbf{Q}}(\lambda,i)\ ;\ i\in[0,\frac{p}{o_\lambda}-1],\ \lambda\in\mathcal{L}\}$$

is a complete set of non isomorphic simple $\mathbf{H}'$*-modules.*

If $\mathbb{T}$ is a standard λ-tableau, then for $j\in[0,\frac{p}{o_\lambda}-1]$, the element

$$(\mathbb{T},j):=\sum_{i=0}^{\frac{p}{o_\lambda}-1}\eta_p^{-o_\lambda ij}\varpi^{o_\lambda i}(\mathbb{T})$$

belongs to the eigenspace $\mathrm{Ker}(h_\lambda^{o_\lambda}-\eta_p^{o_\lambda j}\mathrm{Id}_{S_K^{\mathbf{v},\mathbf{Q}}(\lambda)})$. It comes from the fact that $\mathbb{T}=\dfrac{o_\lambda}{p}\displaystyle\sum_{j=0}^{\frac{p}{o_\lambda}-1}(\mathbb{T},j)$, that for every $j\in[0,\frac{p}{o_\lambda}-1]$, the eigenspace $\mathrm{Ker}(h_\lambda^{o_\lambda}-\eta_p^{o_\lambda j}\mathrm{Id}_{S_K^{\mathbf{v},\mathbf{Q}}(\lambda)})$ is generated by the $(\mathbb{T},j)$, $\mathbb{T}$ a standard λ-tableau. So we have just described the simple $\mathbf{H}'$-modules, see also [2].

The following part is now concerned with the non semi-simple case: simple $\mathfrak{H}_{r,1,n}^{q,Q}(\mathbb{C})$-modules and $\mathfrak{H}_{r,p,n}^{q,x}(\mathbb{C})$-modules.

2.D The non semi-simple case

We keep the notations of §2.B.

2.D.1 Decomposition maps

The aim of this part is to define the decomposition maps for $H:=\mathfrak{H}_{r,1,n}^{q,Q}(\mathbb{C})$ and $H':=\mathfrak{H}_{r,p,n}^{q,x}(\mathbb{C})$. Recall that $A:=\mathbb{C}[\mathbf{x}_1,\mathbf{x}_1^{-1},\dots,\mathbf{x}_d,\mathbf{x}_d^{-1},\mathbf{v},\mathbf{v}^{-1}]$, $K:=\mathbb{C}(\mathbf{x}_1,\dots,\mathbf{x}_d,\mathbf{v})$ and that $\theta:A\to\mathbb{C}$ is a ring homomorphism.

Using [10], there exists a discrete valuation ring $\mathcal{O}$ with maximal ideal $I(\mathcal{O})$ such that $A\subset\mathcal{O}$ and $I(\mathcal{O})\cap A=\ker(\theta)$. Let $k_0:=\mathcal{O}/I(\mathcal{O})$ be the residue field of $\mathcal{O}$. Denote by $^-:\mathcal{O}\to k_0$ the canonical map. We obtain the algebras $\mathfrak{H}_{r,1,n}^{\overline{\mathbf{q}},\overline{\mathbf{Q}}}(k_0)$ and $\mathfrak{H}_{r,p,n}^{\overline{\mathbf{q}},\overline{\mathbf{x}}}(k_0)$. The field k_0 can be considered as an extension of $\mathbb{C}$ that is the quotient field of $\theta(A)$. By [12, Lemma 7.3.4], there is an isomorphism between the Grothendieck groups of finitely generated H-modules and $\mathfrak{H}_{r,1,n}^{\overline{\mathbf{q}},\overline{\mathbf{Q}}}(k_0)$-modules (resp. H'-modules and $\mathfrak{H}_{r,p,n}^{\overline{\mathbf{q}},\overline{\mathbf{x}}}(k_0)$-modules):

$$R_0(H)\simeq R_0(\mathfrak{H}_{r,1,n}^{\overline{\mathbf{q}},\overline{\mathbf{Q}}}(k_0))\qquad\text{and}\qquad R_0(H')\simeq R_0(\mathfrak{H}_{r,p,n}^{\overline{\mathbf{q}},\overline{\mathbf{x}}}(k_0)).$$

As a consequence, we obtain well-defined decomposition maps by choosing $\mathcal{O}$-forms for the simple $\mathfrak{H}_{r,1,n}^{\mathbf{q},\mathbf{Q}}(K)$-modules (resp. $\mathfrak{H}_{r,p,n}^{\mathbf{q},\mathbf{x}}(K)$-modules) and reducing them modulo $I(\mathcal{O})$:

$$d^{r,1,n}:R_0(\mathbf{H})\to R_0(H),$$

$$d^{r,p,n} : R_0(\mathbf{H}') \to R_0(H').$$

For a simple $\mathbf{H}$-module V and W a simple $\mathbf{H}'$-module, there exist non negative integers $d^{(1)}_{V,M}$ with M a simple H-module and $d^{(p)}_{W,N}$ with N a simple H'-module such that:

$$d^{r,1,n}([V]) = \sum_{M\in \mathrm{Irr}(H)} d^{(1)}_{V,M}[M]$$

$$\text{and} \qquad d^{r,p,n}([W]) = \sum_{N\in \mathrm{Irr}(H')} d^{(p)}_{W,N}[N].$$

By using the same argument as in [11, Lemma 5.2] (see also [12, Theorem 7.4.3.c]), we obtain the following result.

Proposition 2.6 *The following diagram commutes:*

$$\begin{array}{ccc} R_0(\mathbf{H}) & \xrightarrow{\mathrm{Res}} & R_0(\mathbf{H}') \\ {\scriptstyle d^{r,1,n}}\downarrow & & \downarrow{\scriptstyle d^{r,p,n}} \\ R_0(H) & \xrightarrow{\mathrm{Res}} & R_0(H') \end{array}$$

Moreover, for any simple $\mathbf{H}$-module V and any simple H-module M, we have

$$d^{(1)}_{V,M} = d^{(1)}_{{}^{f}\mathbf{H}_{\mathbf{H}'}V,\, {}^{f}H_{H'}M},$$

and for any simple $\mathbf{H}'$-module W and any simple H'-module N, we have:

$$d^{(p)}_{W,N} = d^{(p)}_{{}^{g}\mathbf{H}_{\mathbf{H}'}W,\, {}^{g}H_{H'}N}.$$

In the following, we will see that under some additionnal hypotheses, the above property leads to some interesting results about the decomposition map of cyclotomic Hecke algebras of type $G(r,p,n)$ and about the simple H'-modules.

2.D.2 Simple modules of non semi-simple Ariki-Koike algebras

We will work under the following hypothesis. We assume that v is a primitive root of unity,

$$v := \eta_e = \exp(\frac{2i\pi}{e}),$$

for a positive integer $e > 1$. Let

$$f := \gcd(e,p),\ e = fe',\ p = fp',$$

then $\eta_f = \eta_p^{p'} = \eta_e^{e'}$.
By the definition of Q and Theorem 2.1, without lost of generality, we can suppose that there exist integers $v_1 \leq \ldots \leq v_\delta$ that belong to $[0, e'-1]$ such that Q consists of the complex numbers $\eta_e^{v_i}\eta_p^j$, where i ranges over $[1,\delta]$ and j ranges over $[0, p-1]$. Then $r = \delta f p'$ and we can split Q into p' sets:

$$Q = Q^1 \coprod \cdots \coprod Q^{p'},$$

with Q^j formed by the $\eta_e^{v_i}\eta_p^{lp'+j}$, $i \in [1,\delta]$ and $l \in [0, f-1]$. We consider Q^j as an ordered sequence

$$Q^j = (\eta_p^{j-1}\eta_e^{v_1}, \dots, \eta_p^{j-1}\eta_e^{v_\delta}, \eta_p^{p'+j-1}\eta_e^{v_1}, \dots, \eta_p^{p'+j-1}\eta_e^{v_\delta}, \dots \\ \eta_p^{(f-1)p'+j-1}\eta_e^{v_1}, \dots, \eta_p^{(f-1)p'+j-1}\eta_e^{v_\delta}). \tag{9}$$

Remark: First note that, for $j \in [1,p']$, $Q^j = \eta_p^{j-1}Q^1$ and

$$Q^1 = (\eta_e^{v_1}, \dots, \eta_e^{v_\delta}, \eta_e^{v_1+e'}, \dots, \eta_e^{v_\delta+e'}, \dots, \eta_e^{v_1+(f-1)e'}, \dots, \eta_e^{v_\delta+(f-1)e'}),$$

with $v_1 \leq \dots \leq v_\delta \leq v_1 + e' \leq \dots \leq v_\delta + (f-1)e'$.

Note also that the quotient of two elements that belong to Q^j, for $j \in [1,p']$, is a power of $v = \eta_e$ while the quotient of an element of Q^j by an element of Q^l, $l \in [1,p']$, $l \neq j$ is not such a power, by definition of f.

If we order the elements of Q by the ordered elements of Q^1, then the ordered elements of Q^2, etc, Theorem 2.1 implies that the Ariki-Koike algebra H is Morita equivalent to the algebra

$$\bigoplus_{\substack{n_1,\dots,n_{p'}\geq 0 \\ n_1+\dots+n_{p'}=n}} \mathfrak{H}_{f\delta,1,n_1}^{q,Q^1}(\mathbb{C}) \otimes \dots \otimes \mathfrak{H}_{f\delta,1,n_{p'}}^{q,Q^{p'}}(\mathbb{C}).$$

So it is clear that H is also Morita equivalent to the algebra

$$\bigoplus_{\substack{n_1,\dots,n_{p'}\geq 0 \\ n_1+\dots+n_{p'}=n}} \mathfrak{H}_{f\delta,1,n_1}^{q,Q^1}(\mathbb{C}) \otimes \dots \otimes \mathfrak{H}_{f\delta,1,n_{p'}}^{q,Q^1}(\mathbb{C}).$$

We order $\mathbf{Q}$ and we split it into p' sets following what we have done for Q:

$$\mathbf{Q} = \mathbf{Q}^1 \coprod \dots \coprod \mathbf{Q}^{p'}.$$

2.D.3 Parametrization of simple H-modules

We can now give a parametrization for the simple H-modules. By [3, Theorem 2.5], the simple modules of the Ariki-Koike algebra $\mathfrak{H}_{f\delta,1,n_i}^{q,Q_1}(\mathbb{C})$, $i \in [1,p']$, are the quotient modules $D_{\mathbb{C}}^{v,Q^1}(\lambda) := S_{\mathbb{C}}^{\mathbf{v},\mathbf{Q^1}}(\lambda)/J(S_{\mathbb{C}}^{\mathbf{v},\mathbf{Q^1}}(\lambda))$ of the specialisations of the Specht modules $S_{\mathbb{C}}^{\mathbf{v},\mathbf{Q^1}}(\lambda) := \mathbb{C} \otimes S^{\mathbf{v},\mathbf{Q^1}}(\lambda)$ by their radical of Jacobson, with λ a Kleshchev $f\delta$-partition (for the parameters Q^1) of size n_i. So the simple modules of H are labelled by the p'-tuples of Kleshchev $f\delta$-partitions (associated to Q^1) $(\lambda^1, \dots, \lambda^{p'})$ with $\sum_{i=1}^{p'} |\lambda^i| = n$. We will denote this set by Λ^0 and for $\lambda \in \Lambda^0$, $D_{\mathbb{C}}^{v,Q}(\lambda)$ the corresponding simple H-module.

In [20], another parametrization for the simple $\mathfrak{H}^{q,Q_1}_{f\delta,1,n_i}(\mathbb{C})$-modules has been found by using Lusztig's a-function (see Theorem 2.7 below). Following this paper, we will associate to each r-partition an a-value. To do this, we first describe the a-value of a $f\delta$-partition $\mu=(\mu(1),\mu(2),...,\mu(f\delta))$.

For $k=1,...,\delta$, $s=1,...,f$ and $j=(s-1)\delta+k$ we put $w_j:=v_k+(s-1)e'$ so that w_j is the power of the j^{th}-component of Q^1.

Then, for $j:=1,...,f\delta$, we define:

$$m^{(j)}:=w_j-\frac{je}{f\delta}+e$$

and for $s=1,...,n$, we put:

$$B'^{(j)}_s:=\mu(j)_s-s+n+m^{(j)}$$

where we use the convention that $\mu(j)_p:=0$ if p is greater than the height of $\mu(j)$. For $j=1,...,f\delta$, let $B'^{(j)}=(B'^{(j)}_1,...,B'^{(j)}_n)$. Then, we define:

$$a^{(1)}_{f\delta}(\mu):=\sum_{\substack{0\leq i\leq j<f\delta\\ (a,b)\in B'^{(i)}\times B'^{(j)}\\ a>b \text{ if } i=j}}\min\{a,b\}-\sum_{\substack{1\leq i,j\leq f\delta\\ a\in B'^{(i)}\\ 1\leq k\leq a}}\min\{k,m^{(j)}\}+g(n).$$

where $g(n)$ is a rational number which only depends on the $m^{(j)}$ and on n (the expression of g is given in [20]).

Let $\lambda\in\Pi^r_n$, write $\lambda=(\lambda(1),\ldots,\lambda(r))$ as $\lambda=(\lambda^1,\ldots,\lambda^{p'})$ where, for $i\in[1,p']$, $\lambda^i=\big(\lambda(f\delta(i-1)+1),\ldots,\lambda(f\delta(i-1)+f\delta)\big)$.

Finally, we define

$$a^{(1)}_r(\lambda):=\sum_{i=1}^{p'}a^{(1)}_{f\delta}(\lambda^i).$$

To each Specht module $S^{\mathbf{v},\mathbf{Q^1}}_K(\lambda)$ of $\mathfrak{H}^{\mathbf{q},\mathbf{Q_1}}_{f\delta,1,n'}(K)$, $n'\in[0,n]$ and λ an $f\delta$-partition, we also associate an a-value

$$a^{(1)}_{f\delta}\big(S^{\mathbf{v},\mathbf{Q^1}}_K(\lambda)\big):=a^{(1)}_{f\delta}(\lambda).$$

To each simple $\mathfrak{H}^{q,Q_1}_{f\delta,1,n'}(\mathbb{C})$-module $D^{v,Q^1}_{\mathbb{C}}(\lambda)$, $n'\in[0,n]$ and λ a Kleshchev $f\delta$-partition of size n', we again associate an a-value

$$a^{(1)}_{f\delta}(D^{v,Q^1}_{\mathbb{C}}(\lambda)):=\min\{a^{(1)}_{f\delta}(\mu)\mid\mu\in\Pi^{f\delta}_{n'},\ d^{(1)}_{S^{\mathbf{v},\mathbf{Q^1}}_K(\mu),D^{v,Q^1}_{\mathbb{C}}(\lambda)}\neq 0\}.$$

Theorem 2.7 ([19, Theorem 2.3.8]) *Let λ be a Kleshchev $f\delta$-partition of n' for a non-negative integer n'. There exists a unique $\mu:=\kappa(\lambda)$ such that $a^{(1)}_{f\delta}(D^{v,Q^1}_{\mathbb{C}}(\lambda))=a^{(1)}_{f\delta}(\mu)$ and $d^{(1)}_{S^{\mathbf{v},\mathbf{Q^1}}_K(\mu),D^{v,Q^1}_{\mathbb{C}}(\lambda)}\neq 0$. The function κ is a bijection between the set of the Kleshchev $f\delta$-partitions of size n' and the set of the FLOTW $f\delta$-partitions of size n'.*

Recall what is a FLOTW $f\delta$-partition $\lambda = \big(\lambda(1), \dots, \lambda(f\delta)\big)$ of size n' associated to the parameters v and Q^1, see [9]. Denote each partition $\lambda(i)$, $i \in [1, f\delta]$, as $(\lambda(i)_1, \lambda(i)_2, \dots)$. The multipartition λ satisfies

1. for $i \in [1, f]$, $j \in [1, \delta - 1]$, k positive integer,
$$\lambda\big((i-1)\delta + j\big)_k \geq \lambda\big((i-1)m + j + 1\big)_{k+v_{j+1}-v_j},$$
2. for $i \in [1, f-1]$, k positive integer,
$$\lambda(i\delta)_k \geq \lambda(i\delta + 1)_{k+v_1+e'-v_\delta},$$
3. for k positive integer,
$$\lambda(f\delta)_k \geq \lambda(1)_{k+v_1+e'-v_\delta},$$
as well as
4. to each node of the diagram of λ which is located in the a^{th} row and the b^{th} column of the c^{th} partition of λ (for two poditive integers a and b and $1 \leq c \leq f\delta$), we associate its residue $\eta_e^{b-a+v_c}$. Then, for any positive integer k, the cardinality of the set of the residues associated to the nodes in both the k^{th} columns and the right rims of the Young diagrams of λ is not e.

In the same way, to each Specht module $S_K^{\mathbf{v},\mathbf{Q}}(\lambda)$ of $\mathbf{H}$, $\lambda \in \Pi_n^r$, $\lambda = (\lambda^1, \dots, \lambda^{p'})$, we can define $a_r^{(1)}(S_K^{\mathbf{v},\mathbf{Q}}(\lambda)) := a_r^{(1)}(\lambda)$ that is the sum $\sum_{i=1}^{p'} a_{f\delta}^{(1)}(\lambda^i)$. To each simple H-module $D_{\mathbb{C}}^{v,Q}(\lambda)$, $\lambda \in \Lambda^0$, we associate the a-value
$$a_r^{(1)}(D_{\mathbb{C}}^{v,Q}(\lambda)) := \min\{a_r^{(1)}(\mu) \mid \mu \in \Pi_n^r,\ d^{(1)}_{S_K^{\mathbf{v},\mathbf{Q}}(\mu), D_{\mathbb{C}}^{v,Q}(\lambda)} \neq 0\}. \tag{10}$$
With Theorem 2.1, it is clear that there exists a unique $\mu := \kappa'(\lambda)$ such that $a_r^{(1)}(D_{\mathbb{C}}^{v,Q}(\lambda)) = a_r^{(1)}(\mu)$ and $d^{(1)}_{S_K^{\mathbf{v},\mathbf{Q}}(\mu), D_{\mathbb{C}}^{v,Q}(\lambda)} \neq 0$. In fact,
$$\kappa'(\lambda) = (\kappa(\lambda^1), \dots, \kappa(\lambda^{p'}))$$
if $\lambda = (\lambda^1, \dots, \lambda^{p'})$.
The function κ' is a bijection between Λ^0 and Λ^1 the set of the p'-tuples of FLOTW $f\delta$-partitions $(\lambda^1, \dots, \lambda^{p'})$ of size $\sum_{i=1}^{p'} |\lambda^i| = n$.

2.D.4 Preliminary results

In order to prove the main Theorem, we will need some preliminary results. First, recall that we have ordered $\mathbf{Q}$ by the ordered elements of $\mathbf{Q}^1$, then by the ordered elements of $\mathbf{Q}^2$ etc as in §2.D.2. Let ϖ be the permutation of $\mathfrak{S}_r$ defined in Proposition 2.4 and let
$$\lambda := (\lambda^1[1], ..., \lambda^1[f\delta], \lambda^2[1], ..., \lambda^2[f\delta], ..., \lambda^{p'}[1], ..., \lambda^{p'}[f\delta])$$
be an r-partition. Hence, following the notations of the previous paragraph, we have $\lambda^j[i] = \lambda(f\delta(j-1) + i)$. Then, it is easy to verify that

- for $i \in [1,(p'-1)f\delta]$, then $\varpi(i) = i + f\delta$,
- for $i \in [r - f\delta + 1, r - \delta]$, then $\varpi(i) = i - r + (f+1)\delta$,
- for $i \in [r - \delta + 1, r]$, then $\varpi(i) = i - r + \delta$.

Then, by Proposition 2.4, we have:

$$S_K^{\mathbf{v},\mathbf{Q}}(\lambda) \simeq S_K^{\mathbf{v},\mathbf{Q}}\big(\varpi(\lambda)\big)$$

where $\varpi(\lambda) = (\lambda^{p'}[f\delta - \delta + 1], \ldots, \lambda^{p'}[f\delta], \lambda^{p'}[1], \ldots, \lambda^{p'}[f\delta - \delta], \lambda^1[1], \ldots, \lambda^1[f\delta], \ldots, \lambda^{p'-1}[1], \ldots, \lambda^{p'-1}[f\delta])$.

Proposition 2.8 *Let $\lambda \in \Pi_n^r$, then $a_r^{(1)}(S_K^{\mathbf{v},\mathbf{Q}}(\lambda)) = a_r^{(1)}(f_{\mathbf{H}'}^{\mathbf{H}} S_K^{\mathbf{v},\mathbf{Q}}(\lambda))$. Besides, if $\lambda \in \Lambda^0$, then $a_r^{(1)}(D_{\mathbb{C}}^{v,Q}(\lambda)) = a_r^{(1)}(f_{H'}^{H} D_{\mathbb{C}}^{v,Q}(\lambda))$.*

Proof: By the definition of the a-value (see §2.D.3), we have:

$$a_r^{(1)}(S_K^{\mathbf{v},\mathbf{Q}}(\lambda)) := \sum_{i=1}^{p'} a_{f\delta}^{(1)}(\lambda^i[1], ..., \lambda^i[f\delta])$$

Thus, by using the above properties, it is sufficient to show that $a_{f\delta}^{(1)}(\lambda^{p'}[1], \ldots, \lambda^{p'}[f\delta]) = a_{f\delta}^{(1)}(\lambda^{p'}[f\delta - \delta + 1], \ldots, \lambda^{p'}[f\delta], \lambda^{p'}[1], \ldots, \lambda^{p'}[f\delta - \delta])$.

Using the notations of §2.D.3, for $i = 1, ..., \delta$ and $j = 1, ..., f$ we write $m^{(i)}[j] := m^{((j-1)\delta + i)}$. Then, all we have to do is to prove that $m^{(i)}[j]$ doesn't depend on j. We have $m^{(i)}[j] = v_i - \dfrac{((j-1)\delta + i)e}{f\delta} + (j-1)e' = v_i - \dfrac{ie}{f}$ because $e'f = e$. Hence, we obtain the desired result.

With the definition (10), it is trivial that the previous result about Specht modules implies that, for any $\lambda \in \Lambda^0$, $a_r^{(1)}(D_{\mathbb{C}}^{v,Q}(\lambda)) = a_r^{(1)}(f_{H'}^{H} D_{\mathbb{C}}^{v,Q}(\lambda))$. ∎

With Proposition 2.8 and Lemma 2.2, we can associate an a-value to each simple $\mathbf{H}'$-module U. If U appears in the restriction $\operatorname{Res} S_K^{\mathbf{v},\mathbf{Q}}(\lambda)$, for $\lambda \in \Pi_n^r$, put

$$a_r^{(p)}(U) := a_r^{(1)}\big(S_K^{\mathbf{v},\mathbf{Q}}(\lambda)\big).$$

In the same way, we can associate an a-value to each simple H'-module W. If W appears in the restriction $\operatorname{Res} D_{\mathbb{C}}^{v,Q}(\lambda)$, for $\lambda \in \Lambda^0$, then

$$a_r^{(p)}(W) := a_r^{(1)}(D_{\mathbb{C}}^{v,Q}(\lambda)).$$

Another useful proposition is the following one.

Proposition 2.9 *Let $\lambda \in \Lambda^1$, then $\varpi(\lambda) \in \Lambda^1$, where ϖ is defined in the* Proposition 2.4.

Proof: Write $\lambda \in \Lambda^1$ as $\lambda = (\lambda^1, \dots, \lambda^{p'})$ where $\lambda^i = (\lambda^i[1], \dots, \lambda^i[f\delta])$ is a FLOTW $f\delta$-partition for Q^1. We will see that $\varpi(\lambda) \in \Lambda^1$ so the result will be proved. The r-partition $\varpi(\lambda)$ is equals to $(\lambda^{p'}[f\delta - \delta + 1]\ , \dots,\ \lambda^{p'}[f\delta], \lambda^{p'}[1]\ , \dots,\ \lambda^{p'}[f\delta - \delta],\ \lambda^1[1]\ , \dots,\ \lambda^1[f\delta],\ \dots\ \lambda^{p'-1}[1]\ , \dots,\ \lambda^{p'-1}[f\delta])$. It belongs to Λ^1 if and only if $(\lambda^{p'}[f\delta - \delta + 1]\ , \dots,\ \lambda^{p'}[f\delta],\ \lambda^{p'}[1]\ , \dots,\ \lambda^{p'}[f\delta - \delta])$ is a FLOTW $f\delta$-partition. It is easy to verify that the three first conditions hold. Now, the set of the residues associated to the nodes in both the k^{th} columns and the right rims of the Young diagrams of this last multipartition consists of the residues of $\lambda^{p'}$ but multiplied by $\eta_e^{e'}$. So the fourth condition about FLOTW multipartitions hold as well as the Proposition. ∎

3 A parametrization of the simple H'-modules

We keep the above notations. The aim of this section is to describe a parametrization of the simple H'-modules, with $H' = \mathfrak{H}^{v,Q}_{r,p,n}(\mathbb{C})$. The proof is highly inspired by [14, Theorem 2.1].

Theorem 3.1 *For all simple H'-module N, there exists a simple $\mathbf{H}'$-module W_N such that*

$$d^{r,p,n}([W_N]) = [N] + \sum_{a_r^{(p)}(L) < a_r^{(p)}(N)} d_{W_N,L}[L],$$

and $a_r^{(p)}(N) = a_r^{(p)}(W_N)$.
Let $\Lambda^{1'} := \{W_N \mid N \in \mathrm{Irr}(H')\}$. Then $\Lambda^{1'}$ consists in the $S_K^{\mathbf{v},\mathbf{Q}}(\lambda, i)$, with $\lambda \in \Lambda^1 \cap \mathcal{L}$, $i \in [0, \frac{p}{o_\lambda} - 1]$ with the notations of the Proposition 2.5 and the map $N \in \mathrm{Irr}(H') \mapsto W_N \in \Lambda^{1'}$ is bijective. So $\Lambda^{1'}$ is a parametrization of the simple H'-modules.

Proof : Fix a simple H'-module N. By Lemma 2.2, there exists $\lambda \in \Lambda^0$ such that N appears in the restriction $\mathrm{Res} D_{\mathbb{C}}^{v,Q}(\lambda)$. We have

$$d^{r,1,n}([S_{\mathbb{C}}^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda))]) = [D_{\mathbb{C}}^{v,Q}(\lambda)] + \sum_{\substack{a_r^{(p)}(D_{\mathbb{C}}^{v,Q}(\mu)) < \\ a_r^{(p)}(D_{\mathbb{C}}^{v,Q}(\lambda))}} d^{(1)}_{S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda)), D_{\mathbb{C}}^{v,Q}(\mu)} [D_{\mathbb{C}}^{v,Q}(\mu)], \tag{11}$$

and $a_r^{(1)}\big(S_{\mathbb{C}}^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda))\big) = a_r^{(1)}(D_{\mathbb{C}}^{v,Q}(\lambda))$. With the notations of (8) and Proposition 2.6, this equality implies that

$$\sum_{i=0}^{\frac{p}{o_{\kappa(\lambda)}} - 1} d^{r,p,n}([S_{\mathbb{C}}^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda), i)]) = \sum_{i=0}^{\frac{p}{o_{f^H_{H'}, D_{\mathbb{C}}^{v,Q}(\lambda)}} - 1} [{}^{(g^H_{H'})^i}N] + [N'], \tag{12}$$

where N' is a sum of H'-modules, D, with a-value $a_r^{(p)}(D) < a_r^{(p)}(D_{\mathbb{C}}^{v,Q}(\lambda))$ and, for all i, j, $a_r^{(p)}(S_{\mathbb{C}}^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda),i)) = a_r^{(p)}({}^{(g_{H'}^{H})^i}N)$.
Denote by $\tau(\lambda)\in\Lambda^0$, the multipartition defined by $D_{\mathbb{C}}^{v,Q}(\tau(\lambda)) := {}^{f_{H'}^{H}}D_{\mathbb{C}}^{v,Q}(\lambda)$. If we twist the action of H by $f_{H'}^{H}$, the equality (11) gives that $\kappa(\tau(\lambda)) = \varpi(\kappa(\lambda))$ by Proposition 2.9 and by definition of κ. As this map is bijective, it is clear that

$$o_{f_{\mathbf{H'}}^{\mathbf{H}},S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda))} = o_{f_{H'}^{H},D_{\mathbb{C}}^{v,Q}(\lambda)}$$

and therefore for all $i\in[0,\frac{p}{o_{\kappa(\lambda)}}-1]$,

$$o_{g_{\mathbf{H'}}^{\mathbf{H}},S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda),i)} = o_{g_{H'}^{H},N}$$

with the last equality of Lemma 2.2.
Recall that for $i\in[0,\frac{p}{o_{\kappa(\lambda)}}-1]$, $S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda),i)$ is conjugate to $S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda),0)$ by $(g_{\mathbf{H'}}^{\mathbf{H}})^i$. The equality (12) implies that there exists an unique $i\in[0,\frac{p}{o_{\kappa(\lambda)}}-1]$ such that

$$d^{r,p,n}([S_{\mathbb{C}}^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda),i)]) = [N]+[N''], \tag{13}$$

where N'' is a sum of H'-modules, D, with a-value $a_r^{(p)}(D) < a_r^{(p)}(N)$. This construction makes clear all the statements of the Theorem. ∎

Remark: The above theorem is only concerns with the case where v is a primitive e^{th}-root of unity. If v isn't a root of unity, it is readily checked that an analogue of this theorem holds by replacing Λ^1 by the set of Kleshchev multipartitions Λ^0 at $e=\infty$. The proof of this result can be easily obtained following the outline of [14].

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