Blow-up surfaces for nonlinear wave equations, Part II

Satyanad Kichenassamy, Walter Littman

BLOW-UP SURFACES FOR NONLINEAR WAVE EQUATIONS, II[1]

Satyanad Kichenassamy and Walter Littman

School of Mathematics

University of Minnesota

Minneapolis, MN 55455-0487

Abstract

In this second part, we prove that the equation $\Box u = e^u$ has solutions blowing up near a point of any analytic, space-like hypersurface in $\mathbf{R}^n$, without any additional condition; if $(\phi(x,t) = 0)$ is the equation of the surface, $u - \ln(2/\phi^2)$ is not necessarily analytic, and generally contains logarithmic terms. We then construct singular solutions of general semilinear equations which blow-up on a non-characteristic surface, provided that the first term of an expansion of such solutions can be found. We finally list a few other simple nonlinear evolution equations to which our methods apply; in particular, formal solutions of soliton equations given by a number of authors can be shown to be convergent by this procedure.

[1] *Communications in Partial Differential Equations* **18** (11) 1869–1899 (1993)

1. Introduction.

We proved in part I of this paper that

$$\Box u = e^u, \tag{1}$$

where $\Box$ denotes the d'Alembertian in n space dimensions, has a singular solution of the form

$$u(x,t) = \ln(2/\phi^2) + v(x,t) \tag{2}$$

($x \in \mathbf{R}^n$, $t \in \mathbf{R}$, (x,t) close to (x_0, t_0)) where v *has a holomorphic extension* to a complex neighborhood of (x_0, t_0), $\phi(x,t) = t - \psi(x)$ with ψ analytic, and $\phi(x_0, t_0) = 0$, *if and only if* the *blow-up surface* Σ defined by $t = \psi(x)$ is space-like and has zero scalar curvature for the metric induced on Σ by its embedding into Minkowski space. The strategy was to show that if we define w by

$$w(x,t) = [u - \ln(2/\phi^2) - u_0(x) - u_1(x)\phi(x,t)]/\phi(x,t)^2, \tag{3}$$

with a suitable choice of u_0 and u_1, then w solves an equation of *Fuchsian type*; it was then proved that the initial-value problem for this equation, where the restriction of w to Σ is a prescribed holomorphic function, has a local analytic solution if and only if Σ is space-like and the curvature condition on Σ is satisfied; the solution is then unique. This procedure yields infinitely many solutions blowing up on Σ, depending on the choice of one arbitrary function on the surface.

We also recall that one consequence of this result is the *continuation* of the solutions of (1) of the form (2) beyond their blow-up surface. Indeed, if we agree on a continuation of the singular part $\ln(2/\phi^2)$, there is no choice for the continuation of u, since its "regular part" v is analytic.[2] It seems convenient

[2]There are two natural ways of continuing the function $\ln(2/t^2)$for $t < 0$: either by the same expression, which still makes sense and is real, or by $\ln 2 - 2\ln|t| \pm 2k\pi i$, k a non-zero integer.

to assume that the equation holds in a generalized sense, which, however, is *not* the ordinary weak formulation, which does not apply directly.

In this second part, we obtain the following results:

1. We construct singular solutions for (1) near any blow-up surface which is merely assumed to be space-like and analytic. The representation (2) is replaced by

$$u(x,t) = \ln(2/\phi^2) + v(x,t,\phi\ln\phi), \tag{4}$$

where v is analytic in all its $(n+2)$ arguments (§§2 and 3).

2. We prove in §6 that semi-linear systems for which the leading term of the expansion of a singular solution can be found, can be reduced to "generalized Fuchsian equations," introduced in §4, for which an existence theorem is given in §5.

3. We also prove in §6 that our procedure applies to any equation of the form

$$\Box u = P(u),$$

where P is a polynomial, and briefly describe the results for more general nonlinearities. We apply the procedure to soliton equations, and obtain the convergence of the formal solutions generated by the method of Weiss, Tabor, and Carnevale (WTC), as announced in part I.

The reader is referred to part I and to the references in §6 for the notation and a discussion of the literature.

2. Logarithmic expansions.

Sections 2 and 3 are devoted to the proof of the following result.

Theorem 1 *Let Σ be an analytic, space-like, hypersurface in Minkowski space. Fix a point $(x_0, t_0) \in \Sigma$. Then (1) has infinitely many solutions, depending on the choice of one arbitrary holomorphic function on Σ, defined near (x_0, t_0) and blowing up precisely on Σ. These solutions have the form*

$$u(x,t) = \ln(2/\phi^2) + v(x, t, \phi \ln \phi),$$

where v is analytic in all its $(n+2)$ arguments, and $\phi = 0$ is an equation for Σ, with $\nabla\phi \neq 0$.

We reduce in this section Eq. (1) to the form (8) (see below), for which we solve the initial-value problem in §3. We may assume $\phi = t - \psi(x)$.

Recall from Part I that if we define w by (3) off Σ, and v_0, v_1 by

$$v_0 = \ln(1 - |D\psi|^2)$$

and

$$v_1 = -\frac{\Delta\psi}{(1 - |D\psi|^2)},$$

then w solves:

$$\begin{aligned} &T^2 \left\{(1 - |D\psi|^2)w_{TT} - \Delta' w + 2\psi^i \delta_i^{i'} w_{Ti'} + (\Delta\psi)w_T\right\} \\ &+ T\left\{4(1 - |D\psi|^2)w_T + 4\psi^i\delta_i^{i'} w_{i'} + 2w\Delta\psi - \Delta v_1\right\} \\ &+ \{2(1 - |D\psi|^2)w + v_1\Delta\psi + 2\psi^i\partial_i v_1 - \Delta v_0\} \\ &\qquad - \frac{2}{T}\Delta\psi + \frac{2}{T^2}(1 - |D\psi|^2) \\ &= \frac{2}{T^2}e^{v_0}\left\{1 + T(v_1 + Tw) + \frac{1}{2}T^2(v_1 + Tw)^2 \right. \\ &\left. + \tfrac{1}{2}T^3(v_1 + Tw)^3 \textstyle\int_0^1 (1-\sigma)^2 \exp[\sigma T(v_1 + Tw)]\, d\sigma\right\} \end{aligned} \tag{5}$$

where $T = t - \psi(x)$ and $X^i = x^i$ for $1 \leq i \leq n$.

In fact, the choice of v_0 and v_1 above makes the terms in T^{-1} and T^{-2} in (5) vanish, and also causes the two linear terms in w to cancel each other.

As in the first Part, we use the summation convention on repeated indices *one of which is in the upper position and the other in the lower position.* Thus, there is summation in $a^i b_i$ but not in $a_i b_i$. More generally, all indices are raised and lowered using the Kronecker delta. We let $\psi_i = \partial_i \psi, \psi_{ij} = \partial_{ij}\psi \ldots$, and $\psi^i = \delta^{ik}\psi_k, \psi^{ij} = \delta^{ik}\delta^{jl}\psi_{kl} \ldots$ In particular, $\sum_{1\leq i\leq n}(\psi_i)^2 = \psi^i\psi_i$. We denote by Δ' the Laplacian in the X^i variables.

Let us set

$$Y = T \ln T,$$

and

$$w(T, X) = \lambda(X) \ln T + f(T, Y, X),$$

where

$$\lambda(X) = -\frac{2}{3} R(x)(1 - |D\psi|^2)^{-1},$$

R is the scalar curvature of Σ, given in terms of ψ in the Appendix of [3], and f is a new unknown.

We have, since $\partial_T Y = (1 + \ln T)$,

$$Tw(T, X) = \lambda Y + Tf(T, Y, X),$$

$$\begin{aligned} T\partial_T w &= \lambda(X) + Tf_T + T(1 + \ln T) f_Y \\ &= \lambda(X) + Nf, \end{aligned}$$

where

$$N = (T\partial_T + (T + Y)\partial_Y).$$

Equation (5) for w becomes

$$(1 - |D\psi|^2)(N(N + 3)f + 3\lambda) +$$
$$T\left[(\Delta\psi)(\lambda + Nf) + 2\psi^i\delta_i^{i'}\partial_{i'}(\lambda + Nf) - Y\Delta\lambda - T\Delta' f\right] +$$

$$
\begin{aligned}
&+ 4\psi^i \delta_i^{i'} \partial_{i'}(\lambda Y + Tf) + 2(\Delta\psi)(\lambda Y + Tf) \\
&- T\Delta v_1 + v_1 \Delta\psi + 2\psi_i \partial_i v_1 - \Delta v_0 \qquad (6)\\
= \quad & (1 - |D\psi|^2)\{(v_1 + \lambda Y + Tf)^2 \\
&+ T(v_1 + \lambda Y + Tf)^2 \int_0^1 (1-\sigma)^2 \exp(T\sigma(v_1 + \lambda Y + Tf))\, d\sigma\}.
\end{aligned}
$$

Observe that only Y occurs in this equation, not $\ln T$. We also note that if we let $T = 0$, we find, using the calculation of the constant term in Part I, that one must have

$$3\lambda(1 - |D\psi|^2) + 2R(x) = 0. \qquad (7)$$

This justifies our choice of λ. With this choice, all terms except the first on the l.h.s. have either T or Y as a factor.

Eq. (6) is readily converted into a first-order system: let

$$z_1 = f;$$

$$z_2 = Nf;$$

$$z_{2+i} = T\partial_{i'} f \quad (1 \le i' \le n),$$

where $\partial_{i'} = \partial_{X^i}$. Thus, $z_2 = z_3 = \cdots = 0$ for $T = Y = 0$.

We find

$$
\begin{aligned}
(1 - |D\psi|^2)(Nz_2 + 3z_2) &= -T\{\Delta\psi(\lambda + z_2) + 2\psi^i\delta_i^{i'}\partial_{i'} z_2 \\
&\qquad - \delta^{ii'}\partial_{i'}(z_{2+i})\} + O(T) + O(Y); \\
Nz_{2+i} &= (T\partial_T + (T+Y)\partial_Y) T\partial_{i'} f \\
&= T\partial_{i'}(Nf) + T\partial_{i'} f \\
&= T\partial_{i'}(z_2 + z_1).
\end{aligned}
$$

This system has the general form

$$Nz + Az = Th_1(T, Y, X, z, D_X z) + Yh_2(T, Y, X, z, D_X z), \qquad (8)$$

where A is a constant matrix; in our case, A has eigenvalues 3 and 0, with respective multiplicities 1 and $n+1$.

Remark. The preceding considerations also apply to the problem

$$\Box u = a_1 e^u + \sum_{j\geq 0} a_{-j}(x) e^{-ju}. \tag{9}$$

where $a_1(x)$ is bounded away from zero. Indeed, $\tilde{u} := u + \ln(a_1)$ solves then a similar equation with coefficients $\tilde{a}_j$ given by

$$\tilde{a}_1 = 1,$$

$$\tilde{a}_0 = a_0 + \Box \ln(a_1),$$

and for $j \geq 1$,

$$\tilde{a}_{-j} = a_1^j a_{-j}.$$

Performing the reduction of this paragraph with

$$\lambda = -\frac{2}{3}\left(R - \frac{1}{2}[a_0 + \Box \ln(a_1)]\right)(1 - |D\psi|^2)^{-1} \tag{10}$$

leads again to an equation of the form (8). The analysis applies also if terms of the form $\exp\{-(j-1/2)u\}$ with $j \geq 0$ are allowed in the nonlinearity, with minor modifications; this form includes in particular the Mikhailov-Fordy-Gibbons-Dodd-Bullough equation. More general fractional exponentials could presumably be handled by the techniques of §§4 and 6, if desired.

3. Solution of system (8).

We prove in this section that (8) has one solution if $z(0,0,X)$ is prescribed. The relation between z and f implies that the solution we seek must also satisfy $z_{2+i} = z_2 = 0$ for $T = Y = 0$; there is therefore only one arbitrary scalar function, *viz.* $z_1(0,0,X)$, in the data.

This will complete the proof of Theorem 1.

The argument is very similar in spirit to that of Part I, and we refer to it for references; the details are however somewhat different, as we will see. We therefore have included a complete proof.

Let us therefore consider the problem

$$\begin{cases} (N+A)z = f(T,Y,X,z,Dz) \\ z(0,0,X) = z_0(X) \in \mathrm{Ker}(A), \end{cases} \tag{11}$$

where $D = D_X$ and $f \equiv 0$ for $T = Y = 0$. The unknown has m components. Replacing z by $z - z(0,0,X)$, we see that, since here $Az(0,0,X) = 0$, we may assume

$$z(0,0,X) = 0.$$

This means that the solution of (11) depends on the choice of one function, which has been incorporated into the right-hand side. Also, one may, by introducing new dependent variables, assume, as we will, that f is linear in Dz.

We define

$$F[z] := f(T,Y,X,z,Dz).$$

The argument is in five steps.

Step 1. We first observe that

$$\begin{cases} (N+A)\, z(T,Y) = k(T,Y); \\ \qquad z(0,0) = 0, \end{cases} \tag{12}$$

where k is analytic, independent of X, and vanishes for $T = Y = 0$ has a unique analytic solution, given by

$$z(T,Y) = H[k] := \int_0^1 \sigma^{A-1} k(\sigma T, \sigma(T \ln \sigma + Y))\, d\sigma. \tag{13}$$

Indeed, let

$$g(\sigma) = z(\sigma T, \sigma(T \ln \sigma + Y))$$

for $0 < \sigma < 1$. We find

$$\frac{d}{d\sigma}(\sigma^A g(\sigma)) = \sigma^{A-1} k(\sigma T, \sigma(T\ln\sigma + Y)),$$

and since $g(\sigma)$ must tend to zero as σ goes to zero, while σ^A remains bounded, we have

$$\sigma^A g(\sigma) = \int_0^\sigma \tau^{A-1} k(\tau T, \tau(T\ln\tau + Y))\, d\tau,$$

Since k vanishes at the origin, the contribution from k to the integral is $O(\tau^{1-\varepsilon})$ for any $\varepsilon > 0$, and the integral converges.

Equation (13) follows by letting $\sigma = 1$ in the last equation. One checks directly, using the Cauchy-Riemann equations, that this does provide an analytic solution to the problem.

Eq. (11) can therefore be rewritten as the integral equation $z = H[F[z]]$. We let instead $u = F[z]$, and consider

$$u = G[u] := F[H[u]], \tag{14}$$

which will be solved by a fixed point argument. The desired solution will then be given by $z = H[u]$.

Step 2. We define two norms. Assume f is analytic for $X \in \mathbf{C}^n$ and $d(X, \Omega) < 2s_0$ and $u \in \mathbf{C}^m$ with $|u| < 2R$, for some positive constants s_0 and R, where Ω is a bounded open neighborhood of 0.

For any function $u = u(X)$ we define the s-norm

$$\|u\|_s := \sup\{|u(X)| : d(x, \Omega) < s\}. \tag{15}$$

For any function $u = u(T, Y, X)$, and a a sufficiently small positive number, to be chosen later, we define the a-norm

$$|u|_a := \sup_{\substack{\delta_0(T,Y) < a(s_0 - s) \\ 0 \le s < s_0}} \left\{ \delta_0^{-1} \|u\|_s(T, Y)(s_0 - s)\sqrt{1 - \frac{\delta_0}{a(s_0 - s)}} \right\}, \tag{16}$$

where $\delta_0 = \delta_0(T, Y) := |T| + \theta|Y|$, and $0 < \theta < 1$ is fixed. We wrote $\|u\|_s(T, Y)$ for the s-norm of $u(., T, Y)$.

We also let $\delta(\sigma) = \delta_0\sigma(1 - \theta \ln \sigma)$. The main properties of $\delta(\sigma)$ are

1. $\delta(\sigma)$ increases strictly from 0 to δ_0,
2. $\delta_0(\sigma T, \sigma(Y + T \ln \sigma)) \leq \delta(\sigma)$ if $0 < \sigma < 1$,
3. $d\delta(\sigma)/d\sigma \geq \delta(\sigma)/(C_0\sigma)$. One can take $C_0 = 1 - \theta$.

It follows in particular that if $|u|_a < \infty$,

$$\|u\|_s(\sigma T, \sigma(T \ln \sigma + Y)) \leq \frac{\delta(\sigma)|u|_a}{s_0 - s}\left(1 - \frac{\delta(\sigma)}{a(s_0 - s)}\right)^{-1/2}.$$

This is how the a-norm comes into the argument.

Step 3. We prove that one can estimate the s-norm of Hu in terms of the a-norm of u. From the definitions of our various norms, it follows that

$$\|Hu\|_s(T, Y) \leq \frac{|u|_a}{s_0 - s}\int_0^1 |\sigma^A|\frac{\delta(\sigma)}{\sigma}\left\{1 - \frac{\delta(\sigma)}{a(s_0 - s)}\right\}^{-1/2} d\sigma \qquad (17)$$

if $\delta_0 < a(s_0 - s)$. We estimate σ^A by a constant C_1. Let us define

$$\rho = \frac{\delta(\sigma)}{a(s_0 - s)}$$

so that, by property 3. above,

$$\frac{d\sigma}{d\rho} \leq C_0\frac{\sigma}{\delta(\sigma)}a(s_0 - s).$$

It follows that

$$\|Hu\|_s(T, Y) \leq \frac{|u|_a}{s_0 - s}\int_0^1 C_0C_1\frac{a(s_0 - s)\, d\rho}{\sqrt{1 - \rho}} = 2C_0C_1a|u|_a,$$

or

$$\|Hu\|_s(T, Y) \leq C_2a|u|_a. \qquad (18)$$

Step 4. Next, we observe that, since we have taken f to be linear in the spatial derivatives, Cauchy's inequality gives

$$\|F[u] - F[v]\|_{s'}(T, Y) \leq \frac{C_3 \delta_0(T, Y)}{s - s'} \|u - v\|_s \tag{19}$$

for $0 < s' < s < s_0$, if $\|u\|_s \leq R$ and $\|v\|_s \leq R$. We should of course also require $s' < s_0 - \delta_0/a$ if we wish to use the a-norm.

Step 5. Let us now assume $|u|_a$, $|v|_a < R/(2C_2 a)$. Let $G[u] = F[Hu]$. We prove that

$$|G[u] - G[v]|_a \leq C_4 a |u - v|_a \tag{20}$$

for some constant C_4.

To this end, let $\sigma_j = j/n$, for $0 \leq j \leq n$, and

$$w_j = \int_0^{\sigma_j} \sigma^{A-1} u(\sigma T, \sigma(T \ln \sigma + Y))\, d\sigma - \int_{\sigma_j}^1 \sigma^{A-1} v(\sigma T, \sigma(T \ln \sigma + Y))\, d\sigma,$$

and observe that

$$G[u] - G[v] = \sum_{j=1}^n F[w_j] - F[w_{j-1}]. \tag{21}$$

One checks, using the argument of Step 3, that $\|w_j\|_s \leq R$ for $\delta_0(T, Y) < a(s_0 - s)$, so that $F[w_j]$ is indeed defined.

We can now use (19): if $s_j \in (s, s_0 - \delta_0(T, Y)/a)$ for every j, we find from Step 4 that

$$\|F[w_j] - F[w_{j-1}]\|_s \leq \frac{C_3 \delta_0}{s_j - s} \|w_j - w_{j-1}\|_{s_j}.$$

On the other hand,

$$\|w_j - w_{j-1}\|_{s_j} \leq \int_{\sigma_{j-1}}^{\sigma_j} |\sigma^{A-1}|\; \|u - v\|_{s_j}(\sigma T, \sigma(T \ln \sigma + Y))\, d\sigma.$$

This suggests the choice:

$$s_j = \min\{s(\sigma) : \sigma_{j-1} \leq \sigma \leq \sigma_j\},$$

where

$$s(\sigma) = \frac{1}{2}\left[s + s_0 - \frac{\delta(\sigma)}{a}\right].$$

Observe next that $\sum_j s_j \chi_{[\sigma_{j-1},\sigma_j)} \to s(\sigma)$ as j tends to infinity, uniformly and *from below* on $(0,1)$, if $\delta_0(T,Y) < a(s_0 - s)$. Furthermore, if $\sigma_{j-1} \le \sigma \le \sigma_j$,

$$\begin{aligned} \|u-v\|_{s_j}(\sigma T, \sigma(T\ln\sigma + Y)) &\le \|u-v\|_{s(\sigma)}(\sigma T, \sigma(T\ln\sigma + Y)) \\ &\le \frac{\delta(\sigma)|u-v|_a}{s_0 - s(\sigma)}\left(1 - \frac{\delta(\sigma)}{a(s_0 - s(\sigma))}\right)^{-1/2}. \end{aligned}$$

We therefore find, since $|\sigma^{A-1}| \le C_1/\sigma$, letting $j \to \infty$,

$$\begin{aligned} &\|G[u] - G[v]\|_s(T,Y) \\ &\le C_3\delta_0 \int_0^1 \frac{\delta(\sigma)|u-v|_a}{(s(\sigma)-s)(s_0 - s(\sigma))}\left(1 - \frac{\delta(\sigma)}{a(s_0 - s(\sigma))}\right)^{-1/2} C_1 \frac{d\sigma}{\sigma}. \end{aligned}$$

Since

$$s(\sigma) - s = \frac{s_0 - s}{2}(1 - \frac{\delta(\sigma)}{a(s_0 - s)})$$

and

$$s_0 - s(\sigma) = \frac{s_0 - s}{2}(1 + \frac{\delta(\sigma)}{a(s_0 - s)})$$

we let again $\rho = \delta(\sigma)/[a(s_0 - s)]$. As σ varies from 0 to 1, ρ varies from 0 to $\delta_0/[a(s_0 - s)]$ (which is always less than 1); note also that

$$1 - \frac{\delta(\sigma)}{a(s_0 - s(\sigma))} = \frac{1-\rho}{1+\rho}.$$

Performing this change of variables, we find, using $d\sigma/d\rho \le C_0\sigma/\rho$,

$$\begin{aligned} &\|G[u] - G[v]\|_s(T,Y) \\ &\le C_1C_3\delta_0 \int_0^{\delta_0/[a(s_0-s)]} C_0\delta(\sigma)\frac{4|u-v|_a}{(s_0-s)^2}(1-\rho^2)^{-1}\sqrt{\frac{1+\rho}{1-\rho}}\,\frac{a(s_0-s)\,d\rho}{\delta(\sigma)} \\ &= 4a(s_0-s)^{-1}C_0C_1C_3\delta_0|u-v|_a \int_0^{\delta_0/[a(s_0-s)]} d\rho/(1-\rho)^{3/2} \\ &= C_4\delta_0 a(s_0-s)^{-1}|u-v|_a\left(1 - \frac{\delta_0}{a(s_0-s)}\right)^{-1/2}. \end{aligned}$$

Therefore

$$|G[u] - G[v]|_a \le C_4 a|u-v|_a,$$

QED.

End of proof. Let us define $u_0 = 0$ and $u_1 = G[u_0]$. There is a constant R_0 such that

$$\|u_1\|_{s_0} \leq R_0 \delta_0(T, Y)$$

if $|T| + \theta|Y| = \delta_0$. Assume a is chosen so small that

$$C_4 a < 1/2 \quad \text{and} \quad R_0 s_0 < R/(4C_2 a).$$

This ensures in particular $|u_1|_a \leq R_0 s_0 < R/(4C_2 a)$. The mapping G is now a contraction in the a-norm on the set $\{|u|_a \leq R/(2C_2 a)\}$. The existence (and uniqueness) of the desired solution now follow from the contraction mapping principle.

This ends the proof of Theorem 1.

4. General logarithmic expansions.

The above reduction procedure can be applied in principle to equations and systems that are much more general than (1); examples are given in §6. However, in order to treat them, we need a few facts applicable to general classes of equations. Such results are obtained in the present section.

All functions are analytic in their arguments unless otherwise specified.

We consider a system of the form

$$tu_t + Au = tf(t, x, u, Du) \tag{22}$$

where u and f are vector-valued, with m components, and are analytic near $(t, x, u, Du) = (0, 0, 0, 0)$. We do not require f to be linear in Du. The matrix A is constant, which is sufficient for most applications. The case when $P(x)^{-1}AP(x)$ is constant for a suitable matrix P can of course also be handled, after redefining the unknown.

Our result is the following.

Theorem 2 *There is an integer $l \geq 0$ and a formal solution u of* (22) *in increasing powers of $t, t\ln t, \ldots, t(\ln t)^l$, with coefficients analytic in x near the origin. There are infinitely many such solutions if and only if $-p$ is an eigenvalue of A for some nonnegative integer p. These solutions depend on the choice of finitely many arbitrary functions of x.*

Remark 1. If we require that no logarithms, or only a specified number of them, should occur in the solution, the arbitrary functions and the right-hand side f must satisfy certain conditions. This is precisely how the curvature condition for (1) arises. It may be appropriate to view such conditions as smoothness assumptions on the regular part of the solution.

Remark 2. As we show on the examples of §6, one can in practice be much more specific about the conditions and arbitrary functions involved, although this is often at the expense of very lengthy calculations. Nevertheless, this theorem, combined with the arguments of §6, enables one to check the existence of a formal solution without constructing it. The convergence of such formal solutions is dealt with in §5.

Remark 3. A very conservative estimate of the number l of logarithms one needs to introduce is obtained as follows: Let $l_{-1} = 0$, $l_0 = m$, and $l_{j+1} = l_j(l_j + 1)$ for $j \geq 0$. Then, if all the eigenvalues of A have real parts $\geq -P$, with $P \in \mathbf{N}$, we may take $l = l_{P-1}$. Note that one can often take a much smaller value for l; thus, $l = 0$ is enough if no nonpositive integer is an eigenvalue of A. It may even happen that $l = 0$ be permissible even if such integers are eigenvalues; this is the case when an analogue of the curvature condition holds (see Rem. 1).

Proof: After a linear change on the u's, we may assume that A is in Jordan form. Let us write t_k for $t(\ln t)^k$, $0 \leq k \leq l$, where l will be specified later. Let us seek u as a function of the $l+1+n$ variables (t', x), where $t' = (t_0, \dots, t_l)$. Equation (22) now becomes

$$Nu + Au = t_0 f(t', x, u, Du), \tag{23}$$

where

$$N = N_l := \sum_{k=0}^{l} (t_k + kt_{k-1})\partial/\partial t_k.$$

We call equations such as (23) "generalized Fuchsian" since they reduce to Fuchsian equations for $l = 0$. Note that we are writing f as a function of t' even though it involves only t_0; this is for later convenience.

Let u_0 be an arbitrary function such that $Au_0 = 0$. Let us seek u in the form

$$u = u_0 + t'.v := u_0 + t_0 v_0 + \cdots + t_l v_l,$$

and let us write an equation for v, the solutions of which generate, via the above formula, solutions of (23). Note that $v_0, \dots, v_l$ are *new dependent variables*, which are not, if $l \neq 0$, uniquely determined by u; only their values for $t' = 0$ are.

We have

$$\begin{aligned} &(N + A)\sum_{j=0}^{l} t_j v_j \\ &= \sum_{j=0}^{l} [t_j(N + A)v_j + (t_j + jt_{j-1})v_j] \\ &= \sum_{j=0}^{l} t_j\{(N + A)v_j + v_j + (j + 1)v_{j+1}\}, \end{aligned}$$

where t_{-1} and v_{l+1} are taken to be zero.

On the other hand,

$$\begin{aligned} &t_0 f(t', x, u_0 + t'.v, D(u_0 + t'.v)) \\ &= t_0(f(0, x, u_0, Du_0) + \sum_{j=0}^{l} t_j g_j(t', x, v, Dv)), \end{aligned}$$

for suitable functions g_j.

This suggests the following system for v, where again $v_{l+1} = 0$, and δ_{j0} is a Kronecker symbol:

$$(N + A + 1)v_j + (j + 1)v_{j+1} = \delta_{j0} f(0, x, u_0, Du_0) + t_0 g_j(t', x, v, Dv). \quad (24)$$

The left-hand side is the j^{th} component of

$$(N + B)v,$$

where B is a block upper triangular matrix with $m \times m$ blocks equal to $A + 1$ on the diagonal. The eigenvalues of B are therefore just those of A shifted by one.

We now claim that there is a vector $\tilde{v}(x)$ such that

$$(B\tilde{v})_j = \delta_{j0}\varphi(x) := \delta_{j0} f(0, x, u_0, Du_0),$$

provided that l is not smaller than the size of the largest Jordan block of $A+1$ for the eigenvalue 0 (thus, if $A + 1$ is invertible, we may take $l = 0$). Indeed, note first that we may decompose φ according to the invariant subspaces of A and thereby assume that A consists of a single Jordan block. In that case, *either* $A + 1$ is invertible, and we may take $\tilde{v}_0 = (A + 1)^{-1}\varphi$, and $\tilde{v}_j = 0$ for $j \geq 1$, *or* $(A + 1)^l = 0$ and we may take $\tilde{v}_0 = 0$, $\tilde{v}_1 = \varphi$, and

$$j\tilde{v}_j = -(A + 1)\tilde{v}_{j-1}$$

for $j \geq 2$. The $(l+1)^{\text{st}}$ component of the equation for $\tilde{v}$ reduces to the identity $(A + 1)^l \tilde{v}_l = 0$. This proves our claim.

We may now replace v by $v - \tilde{v}$ to obtain a system of the form (23), but with more dependent variables, and where A has been replaced by B. We may now repeat the procedure, which may require a larger value of l. But at any rate, since the eigenvalues of A are increased by one at each step, we know that after finitely many steps, we will obtain an equation where A has all its eigenvalues in the right complex half-plane, and no further increase of l is necessary from then on.

If we truncate this series after K operations, we obtain a polynomial in t' of degree K; when inserted in (23), it makes all terms of degree K vanish, by construction.

We have therefore achieved a solution of the desired form. Theorem 2 is thus proved.

The estimate of l in Remark 3 follows, since the worst case is that in which l equals the size of A; the number l_j is an upper bound on the size of the matrix A one obtains after j operations.

Remark. One might have presented these results in a less deductive fashion as follows: starting with the original equation (22), insert $u = u_0 + tv(x,t)$. One then finds a Fuchsian equation for v which has, in general, no power series formal solution. It is, however, always satisfied at lowest order by a combination of logarithms. But these are not bounded near 0. Therefore one should back up one step and let $u = u(t, t\ln t, \ldots)$. One is then naturally led to Eq. (23). In general terms, each obstruction in the formal process is removed by the introduction of a new *independent variable*, while higher order terms in the formal expansion are computed by introducing new *dependent variables.*

The next section proves the existence of a convergent series solution of the above form. The last section is devoted to examples.

5. Generalized Fuchsian Equations.

We consider equations of the form

$$Nu + Au = f(t', x, u, Du), \tag{25}$$

where $N = N_l = \sum_{k=0}^{l}(t_k + kt_{k-1})\partial/\partial t_k$, A is constant, and f is analytic near (0,0,0,0) without constant term in t'. Recall that $t' = (t_0, \dots, t_l)$. Again, all functions are analytic in their arguments unless otherwise specified.

We prove the following result.

Theorem 3 *If A has no eigenvalue with negative real part, (25) has near the origin exactly one analytic solution which vanishes for $t' = 0$.*

REMARK 1. This proves the convergence of the expansions of the previous section, since we saw that one could always reduce the problem to a system in which A has eigenvalues with positive real parts. Note also that the result of §3 is a special case of this theorem.

REMARK 2. Instead of requiring A to be constant, one may ask that $P(x)^{-1}AP(x)$ be constant, for a suitable matrix P, since the latter case reduces to the former by a redefinition of u. It is likely that N may be replaced by a more general first-order operator, with similar proofs, but the present set-up is sufficient for the applications of §6.

REMARK 3. The equations we consider are related to equations with several Fuchsian variables for which some existence results are available (see [1], [4] and more references therein); however the result we need does not seem to follow from the existing literature.

Proof: The proof parallels that of §3 and we therefore only indicate the differences.

We write

$$N = \sum_j m_{ij} t_j \partial_i$$

for suitable coefficients m_{ij} forming a matrix M. One must then replace $H[k]$ by

$$\int_0^1 \sigma^{A-1} k(\sigma^M t')\, d\sigma,$$

and use $\delta_0(t_0,\ldots,t_l)=\sum_{k=0}^{l}\theta^k|t_k|$, where $\theta\in(0,1/l)$ is fixed. We then use for $\delta(\sigma)$ the quantity $\delta_0\sigma(1-\theta\ln\sigma)^l$.

We need to check the three properties of δ in Step 2, §3.

The first follows from the assumption $0<\theta<1/l$.

The second is checked as follows: Since $t'(\sigma):=\sigma^M t'$ solves $\sigma\, dt'/d\sigma = Mt'$, $t'(1)=t'$, we find

$$(\sigma^M t')_k=\sum_{j=0}^{k} t_j\binom{k}{j}\sigma(\ln\sigma)^{k-j}.$$

We then compute

$$\begin{aligned}
\delta_0(\sigma^M t') &= \sum_k \theta^k|t_k(\sigma)|\le\sigma\sum_{0\le j\le k\le l} t_j\theta^k\binom{k}{j}|\ln\sigma|^{k-j}\\
&= \sigma\sum_{0\le j\le k\le l}\theta^j|t_j|\binom{k}{j}(-\theta\ln\sigma)^{k-j}\\
&= \sigma\sum_{0\le j\le k\le l}(-\theta\ln\sigma)^j\binom{k}{j}\theta^{k-j}|t_{k-j}|\\
&\le \sigma\sum_{j=0}^{l}(-\theta\ln\sigma)^j\binom{l}{j}\sum_{k\ge j}\theta^{k-j}|t_{k-j}|\\
&\le \sigma\delta_0(t')\sum_{j=0}^{l}(-\theta\ln\sigma)^j\binom{l}{j}\\
&= \sigma\delta_0(t')(1-\theta\ln\sigma)^l,
\end{aligned}$$

QED.

The third property follows from

$$\frac{\delta'(\sigma)}{\delta(\sigma)}\ge\frac{1-l\theta}{\sigma}.$$

The rest of the proof proceeds verbatim. We obtain existence on domains of the form

$$\{\delta(t')<a(s_0-s);\quad d(x,\Omega)<s\}.$$

6. Examples.

This section is devoted to applications of the results of §§4 and 5.

We first give, in §6.1, a rather general criterion to check the existence of singular solutions of the type considered here, for semilinear systems. Roughly speaking, any semilinear equation, irrespective of its type, has solutions blowing up near any open and bounded portion of any surface such that the *first term* of an appropriate formal solution exists. Two applications are then treated in some detail (§§6.2 and 6.3).

We consider, in §6.2, the case of

$$\Box u = P(u),$$

where P is a polynomial. The assumption on the existence of the first term of a (non-zero) formal solution translates into the condition that the putative blow-up surface be space-like, if $\deg P$ is odd and the leading term of P has a positive coefficient (resp. time-like if this coefficient is negative), or that it be merely non-characteristic if $\deg P$ is even. We also write out the condition for the absence of logarithmic terms in the case $P(u) = 6u^2$, which has been extensively studied from other perspectives in the literature. Further remarks on related equations are also included.

We then turn in §6.3 to the justification of a number of formal expansions for other semilinear equations which are to be found in the literature, in particular those obtained in connection with the Painlevé test and its generalization, the "WTC method" which we described in [3]; those expansions correspond to solutions where logarithmic terms are absent: logarithmic terms were sometimes suspected, but were generally not studied in any detail since their appearance was interpreted as indicating that the equation under consideration was not integrable. For very special equations, conditions for the disappearance of logarithmic terms have been found. Our results prove in addition that

1. Formal singular solutions can always be found to all orders (generally without logarithmic terms, due to the special form of the equations).

2. They converge and define solutions which are meromorphic, or exhibit branching, near the singularity surface. Note that when no logarithms are present, the convergence of these expansions actually follows from Theorem 1 in Part I [3].

Section 6.4 explains the consequences of these results for the problem of continuation of singular solutions after blow-up.

Since calculations generally become quite complicated in specific examples, the following remarks may be helpful in the practical application of the general results:

1. The early reduction of a single equation to a first-order system may make the formal calculations more cumbersome, an example being the case of §6.2. In particular, a careful choice of an associated first-order system helps find the minimal number of logarithms that are needed.

2. In an equation of the form $\partial_t^m u = \text{r.h.s.}$, it is useful to introduce $(t\partial_t)^j u$ instead of $\partial_t^j u$ as new unknowns, since the former blow-up at the same rate as u.

3. If a formal solution is already known, one can substract off its first few terms, and divide by an appropriate power of T; this generally helps ensure that A has no negative eigenvalues.

6.1. Semilinear systems. Let us consider a semilinear system, in m unknowns, of the form

$$u_t = \sum_{j=1}^{n} a^j D_j u + b(u), \tag{26}$$

where $a^j = a^j(x,t) = \sum_{k\geq 0} a^j_k(x)t^k$. We also assume b to be independent of x. Some x, t dependence could be allowed, but would make the results slightly more cumbersome to state; we did not strive for maximum generality.

We find conditions on (26) which ensure the existence of a singular solution which blows up for $t = \psi(x)$, like a rational power of $(t - \psi)$.

Our assumptions are

1. There are integers p and q, with $q > 0$, such that $\tau^{p+q} b(\tau^{-p}\xi)$ is analytic in $\tau \in \mathbf{C}$ and $\xi \in \mathbf{C}^m$, near $\tau = 0$, $\xi = 0$. We write

$$\tau^{p+q} b(\tau^{-p}\xi) = c(\tau, \xi) := \sum_{j\geq 0} c_j(\xi)\tau^j.$$

2. There is an analytic function $v_0(x)$ such that

$$-pv_0 = qQ(x)^{-1}c_0(v_0),$$

   where $Q(x) = (1 + \sum_j a^j_0 D_j\psi)$ is the characteristic matrix corresponding to the blow-up surface. We therefore require the singular surface to be non-characteristic.

3. There is a matrix-valued function $P(x)$ such that $P^{-1}Q^{-1}c_0'(v_0)P$ is constant.

The result is

Theorem 4 *If the above conditions are satisfied, (26) has a solution such that $u(x,t)(t - \psi(x))^{p/q}$ is an analytic function of the $l + 1 + n$ variables $x, \tau = (t - \psi(x))^{1/q}, \tau \ln \tau, \ldots, \tau(\ln \tau)^l$, for a suitable integer l, and which reduces to $v_0(x)$ for $\tau = 0$.*

Remark 1. The solution is generally not unique, and the discussion in §4 on the form and degree of arbitrariness of the solution apply here; those remarks are therefore not repeated.

Remark 2. The assumptions are verified on a remarkably large number of equations and systems of practical interest. They have the following meaning: The first gives the rate of blow-up; it requires that singular solutions where the derivatives and the nonlinear terms blow-up in the same way are possible. The second and the third give the first term of the expansion of the putative solution. In many examples, $a_0^j = 0$ for every j, and v_0 is constant, so that the third condition is actually vacuous (i.e., satisfied with $P = 1$).

Remark 3. The case $p = 0$, $q = 1$ leads in fact to the solution of the Cauchy problem, v_0 being simply the initial value of u. In that case, $c_0 \equiv 0$. Thus, singular as well as regular solutions are encompassed by a single procedure.

Proof: We reduce the problem to a generalized Fuchsian equation to which the results of §§4 and 5 apply.

Let us first take $t - \psi(x)$ as new time variable, still called t for convenience. We find

$$(1 + a_0(D\psi))u_t = a(Du) + b(u) + (a_0 - a)(D\psi)u_t,$$

where $a(Du) = \sum_j a^j D_j u$, and $a_0(Du) = \sum_j a_0^j D_j u$. Multiply by t, and let $t = \tau^q$, so that $t\partial_t = (1/q)\tau\partial_\tau$. Let also $u = v\tau^{-p}$. We find, using the first assumption,

$$Q(\tau v_\tau - pv)/q = \tau^q a(Dv) + c(\tau, v) + (a_0 - a)(D\psi)(\tau v_\tau - pv)/q.$$

Let $v = v_0 + \tau w$. We obtain, since

$$Q(Q - (a_0 - a)(D\psi))^{-1} = (1 + O(\tau^q)),$$

the following equation for w, where φ_0 denotes a function of x alone ($\equiv 0$ if $q > 1$):

$$
\begin{aligned}
&Q(\tau\partial_\tau - p)(\tau w)/q - pQv_0/q \\
&\quad = \tau^q(1+O(\tau^q))[a(Dv_0) + \tau a(Dw)] \\
&\qquad + c_0(v_0) + \tau(c_0'(v_0)w + c_1(v_0) + \varphi_0) + \tau^2 g_1(\tau, x, w).
\end{aligned}
$$

The two constant terms cancel, thanks to the second assumption.

This equation has the form

$$[\tau\partial_\tau - p + 1 - qQ^{-1}c_0'(v_0)]w = \varphi(x) + \tau G(\tau, x, w, Dw),$$

where $\varphi(x)$ depends on x alone. Now let $w = Pz$ and multiply each component of the system by P^{-1}. Using the third assumption, we obtain a system of the form studied in §§4 and 5 (see especially (24)). By introducing logarithms as in Theorem 2, $\varphi(x)$ can be absorbed into w. The theorem therefore follows from Theorems 2 and 3.

The proof is complete.

6.2. Nonlinear hyperbolic equations. For simplicity we restrict ourselves to one example, and briefly comment on more general ones. Our first example is

$$\Box u = \frac{2(m+1)}{(m-1)^2}u^m + \sum_{j=-\infty}^{m-1} a_j(x,t)u^j, \tag{27}$$

where m is an integer greater than or equal to 2. Fix a hypersurface Σ with equation $t = \psi(x)$. Σ is assumed to be space-like is m is odd, non-characteristic otherwise. The functions ψ and a_j are analytic.

Theorem 5 *Equation (27) has singular solutions which, near the origin, blow up precisely on Σ. These solutions involve one arbitrary function, and satisfy $(t-\psi)^{2/(m-1)}u = (1-|D\psi|^2)^{1/(m-1)}$ on Σ. They are given by convergent series in x, τ and $\tau\ln\tau$, where $\tau = (t-\psi)^{1/(m-1)}$.*

As usual, the logarithmic terms are absent whenever ψ satisfies an appropriate condition. This equation may be non-trivial, even in one space dimension.

Remark 1. If m is even, Σ may be allowed to be time-like. This is because

$$u = \tau^{-2}(u_0(x) + \tau'.u_1(x) + \cdots), \text{ with } \tau' = (\tau, \tau \ln \tau), \tag{28}$$

where u_0^{m-1} is equal to $(1 - |D\psi|^2)$; only for odd m does this force Σ to be space-like. It is conceivable, however, that blow-up surfaces generated by smooth data on a *space-like* initial hypersurface cannot be time-like anywhere. Observe also that if $m = 2$, branching does not occur in the solution.

Remark 2. The constant in front of u^m in (27) has been chosen only for convenience in explicit calculations. One can change it by scaling independent variables.

Proof: This result is, like Theorem 1, obtained by reduction to a generalized Fuchsian equation. The algebra being very similar to that of Part I and §2, we give below only the main steps of the proof. One could try to reduce the equation to a system and mimic the procedure of §6.1, but the calculations are a little simpler if we argue as follows:

Let us first write the r.h.s. as

$$\frac{2(m+1)}{(m-1)^2}u^m + f(x, t - \psi(x), u)/(m-1)^2,$$

where

$$\tau^{2m-2} f(x, \tau^{m-1}, v/\tau^2) = b(x, \tau, v)$$

is analytic in (τ, v).

Introduce, as in §2, the new time variable $T = t - \psi(x)$, while $X^i = x^i$; we still write x for X, for convenience. This change of variables produces

$$(1 - |D\psi|^2)u_{TT} - \Delta u + 2D\psi.Du_T + (\Delta\psi)u_T = \text{r.h.s.}$$

Next, multiply by T^2, and replace T by τ^{m-1}, so that $T\partial_T$ becomes $(m-1)^{-1}\tau\partial_\tau$. We get

$$(1 - |D\psi|^2)(\tau\partial_\tau)(\tau\partial_\tau - m + 1)u - 2(m+1)\tau^{2m-2}u^m$$

$$
\begin{aligned}
&= \tau^{2m-2} f(x, \tau^{m-1}, u) + (m-1)^2 \tau^{2m-2} \Delta u \\
&\quad - (m-1)(2\tau^{m-1} D\psi . D(\tau u_\tau) + (\Delta\psi)\tau^m u_\tau).
\end{aligned}
$$

Finally, let $u = \tau^{-2} v(\tau, x)$. We find an equation of the form

$$
\begin{aligned}
&(1 - |D\psi|^2)(\tau\partial_\tau - 2)(\tau\partial_\tau - m - 1)v - 2(m+1)v^m \\
&= \tau^2 b(x, \tau, v) + (m-1)^2 \tau^{2m-2} \Delta v \\
&\quad - (m-1)(2\tau^{m-1} D\psi . D(\tau v_\tau - 2v) + (\Delta\psi)\tau^{m-1}(\tau v_\tau - 2v)) \\
&= \tau g(x, \tau, v, \tau v_\tau, D\tau v_\tau) + (m-1)^2 \tau^{2m-2} \Delta v,
\end{aligned}
$$

because $m - 1 \geq 1$.

This suggests seeking v in increasing powers of τ. In fact, one can easily check that upon inserting $v = v_0 + \tau v_1 + \ldots$, one can compute $v_0, \ldots, v_{2m+1}$, but not the next term. In particular, $v_0^{m-1} = (1 - |D\psi|^2)$. This suggests the substitution

$$
v = \sum_{j=0}^{2m+1} v_j(x)\tau^j + \tau^{2m+2} w(x, \tau).
$$

A careful inspection of the equation shows that the equation for w has the form

$$
\begin{aligned}
&(1 - |D\psi|^2)(\tau\partial_\tau)(\tau\partial_\tau + 3m + 1)w \\
&= \varphi(x) + h(x, \tau, \tau w, \tau^2 w_\tau, \tau D(\tau w_\tau)) + (m-1)^2 \tau^{2m-2} \Delta w,
\end{aligned}
$$

where h vanishes when $\tau = 0$.

This suggests

$$
w = \frac{\varphi \ln \tau}{(3m+1)(1 - |D\psi|^2)} + z,
$$

where $z = z(x, \tau, \tau \ln \tau)$. If we let $u_0 = z$, $u_1 = \tau\partial_\tau z$ and $u_{2+i} = \tau\partial_i z$, and $\eta = \tau \ln \tau$, the u_k being considered as functions of x, τ, η, we find a generalized Fuchsian equation to which the result of §5 (or §3) applies.

This proves Theorem 5.

REMARK: A brutal application of §6.1 is not only more lengthy here: it makes it more difficult to realize that there is in fact only *one* logarithm in the solution.

We include below the condition required in order that the logarithmic terms be absent, for the case $m = 2$, $a_j \equiv 0$. If $T = t - \psi$, and $u = (v_0 + Tv_1 + \cdots)/T^2$, the condition reads

$$3(2\psi^i \partial_i + \Delta\psi - 4v_1)v_5 = \Delta v_4 + (v_3^2 + 2v_2 v_4), \tag{29}$$

with

$$\begin{aligned}
10v_0v_1 &= -4\psi^i\partial_i v_0 - 2v_0\Delta\psi \\
12v_0v_2 &= -\Delta v_0 - v_1\Delta\psi - 2\psi^i\partial_i v_1 - 6v_1^2 \\
12v_0v_3 &= -\Delta v_1 - 12v_1v_2 \\
10v_0v_4 &= -\Delta v_2 - v_3\Delta\psi + 2\psi^i\partial_i v_3 - 6(2v_1v_3 - v_2^2) \\
6v_0v_5 &= -\Delta v_3 + 2v_4\Delta\psi - 4\psi^i\partial_i v_4 - 12(v_1v_4 + v_2v_3).
\end{aligned}$$

One checks that any linear ψ satisfies this condition.

If this condition holds, v_6 is arbitrary and $(t - \psi(x))^2 u(x,t)$ is analytic across Σ.

Despite its complexity, the case $m = 2$ is actually the simplest among polynomial nonlinearities. If $m > 2$, the condition for the absence of logarithms is found only after $2m + 1$ terms have been computed.

REMARK. The above results extend with minor modifications to

$$\Delta u = P(u).$$

Replacing t by it in the hyperbolic results is permissible, since we are throughout dealing with holomorphic solutions; the condition that ψ be real when x is was imposed only in order to make the interpretation of the final results simpler.

6.3. The Painlevé test. We consider evolution equations of the form

$$u_t = f(u, D_x u, \dots, D_x^k u) \tag{30}$$

in one or two space dimensions. Weiss, Tabor and Carnevale (WTC) [6] have shown that for a number of equations integrable by Inverse Scattering, (the first terms of) an expansion of the form

$$\sum_{k\geq 0} u_k \phi(x,t)^{k+\nu} \tag{31}$$

which formally solves the equation, can be found without any condition on ϕ other than the non-vanishing of ϕ_x;[3] ν is generally a negative integer. As pointed out in [3], these authors have focused on terminating series of this form which, in addition to providing closed form solutions, can often be related to the eigenvalue problem of which the evolution equation represents an iso-spectral deformation.

It follows from the results of §§4 and 5 that (i) such series can, as one may expect, be computed to all orders, and (ii) they *converge* in the vicinity of the set where ϕ vanishes. The special form of the equations enables one to avoid logarithmic terms. This means that the existence theorem of the first part is actually sufficient to prove the convergence of these series.

We list a few of the simplest examples, without aiming at completeness; we indicate briefly for the first example the associated Fuchsian equation, and how our general results apply here:

The *Korteweg-de Vries* equation (suitably scaled)

$$u_t + uu_x - u_{xxx} = 0$$

has $\nu = -2$, and u_4 and u_6 are arbitrary.

[3]This means that the singular surface defined by $\phi = 0$ is non-characteristic.

Let $u_j = (x\partial_x)^j u$ for $j = 0, 1, 2$. We then find

$$\begin{aligned} x\partial_x u_0 &= u_1; \\ x\partial_x u_1 &= u_2; \\ x\partial_x u_2 &= 3u_2 - 2u_1 + x^3 u_{0t} + x^2 u_0 u_1. \end{aligned}$$

Assume the singularity surface is given by $x = \psi(t)$. Take $x - \psi(t)$ as new space variable, still denoted by x for convenience, so that we should replace u_{0t} by $u_{0t} - \psi'(t)u_{0x} = u_{0t} - \psi' u_1/x$. Now, if the u_j's behave like x^ν, we find that both sides of the equation balance each other at leading order if $\nu = -2$. This suggests the substitution $u_j = v_j x^{-2}$. We find for v the system

$$\begin{aligned} (x\partial_x - 2)v_0 &= v_1; \\ (x\partial_x - 2)v_1 &= v_2; \\ (x\partial_x - 2)v_2 &= 3v_2 - 2v_1 + v_0 v_1 - \psi' x^2 v_1 + x^3 v_{0t}. \end{aligned}$$

Following the idea of §6.1, we find that for $x = 0$, we must take either $v = (12, -24, 48)$ or $v = 0$. We of course consider only the former. Since the function called v_0 in §6.1 is here constant, Theorem 4 applies. It is however not difficult to reprove it in this special case: setting $v_0 = 12 + xw_0$, $v_1 = -24 + xw_1$, $v_2 = 48 + xw_2$, we obtain

$$\begin{aligned} (x\partial_x - 1)w_0 &= w_1; \\ (x\partial_x - 1)w_1 &= w_2; \\ (x\partial_x - 4)w_2 &= 10w_1 - 24w_0 + xw_0 w_1 - \psi' x^2 w_1 + 24x\psi' + x^3 w_{0t}. \end{aligned}$$

If we now perform the reduction of §4, it turns out that one may take $l = 0$ (no logarithms); Theorem 3 now guarantees the convergence of the expansion.

Two other examples, among many others, are

1. The *modified Korteweg-de Vries* equation

$$u_t - 3u^2 u_x + 2\sigma^2 u_{xxx} = 0,$$

for which $\nu = -1$, and u_3 and u_4 are arbitrary.

2. The *Kadomtsev-Petviashvili* equation

$$(u_t + uu_x + \delta u_{xxx})_x + u_{yy} = 0,$$

for which $\nu = -2$, and u_4, u_5 and u_6 are now arbitrary.

A few equations that are not known to be integrable by Inverse Scattering have been tested by this "generalized Painlevé test;" it is in particular known (Olver and McLeod [5], Clarkson, McLeod, Olver and Ramani [2]) that up to scaling, the only equations

$$u_{tt} - u_{xx} = f(u)$$

for which all such singular solutions are free of branching or essential singularities *in the whole complex plane* are linear combinations of $e^{\pm u}$ and $e^{\pm 2u}$. Indeed, even the equation for traveling wave solutions has otherwise movable critical points. One can further rule out all the equations that are not known to be integrable by Inverse Scattering since the other equations in this class do not have WTC expansions that are free from logarithms [2].

As we have shown above, any equation with

$$f(u) = e^u + g(e^{-u})e^{-u}$$

possesses singular solutions such that e^u is meromorphic near the blow-up surface, and no logarithmic terms appear, since we are in one space dimension; this does not preclude the existence of more complicated singularities far from the blow-up set under consideration.

6.4. Continuation. Just as in the case of Eq. (1), the present results imply that all solutions constructed can be continued beyond their blow-up set, possibly after the introduction of a uniformizing variable. In general, branching cannot be avoided. It is due either to the presence of a fractional power of $(t - \psi(x))$ in the leading term, or to the appearance of logarithmic terms further down in the expansion. One branch might be preferable to another in a specific application; thus, one can wish to choose a real branch. One may also wish to use different determinations on the two sides of the singularity surface.

The possibility of continuation is quite unavoidable, since the solution involves an *analytic* function which is at once defined on both sides of the singular set.

In some special cases, branching may be avoided, and even the above ambiguity disappears. Thus, in the case of

$$\Box u = 6u^2,$$

one finds that the solution is, when logarithmic terms are absent, actually meromorphic, and the continuation *is* uniquely determined. Similarly, for (1), we observe that e^u does not exhibit any branching.

References.

1. G. Bengel and R. Gérard, Formal and convergent solutions of singular partial differential equations, *Man. Math.,* **38**, (1982) 343–373.

2. P. A. Clarkson, J. B. McLeod P. J. Olver and A. Ramani, Integrability of Klein-Gordon equations, *SIAM J. Math. Anal.,* **17**, No. 4 (1986) 798–802.

3. S. Kichenassamy and W. Littman, Blow-up surfaces for nonlinear wave equations, I, *Comm. P. D. E.*, **18** (3&4) (1993) 431–452.

4. N.-S. Madi,

   (a) Sur certains problèmes de Goursat Fuchsiens à données initiales sur les hypersurfaces caracteristiques, *Bull. Sc. Math.,* 2ème série, **112** (1988) 325–343.

(b) Problème de Goursat holomorphe à variables Fuchsiennes, *Bull. Sc. Math.,* 2ème série, **111** (1987) 291–312.

5. J. B. McLeod and P. J. Olver, The connection between partial differential equations soluble by inverse scattering and ordinary differential equations of Painlevé type, *SIAM J. Math. Anal.,* **14**, No. 3 (1983) 488–506.

6. J. Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations, *J. Math. Phys.* **24** (1983) 522–526.