The roots of any polynomial equation

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Abstract
We provide a method for solving the roots of the general polynomial equation

$$a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \ .+ a_1 \cdot x + s = 0 \tag{1}$$

To do so, we express x as a powerseries of s, and calculate the first n-2 coefficients. We turn the polynomial equation into a differential equation that has the roots as solutions. Then we express the powerseries' coefficients in the first n-2 coefficients. Then the variable s is set to a0. A free parameter is added to make the series convergent.

Keywords: Algebraic equation

The method

The method is based on [1]. Let's take the first n-1 derivatives of (1) to s. Equate these derivatives to zero. Then find $\frac{d^i}{ds^i} x(s)$ in terms of x(s) for i from 1 to n-1. Now make a new differential equation

$$m1 \cdot \frac{d^{n-1}}{ds^{n-1}} x(s) + m2 \cdot \frac{d^{n-2}}{ds^{n-2}} x(s) + \ .+ m_n \cdot x(s) + m_{n+1} = 0 \tag{2}$$

and fill in our $\frac{d^i}{ds^i} x(s)$ in (2). Multiply by the denominator of the expression. Now we have a polynomial in x(s) of degree higher then n. Using (1) as property, we simplify this polynomial to the degree of n. Set it equal to (1) and solve m_1 .. m_{n+1} in terms of s and a_1 .. a_n Substituting these in (2) gives a differential equation that has the zeros of (1) among its solutions. We then insert

$$x(s) = y(s) - \frac{a_{n-1}}{n \cdot a_n} \tag{3}$$

in (2). Multiplying by the denominator we get a differential equation of the linear form:

$$p1 \cdot \frac{d^{n-1}}{ds^{n-1}} y(s) + p2 \cdot \frac{d^{n-2}}{ds^{n-2}} y(s) + \ .+ p_n \cdot y(s) = 0 \tag{4}$$

With $p_1(s)..p_n(s)$ polynomials in s. If we substitute our powerseries, all the coefficients are determined by the first n-1 coefficients. The first coefficients are calculated as follows: A powerseries is filled in in (1).

$$x(s) = \sum_{i=0}^{n-2} b_i \cdot s^i - \frac{a_n}{n \cdot a_{n-1}} \tag{5}$$

and it should be zero for all s. From this, we calculate b_i for i from 0 to n-2.b_0 Is a root af an n-1 degree polynomial and the other b_i are expressed in b_0 Now a powerseries is inserted in (4):

$$y(s) = \sum_{i=0}^{\infty} b_i \cdot s^i \tag{6}$$

and we get an equation of the form:

$$q_1(i)\cdot c_1 \cdot b_i + q_2(i)\cdot c_2 \cdot b_{i+1} + \ .+ q_n(i)\cdot c_n \cdot b_{i+n-1} = 0 \tag{7}$$

where q_m (i) are polynomials in i of degree n-1. c_m Are constants.
We define b_{n-1} as the determinant of a matrix A

$$A = \begin{bmatrix} \frac{c_{n-1}\cdot q_{n-1}(0)}{c_n \cdot q_n(0)} & .. & \frac{c_2 \cdot q_2(0)}{c_n \cdot q_n(0)} & \frac{c_1 \cdot q_1(0)}{c_n \cdot q_n(0)} & 0 & 0 & 0 & .. & 0 & 0 \\ 0 & .. & 0 & 0 & 1 & 1 & 0 & .. & 0 & 0 \\ 0 & .. & 0 & 0 & 0 & 1 & 1 & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\ 0 & .. & 0 & 0 & 0 & 0 & 0 & .. & 1 & 1 \\ 0 & .. & 0 & 1 & b_0 & 0 & 0 & .. & 0 & 0 \\ 0 & .. & 1 & 0 & 0 & -b_1 & 0 & .. & 0 & 0 \\ 0 & .. & 0 & 0 & 0 & 0 & b2 & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & .. & 0 & -1^{n-2}\cdot b_{n-2} \end{bmatrix} \tag{8}$$

and for the rest of the coefficients

$$b_{i+n-1} = \begin{vmatrix} \frac{-c_{n-1}\cdot q_{n-1}(i)}{c_n \cdot q_n(i)} & \frac{-c_{n-2}\cdot q_{n-2}(i)}{c_n \cdot q_n(i)} & \frac{-c_{n-3}\cdot q_{n-3}(i)}{c_n \cdot q_n(i)} & . & \frac{-c_1 \cdot q_1(i)}{c_n \cdot q_n(i)} & 0 & 0 & 0 & 0 & .. \\ -1 & \frac{-c_{n-1}\cdot q_{n-1}(i-1)}{c_n \cdot q_n(i-1)} & \frac{-c_{n-2}\cdot q_{n-2}(i-1)}{c_n \cdot q_n(i-1)} & . & \frac{-c_2 \cdot q_2(i-1)}{c_n \cdot q_n(i-1)} & \frac{c_1 \cdot q_1(i-1)}{c_n \cdot q_n(i-1)} & 0 & 0 & 0 & .. \\ .. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & -1 & \frac{-c_{n-1}\cdot q_{n-1}(1)}{c_n \cdot q_n(1)} & \frac{c_{n-2}\cdot q_{n-2}(1)}{c_n \cdot q_n(1)} & . & \frac{c_1 \cdot q_1(1)}{c_n \cdot q_n(1)} & 0 & .. \\ 0 & 0 & 0 & 0 & -1 & & & & & \\ 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & 0 & & A & & & \\ 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & 0 & & & & & \\ .. & .. & .. & .. & .. & & & & & \end{vmatrix} \tag{9}$$

The series (6) can be proven to be convergent [2] if for a constant E

$$s \le \frac{1}{E}\cdot \left| c_n \right| \tag{10}$$

and if all the absolute values of the coefficients of (1) are chosen smaller then 1. This is done by dividing a polynomial by (more than) the maximum of the absolute values of the coefficients.

To make the series convergent, we transform s to $e \cdot s$. That is, if we insert the powerseries it is not in s but in $e \cdot s$. Writing out the terms of the sum, we find that each term d_i has a factor $s^i \cdot e^{i-n+1}$.Setting

$$e = \frac{\left|c_n\right|}{E} \tag{12}$$

We still need $s<1$, which is why we set s to a_0 and $a_0<1$.

References

[1] R. Harley, On the theory of the Transcendental Solution of Algebraic Equations, Quart. Journal of Pure and Applied Math, Vol. 5 p.337. 1862

[2] E. Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons, Inc. p.789. 1993.