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PURELY PERIODIC \( \beta \)-EXPANSIONS IN THE PISOT NON-UNIT CASE

VALERIE BERTHÉ AND ANNE SIEGEL

Abstract. It is well known that real numbers with a purely periodic decimal expansion are the rationals having, when reduced, a denominator coprime with 10. The aim of this paper is to extend this result to beta-expansions with a Pisot base beta which is not necessarily a unit: we characterize real numbers having a purely periodic expansion in such a base; this characterization is given in terms of an explicit set, called generalized Rauzy fractal, which is shown to be a graph-directed self-affine compact subset of non-zero measure which belongs to the direct product of Euclidean and p-adic spaces.

Keywords: expansion in a non-integral base, Pisot number, beta-shift, beta-numeration, purely periodic expansion, self-affine set.

Let \( \beta \) be a Pisot number and \( T_\beta : x \mapsto \beta x \pmod{1} \) be the associated \( \beta \)-transformation. The aim of this paper is to characterize the real numbers \( x \) in \( \mathbb{Q}(\beta) \cap [0,1) \) having a purely periodic \( \beta \)-expansion.

It is well known that if \( \beta \) is a Pisot number, then the real numbers that have a ultimately periodic \( \beta \)-expansion are the elements of \( \mathbb{Q}(\beta) \) \cite{Ber77, Sch80}. Thus real numbers \( x \) that have a purely periodic beta-expansion belong to \( \mathbb{Q}(\beta) \). We present a characterization that involves the conjugates of the algebraic number \( x \), and can be compared to Galois’ theorem for classical continuous fractions.

**Theorem 1.** Let \( \beta \) be a Pisot number. A real number \( x \in \mathbb{Q}(\beta) \cap [0,1) \) has a purely periodic \( \beta \)-expansion if and only if \( x \) and its conjugates belong to an explicit subset in the product of Euclidean and \( p \)-adic spaces (see Figure 2.2 below); this set (denoted \( \tilde{R}_\beta \) and called generalized Rauzy fractal) is a graph-directed self-affine compact subset in the sense of \( \text{MW88} \) of non-zero measure; the primes \( p \) that occur are the prime divisors of the norm of \( \beta \).

The scheme of the proof is based on a realization of the natural extension of the \( \beta \)-transformation \( T_\beta \) extended to a geometric space of representation for the two-sided \( \beta \)-shift \( (X_\beta, S) \). Our results and our proof is inspired by \cite{IR02, IS01, San02} which presents a similar characterization of purely periodic expansions in the case where \( \beta \) is a Pisot unit.

The construction of the set \( \tilde{R}_\beta \) (introduced in Theorem 1) is inspired by the geometric representation as generalized Rauzy fractals (also called atomic surfaces) of substitutive symbolic dynamical systems in the non-unimodular case developed in \cite{Sie03}. In fact, a substitution \( \sigma \) is a non-erasing morphism of the free monoid \( \mathcal{A}^* \) and a substitutive dynamical system is a symbolic dynamical system generated by an infinite sequence which is a fixed point of a substitution. Furthermore, if the \( \beta \)-expansion of 1 in base \( \beta \) is finite (\( \beta \) is said to be a simple Parry number) and if its length coincides with the degree of \( \beta \), then the set \( \tilde{R}_\beta \) involved in our characterization is exactly the generalized Rauzy fractal that is associated in \cite{Sie02} with the underlying \( \beta \)-substitution (in the sense of \cite{Thu89, Fab95}).

Rauzy fractals have been widely studied; for more details, see for instance \cite{Sie02}. There are mainly two methods of construction for Rauzy fractals. One first approach inspired by the seminal paper \cite{Kau82}, is based on formal power series, and is developed in \cite{Mes98a, Mes98b}, or in \cite{CS01a}.

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A second approach via nested fractals and generalized substitutions has been developed following ideas from [IK91] in [AI01, SAI01, HZ98, SW02] with special focus on the self-similar properties of the Rauzy fractals. We combine here both approaches: we define the set $\tilde{R}_\beta$ by introducing a representation map of the two-sided shift $(X_\beta, S)$ based on formal power series, and prove that this set has non-zero Haar measure by cutting it into pieces that are solutions of an IFS. Similar sets have also been introduced and studied in the framework of $\beta$-numeration by S. Akiyama in [Aki98, AS98, Aki99, Aki00], inspired by [Thu89]. As an application of generalized Rauzy fractals, let us mention that they provide Markov partitions for toral automorphisms of the torus, as illustrated in [IO93, KV98, Pra99, Sch00, Sie00]. Furthermore, there are numerous relations between generalized Rauzy fractals and discrete planes as studied for instance in [ABI02, ABS04].

The aim of this paper is twofold. We first want to characterize real numbers having a purely periodic $\beta$-expansion; we second try to settle the first steps of a study of the geometric representation of $\beta$-shifts in the Pisot non-unit case, generalizing the results of [Aki98, AS98, Aki99, Aki00], based on the formalism introduced in the substitutive case in [Sie03].

This paper is organized as follows. We first recall in Section 1 the basic elements needed on $\beta$-expansions. We then associate in Section 2 with the two-sided $\beta$-shift $(X_\beta, S)$ formal power series in $\mathbb{Q}[[X]]$; we obtain in Section 2.3 a representation map for the two-sided $\beta$-shift by gathering the set of finite values which can be taken for any topology (Archimedean or not) by these formal power series when specializing them in $\mathbb{Q}[[\beta]]$. This allows to prove that this set has non-zero Haar measure by cutting it into pieces that are solutions of an IFS.

Section 3 is devoted to the study of the properties of the set $\tilde{R}_\beta$. We then prove Theorem 4 in Section 4.

1. $\beta$-numeration

Let $\beta > 1$ be a real number. In all that follows, $\beta$ is assumed to be a Pisot number. The Renyi $\beta$-expansion of a real number $x \in [0, 1)$ is defined as the sequence $(x_i)_{i \geq 1}$ with values in $\mathcal{A}_\beta := \{0, 1, \ldots, [\beta]\}$ produced by the $\beta$-transformation $T_\beta : x \mapsto \beta x \mod 1$ as follows

$$\forall i \geq 1, \: u_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor,$$

and thus $x = \sum_{i \geq 1} u_i \beta^{-i}$.

Let $d_{\beta}(1) = (t_i)_{i \geq 1}$ denote the $\beta$-expansion of 1. Numbers $\beta$ such that $d_{\beta}(1)$ is ultimately periodic are called Parry numbers and those such that $d_{\beta}(1)$ is finite are called simple Parry numbers. Since $\beta$ is assumed to be Pisot, then $\beta$ is either a Parry number or a simple Parry number [BMS89]. Let $d_{\beta}^*(1) = d_{\beta}(1)$, if $d_{\beta}(1)$ is infinite, and $d_{\beta}^*(1) = (t_1 \ldots t_{n-1}(t_{n-1} - 1))^{\infty}$, if $d_{\beta}(1) = t_1 \ldots t_{n-1} + n$ is finite ($t_n \neq 0$). The set of $\beta$-expansions of real numbers in $[0, 1)$ is exactly the set of sequences $(u_i)_{i \geq 1}$ in $\mathcal{A}_\beta^N$ such that

$$\forall k \geq 1, \: (u_i)_{i \geq k} \leq_{\text{lex}} d_{\beta}^*(1).$$

For more details on the $\beta$-numeration, see for instance [Fro02, Fro00].

The (two-sided symbolic) $\beta$-shift. Let $(X_\beta, S)$ denote the two-sided symbolic dynamical system associated with $\beta$, where the shift map $S$ maps the sequence $(y_i)_{i \in \mathbb{Z}}$ onto $(y_{i+1})_{i \in \mathbb{Z}}$. The set $X_\beta$ is defined as the set of two-sided sequences $(y_i)_{i \in \mathbb{Z}}$ in $\mathcal{A}_\beta^\mathbb{Z}$ such that each left truncated sequence is less than or equal to $d_{\beta}^*(1)$, that is, $\forall k \in \mathbb{Z}, \: (y_i)_{i \geq k} \leq_{\text{lex}} d_{\beta}^*(1)$.

We will use the following notation for the elements of $X_\beta$: if $y = (y_i)_{i \in \mathbb{Z}} \in X_\beta$, define $u = (u_i)_{i \geq 1} = (y_i)_{i \geq 1}$ and $w = (w_i)_{i \geq 0} = (y_{-i})_{i \geq 0}$. One thus gets a two-sided sequence of the form

$$\ldots u_3 u_2 u_1 w_{0} u_{1} u_{2} u_{3} \ldots$$
and write it as $y = \beta_{(\beta_1, \beta_2, \ldots, \beta_n)}(u_1, u_2, \ldots, u_n)$. In other words, we will use the letters $(u_i)$ for denoting the “past” and $(\beta_i)$ for the “future” of the element $y = (w, u)$ of the two-sided shift $X_\beta$.

One-sided $\beta$-shifts. We denote by $X^1_\beta$ the set of one-sided sequences $u = (u_i)_{i \geq 1}$ such that there exists $w = (w_i)_{i \geq 0}$ with $(w, u) \in X_\beta$. This set is called the right one-sided $\beta$-shift. It coincides with the usual one-sided $\beta$-shift and is equal to the set of sequences $(u_i)_{i \geq 1}$ which satisfy

$$\forall k \geq 1, \ (u_i)_{i \geq k} \leq_{\text{lex}} d^1_\beta(1).$$

One similarly defines $X^0_\beta$ as the set of one-sided sequences $w = (w_i)_{i \geq 0}$ such that there exists $u = (u_i)_{i \geq 1}$ with $(w, u) \in X_\beta$. We call it the left one-sided $\beta$-shift.

Sofic shift. Since $\beta$ is a Parry number (simple or not), then $(X_\beta, S)$ is sofic [BM86]. We denote by $F(X_\beta)$ the set of finite factors of the sequences in $X_\beta$; the minimal automaton $\mathcal{M}_\beta$ recognizing the set of factors of $F(X_\beta)$ can easily be constructed (see Figure 1.1). The number of states $d$ of this automaton is equal to the length of the period $n$ of $d^1_\beta(1)$ if $\beta$ is a simple Parry number with $d(1) = n_1 \ldots n_{t} t_n$, $t_n \neq 0$, and to the sum of its preperiod $n$ plus its period $p$, if $\beta$ is a non-simple Parry number with $d(1) = n_1 \ldots n_t (n_{t+1} \ldots n_{t+p})^\infty$ ($t_n \neq t_{n+p}$, $t_{n+1} \ldots t_{n+p} \neq 0^p$).

![Figure 1.1. The automata $\mathcal{M}_\beta$ for $\beta$ simple Parry number ($d_\beta(1) = n_1 \ldots n_{t-1} t_n$) and for $\beta$ non-simple Parry number ($d_\beta(1) = n_1 \ldots n_t (n_{t+1} \ldots n_{t+p})^\infty$).](Image)

$\beta$-substitutions. Let us recall that a substitution $\sigma$ is a morphism of the free monoid $\mathcal{A}^*$, such that the image of each letter of $\mathcal{A}$ is non-empty. As introduced for instance in [Thu89] and in [Fab95], one can associate in a natural way a substitution $\sigma_\beta$ (called $\beta$-substitution) with $(X_\beta, S)$ over the alphabet $\{1, \ldots, d\}$, where $d$ denotes the number of states of the automaton $\mathcal{M}_\beta$: $j$ is the $k$-th letter occurring in $\sigma_\beta(i)$ (that is, $\sigma_\beta(i) = p j s$, where $p, s \in \{1, \ldots, d\}$ and $|p| = k – 1$) if and only if there is an arrow in $\mathcal{M}_\beta$ from the state $i$ to the state $j$ labeled by $k – 1$. One easily checks that this definition is consistent.

An explicit formula for $\sigma_\beta$ can be computed by considering the two different cases, $\beta$ simple and $\beta$ non-simple Parry number.

- Assume $d_\beta(1) = n_1 \ldots n_{t-1} t_n$ is finite, with $t_n \neq 0$. Thus $d^1_\beta(1) = n_1 \ldots n_{t-1} (t_n - 1)^\infty$. One defines $\sigma_\beta$ over the alphabet $\{1, 2, \ldots, n\}$ as shown in (1.3).
- Assume $d_\beta(1)$ is infinite. Then it cannot be purely periodic (according to Remark 7.2.5 [Fro02]). Hence $d_\beta(1) = d^1_\beta(1) = n_1 \ldots n_t (n_{t+1} \ldots n_{t+p})^\infty$, with $n \geq 1$, $t_n \neq t_{n+p}$ and $t_{n+1} \ldots t_{n+p} \neq 0^p$. One defines $\sigma_\beta$ over the alphabet $\{1, 2, \ldots, n + p\}$ as shown in (1.3).
We will use $\beta$-substitutions in Section 3 in order to describe properties of the set $\tilde{K}_\beta$. Notice that the automaton $M_\beta$ is exactly the prefix-suffix automaton of the substitution $\sigma_\beta$ considered for instance in [CS01a, CS01b], after reversing all the edges and replacing the labels in the prefix-suffix automaton by the lengths of the prefixes.

The incidence matrix of the substitution $\sigma_\beta$ is defined as the $d \times d$ matrix whose entry of index $(i, j)$ counts the number of occurrences of the letter $i$ in $\sigma(j)$. As a consequence of the definition, the incidence matrix of $\sigma_\beta$ coincides with the transpose of the adjacency matrix of the automaton $\mathcal{M}_\beta$. By eigenvalue of a substitution $\sigma$, we mean in all that follows an eigenvalue of the characteristic polynomial of the incidence matrix of $\sigma$. A substitution is said to be of Pisot type if all its eigenvalues except its largest one which is assumed to be simple, are non-zero and of modulus smaller than 1.

2. Representation of $\beta$-shifts

The right one-sided shift $X^r_\beta$ admits as a natural geometric representation the interval $[0, 1]$; namely, one associates with a sequence $(u_i)_{i \geq 1} \in X^r_\beta$ its real value $\sum_{i \geq 1} u_i \beta^{-i}$. We even have a measure-theoretical isomorphism between $X^r_\beta$ endowed with the shift, and $[0, 1]$ endowed with the map $T_\beta$. We want now to give a similar geometric interpretation of the set $X^l_\beta$; for that purpose, we first give a geometric representation of $X^l_\beta$ as a generalized Rauzy fractal.

2.1. Representation of the left one-sided shift $X^l_\beta$. The aim of this section is to introduce first a formal power series, called formal representation of $X^l_\beta$ and second a geometrical representation as an explicit compact set of in the product of Euclidean and $p$-adic spaces following [Sie03]; the primes which appear as $p$-adic spaces here will be the prime factors of the norm of $\beta$.

*Formal representation of the symbolic dynamical system $X^l_\beta$.*

**Definition 1.** The formal representation of $X^l_\beta$ is denoted

$$\varphi_X : X_\beta \to \mathbb{Q}[[X]]$$

where $\mathbb{Q}[[X]]$ is the ring of formal power series with coefficients in $\mathbb{Q}$, and defined by:

for all $(w_i)_{i \geq 0} \in X^l_\beta$, $\varphi_X(w_i) = \sum_{i \geq 0} w_i X^i \in \mathbb{Q}[[X]]$.

*Topologies over $\mathbb{Q}(\beta)$.* We now want to specialize these formal power series by giving to the indeterminate $X$ the value $\beta$, and associating with them values by making them converge. We thus want to find a topological framework in which all the series $\sum_{i \geq 0} w_i \beta^i$ for $(w_i)_{i \geq 0} \in X^l_\beta$, would converge; in fact, this boils down to find all the Archemedean and non-Archemedean metrizable topologies on $\mathbb{Q}(\beta)$ for which these series converge in a suitable completion; they are of two types.
• Suppose that the topology (with absolute value \( | \cdot | \)) is Archimedean: its restriction to \( \mathbb{Q} \) corresponds to the usual absolute value on \( \mathbb{Q} \) and there exists a \( \mathbb{Q} \)-isomorphism \( \tau_i \) such that \( |x| = |\tau_i(x)|_\mathbb{C} \), for \( x \in \mathbb{Q}(\beta) \). The series \( \varphi_X \) specialized in \( \beta \) converge in \( \mathbb{C} \) if and only if \( \tau_i \) is associated with a conjugate \( \beta \) of modulus strictly smaller than one.

• Assume that the topology is non-Archimedean: there exists a prime ideal \( I \) of the integer ring \( \mathcal{O}_\mathbb{Q}(\beta) \) of \( \mathbb{Q}(\beta) \) for which the topology coincides with the the \( I \)-adic topology; let \( p \) be the prime number defined by \( I \cap \mathbb{Z} = p\mathbb{Z} \); the restriction of the topology to \( \mathbb{Q} \) is the \( p \)-adic topology. The series \( \varphi_X \) specialized in \( \beta \) take finite values in the completion \( \mathbb{K}_I \) of \( \mathbb{Q}(\beta) \) for the \( I \)-adic topology if and only if \( \beta \in I \), i.e., \( |\beta|_I < 1 \).

**Representation space** \( \mathbb{K}_β \) of \( X^l_β \). We assume now that \( \beta \) be a Pisot number of degree \( d \), say. Let \( β_1, \ldots, β_r \) be the real conjugates of \( β \) (they all have modulus strictly smaller than 1, since \( β \) is Pisot), and let \( β_{r+1}, \ldots, β_{r+s} \) be its complex conjugates. For \( 2 \leq j \leq r \), let \( \mathbb{K}_{β_j} \) be equal to \( \mathbb{R} \), and for \( r+1 \leq j \leq r+s \), let \( \mathbb{K}_{β_j} \) be equal to \( \mathbb{C} \), \( \mathbb{R} \) and \( \mathbb{C} \) being endowed with the usual topology. For \( i = 1 \) to \( d \), let \( τ_i \) be a \( \mathbb{Q} \)-automorphism of \( \mathbb{K} = \mathbb{Q}(β_1, \ldots, β_d) \) which sends \( β \) on its algebraic conjugate \( β_i \). For a given \( i \) and for every element \( Q(β) \) of \( \mathbb{Q}(β) \), then \( τ_i(Q(β)) = Q(β_i) \).

We first gather the complex representations by omitting the ones which are conjugate in the complex case. This representation contains all the possible Archimedean values for \( \varphi_X \). It takes values in

\[
\mathbb{K}_∞ = \mathbb{K}_{β_2} \times \cdots \times \mathbb{K}_{β_{r+s}}.
\]

Let \( I_1, \ldots, I_ν \) be the prime ideals in the integer ring \( \mathcal{O}_\mathbb{Q}(β) \) of \( \mathbb{Q}(β) \) that contain \( β \), that is,

\[
β\mathcal{O}_\mathbb{Q}(β) = \prod_{i=1}^ν I_i^{n_i}.
\]

Recall that \( \mathbb{K}_I \) denotes the completion of \( \mathbb{Q}(β) \) for the \( I \)-adic topology. We then gather the representations in the completions of \( \mathbb{Q}(β) \) for the non-Archimedean topologies. Hence one defines the representation space of \( X_β \) as the direct product \( \mathbb{K}_β \) of all these fields:

\[
\mathbb{K}_β = \mathbb{K}_{β_2} \times \mathbb{K}_{β_{r+s}} \times \mathbb{K}_{I_1} \times \cdots \times \mathbb{K}_{I_ν} \simeq \mathbb{R}^{r-1} \times \mathbb{C}^s \times \mathbb{K}_{I_1} \times \cdots \times \mathbb{K}_{I_ν}.
\]

The field \( \mathbb{K}_I \) is a finite extension of the \( p_I \)-adic field \( \mathbb{Q}_{p_I} \) where \( I \cap \mathbb{Z} = p_I\mathbb{Z} \). For a given prime \( p \), the fields \( \mathbb{K}_I \) that are \( p \)-adic fields are the ones for which \( I \) contains simultaneously \( p \) and \( β \). Furthermore, the prime numbers \( p \) for which there exists a prime ideal of \( \mathcal{O}_\mathbb{Q}(β) \) which contains simultaneously \( p \) and \( β \) are exactly the prime divisors of the constant term of the minimal polynomial of \( β \) (see Lemma 4.2 [Sie(3)]. In particular, \( \mathbb{K}_β \) is a Euclidean space if and only if \( β \) is a unit. Endowed with the product of the topologies of each of its elements, \( \mathbb{K}_β \) is a metric abelian group.

The **canonical embedding** of \( \mathbb{Q}(β) \) into \( \mathbb{K}_β \) is defined by the following morphism:

\[
d_β : P(β) ∈ \mathbb{Q}(β) \mapsto (P(β_2), \ldots, P(β_{r+s}), P(β), \ldots, P(β) \in \mathbb{K}_β.
\]

Since the topology on \( \mathbb{K}_β \) has been chosen so that the formal power series \( \lim_{n→∞} d_β(\sum_{i=0}^n w_i β^i) = \sum_{i≥0} w_i d_β(β)^i \) are convergent in \( \mathbb{K}_β \) for every \( (w_i)_{i≥0} ∈ X^l_β \), one defines the following, where the notation \( d_β(\sum_{i≥0} w_i β^i) \) stands for \( \sum_{i≥0} w_i d_β(β)^i \).

**Definition 2.** The **representation map** of \( X^l_β \), called one-sided representation map, is defined by

\[
φ_β : X^l_β → \mathbb{K}_β, \quad (w_i)_{i≥0} → d_β(\sum_{i≥0} w_i β^i).
\]

We set \( R_β := φ_β(X^l_β) \) and call it the generalized Rauzy fractal or geometric representation of the left one-sided shift \( X^l_β \).
2.2. Examples. Valérie Berthé and Anne Siegel

The golden ratio. Let $\beta = (1 + \sqrt{5})/2$ be the golden ratio, that is, the largest root of $X^2 - X - 1$. One has $d_\beta(1) = 11$ ($\beta$ is a simple Parry number) and $d_\beta^*(1) = (10)\infty$. Hence $X_\beta$ is the set of sequences in $\{0, 1\}^\mathbb{Z}$ in which there are no two consecutive 1’s. Furthermore, the associated $\beta$-substitution is the Fibonacci substitution: $\sigma_\beta : 1 \mapsto 12, 2 \mapsto 1$. One has $\mathbb{K}_\beta = \mathbb{R}$; the canonical embedding $\delta_\beta$ is reduced to the map $\tau_{(1-\sqrt{5})/2}$ (that is, the $\mathbb{Q}$-automorphism of $\mathbb{Q}(\beta)$ which maps $\beta$ on its conjugate), and $\delta_\beta(Q(\beta)) = Q(\beta)$. The set $\mathcal{R}_\beta$ is an interval.

The Tribonacci number. Let $\beta$ be the Tribonacci number, that is, the Pisot root of the polynomial $X^3 - X^2 - X - 1$. One has $d_\beta(1) = 111$ ($\beta$ is a simple Parry number) and $d_\beta^*(1) = (110)\infty$. Hence $X_\beta$ is the set of sequences in $\{0, 1\}^\mathbb{Z}$ in which there are no three consecutive 1’s. Furthermore, $\sigma_\beta : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$. One has $\mathbb{K}_\beta = \mathbb{C}$; the canonical embedding is reduced to the $\mathbb{Q}$-isomorphism $\tau_\alpha$ which maps $\beta$ on $\alpha$, where $\alpha$ is one of the complex roots of $X^3 - X^2 - X - 1$. The set $\mathcal{R}_\beta$ which satisfies

$$\mathcal{R}_\beta = \left\{ \sum_{i\geq 0} w_i \alpha^i \; ; \; \forall i, \; w_i \in \{0, 1\}, \; w_iw_{i+1}w_{i+2} \neq 0 \right\}$$

is a compact subset of $\mathbb{C}$ called the Rauzy fractal. This set was introduced in [Rau82], see also [K91, Mes98, Mes00]. It is shown in Fig. 2.2 with a division into three pieces indicated by different shades. They correspond to the sequences $(w_i)_{i\geq 0}$ such that either $w_0 = 0$, or $w_0w_1 = 10$, or $w_0w_1 = 11$. There are as many pieces as the length of $d_\beta(1)$, which is also equal here to the degree of $\beta$. We will come back to the interest of this division of the Rauzy fractals into smaller pieces in Section 3.

The smallest Pisot number. Let $\beta$ be the Pisot root of $X^3 - X^2 - X - 1$. One has $d_\beta(1) = 10001$ ($\beta$ is a simple Parry number) and $d_\beta^*(1) = (10000)\infty$; $\sigma_\beta : 1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 1$; the characteristic polynomial of its incidence matrix is $(X^3 - X - 1)(X^2 - X + 1)$, hence $\sigma_\beta$ is not a substitution of Pisot type. A self-similar tiling generated by it has been studied in details in [AS98]; some connected surprising tilings have also been introduced in [E102]. One has $\mathbb{K}_\beta = \mathbb{C}$; the canonical embedding is also reduced to the $\mathbb{Q}$-isomorphism $\tau_\alpha$ which maps $\beta$ on $\alpha$, where $\alpha$ is one of the complex roots of $X^3 - X^2 - X - 1$. The set $\mathcal{R}_\beta$ is shown in Fig. 2.2 with a division into five pieces corresponding respectively to the sequences $(w_i)_{i\geq 0}$ such that either $w_0w_1w_2w_3 = 0000$, or $w_0 = 1, w_0w_1 = 01, w_0w_1w_2 = 001$, or $w_0w_1w_2w_3 = 0001$. The number of different pieces is equal to the length of $d_\beta(1)$; there are here 5 pieces whereas the degree of $\beta$ is 3.

A non-unit example. Let $\beta = 2 + \sqrt{2}$ be the dominant root of the polynomial $X^2 - 4X + 2$. The other root is $2 - \sqrt{2}$. One has $d_\beta(1) = d_\beta^*(1) = 31\infty$; $\beta$ is not a simple Parry number; $\sigma_\beta : 1 \mapsto 1112, 2 \mapsto 12$. The ideal $2\mathbb{Z}$ is ramified in $\mathcal{O}_{\mathbb{Q}(\sqrt{2})}$, that is, $2\mathbb{Z} = \mathbb{T}^2$. Hence there exists only one ideal which contains $\sqrt{2}$; its index of ramification is 2; the degree of the extension $\mathbb{K}_\beta$ over $\mathbb{Q}_2$ has degree 2. Hence the geometric one-sided representation $\mathcal{R}_\beta$ is a subset of $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_2$, shown in Fig. 2.2. The division of the Rauzy fractal cannot be expressed as in the previous examples as finite conditions on the prefixes of the sequences $(w_i)_{i\geq 0}$ (for more details, see Section 3).

2.3. Representation of the two-sided shift $(X_\beta, S)$. 

Definition 3. The representation map $\tilde{\varphi}_\beta : X_\beta \to \mathbb{K}_\beta$ of $X_\beta$ is defined for all $(w, u) = ((w_i)_{i \geq 0}, (u_i)_{i \geq 1}) \in X_\beta$ by:

$$\tilde{\varphi}_\beta(w, u) = (-\varphi_\beta(w_i), \sum_{i \geq 1} u_i \beta^{-i}) = (-\delta_\beta(\sum_{i \geq 0} w_i \beta^i), \sum_{i \geq 1} u_i \beta^{-i}) \in \mathbb{K}_\beta \times \mathbb{R}.$$ 

The set $\mathbb{R}_\beta := \tilde{\varphi}_\beta(X_\beta)$ is called the Rauzy fractal or geometric representation of the two-sided $\beta$-shift. This set is easily seen to be bounded and hence compact.

We will see below (see (2.4) and (4.1)) the interest of introducing the sign minus before $\delta_\beta$ in the definition of the map $\tilde{\varphi}_\beta$.

Extension $\tilde{T}_\beta$ of the $\beta$-transformation. One can extend in a natural way the definition of $T_\beta$ to the product of the representation space $\mathbb{K}_\beta$ by $\mathbb{R}$ as follows, in order to obtain a realization of its natural extension.

Definition 4. Let $h_\beta : \mathbb{K}_\beta \to \mathbb{K}_\beta$ stands for the multiplication map in $\mathbb{K}_\beta$ by the diagonal matrix whose diagonal coefficients are given by $\delta_\beta(\beta)$. One thus defines

$$\tilde{T}_\beta : \mathbb{K}_\beta \times \mathbb{R} \to \mathbb{K}_\beta \times \mathbb{R}, \quad (a, b) \mapsto (h_\beta(a) - [\beta b] \delta_\beta(1), \beta b - [\beta b]).$$

In particular, the following commutation relations hold, where $Id$ denotes the identity map over $\mathbb{R}$; recall that $S$ denotes the shift over $X_\beta$:

(2.3) $\tilde{T}_\beta \circ (\delta_\beta, Id) = (\delta_\beta, Id) \circ T_\beta$ over $\mathbb{Q}(\beta)$.

(2.4) $\forall (w, u) \in X_\beta,$ if $u \neq d_0^\beta(1)$, $\tilde{\varphi}_\beta \circ S(w, u) = \tilde{T}_\beta \circ \tilde{\varphi}_\beta(w, u).$

The proof of (2.3) is immediate. Let us prove (2.4). Take an element $(w, u) = ((w_i)_{i \geq 0}, (u_i)_{i \geq 1}) \in X_\beta$ with $u \neq d_0^\beta(1)$. One has $\tilde{\varphi}_\beta(w, u) = (-\delta_\beta(\sum_{i \geq 0} w_i \beta^i), \sum_{i \geq 1} u_i \beta^{-i})$. Since $u$ satisfies (1.1), then the integer part of $\beta(\sum_{i \geq 1} u_i \beta^{-i})$ is equal to $u_1$, hence $T_\beta(\sum_{i \geq 1} u_i \beta^{-i}) = \sum_{i \geq 2} u_i \beta^{-i}$, and the assertion follows.
Examples. The geometric representation map $\tilde{\varphi}_\beta$ of the golden ratio-shift maps to $\mathbb{R}^2$. The ones for the Tribonacci number and the smallest Pisot number map to $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. They are shown (up to a change of coordinates) in Fig. 2.2. The sets $\tilde{\mathcal{R}}_\beta$ are unions of products of the different pieces of the Rauzy fractal $\mathcal{R}_\beta$ by finite real intervals of different heights. For instance, in the Tribonacci case, since the different pieces in $\mathcal{R}_\beta$ correspond to the sequences $(w_i)_{i \geq 0}$ such that either $w_0 = 0$, or $w_0w_1 = 10$, or $w_0w_1 = 11$, this gives different constraints on the sequences $(u_i)_{i \geq 1}$ which produce the real component; for instance, the piece which corresponds to the sequences $(w_i)_{i \geq 0}$ such that $w_0w_1 = 11$ implies the following constraint on the sequences $(u_i)_{i \geq 1}$: $u_0$ has to be equal to 0.

![Figure 2.2. The geometric representation of the two-sided $\beta$-shift for the Fibonacci-shift, the Tribonacci-shift, and the smallest Pisot number-shift.](image)

Remark 1. The definition of the representation map $\tilde{\varphi}_\beta$ is inspired by [Sie03] in the substitutive case. We cannot apply here directly the substitutive formalism to the substitution $\sigma_\beta$ since it is generally not Pisot: the dominant eigenvalue of $\sigma_\beta$ is still a Pisot number but other eigenvalues may occur, as in the smallest Pisot case. The main difference here is that we do not take those extra eigenvalues into account in our definition of $\tilde{\varphi}_\beta$.

3. Geometric properties of the Rauzy fractal of the $\beta$-numeration

The aim of this section is to prove some geometric properties of the Rauzy fractal associated with the $\beta$-numeration:

**Theorem 2.** Let $\beta$ be a Pisot number.

1. The Rauzy fractal $\mathcal{R}_\beta$ of the one-sided $\beta$-shift has a graph directed self-affine structure in the sense of [MWS88]. More precisely:
   - it has a non-zero measure for the Haar measure;
   - there are $d$ pieces which form the self-affine structure, where $d$ is equal to the period of $d_\beta(1)^*$ if $\beta$ is a simple Parry number or to the sum of its preperiod plus its period, otherwise.

2. The Rauzy fractal $\tilde{\mathcal{R}}_\beta$ of the two-sided $\beta$-shift has non-zero measure for the Haar measure $\mu_{\mathbb{K}_\beta}$ over $\mathbb{K}_\beta \times \mathbb{R}$. It is the disjoint union of $d$ cylinders obtained as the product of each piece of the one-sided Rauzy fractal by a finite interval of $\mathbb{R}$.

3.1. **Proof of Theorem 2.** The proof of the first point of this theorem is divided into several steps; roughly, we use the self-affine structure of $\mathcal{R}_\beta$ to deduce that it has non-zero measure. Similar statements are proved in the framework of Pisot substitutions in [Sie03], but we cannot use here directly these statements since the $\beta$-substitutions (Section []) used here are not necessarily Pisot according to Remark [], as for instance in the smallest Pisot case.
Division of $\mathcal{R}_\beta$ into subpieces. Let us first divide (as illustrated in Figure 2.3) the set $\mathcal{R}_\beta$ into $d$ pieces where $d$ denotes the number of states in the minimal automaton $\mathcal{M}_\beta$ which recognizes the set of factors $F(X_\beta)$ of the two-sided shift $X_\beta$.

Let $\mathcal{M}_\beta$ denote the (non-deterministic) automaton obtained from $\mathcal{M}_\beta$ by reversing the orientation of the edges. The set of sequences in the left one-sided shift is equal to the set of labels of infinite one-sided paths in the automaton $\mathcal{M}_\beta$. We label the states of the automaton $\mathcal{M}_\beta$ by $a_1, \cdots, a_d$. For $i = 1, \cdots, d$, one defines

$$\mathcal{R}_\beta(i) := \left\{ \varphi_\beta((w_k)_{k \geq 0}) : (w_k)_{k \geq 0} \in A_\beta^\mathbb{N} : (w_k)_{k \geq 0} \text{ is a path from the state } a_i \text{ in } \mathcal{M}_\beta \right\}.$$ 

Let us recall that $d$ is larger than or equal to the degree $r$ of the minimal polynomial of $\beta$. For instance, $r = 3$ and $d = 5$ in the smallest Pisot case. In particular, $d$ can be arbitrarily large when $\beta$ is cubic, according to [Bas02].

Self-affine decomposition. We recall that the map $h_\beta : \mathbb{K}_\beta \to \mathbb{K}_\beta$ stands for the multiplication map in $\mathbb{K}_\beta$ by the diagonal matrix whose diagonal coefficients are given by $\delta_\beta(\beta)$ (Definition 4). Hence $\delta_\beta(\beta x) = h_\beta \delta_\beta(x)$ for every $x \in \mathbb{Q}(\beta)$.

Let us first prove that for $i = 1, \cdots, d$:

$$(3.1) \quad \mathcal{R}_\beta(i) = \bigcup_{1 \leq j \leq d \cup_p, \sigma_\beta(j) = pis} h_\beta(\mathcal{R}_\beta(j)) + \delta_\beta(|p|).$$

Let $i \in \{1, \cdots, d\}$ be given. Let $(w_k)_{k \geq 0} \in \mathcal{R}_\beta(i)$. One has

$$\varphi_\beta((w_k)_{k \geq 0}) = \delta_\beta(\sum_{k \geq 1} w_k \beta^k) + \delta_\beta(w_0) = h_\beta \delta_\beta(\sum_{k \geq 1} w_k \beta^{k-1}) + \delta_\beta(w_0) = h_\beta \varphi_\beta((w_k)_{k \geq 1}) + \delta_\beta(w_0).$$

Let $a_j$ denote the state in $\mathcal{M}_\beta$ obtained by reading the label $w_0$ from the state $a_i$. By the definition of $\sigma_\beta$, there exist $p = 1^{w_0}$ and $s$ such that $\sigma_\beta(j) = pis$, that is, $w_0 = |p|$. Hence

$$\varphi_\beta((w_k)_{k \geq 0}) \in h_\beta(\mathcal{R}_\beta(j)) + \delta_\beta(|p|),$$

which provides one inclusion for the equality \((3.1)\). The other inclusion is then immediate.

The Rauzy fractal has non-zero measure. The proof is an adaptation of the corresponding proof of [Sie03] which is done in the framework of Pisot substitution dynamical systems; two properties are recalled below without a proof, the first one describing the action of the multiplication map $h_\beta$ on the Haar measure $\mu_{\mathbb{K}_\beta}$ of $\mathbb{K}_\beta$:

Lemma 1 ([Sie03]). (1) For every Borelian set $B$ of $\mathbb{K}_\beta$

$$\mu_{\mathbb{K}_\beta}(h_\beta(B)) = \frac{1}{\beta} \mu_{\mathbb{K}_\beta}(B).$$

(2) Let $\mathcal{S}$ be a finite set included in $\mathbb{Q}(\beta)$. The set of points $\{\delta(P(1/\beta)) : P \in \mathcal{S} \{X\} \}$ is a discrete set in $\mathbb{K}_\beta$.

Let $(U_N)_{N \in \mathbb{N}}$ denote the linear canonical numeration system associated with $\beta$ according to [BM89], that is, $U_0 = 1$, and for all $k$, $U_k = t_kU_{k-1} + \cdots + t_1U_0 + 1$, where $d_\beta(1) = (t_k)_{k \in \mathbb{N}}$. Let us expand every integer $i = 0, 1, \cdots, U_N - 1$ in this system according to the greedy algorithm (this expansion being unique):

$$i = \sum_{0 \leq k \leq N-1} w_k^{(i)} U_k.$$
According to [BM89], see also [Fro02], the finite words \(w_0^{(i)} \ldots w_{N-1}^{(i)}\), for \(i = 0, 1, \ldots, U_N - 1\) are all distinct and all belong to the set \(F\). Hence the sequences \((w_0^{(i)} \ldots w_{N-1}^{(i)})\), for \(i = 0, 1, \ldots, U_N - 1\), all belong to the left one-sided \(\beta\)-shift \(X_\beta^L\).

Let \(E_N\) denote the image under the action of \(\varphi_\beta\) of this set of points. One has for \(i = 0, 1, \ldots, U_N - 1\),

\[
\varphi_\beta((w_0^{(i)} \ldots w_{N-1}^{(i)})0^\infty) = \delta_\beta\left(\sum_{0 \leq k \leq N-1} w_k^{(i)} \beta^i\right).
\]

The points in \(E_N\) are all distinct since for \(i = 0, 1, \ldots, U_N - 1\), the finite words \(w_0^{(i)} \ldots w_{N-1}^{(i)}\) are all distinct. There thus exists \(B > 0\) such that \(\text{Card } E_N > U_N > B\beta^N\), since \(\beta\) is a Pisot number.

Let us apply now Lemma 1 with \(S = \{0, 1, \ldots, [\beta]\}\). Hence there exists a constant \(A > 0\) such that the distance between two elements of \(E_N\) is larger than \(A\). Let us define now for every non-negative integer \(N\), the set

\[
B_N = \bigcup_{z \in E_N} (h_\beta)^N B(z, A/2),
\]

where \(B(z, A/2)\) denotes the closed ball in \(K_\beta\) of center \(z\) and radius \(A/2\). According to Lemma 1 and to the fact that \(\text{Card } E_N > B\beta^N\), for all \(N\), there exists \(C > 0\) such that \(\mu_{K_\beta}(B_N) > C\), for all \(N\).

The main point now is that the sequence of compact sets \((B_N)_{N \in \mathbb{N}}\) converges toward a subset of \(R_\beta\) with respect to the Hausdorff metric; indeed for a fixed positive integer \(p\), for \(N\) large enough, \(B_N \subset R_\beta(1/p) := \{x \in K_\beta; d(x, R_\beta) \leq 1/p\}\). Independently, fix \(\varepsilon > 0\); since the sequence \((R_\beta(1/p))_p\) converges toward \(R_\beta\), there exists \(p > 0\) such that \(\mu_{K_\beta}(R_\beta(1/p)) \leq \mu_{K_\beta}(R_\beta) + \varepsilon\). This finally implies \(\liminf \mu_{K_\beta}(B_N) \leq \mu_{K_\beta}(R_\beta) + \varepsilon\). Since this holds for every \(\varepsilon > 0\), one obtains \(\mu_{K_\beta}(R_\beta) \geq C > 0\).

**Computation of the measures and self-affine structure.** Let us prove now that the union in (3.1) is a disjoint union up to sets of zero measure. One has for a given \(i \in \{1, \ldots, d\}\) according to (3.1) and to Lemma 1

\[
\mu_{K_\beta}(R_\beta(i)) \leq \sum_{j=1, \ldots, d, \sigma(j)=\pi a} \mu_{K_\beta}(h_\beta(R_\beta(j))) \\
\leq 1/\beta \sum_{j=1, \ldots, d, \sigma(j)=\pi a} \mu_{K_\beta}(R_\beta(j)).
\]

Let \(m = (\mu_{K_\beta}(R_\beta(i)))_{i=1, \ldots, d}\) denote the vector in \(\mathbb{R}^d\) of measures in \(K_\beta\) of the pieces of the Rauzy fractal. We know from what precedes that \(m\) is a non-zero vector with non-negative entries. According to Perron-Frobenius theorem, the previous equality implies that \(m\) is an eigenvector of the primitive incidence matrix of the substitution \(\sigma\). We thus have equality in (3.2) which implies that the unions are disjoint up to sets of zero measure. One similarly proves that this equality in measure still holds by replacing \(\sigma\) by \(\sigma^n\), for every \(n\).

Now take two distinct pieces, \((R_\beta(j))\) and \((R_\beta(k))\) say, with \(j \neq k\). There exists \(n\) such that both \(\sigma^n(j)\) and \(\sigma^n(k)\) admit as first letter \(1\). Hence they both occur in (3.1) for \(i = 1\) with the same translation term (which is indeed equal to 0) and they are thus distinct. We hence have proved that the \(d\) pieces of the Rauzy fractal \((R_\beta(j))\) are disjoint up to sets of zero measure, which ends the proof of Assertion (i), that is, \(R_\beta\) has a self-affine structure.

**\(R_\beta\) has non-zero measure.** It remains now to prove the second point of the theorem. The assertion that \(R_\beta\) has non-zero measure is a direct consequence of the structure of the Rauzy fractal \(\widetilde{R}_\beta\) studied above since \(\widetilde{R}_\beta\) can be decomposed as the disjoint (in measure) union of \(\widetilde{R}_\beta(i), i = 1, \ldots, d\), where

\[
\widetilde{R}_\beta(i) := \left\{ \tilde{\varphi}_\beta(u, w) \colon (u, w) \text{ is a two-sided path in } \widetilde{M}_\beta \text{ such that } w \text{ starts from the state } a_i \right\}.
\]
which the product of $\mathcal{R}_\beta(n)$ by a finite interval of $x$ of non-zero measure.

3.2. Examples. Let us pursue the study of three of the examples of Section 2.2.

The Tribonacci number. Let us recall that

$$\mathcal{R}_\beta = \{ \sum_{i \geq 0} w_i \alpha^i ; \forall i, w_i \in \{0, 1\}, w_i w_{i+1} w_{i+2} \neq 0 \}.$$  

One easily checks thanks to the automaton $\tilde{M}_\beta$ shown in Fig. 8.1 that $\mathcal{R}_\beta(1)$ corresponds to the sequences $(w_i)_{i \geq 0}$ such that $w_0 = 1$, $\mathcal{R}_\beta(2)$ corresponds to the set of sequences $(w_i)_{i \geq 0}$ such that $w_0 w_1 = 10$, and lastly, $\mathcal{R}_\beta(3)$ corresponds to the sequences $(w_i)_{i \geq 0}$ such that $w_0 w_1 = 11$. One has

$$\mathcal{R}_\beta(1) = \alpha(\mathcal{R}_\beta(1) \cup \mathcal{R}_\beta(2) \cup \mathcal{R}_\beta(3)) \cup \mathcal{R}_\beta(2) = \alpha(\mathcal{R}_\beta(1)) + 1 \cup \mathcal{R}_\beta(3) = \alpha(\mathcal{R}_\beta(2)) + 1.$$  

The smallest Pisot number. One has

$$\mathcal{R}_\beta = \{ \sum_{i \geq 0} w_i \alpha^i ; \forall i, w_i \in \{0, 1\}, \text{ if } w_i = 1, \text{ then } w_{i+1} = w_{i+2} = w_{i+3} = w_{i+4} = 0 \}.$$  

One easily checks that $\mathcal{R}_\beta(1)$ corresponds to the sequences $(w_i)_{i \geq 0}$ such that $w_0 = 0000$, $\mathcal{R}_\beta(2)$ corresponds to the set of sequences $(w_i)_{i \geq 0}$ such that $w_0 w_1 = 1$, $\mathcal{R}_\beta(3)$ corresponds to the sequences $(w_i)_{i \geq 0}$ such that $w_0 w_1 = 01$, $\mathcal{R}_\beta(4)$ corresponds to the sequences $(w_i)_{i \geq 0}$ such that $w_0 w_1 = 001$, and lastly $\mathcal{R}_\beta(5)$ corresponds to the sequences $(w_i)_{i \geq 0}$ such that $w_0 w_1 = 0001$.

One has

$$\mathcal{R}_\beta(1) = \alpha(\mathcal{R}_\beta(1) \cup \mathcal{R}_\beta(5)) \cup \mathcal{R}_\beta(2) = \alpha(\mathcal{R}_\beta(1)) + 1 \cup \mathcal{R}_\beta(3) = \alpha(\mathcal{R}_\beta(2)) \cup \mathcal{R}_\beta(4) = \alpha(\mathcal{R}_\beta(3)) \cup \mathcal{R}_\beta(5) = \alpha(\mathcal{R}_\beta(4)).$$  

The $(2 + \sqrt{2})$-shift. One has

$$\mathcal{R}_\beta = \{ \sum_{i \geq 0} w_i (2 - \sqrt{2})^i ; \forall i, w_i \in \{0, 1, 2\}, (w_i)_{i \geq 0} \leq \text{lex } (31^\infty) \}.$$  

In this non-simple Parry case, we cannot express as easily as previously the sets $\mathcal{R}_\beta(1)$ and $\mathcal{R}_\beta(2)$: one checks in Figure 8.4 that there exist cycles from $a_1$ to $a_1$ and from $a_2$ to $a_2$, which implies that both $\mathcal{R}_\beta(1)$ and $\mathcal{R}_\beta(2)$ contain sequences with arbitrarily long common prefixes, such as $1^n$ for every $n$.

One has

$$\mathcal{R}_\beta(1) = (2 - \sqrt{2})(\mathcal{R}_\beta(1)) \cup (2 - \sqrt{2})(\mathcal{R}_\beta(1)) + 1 \cup (2 - \sqrt{2})(\mathcal{R}_\beta(1)) + 2 \cup (2 - \sqrt{2})(\mathcal{R}_\beta(2)) \cup (2 - \sqrt{2})(\mathcal{R}_\beta(1)) + 3 + (2 - \sqrt{2})(\mathcal{R}_\beta(2)) + 1.$$  

4. Characterization of purely periodic points

We can now state the main theorem of this paper.

**Theorem 3.** Let $\beta$ be a Pisot number. For all $x \in \mathbb{Q}(\beta) \cap [0, 1)$, the $\beta$-expansion of $x$ is purely periodic if and only if $(\delta(x), x) \in \mathcal{R}_\beta = \varphi(\mathcal{X}_\beta)$.
belongs to \( Q \) do not appear in the decomposition (2.1). These topologies are the extensions on \( T \) involving the primes that were not already considered in \( \beta \) of \( I \), and which is stable under the action of the map \( \tilde{\varphi} \). We thus have

\[
\sum_{i \geq 1} u_i \beta^{-i} = \frac{a_1 \beta^{-1} + \cdots + a_L \beta^{-L}}{1 - \beta^{-L}} = \frac{a_1 \beta^{L-1} + \cdots + a_L}{\beta^L - 1} = x.
\]

Furthermore, \( \lim_{n \to -\infty} \delta_\beta(\beta^n) = 0 \) in \( K_\beta \). We thus have

\[
\tilde{\varphi}_\beta(w, u) = \left( -\delta_\beta\left( \sum_{i \geq 0} w_i \beta^i \right), \sum_{i \geq 1} u_i \beta^{-i} \right)
\]

\[
= \left( -\lim_{n \to -\infty} \delta_\beta((a_L + \cdots + a_1 \beta^{L-1})(1 + \beta^L + \cdots + \beta^n L)), x \right)
\]

\[
= \left( \lim_{n \to -\infty} \delta_\beta(\cdots(a_1 \beta^{L-1} + \cdots + a_L 1 - \beta^L \beta^L), x \right)
\]

\[
= \delta_\beta\left( \frac{a_1 \beta^{L-1} + \cdots + a_L}{\beta^L - 1}, x \right)
\]

\[
= (\delta_\beta(x), x),
\]

hence \( (\delta_\beta(x), x) \in \hat{R}_\beta \).

Consider now the reciprocal and let \( x \in Q(\beta) \) such that \( (\delta_\beta(x), x) \in \hat{R}_\beta \).

The sketch of the proof is the following and is inspired by a similar discrete argument in [IR02], dealing with the case \( \beta \) is a unit. We introduce below a finite subset \( T_x \) of \( \hat{R}_\beta \), which depends on \( x \), and which is stable under the action of the map \( \tilde{\varphi}_\beta \) which is onto on it. In order to define this finite set, we take into account all the \( \mathcal{I} \)-adic topologies which correspond to prime ideals \( \mathcal{I} \) which do not appear in the decomposition (2.3). These topologies are the extensions on \( Q(\beta) \) of \( p \)-adic topologies for the primes \( p \) which do not divide the constant term of the minimal polynomial \( P_\beta \) of \( \beta \) according to [Sie03]. Roughly speaking, one introduces further restrictions over \( Q(\beta) \) which involve the primes that were not already considered in \( K_\beta \). We will prove below that the set \( T_x \subset \hat{R}_\beta \subset K_\beta \times Q(\beta) \) is finite by using the fact that the second coordinate of its elements, which belongs to \( Q(\beta) \), is bounded for all the topologies on \( Q(\beta) \).

**Figure 3.1.** Reversed minimal automaton \( \tilde{M}_\beta \) describing the structure of the \( \beta \)-shift.
Let us first introduce the following set \( S_x \):

\[
S_x = \{ z \in \mathbb{Q}(\beta), \ |z|_I \leq \max(|x|_I, 1) \},
\]

for every prime ideal \( I \) such that \(|\beta|_I = 1\).

One has

- \( x \in S_x \) since \(|x|_I \leq \max(|x|_I, 1) \) for every prime ideal \( I \).
- \( \mathbb{Z} \subset S_x \) since for every integer \( N \) and for every ideal \( I \), one has \(|N|_I \leq 1 \leq \max(|x|_I, 1) \).
- \( \beta S_x \subset S_x \) and \( \beta^{-1} S_x \subset S_x \) since \(|\beta z|_I = |\beta|_I |z|_I = |z|_I \) if \(|\beta|_I = 1 \), and as well \(|\beta^{-1} z|_I = |z|_I \).
- \((S_x, +)\) is a group since the \( I \)-adic valuations are ultrametric.

Hence one deduces that:

- \( S_x \) is stable under the action of \( T_\beta \); indeed
  \[
  T_\beta S_x \subset \beta S_x - \mathbb{Z} \subset S_x - \mathbb{Z} \subset S_x - S_x \subset S_x.
  \]
- For every integer \( N \), one has \( \beta^{-1}(S_x + N) \subset \beta^{-1} S_x \subset S_x \).

In the original proof of [1R02], the analogous of the set \( S_x \) is \( \mathbb{Z}[\beta]/q \) for an integer \( q \) which depends on \( x \), this set being no more stable under the multiplication by \( 1/\beta \) in the non-unit case. The keypoint of the present proof is that \( S_x \) is not only stable under the action of \( T_\beta \) but also under the multiplication by \( 1/\beta \), even when \( \beta \) is not a unit.

Let us consider now the following subset of \( \tilde{\mathcal{R}}_\beta \) obtained by first embedding the points of \( S_x \) into \( \mathbb{K}_\beta \times \mathbb{R} \), and then intersecting it with the bounded set \( \tilde{\mathcal{R}}_\beta \):

\[
T_x = (\delta_\beta, Id)(S_x) \cap \tilde{\mathcal{R}}_\beta \subset \mathbb{K}_\beta \times \mathbb{Q}(\beta).
\]

Let us first observe that \((\delta_\beta(x), x) \in T_x\).

- **The set \( T_x \) is finite.** Indeed let us prove that all the absolute values of \( \mathbb{Q}(\beta) \) take bounded values on the last coordinate of the elements of \( T_x \). Let \( z \in S_x \) such that \((\delta_\beta(z), z) \in \tilde{\mathcal{R}}_\beta \). Note first that the usual absolute value of \( z \) is bounded. Furthermore
  - if \( |\cdot| \) is an Archimedean valuation or if \( |\cdot|_I \) is an ultrametric valuation satisfying \(|\beta|_I \neq 1 \), this valuation appears by construction in \( \mathbb{K}_\beta \), and thus \(|z|_I \) is bounded since \((\delta_\beta(x), x) \) belongs to the bounded set \( \tilde{\mathcal{R}}_\beta \).
  - if \(|\cdot|_I \) is an ultrametric valuation satisfying \(|\beta|_I = 1 \), then \(|z|_I \) is bounded by definition of \( S_x \).

- **The map \( \tilde{T}_\beta \) is onto \( T_x \).** Let \((\delta_\beta(z), z) \in T_x \), with \( z \in S_x \). By definition, there exists \((w, u) = ((w_i)_{i \geq 0}, (u_i)_{i \geq 1}) \in \tilde{X}_\beta \) such that \( \tilde{\varphi}_\beta(w, u) = (\delta_\beta(z), z) \). Hence \( z = \sum_{i \geq 1} u_i \beta^{-i} \) and \( \delta_\beta(z) = -\delta_\beta(\sum_{i \geq 0} w_i \beta^i) \). An easy computation shows that

\[
\begin{align*}
\tilde{\varphi}_\beta \circ S^{-1}(w, u) &= \left( \delta_\beta \left( \frac{z + w_0}{\beta} \right), \frac{z + w_0}{\beta} \right).
\end{align*}
\]

Let us note that this computation works thanks to the introduction of a sign minus before \( \delta_\beta \) in the expression of \( \tilde{\varphi}_\beta \).

Let us first assume that \( w_0 u_1 u_2 \cdots \neq d'_\beta(1) \). One thus deduces according to(2.4), that

\[
(\delta_\beta(z), z) = \tilde{\varphi}_\beta(w, u) = \tilde{T}_\beta \circ S \circ S^{-1}(w, u) = \tilde{T}_\beta \circ \tilde{\varphi}_\beta \circ S^{-1}(w, u) = \tilde{T}_\beta \circ (\delta_\beta, Id) \left( \frac{z + w_0}{\beta} \right).
\]

On the other hand \((\delta_\beta, Id) \left( \frac{z + w_0}{\beta} \right) = \tilde{\varphi}_\beta \circ S^{-1}(w, u) = \tilde{\mathcal{R}}_\beta \), and on the other hand \( \frac{z + w_0}{\beta} \in \tilde{S}_x + w_0 \subset \beta^{-1}(S_x + \mathbb{Z}) \subset S_x \). Hence, \((\delta_\beta(z), z) \in \tilde{T}_\beta(T_x) \).
Assume now that \( w_0 u_1 a_1 + \cdots + a_n \beta^n \). By definition \( \beta \neq \beta_0 \). Moreover, one checks that \( \beta = \beta - w_0 \). One has

\[
\tilde{T}_\beta \circ (\delta_\beta, Id)(1) = (h_\beta(1) - [\beta] \delta_\beta(1), \beta - [\beta]) = (\delta_\beta(\beta) - \delta_\beta([\beta]), \beta - [\beta]) = (\delta_\beta(z), z).
\]

Since \( \frac{\beta}{\beta} \) and from (1.1), one has \( (\delta_\beta, Id)(1) \in \tilde{R}_\beta \) so that \( \delta_\beta(z), z \in \tilde{T}_\beta(T_x) \).

- **The set** \( T_x \) **is stable by** \( \tilde{T}_\beta \). Let \( (\delta_\beta(z), z) \in T_x \), with \( z \in S_x \). By definition, there exists \( (w, u) = ((w_1)_{i\geq 0}, (u_i)_{i\geq 1}) \in \tilde{X}_\beta \) such that \( \tilde{\varphi}_\beta(w, u) = (\delta_\beta(z), z) \). One has according to (2.3)

\[
\tilde{T}_\beta(\delta_\beta(z), z) = (\delta_\beta, Id) \circ T_\beta(z).
\]

Let us first suppose that \( u \neq d_\beta^n(1) \). One deduces from (2.4)

\[
\tilde{T}_\beta \circ \tilde{\varphi}_\beta(w, u) = \tilde{\varphi}_\beta \circ S(w, u).
\]

Now if \( u = d_\beta^n(1) \), then \( z = \sum_{i\geq 1} u_i \beta^{-i} = 1 \), \( T_\beta(z) = \beta - u_1 \) and \( \delta_\beta(\sum_{i\geq 0} w_i \beta^i) = \delta_\beta(1) \) so that

\[
\tilde{\varphi}_\beta \circ S(w, u) = (-\delta_\beta(u_1 + \sum_{i\geq 0} w_i \beta^{i+1}), \sum_{i\geq 2} u_i \beta^{-i+1}) = (-\delta_\beta(u_1) + \delta_\beta(\beta), \beta - u_1) = (\delta_\beta, Id) \circ T_\beta(z).
\]

Hence, since \( \tilde{T}_\beta \) is onto over a stable finite set, it is one-to-one over this set \( T_x \). By construction, \( (\delta_\beta(x), x) \in T_x \). Hence there exists an integer \( n \) such that

\[
(\delta_\beta(x), x) = \tilde{T}_\beta^{-n}(\delta_\beta(x), x) = (\delta_\beta(T_\beta^n(x)), T_\beta^n(x)).
\]

We thus deduce that the expansion of \( x \) is purely periodic.

5. Conclusion

One can also introduce a notion of generalized Rauzy fractal in the non-Pisot case. Namely, a similar characterization holds in the non-Pisot case by introducing in the representation map \( \tilde{\varphi}_\beta \) the coordinate \( \sum_{i\geq 1} u_i \lambda^{-i} \) for the conjugates \( \lambda \) of modulus strictly larger that 1 (see for instance [KV98] for a similar description). Yet we are not able to prove that the union in Equation (1.1) is disjoint up to sets of zero measure and that the Haar measure of the generalized Rauzy fractal is non-zero.

It remains now to use this formalism to study further topological or metrical properties of the sets \( R_\beta \) and \( \tilde{R}_\beta \), even in the non-Pisot case, which will be the object of a subsequent paper. Our motivations are the following: first the construction of explicit Markov partitions of endomorphisms of the torus as initiated in [Sie00], second, the study of rational numbers having a purely periodic expansion in the flavour of [Aki98, Sch80], and third, the spectral study of \( \beta \)-shifts in the Pisot non-unit case according to [Aki99].

**References**


