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VALERIE BERTHÉ AND ANNE SIEGEL

Abstract. It is well known that real numbers with a purely periodic decimal expansion are the rationals having, when reduced, a denominator coprime with 10. The aim of this paper is to extend this result to beta-expansions with a Pisot base beta which is not necessarily a unit: we characterize real numbers having a purely periodic expansion in such a base; this characterization is given in terms of an explicit set, called generalized Rauzy fractal, which is shown to be a graph-directed self-affine compact subset of non-zero measure which belongs to the direct product of Euclidean and p-adic spaces.

Keywords: expansion in a non-integral base, Pisot number, beta-shift, beta-numeration, purely periodic expansion, self-affine set.

Let β be a Pisot number and $T_\beta : x \mapsto \beta x \pmod 1$ be the associated β-transformation. The aim of this paper is to characterize the real numbers x in $\mathbb{Q}(\beta) \cap [0,1)$ having a purely periodic β-expansion.

It is well known that if β is a Pisot number, then the real numbers that have a ultimately periodic β-expansion are the elements of $\mathbb{Q}(\beta)$ [Ber77, Sch80]. Thus real numbers x that have a purely periodic beta-expansion belong to $\mathbb{Q}(\beta)$. We present a characterization that involves the conjugates of the algebraic number x, and can be compared to Galois' theorem for classical continuous fractions.

Theorem 1. *Let β be a Pisot number. A real number $x \in \mathbb{Q}(\beta) \cap [0,1)$ has a purely periodic beta-expansion if and only if x and its conjugates belong to an explicit subset in the product of Euclidean and p-adic spaces (see Figure 2.2 below); this set (denoted $\widetilde{\mathcal{R}_\beta}$ and called generalized Rauzy fractal) is a graph-directed self-affine compact subset in the sense of* [MW88] *of non-zero measure; the primes p that occur are the prime divisors of the norm of β.*

The scheme of the proof is based on a realization of the natural extension of the β-transformation T_β extended to a geometric space of representation for the two-sided β-shift (X_β, S). Our results and our proof is inspired by [IR02, IS01, San02] which presents a similar characterization of purely periodic expansions in the case where β is a Pisot unit.

The construction of the set $\widetilde{\mathcal{R}_\beta}$ (introduced in Theorem 1) is inspired by the geometric representation as generalized Rauzy fractals (also called atomic surfaces) of substitutive symbolic dynamical systems in the non-unimodular case developed in [Sie03]. In fact, a substitution σ is a non-erasing morphism of the free monoïd $\mathcal{A}^*$ and a substitutive dynamical system is a symbolic dynamical system generated by an infinite sequence which is a fixed point of a substitution. Furthermore, if the β-expansion of 1 in base β is finite (β is said to be a simple Parry number) and if its length coincides with the degree of β, then the set $\widetilde{\mathcal{R}_\beta}$ involved in our characterization is exactly the generalized Rauzy fractal that is associated in [Sie03] with the underlying β-substitution (in the sense of [Thu89, Fab95]).

Rauzy fractals have been widely studied; for more details, see for instance [Sie02]. There are mainly two methods of construction for Rauzy fractals. One first approach inspired by the seminal paper [Rau82], is based on formal power series, and is developed in [Mes98, Mes00], or in [CS01a,

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CS01b]. A second approach via iterated function systems (IFS) and generalized substitutions has been developed following ideas from [IK91] in [AI01, SAI01, HZ98, SW02] with special focus on the self-similar properties of the Rauzy fractals. We combine here both approaches: we define the set $\widetilde{\mathcal{R}_\beta}$ by introducing a representation map of the two-sided shift (X_β, S) based on formal power series, and prove that this set has non-zero Haar measure by cutting it into pieces that are solutions of an IFS. Similar sets have also been introduced and studied in the framework of β-numeration by S. Akiyama in [Aki98, AS98, Aki99, Aki00], inspired by [Thu89]. As an application of generalized Rauzy fractals, let us mention that they provide Markov partitions for toral automorphisms of the torus, as illustrated in [IO93, KV98, Pra99, Sch00, Sie00]. Furthermore, there are numerous relations between generalized Rauzy fractals and discrete planes as studied for instance in [ABI02, ABS04].

The aim of this paper is twofold. We first want to characterize real numbers having a purely periodic β-expansion; we second try to settle the first steps of a study of the geometric representation of β-shifts in the Pisot non-unit case, generalizing the results of [Aki98, AS98, Aki99, Aki00], based on the formalism introduced in the substitutive case in [Sie03].

This paper is organized as follows. We first recall in Section 1 the basic elements needed on β-expansions. We then associate in Section 2 with the two-sided β-shift (X_β, S) formal power series in $\mathbb{Q}[[X]]$; we obtain in Section 2.3 a representation map for the two-sided β-shift by gathering the set of finite values which can be taken for any topology (Archimedean or not) by these formal power series when specializing them in β: in fact, we take the completion of $\mathbb{Q}(\beta)$ with respect to all the absolute values on $\mathbb{Q}(\beta)$ which take a value different from 1 on β (this value is thus smaller than 1 since β is Pisot). We are then able to define the Rauzy geometric representation of the two-sided β-shift (Definition 3). Section 3 is devoted to the study of the properties of the set $\widetilde{\mathcal{R}_\beta}$. We then prove Theorem 1 in Section 4.

1. β-numeration

Let $\beta > 1$ be a real number. *In all that follows, β is assumed to be a Pisot number.* The Renyi β-expansion of a real number $x \in [0,1)$ is defined as the sequence $(x_i)_{i\geq 1}$ with values in $\mathcal{A}_\beta := \{0, 1, \dots, [\beta]\}$ produced by the β-transformation $T_\beta :\ x \mapsto \beta x \pmod 1$ as follows

$$\forall i \geq 1,\ u_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor, \text{ and thus } x = \sum_{i\geq 1} u_i \beta^{-i}.$$

Let $d_\beta(1) = (t_i)_{i\geq 1}$ denote the β-expansion of 1. Numbers β such that $d_\beta(1)$ is ultimately periodic are called *Parry numbers* and those such that $d_\beta(1)$ is finite are called *simple Parry numbers*. Since β is assumed to be Pisot, then β is either a Parry number or a simple Parry number [BM86]. Let $d^*_\beta(1) = d_\beta(1)$, if $d_\beta(1)$ is infinite, and $d^*_\beta(1) = (t_1 \dots t_{n-1}(t_n - 1))^\infty$, if $d_\beta(1) = t_1 \dots t_{n-1} t_n$ is finite ($t_n \neq 0$). The set of β-expansions of real numbers in $[0,1)$ is exactly the set of sequences $(u_i)_{i\geq 1}$ in $\mathcal{A}_\beta^{\mathbb{N}}$ such that

$$\forall k \geq 1,\ (u_i)_{i\geq k} <_{\text{lex}} d^*_\beta(1). \tag{1.1}$$

For more details on the β-numeration, see for instance [Fro02, Fro00].

The (two-sided symbolic) β-shift. Let (X_β, S) denote the two-sided symbolic dynamical system associated with β, where the shift map S maps the sequence $(y_i)_{i\in\mathbb{Z}}$ onto $(y_{i+1})_{i\in\mathbb{Z}}$. The set X_β is defined as the set of two-sided sequences $(y_i)_{i\in\mathbb{Z}}$ in $\mathcal{A}_\beta^{\mathbb{Z}}$ such that each left truncated sequence is less than or equal to $d^*_\beta(1)$, that is, $\forall k \in \mathbb{Z},\ (y_i)_{i\geq k} \leq_{\text{lex}} d^*_\beta(1)$.

We will use the following notation for the elements of X_β: if $y = (y_i)_{i\in\mathbb{Z}} \in X_\beta$, define $u = (u_i)_{i\geq 1} = (y_i)_{i\geq 1}$ and $w = (w_i)_{i\geq 0} = (y_{-i})_{i\geq 0}$. One thus gets a two-sided sequence of the form

$$\dots w_3 w_2 w_1 w_0 u_1 u_2 u_3 \dots$$

and write it as $y = ((w_i)_{i\geq 0}, (u_i)_{i\geq 1}) = (w, u)$. In other words, we will use the letters (w_i) for denoting the "past" and (u_i) for the "future" of the element $y = (w, u)$ of the two-sided shift X_β.

One-sided β-shifts. We denote by X_β^r the set of one-sided sequences $u = (u_i)_{i\geq 1}$ such that there exists $w = (w_i)_{i\geq 0}$ with $(w, u) \in X_\beta$. This set is called the *right one-sided β-shift.* It coincides with the usual one-sided β-shift and is equal to the set of sequences $(u_i)_{i\geq 1}$ which satisfy

$$\forall k \geq 1,\ (u_i)_{i\geq k} \leq_{\text{lex}} d_\beta^*(1). \tag{1.2}$$

One similarly defines X_β^l as the set of one-sided sequences $w = (w_i)_{i\geq 0}$ such that there exists $u = (u_i)_{i\geq 1}$ with $(w, u) \in X_\beta$. We call it the *left one-sided β-shift.*

Sofic shift. Since β is a Parry number (simple or not), then (X_β, S) is sofic [BM86]. We denote by $F(X_\beta)$ the set of finite factors of the sequences in X_β; the minimal automaton $\mathcal{M}_\beta$ recognizing the set of factors of $F(D_\beta)$ can easily be constructed (see Figure 1.1). The number of states d of this automaton is equal to the length of the period n of $d_\beta^*(1)$ if β is a simple Parry number with $d_\beta(1) = t_1 \dots t_{n-1}t_n$, $t_n \neq 0$, and to the sum of its preperiod n plus its period p, if β is a non-simple Parry number with $d_\beta(1) = t_1 \dots t_n(t_{n+1} \dots t_{n+p})^\infty$ ($t_n \neq t_{n+p}$, $t_{n+1} \cdots t_{n+p} \neq 0^p$).

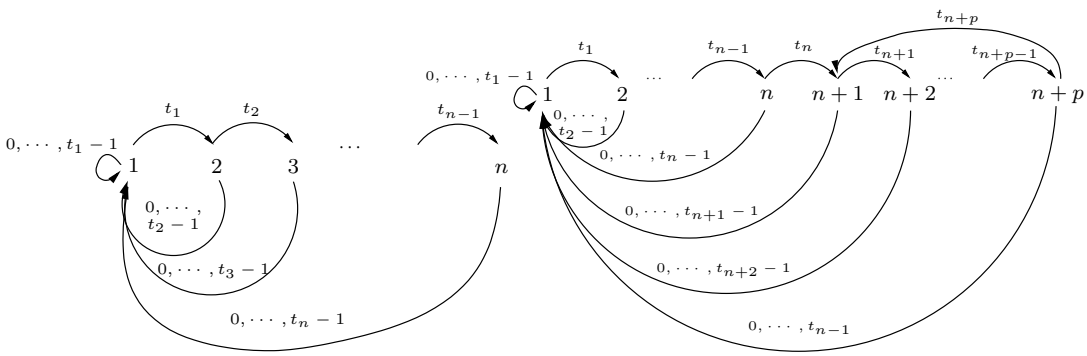

FIGURE 1.1. The automata $\mathcal{M}_\beta$ for β simple Parry number ($d_\beta(1) = t_1 \dots t_{n-1}t_n$) and for β non-simple Parry number ($d_\beta(1) = t_1 \dots t_n(t_{n+1} \dots t_{n+p})^\infty$).

β-substitutions. Let us recall that a *substitution* σ is a morphism of the free monoïd $\mathcal{A}^*$, such that the image of each letter of $\mathcal{A}$ is non-empty. As introduced for instance in [Thu89] and in [Fab95], one can associate in a natural way a substitution σ_β (called β-substitution) with (X_β, S) over the alphabet $\{1, \cdots, d\}$, where d denotes the number of states of the automaton $\mathcal{M}_\beta$: j is the k-th letter occuring in $\sigma_\beta(i)$ (that is, $\sigma_\beta(i) = pjs$, where $p, s \in \{1, \cdots, d\}^*$ and $|p| = k - 1$) if and only if there is an arrow in $\mathcal{M}_\beta$ from the state i to the state j labeled by $k - 1$. One easily checks that this definition is consistent.

An explicit formula for σ_β can be computed by considering the two different cases, β simple and β non-simple Parry number.

- Assume $d_\beta(1) = t_1 \dots t_{n-1}t_n$ is finite, with $t_n \neq 0$. Thus $d_\beta^*(1) = (t_1 \dots t_{n-1}(t_n - 1))^\infty$. One defines σ_β over the alphabet $\{1, 2, \dots, n\}$ as shown in (1.3).
- Assume $d_\beta(1)$ is infinite. Then it cannot be purely periodic (according to Remark 7.2.5 [Fro02]). Hence $d_\beta(1) = d_\beta^*(1) = t_1 \dots t_n(t_{n+1} \dots t_{n+p})^\infty$, with $n \geq 1$, $t_n \neq t_{n+p}$ and $t_{n+1} \dots t_{n+p} \neq 0^p$. One defines σ_β over the alphabet $\{1, 2, \dots, n + p\}$ as shown in (1.3).

$$\sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1}2 \\ 2 & \mapsto 1^{t_2}3 \\ \vdots & \vdots \\ n-1 & \mapsto 1^{t_{n-1}}n \\ n & \mapsto 1^{t_n}. \end{cases} \qquad \sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1}2 \\ 2 & \mapsto 1^{t_2}3 \\ \vdots & \vdots \\ n+p-1 & \mapsto 1^{t_{n+p-1}}(n+p) \\ n+p & \mapsto 1^{t_{n+p}}(n+1). \end{cases} \tag{1.3}$$

Substitution associated with a simple Parry number

Substitution associated with a non-simple Parry number

We will use β-substitutions in Section 3 in order to describe properties of the set $\widetilde{\mathcal{R}_\beta}$. Notice that the automaton $\mathcal{M}_\beta$ is exactly the prefix-suffix automaton of the substitution σ_β considered for instance in [CS01a, CS01b], after reversing all the edges and replacing the labels in the prefix-suffix automaton by the lengths of the prefixes.

The *incidence matrix* of the substitution σ_β is defined as the $d \times d$ matrix whose entry of index (i, j) counts the number of occurrences of the letter i in $\sigma(j)$. As a consequence of the definition, the incidence matrix of σ_β coincides with the transpose of the adjacency matrix of the automaton $\mathcal{M}_\beta$. By eigenvalue of a substitution σ, we mean in all that follows an eigenvalue of the characteristic polynomial of the incidence matrix of σ. A substitution is said to be of *Pisot type* if all its eigenvalues except its largest one which is assumed to be simple, are non-zero and of modulus smaller than 1.

2. Representation of β-shifts

The right one-sided shift X_β^r admits as a natural geometric representation the interval $[0, 1]$; namely, one associates with a sequence $(u_i)_{i\geq 1} \in X_\beta^r$ its real value $\sum_{i\geq 1} u_i\beta^{-i}$. We even have a measure-theoretical isomorphism between X_β^r endowed with the shift, and $[0, 1]$ endowed with the map T_β. We want now to give a similar geometric interpretation of the set X_β; for that purpose, we first give a geometric representation of X_β^l as a generalized Rauzy fractal.

2.1. Representation of the left one-sided shift X_β^l.

The aim of this section is to introduce first a formal power series, called formal representation of X_β^l and second a geometrical representation as an explicit compact set of in the product of Euclidean and p-adic spaces following [Sie03]; the primes which appear as p-adic spaces here will be the prime factors of the norm of β.

Formal representation of the symbolic dynamical system X_β^l.

Definition 1. The *formal representation* of X_β^l is denoted

$$\varphi_X : X_\beta \to \mathbb{Q}[[X]]$$

where $\mathbb{Q}[[X]]$ is the ring of formal power series with coefficients in $\mathbb{Q}$, and defined by:

$$\text{for all } (w_i)_{i\geq 0} \in X_\beta^l, \quad \varphi_X(w_i) = \sum_{i\geq 0} w_i X^i \in \mathbb{Q}[[X]].$$

Topologies over $\mathbb{Q}(\beta)$. We now want to specialize these formal power series by giving to the indeterminate X the value β, and associating with them values by making them converge. We thus want to find a topological framework in which all the series $\sum_{i\geq 0} w_i\beta^i$ for $(w_i)_{i\geq 0} \in X_\beta^l$, would converge; in fact, this boils down to find all the Archemedean and non-Archemedean metrizable topologies on $\mathbb{Q}(\beta)$ for which these series converge in a suitable completion; they are of two types.

- Suppose that the topology (with absolute value $|\cdot|$) is Archemedean: its restriction to $\mathbb{Q}$ corresponds to the usual absolute value on $\mathbb{Q}$ and there exists a $\mathbb{Q}$-isomorphism τ_i such that $|x| = |\tau_i(x)|_{\mathbb{C}}$, for $x \in \mathbb{Q}(\beta)$. The series φ_X specialized in β converge in $\mathbb{C}$ if and only if τ_i is associated with a conjugate β_i of modulus strictly smaller than one.
- Assume that the topology is non-Archimedean: there exists a prime ideal $\mathcal{I}$ of the integer ring $\mathcal{O}_{\mathbb{Q}(\beta)}$ of $\mathbb{Q}(\beta)$ for which the topology coincides with the the $\mathcal{I}$-adic topology; let p be the prime number defined by $\mathcal{I} \cap \mathbb{Z} = p\mathbb{Z}$; the restriction of the topology to $\mathbb{Q}$ is the p-adic topology. The series φ_X specialized in β take finite values in the completion $\mathbb{K}_{\mathcal{I}}$ of $\mathbb{Q}(\beta)$ for the $\mathcal{I}$-adic topology if and only if $\beta \in \mathcal{I}$, i.e., $|\beta|_{\mathcal{I}} < 1$.

Representation space $\mathbb{K}_\beta$ of X^l_β. We assume now that β be a Pisot number of degree d, say. Let β_2, $\ldots$, β_r be the real conjugates of β (they all have modulus strictly smaller than 1, since β is Pisot), and let β_{r+1}, $\overline{\beta_{r+1}}$, $\ldots$, β_{r+s}, $\overline{\beta_{r+s}}$ be its complex conjugates. For $2 \leq j \leq r$, let $\mathbb{K}_{\beta_j}$ be equal to $\mathbb{R}$, and for $r+1 \leq j \leq r+s$, let $\mathbb{K}_{\beta_j}$ be equal to $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{C}$ being endowed with the usual topology. For $i = 1$ to d, let τ_i be a $\mathbb{Q}$-automorphism of $\mathbb{K} = \mathbb{Q}(\beta_1, \ldots, \beta_d)$ which sends β on its algebraic conjugate β_i. For a given i and for every element $Q(\beta)$ of $\mathbb{Q}(\beta)$, then $\tau_i(Q(\beta)) = Q(\beta_i)$.

We first gather the complex representations by omitting the ones which are conjugate in the complex case. This representation contains all the possible Archemedean values for φ_X. It takes values in

$$\mathbb{K}_\infty = \mathbb{K}_{\beta_2} \times \cdots \times \mathbb{K}_{\beta_{r+s}}.$$

Let $\mathcal{I}_1, \ldots, \mathcal{I}_\nu$ be the prime ideals in the integer ring $\mathcal{O}_{\mathbb{Q}(\beta)}$ of $\mathbb{Q}(\beta)$ that contain β, that is,

$$\beta\mathcal{O}_{\mathbb{Q}(\beta)} = \prod_{i=1}^{\nu} \mathcal{I}_i^{n_i}. \tag{2.1}$$

Recall that $\mathbb{K}_{\mathcal{I}}$ denotes the completion of $\mathbb{Q}(\beta)$ for the $\mathcal{I}$-adic topology. We then gather the representations in the completions of $\mathbb{Q}(\beta)$ for the non-Archemedean topologies. Hence one defines the *representation space of X_β* as the direct product $\mathbb{K}_\beta$ of all these fields:

$$\mathbb{K}_\beta = \mathbb{K}_{\beta_2} \times \ldots \mathbb{K}_{\beta_{r+s}} \times \mathbb{K}_{\mathcal{I}_1} \times \ldots \mathbb{K}_{\mathcal{I}_\nu} \simeq \mathbb{R}^{r-1} \times \mathbb{C}^s \times \mathbb{K}_{\mathcal{I}_1} \times \cdots \times \mathbb{K}_{\mathcal{I}_\nu}.$$

The field $\mathbb{K}_{\mathcal{I}}$ is a finite extension of the $p_{\mathcal{I}}$-adic field $\mathbb{Q}_{p_{\mathcal{I}}}$ where $\mathcal{I} \cap \mathbb{Z} = p_{\mathcal{I}}\mathbb{Z}$. For a given prime p, the fields $\mathbb{K}_{\mathcal{I}}$ that are p-adic fields are the ones for which $\mathcal{I}$ contains simultaneously p and β. Furthermore, the prime numbers p for which there exists a prime ideal of $\mathcal{O}_{\mathbb{Q}(\beta)}$ which contains simultaneously p and β are exactly the prime divisors of the constant term of the minimal polynomial of β (see Lemma 4.2 [Sie03]). In particular, $\mathbb{K}_\beta$ is a Euclidean space if and only if β is a unit. Endowed with the product of the topologies of each of its elements, $\mathbb{K}_\beta$ is a metric abelian group.

The *canonical embedding* of $\mathbb{Q}(\beta)$ into $\mathbb{K}_\beta$ is defined by the following morphism:

$$\delta_\beta : P(\beta) \in \mathbb{Q}(\beta) \mapsto (\underbrace{P(\beta_2)}_{\in \mathbb{K}_{\beta_2}}, \ldots, \underbrace{P(\beta_{r+s})}_{\in \mathbb{K}_{\beta_{r+s}}}, \underbrace{P(\beta)}_{\in \mathbb{K}_{\mathcal{I}_1}}, \ldots, \underbrace{P(\beta)}_{\in \mathbb{K}_{\mathcal{I}_\nu}}) \in \mathbb{K}_\beta. \tag{2.2}$$

Since the topology on $\mathbb{K}_\beta$ has been chosen so that the formal power series $\lim_{n\to\infty} \delta_\beta(\sum_{i=0}^n w_i\beta^i)$ $= \sum_{i\geq 0} w_i \delta_\beta(\beta)^i$ are convergent in $\mathbb{K}_\beta$ for every $(w_i)_{i\geq 0} \in X^l_\beta$, one defines the following, where the notation $\delta_\beta(\sum_{i\geq 0} w_i\beta^i)$ stands for $\sum_{i\geq 0} w_i \delta_\beta(\beta)^i$.

Definition 2. The *representation map of X^l_β*, called *one-sided representation map*, is defined by

$$\varphi_\beta : \ X^l_\beta \to \mathbb{K}_\beta, \ (w_i)_{i\geq 0} \mapsto \delta_\beta(\sum_{i\geq 0} w_i\beta^i).$$

We set $\mathcal{R}_\beta := \varphi_\beta(X^l_\beta)$ and call it the *generalized Rauzy fractal* or *geometric representation* of the left one-sided shift X^l_β.

2.2. **Examples.**

The golden ratio. Let $\beta = (1+\sqrt{5})/2$ be the golden ratio, that is, the largest root of $X^2 - X - 1$. One has $d_\beta(1) = 11$ (β is a simple Parry number) and $d^*_\beta(1) = (10)^\infty$. Hence X_β is the set of sequences in $\{0,1\}^{\mathbb{Z}}$ in which there are no two consecutive 1's. Furthermore, the associated β-substitution is the Fibonacci substitution: $\sigma_\beta :\ 1 \mapsto 12,\ 2 \mapsto 1$. One has $\mathbb{K}_\beta = \mathbb{R}$; the canonical embedding δ_β is reduced to the map $\tau_{(1-\sqrt{5})/2}$ (that is, the $\mathbb{Q}$-automorphism of $\mathbb{Q}(\beta)$ which maps β on its conjugate), and $\delta_\beta(\mathbb{Q}(\beta)) = \mathbb{Q}(\beta)$. The set $\mathcal{R}_\beta$ is an interval.

The Tribonacci number. Let β be the *Tribonacci number*, that is, the Pisot root of the polynomial $X^3 - X^2 - X - 1$. One has $d_\beta(1) = 111$ (β is a simple Parry number) and $d^*_\beta(1) = (110)^\infty$. Hence X_β is the set of sequences in $\{0,1\}^{\mathbb{Z}}$ in which there are no three consecutive 1's. Furthermore, $\sigma_\beta :\ 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$. One has $\mathbb{K}_\beta = \mathbb{C}$; the canonical embedding is reduced to the $\mathbb{Q}$-isomorphism τ_α which maps β on α, where α is one of the complex roots of $X^3 - X^2 - X - 1$. The set $\mathcal{R}_\beta$ which satisfies

$$\mathcal{R}_\beta = \{\sum_{i\geq 0} w_i\alpha^i;\ \forall i,\ w_i \in \{0,1\},\ w_i w_{i+1} w_{i+2} \neq 0\}$$

is a compact subset of $\mathbb{C}$ called the *Rauzy fractal*. This set was introduced in [Rau82], see also [IK91, Mes98, Mes00]. It is shown in Fig. 2.2 with a division into three pieces indicated by different shades. They correspond to the sequences $(w_i)_{i\geq 0}$ such that either $w_0 = 0$, or $w_0w_1 = 10$, or $w_0w_1 = 11$. There are as many pieces as the length of $d_\beta(1)$, which is also equal here to the degree of β. We will come back to the interest of this division of the Rauzy fractals into smaller pieces in Section 3.

The smallest Pisot number. Let β be the Pisot root of $X^3 - X - 1$. One has $d_\beta(1) = 10001$ (β is a simple Parry number) and $d^*_\beta(1) = (10000)^\infty$; $\sigma_\beta :\ 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 5,\ 5 \mapsto 1$; the characteristic polynomial of its incidence matrix is $(X^3 - X - 1)(X^2 - X + 1)$, hence σ_β is not a substitution of Pisot type. A self-similar tiling generated by it has been studied in details in [AS98]; some connected surprising tilings have also been introduced in [EI02]. One has $\mathbb{K}_\beta = \mathbb{C}$; the canonical embedding is also reduced to the $\mathbb{Q}$-isomorphism τ_α which maps β on α, where α is one of the complex roots of $X^3 - X - 1$. The set $\mathcal{R}_\beta$ is shown in Fig. 2.2 with a division into five pieces corresponding respectively to the sequences $(w_i)_{i\geq 0}$ such that either $w_0w_1w_2w_3 = 0000$, or $w_0 = 1$, $w_0w_1 = 01$, $w_0w_1w_2 = 001$, or $w_0w_1w_2w_3 = 0001$. The number of different pieces is equal to the length of $d_\beta(1)$; there are here 5 pieces whereas the degree of β is 3.

A non-unit example. Let $\beta = 2 + \sqrt{2}$ be the dominant root of the polynomial $X^2 - 4X + 2$. The other root is $2-\sqrt{2}$. One has $d_\beta(1) = d^*_\beta(1) = 31^\infty$; β is not a simple Parry number; $\sigma_\beta :\ 1 \mapsto 1112$, $2 \mapsto 12$. The ideal $2\mathbb{Z}$ is ramified in $\mathcal{O}_{\mathbb{Q}(\sqrt{2})}$, that is, $2\mathbb{Z} = \mathcal{I}^2$. Hence there exists only one ideal which contains $\sqrt{2}$; its index of ramification is 2; the degree of the extension $\mathbb{K}_{\mathcal{I}}$ over $\mathbb{Q}_2$ has degree 2. Hence the geometric one-sided representation $\mathcal{R}_\beta$ is a subset of $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_2$, shown in Fig. 2.2. The division of the Rauzy fractal cannot be expressed as in the previous examples as finite conditions on the prefixes of the sequences $(w_i)_{i\geq 0}$ (for more details, see Section 3.2).

2.3. **Representation of the two-sided shift** (X_β, S)**.**

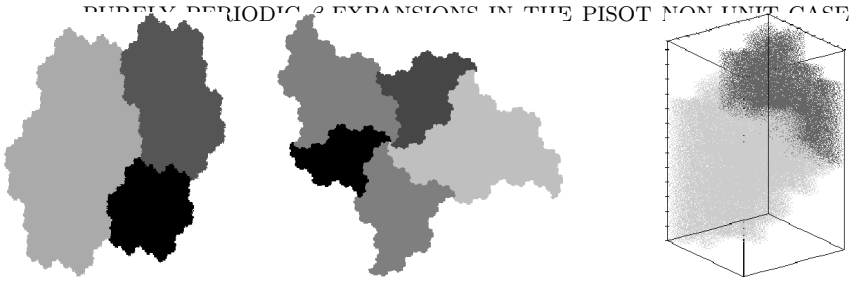

FIGURE 2.1. The Rauzy geometric one-sided representation for the Tribonacci-shift, the smallest Pisot number-shift and the $(2+\sqrt{2})$-shift.

Representation space $\widetilde{\mathbb{K}_\beta}$ of the two-sided β-shift X_β and representation map $\widetilde{\varphi_\beta}$. We define now the *representation map* $\widetilde{\varphi_\beta}$ of X_β: it takes its values in $\mathbb{K}_\beta \times \mathbb{R}$ and maps a point $(w,u) = ((w_i)_{i\geq 0},(u_i)_{i\geq 1})$ in X_β to the point obtained by introducing on the last coordinate (in $\mathbb{R}$) the real number whose β-expansion is given by $(u_i)_{i\geq 1}$, and by gathering on the first coordinate (in $\mathbb{K}_\beta$) the set of finite values which can be taken by the formal power series $\varphi_X(w)$ specialized in β for all the topologies that exist on $\mathbb{Q}(\beta)$. Let us define $\widetilde{\mathbb{K}_\beta}$ as $\mathbb{K}_\beta \times \mathbb{R}$.

Definition 3. The *representation map $\widetilde{\varphi_\beta} : X_\beta \to \widetilde{\mathbb{K}_\beta}$ of X_β*, called *two-sided representation map*, is defined for all $(w,u) = ((w_i)_{i\geq 0},(u_i)_{i\geq 1}) \in X_\beta$ by:

$$\widetilde{\varphi_\beta}(w,u) = (-\varphi_\beta(w_i), \sum_{i\geq 1} u_i\beta^{-i}) = (-\delta_\beta(\sum_{i\geq 0} w_i\beta^i), \sum_{i\geq 1} u_i\beta^{-i}) \in \mathbb{K}_\beta \times \mathbb{R}.$$

The set $\widetilde{\mathcal{R}_\beta} := \widetilde{\varphi_\beta}(X_\beta)$ is called the *Rauzy fractal* or *geometric representation* of the two-sided β-shift. This set is easily seen to be bounded and hence compact.

We will see below (see (2.4) and (4.1)) the interest of introducing the sign minus before δ_β in the definition of the map $\widetilde{\varphi_\beta}$.

Extension $\widetilde{T_\beta}$ of the β-transformation. One can extend in a natural way the definition of T_β to the product of the representation space $\mathbb{K}_\beta$ by $\mathbb{R}$ as follows, in order to obtain a realization of its natural extension.

Definition 4. Let $h_\beta :\ \mathbb{K}_\beta \to \mathbb{K}_\beta$ stands for the multiplication map in $\mathbb{K}_\beta$ by the diagonal matrix whose diagonal coefficients are given by $\delta_\beta(\beta)$. One thus defines

$$\widetilde{T_\beta} :\ \mathbb{K}_\beta \times \mathbb{R} \to \mathbb{K}_\beta \times \mathbb{R},\ (a,b) \mapsto (h_\beta(a) - [\beta b]\delta_\beta(1), \beta b - [\beta b]).$$

In particular, the following commutation relations hold, where Id denotes the identity map over $\mathbb{R}$; recall that S denotes the shift over X_β:

$$\widetilde{T_\beta} \circ (\delta_\beta, Id) = (\delta_\beta, Id) \circ T_\beta \text{ over } \mathbb{Q}(\beta). \tag{2.3}$$

$$\forall\, (w,u) \in X_\beta, \quad \text{if } u \neq d^*_\beta(1), \quad \widetilde{\varphi_\beta} \circ S(w,u) = \widetilde{T_\beta} \circ \widetilde{\varphi_\beta}(w,u). \tag{2.4}$$

The proof of (2.3) is immediate. Let us prove (2.4). Take an element $(w,u) = ((w_i)_{i\geq 0},(u_i)_{i\geq 1}) \in X_\beta$ with $u \neq d^*_\beta(1)$. One has $\widetilde{\varphi_\beta}(w,u) = (-\delta_\beta(\sum_{i\geq 0} w_i\beta^i), \sum_{i\geq 1} u_i\beta^{-i})$. Since u satisfies (1.1), then the integer part of $\beta(\sum_{i\geq 1} u_i\beta^{-i})$ is equal to u_1, hence $T_\beta(\sum_{i\geq 1} u_i\beta^{-i}) = \sum_{i\geq 2} u_i\beta^{-i}$, and the assertion follows.

Examples. The geometric representation map φ_β of the golden ratio-shift maps to $\mathbb{R}^2$. The ones for the Tribonacci number and the smallest Pisot number map to $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. They are shown (up to a change of coordinates) in Fig. 2.2. The sets $\widetilde{\mathcal{R}_\beta}$ are unions of products of the different pieces of the Rauzy fractal $\mathcal{R}_\beta$ by finite real intervals of different heights. For instance, in the Tribonacci case, since the different pieces in $\mathcal{R}_\beta$ correspond to the sequences $(w_i)_{i\geq 0}$ such that either $w_0 = 0$, or $w_0w_1 = 10$, or $w_0w_1 = 11$, this gives different constraints on the sequences $(u_i)_{i\geq 1}$ which produce the real component; for instance, the piece which corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 11$ implies the following constraint on the sequences $(u_i)_{i\geq 1}$: u_0 has to be equal to 0.

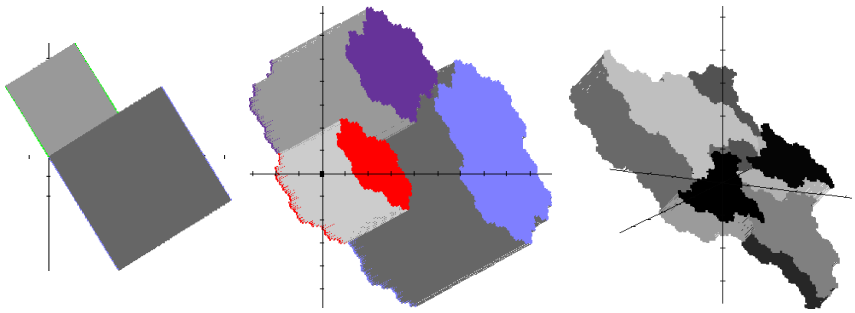

FIGURE 2.2. The geometric representation of the two-sided β-shift for the Fibonacci-shift, the Tribonacci-shift, and the smallest Pisot number-shift.

Remark 1. The definition of the representation map $\widetilde{\varphi_\beta}$ is inspired by [Sie03] in the substitutive case. We cannot apply here directly the substitutive formalism to the substitution σ_β since it is generally not Pisot : the dominant eigenvalue of σ_β is still a Pisot number but other eigenvalues may occur, as in the smallest Pisot case. The main difference here is that we do not take those extra eigenvalues into acount in our definition of $\widetilde{\varphi_\beta}$.

3. Geometric properties of the Rauzy fractal of the β-numeration

The aim of this section is to prove some geometric properties of the Rauzy fractal associated with the β-numeration:

Theorem 2. *Let β be a Pisot number.*

(1) *The Rauzy fractal $\mathcal{R}_\beta$ of the one-sided β-shift has a graph directed self-affine structure in the sense of* [MW88]. *More precisely:*
   - *it has a non-zero measure for the Haar measure;*
   - *there are d pieces which form the self-affine structure, where d is equal to the period of $d_\beta(1)^*$ if β is a simple Parry number or to the sum of its preperiod plus its period, otherwise.*

(2) *The Rauzy fractal $\widetilde{\mathcal{R}_\beta}$ of the two-sided β-shift has non-zero measure for the Haar measure $\mu_{\mathbb{K}_\beta}$ over $\mathbb{K}_\beta \times \mathbb{R}$. It is the disjoint union of d cylinders obtained as the product of each piece of the one-sided Rauzy fractal by a finite interval of $\mathbb{R}$.*

3.1. **Proof of Theorem 2.** The proof of the first point of this theorem is divided into several steps; roughly, we use the self-affine structure of $\mathcal{R}_\beta$ to deduce that it has non-zero measure. Similar statements are proved in the framework of Pisot substitutions in [Sie03], but we cannot use here directly these statements since the β-substitutions (Section 1) used here are not necessarily Pisot according to Remark 1, as for instance in the smallest Pisot case.

Division of $\mathcal{R}_\beta$ into subpieces. Let us first divide (as illustrated in Figure 2.2) the set $\mathcal{R}_\beta$ into d pieces where d denotes the number of states in the minimal automaton $\mathcal{M}_\beta$ which recognizes the set of factors $F(X_\beta)$ of the two-sided shift X_β.

Let $\widetilde{\mathcal{M}_\beta}$ denote the (non-deterministic) automaton obatined from $\mathcal{M}_\beta$ by reversing the orientation of the edges. The set of sequences in the left one-sided shift X_β^l is equal to the set of labels of infinite one-sided paths in the automaton $\widetilde{\mathcal{M}_\beta}$. We label the states of the automaton $\widetilde{\mathcal{M}_\beta}$ by $a_1, \cdots, a_d$. For $i = 1, \cdots, d$, one defines

$$\mathcal{R}_\beta(i) := \left\{ \begin{array}{c} \varphi_\beta((w_k)_{k\geq 0});\ (w_k)_{k\geq 0} \in \mathcal{A}_\beta^{\mathbb{N}^*}; (w_k)_{k\geq 0} \\ \text{is a path from the state } a_i \text{ in } \widetilde{\mathcal{M}_\beta} \end{array} \right\}.$$

Let us recall that d is larger than or equal to the degree r of the minimal polynomial of β. For instance, $r = 3$ and $d = 5$ in the smallest Pisot case. In particular, d can be arbitrarily large when β is cubic, according to [Bas02].

Self-affine decomposition. We recall that the map $h_\beta : \mathbb{K}_\beta \to \mathbb{K}_\beta$ stands for the multiplication map in $\mathbb{K}_\beta$ by the diagonal matrix whose diagonal coefficients are given by $\delta_\beta(\beta)$ (Definition 4). Hence $\delta_\beta(\beta x) = h_\beta \delta_\beta(x)$ for every $x \in \mathbb{Q}(\beta)$.

Let us first prove that for $i = 1, \cdots, d$:

$$\mathcal{R}_\beta(i) = \cup_{1\leq j\leq d} \cup_{p,\ \sigma_\beta(j)=pis} h_\beta(\mathcal{R}_\beta(j)) + \delta_\beta(|p|). \tag{3.1}$$

Let $i \in \{1, \cdots, d\}$ be given. Let $(w_k)_{k\geq 0} \in \mathcal{R}_\beta(i)$. One has

$$\begin{aligned} \varphi_\beta((w_k)_{k\geq 0}) &= \delta_\beta(\textstyle\sum_{k\geq 1} w_k \beta^k) + \delta_\beta(w_0) = h_\beta \delta_\beta(\textstyle\sum_{k\geq 1} w_k \beta^{k-1}) + \delta_\beta(w_0) \\ &= h_\beta \varphi_\beta((w_k)_{k\geq 1}) + \delta_\beta(w_0). \end{aligned}$$

Let a_j denote the state in $\widetilde{\mathcal{M}_\beta}$ obtained by reading the label w_0 from the state a_i. By the definition of σ_β, there exist $p = 1^{w_0}$ and s such that $\sigma_\beta(j) = pis$, that is, $w_0 = |p|$. Hence

$$\varphi_\beta((w_k)_{k\geq 0}) \in h_\beta(\mathcal{R}_\beta(j)) + \delta_\beta(|p|),$$

which provides one inclusion for the equality (3.1). The other inclusion is then immediate.

The Rauzy fractal has non-zero measure. The proof is an adaption of the corresponding proof of [Sie03] which is done in the framework of Pisot substitution dynamical systems; two properties are recalled below without a proof, the first one describing the action of the multiplication map h_β on the Haar measure $\mu_{\mathbb{K}_\beta}$ of $\mathbb{K}_\beta$:

Lemma 1 ([Sie03]). (1) *For every Borelian set B of $\mathbb{K}_\beta$*

$$\mu_{\mathbb{K}_\beta}(h_\beta(B)) = \frac{1}{\beta}\mu_{\mathbb{K}_\beta}(B).$$

(2) *Let $\mathcal{S}$ be a finite set included in $\mathbb{Q}(\beta)$. The set of points $\{\delta(P(1/\beta));\ P \in \mathcal{S}[X]\}$ is a discrete set in $\mathbb{K}_\beta$.*

Let $(U_N)_{N\in\mathbb{N}}$ denote the linear canonical numeration system associated with β according to [BM89], that is, $U_0 = 1$, and for all k, $U_k = t_1 U_{k-1} + \cdots + t_k U_0 + 1$, where $d_\beta^*(1) = (t_k)_{k\in\mathbb{N}}$. Let us expand every integer $i = 0, 1, \cdots, U_N - 1$ in this system according to the greedy algorithm (this expansion being unique):

$$i = \sum_{0\leq k\leq N-1} w_k^{(i)} U_k.$$

According to [BM89], see also [Fro02], the finite words $w_{N-1}^{(i)} \cdots w_0^{(i)}$, for $i = 0, 1, \cdots, U_N - 1$ are all distinct and all belong to the set $F(X_\beta)$ of factors of elements of X_β. Hence the sequences $((w_0^{(i)} \cdots w_{N-1}^{(i)})0^\infty)$, for $i = 0, 1, \cdots, U_N - 1$, all belong to the left one-sided β-shift X_β^l.

Let $\mathcal{E}_N$ denote the image under the action of φ_β of this set of points. One has for $i = 0, 1, \cdots, U_N - 1$,

$$\varphi_\beta((w_0^{(i)} \cdots w_{N-1}^{(i)})0^\infty) = \delta_\beta(\sum_{0 \le k \le N-1} w_k^{(i)} \beta^i).$$

The points in $\mathcal{E}_N$ are all distinct since for $i = 0, 1, \cdots, U_N - 1$, the finite words $w_{N-1}^{(i)} \cdots w_0^{(i)}$ are all distinct. There thus exists $B > 0$ such that Card $\mathcal{E}_N = U_N > B\beta^N$, since β is a Pisot number.

Let us apply now Lemma 1 with $\mathcal{S} = \{0, 1, \cdots, [\beta]\}$. Hence there exists a constant $A > 0$ such that the distance between two elements of $\mathcal{E}_N$ is larger than A. Let us define now for every non-negative integer N, the set

$$\mathcal{B}_N = \cup_{z \in \mathcal{E}_N} (h_\beta)^N B(z, A/2),$$

where $B(z, A/2)$ denotes the closed ball in $\mathbb{K}_\beta$ of center z and radius $A/2$. According to Lemma 1 and to the fact that Card $\mathcal{E}_N > B\beta^N$, for all N, there exists $C > 0$ such that $\mu_{\mathbb{K}_\beta}(\mathcal{B}_N) > C$, for all N.

The main point now is that the sequence of compact sets $(\mathcal{B}_N)_{N \in \mathbb{N}}$ converges toward a subset of $\mathcal{R}_\beta$ with respect to the Hausdorff metric; indeed for a fixed positive integer p, for N large enough, $\mathcal{B}_N \subset \mathcal{R}_\beta(1/p) := \{x \in \mathbb{K}_\beta;\ d(x, \mathcal{R}_\beta) \le 1/p\}$. Independently, fix $\varepsilon > 0$; since the sequence $(\mathcal{R}_\beta(1/p))_p$ converges toward $\mathcal{R}_\beta$, there exists $p > 0$ such that $\mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta(1/p)) \le \mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta) + \varepsilon$. This finally implies $\liminf \mu_{\mathbb{K}_\beta}(\mathcal{B}_N) \le \mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta) + \varepsilon$. Since this holds for every $\varepsilon > 0$, one obtains $\mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta) \ge C > 0$.

Computation of the measures and self-affine structure. Let us prove now that the union in (3.1) is a disjoint union up to sets of zero measure. One has for a given $i \in \{1, \cdots, d\}$ according to (3.1) and to Lemma 1

$$\begin{aligned} \mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta(i)) &\le \textstyle\sum_{j=1,\cdots,d,\ \sigma_\beta(j)=pis} \mu_{\mathbb{K}_\beta}(h_\beta(\mathcal{R}_\beta(j))) \\ &\le 1/\beta \textstyle\sum_{j=1,\cdots,d,\ \sigma_\beta(j)=pis} \mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta(j)). \end{aligned} \tag{3.2}$$

Let $\mathbf{m} = (\mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta(i)))_{i=1,\cdots,d}$ denote the vector in $\mathbb{R}^d$ of measures in $\mathbb{K}_\beta$ of the pieces of the Rauzy fractal. We know from what precedes that $\mathbf{m}$ is a non-zero vector with non-negative entries. According to Perron-Frobenius theorem, the previous equality implies that $\mathbf{m}$ is an eigenvector of the primitive incidence matrix of the substitution σ_β. We thus have equality in (3.2) which implies that the unions are disjoint up to sets of zero measure. One similarly proves that this equality in measure still holds by replacing σ by σ^n, for every n.

Now take two distinct pieces, $(\mathcal{R}_\beta(j))$ and $(\mathcal{R}_\beta(k))$ say, with $j \ne k$. There exists n such that both $\sigma^n(j)$ and $\sigma^n(k)$ admit as first letter 1. Hence they both occur in (3.1) for $i = 1$ with the same translation term (which is indeed equal to 0) and they are thus distinct. We hence have proved that the d pieces of the Rauzy fractal $(\mathcal{R}_\beta(j))$ are disjoint up to sets of zero measure, which ends the proof of Assertion (i), that is, $\mathcal{R}_\beta$ has a self-affine structure.

$\widetilde{\mathcal{R}_\beta}$ has non-zero measure. It remains now to prove the second point of the theorem. The assertion that $\widetilde{\mathcal{R}_\beta}$ has non-zero measure is a direct consequence of the structure of the Rauzy fractal $\widetilde{\mathcal{R}_\beta}$ studied above since $\widetilde{\mathcal{R}_\beta}$ can be decomposed as the disjoint (in measure) union of $\widetilde{\mathcal{R}_\beta}(i)$, $i = 1, \cdots, d$, where

$$\widetilde{\mathcal{R}_\beta}(i) := \left\{ \begin{array}{l} \widetilde{\varphi_\beta}(u, w);\ (u, w) \text{ is a two-sided path in } \widetilde{\mathcal{M}_\beta} \\ \text{such that } w \text{ starts from the state } a_i \end{array} \right\},$$

which the product of $\mathcal{R}_\beta(i)$ by a finite interval of $\mathbb{R}$ of non-zero measure.

3.2. **Examples.** Let us pursue the study of three of the examples of Section 2.2.

The Tribonacci number. Let us recall that

$$\mathcal{R}_\beta = \{\sum_{i\geq 0} w_i\alpha^i;\ \forall i,\ w_i \in \{0,1\},\ w_iw_{i+1}w_{i+2} \neq 0\}.$$

One easily checks thanks to the automaton $\widetilde{\mathcal{M}_\beta}$ shown in Fig. 3.1 that $\mathcal{R}_\beta(1)$ corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0 = 1$, $\mathcal{R}_\beta(2)$ corresponds to the set of sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 10$, and lastly, $\mathcal{R}_\beta(3)$ corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 11$. One has

$$\begin{cases} \mathcal{R}_\beta(1) = \alpha(\mathcal{R}_\beta(1) \cup \mathcal{R}_\beta(2) \cup \mathcal{R}_\beta(3)) \\ \mathcal{R}_\beta(2) = \alpha(\mathcal{R}_\beta(1)) + 1 \\ \mathcal{R}_\beta(3) = \alpha(\mathcal{R}_\beta(2)) + 1. \end{cases}$$

The smallest Pisot number. One has

$$\begin{aligned} \mathcal{R}_\beta = \ & \{\textstyle\sum_{i\geq 0} w_i\alpha^i;\ \forall i,\ w_i \in \{0,1\},\ \text{ if } w_i = 1, \\ & \text{then } w_{i+1} = w_{i+2} = w_{i+3} = w_{i+4} = 0\}. \end{aligned}$$

One easily checks that $\mathcal{R}_\beta(1)$ corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0 = 0000$, $\mathcal{R}_\beta(2)$ corresponds to the set of sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 1$, $\mathcal{R}_\beta(3)$ corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 01$, $\mathcal{R}_\beta(4)$ corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 001$, and lastly $\mathcal{R}_\beta(5)$ corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 0001$.

One has

$$\begin{cases} \mathcal{R}_\beta(1) = \alpha(\mathcal{R}_\beta(1) \cup \mathcal{R}_\beta(5)) \\ \mathcal{R}_\beta(2) = \alpha(\mathcal{R}_\beta(1)) + 1 \\ \mathcal{R}_\beta(3) = \alpha(\mathcal{R}_\beta(2)) \\ \mathcal{R}_\beta(4) = \alpha(\mathcal{R}_\beta(3)) \\ \mathcal{R}_\beta(5) = \alpha(\mathcal{R}_\beta(4)). \end{cases}$$

The $(2+\sqrt{2})$*-shift.* One has

$$\mathcal{R}_\beta = \{\sum_{i\geq 0} w_i(2-\sqrt{2})^i;\ \forall i,\ w_i \in \{0,1,2\},\ (w_i)_{i\geq 0} \leq_{\text{lex}} (31^\infty)\}.$$

In this non-simple Parry case, we cannot express as easily as previously the sets $\mathcal{R}_\beta(1)$ and $\mathcal{R}_\beta(2)$: one checks in Figure 3.1 that there exist cycles from a_1 to a_1 and from a_2 to a_2, which implies that both $\mathcal{R}_\beta(1)$ and $\mathcal{R}_\beta(2)$ contain sequences with arbitrarily long common prefixes, such as 1^n for every n.

One has

$$\begin{cases} \mathcal{R}_\beta(1) = & (2-\sqrt{2})(\mathcal{R}_\beta(1)) \cup ((2-\sqrt{2})(\mathcal{R}_\beta(1)) + 1) \cup ((2-\sqrt{2})(\mathcal{R}_\beta(1)) + 2) \\ & \cup (2-\sqrt{2})(\mathcal{R}_\beta(2)) \\ \mathcal{R}_\beta(2) = & (2-\sqrt{2})(\mathcal{R}_\beta(1)) + 3 + (2-\sqrt{2})(\mathcal{R}_\beta(2)) + 1. \end{cases}$$

4. Characterization of purely periodic points

We can now state the main theorem of this paper.

Theorem 3. *Let β be a Pisot number. For all $x \in \mathbb{Q}(\beta) \cap [0,1)$, the β-expansion of x is purely periodic if and only if $(\delta_\beta(x), x) \in \widetilde{\mathcal{R}_\beta} = \widetilde{\varphi_\beta}(X_\beta)$.*

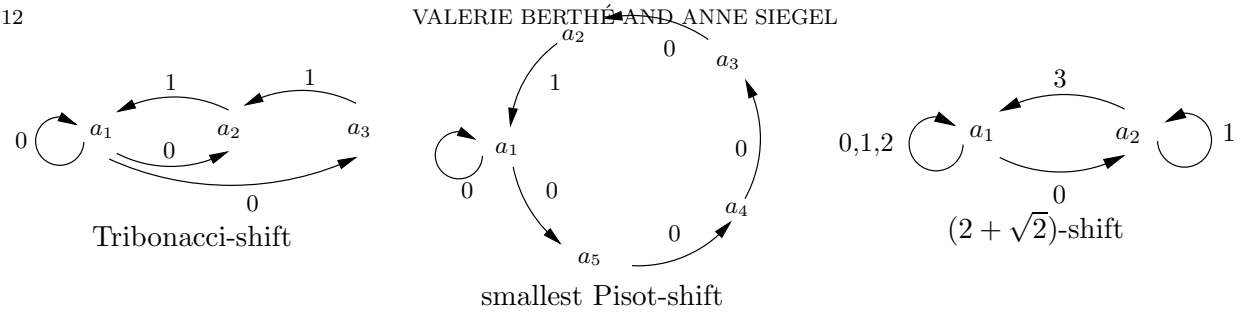

FIGURE 3.1. Reversed minimal automaton $\widetilde{\mathcal{M}_\beta}$ describing the structure of the β-shift.

Proof

Let us assume that the β-expansion of x is purely periodic. Write x as $x = 0.(a_1 \dots a_L)^\infty$, and set $w =^\infty(a_1 \dots a_L)$ and $u = (a_1 \dots a_L)^\infty$ (that is, $w = (w_i)_{i\geq 0}$ with $w_0 \cdots w_{L-1} = a_L \dots a_1$, and $w_{i+L} = w_i$ for all i, and $u = (u_i)_{i\geq 1}$, with $u_1 \cdots u_L = a_1 \dots a_L$, and $u_{i+L} = u_i$ for all i). One has $(w, u) \in X_\beta$ acording to (1.1). Let us compute $\widetilde{\varphi_\beta}(w, u)$. Note that the second coordinate of $\widetilde{\varphi_\beta}(w, u)$ is x by definition:

$$\sum_{i\geq 1} u_i \beta^{-i} = \frac{a_1\beta^{-1} + \cdots + a_L\beta^{-L}}{1-\beta^{-L}} = \frac{a_1\beta^{L-1} + \cdots + a_L}{\beta^L - 1} = x.$$

Futhermore, $\lim_{n\to\infty} \delta_\beta(\beta^n) = 0$ in $\mathbb{K}_\beta$. We thus have

$$\begin{aligned}
\widetilde{\varphi_\beta}(w, u) &= \left(-\delta_\beta(\sum_{i\geq 0} w_i\beta^i), \sum_{i\geq 1} u_i\beta^{-i}\right) \\
&= \left(-\lim_{n\to\infty} \delta_\beta((a_L + \cdots + a_1\beta^{L-1})(1 + \beta^L + \cdots + \beta^{nL})), x\right) \\
&= \left(\lim_{n\to\infty} \delta_\beta\left(-(a_1\beta^{L-1} + \cdots + a_L)\frac{1-\beta^{nL}}{1-\beta^L}\right), x\right) \\
&= \left(\delta_\beta\left(\frac{a_1\beta^{L-1} + \cdots + a_L}{\beta^L - 1}\right), x\right) \\
&= (\delta_\beta(x), x),
\end{aligned}$$

hence $(\delta_\beta(x), x) \in \widetilde{\mathcal{R}_\beta}$.

Consider now the reciprocal and let $x \in \mathbb{Q}(\beta)$ such that $(\delta_\beta(x), x) \in \widetilde{\mathcal{R}_\beta}$.

The sketch of the proof is the following and is inspired by a similar discrete argument in [IR02], dealing with the case β is a unit. We introduce below a finite subset $\mathcal{T}_x$ of $\widetilde{\mathcal{R}_\beta}$, which depends on x, and which is stable under the action of the map $\widetilde{T}_\beta$ which is onto on it. In order to define this finite set, we take into account all the $\mathcal{I}$-adic topologies which correspond to prime ideals $\mathcal{I}$ which do not appear in the decomposition (2.1). These topologies are the extensions on $\mathbb{Q}(\beta)$ of p-adic topologies for the primes p which do not divide the constant term of the minimal polynomial P_β of β according to [Sie03]. Roughly speaking, one introduces further restrictions over $\mathbb{Q}(\beta)$ which involve the primes that were not already considered in $\mathbb{K}_\beta$. We will prove below that the set $\mathcal{T}_x \subset \widetilde{\mathcal{R}_\beta} \subset \mathbb{K}_\beta \times \mathbb{Q}(\beta)$ is finite by using the fact that the second coordinate of its elements, which belongs to $\mathbb{Q}(\beta)$, is bounded for all the topologies on $\mathbb{Q}(\beta)$.

Let us first introduce the following set $\mathcal{S}_x$:

$$\mathcal{S}_x = \{z \in \mathbb{Q}(\beta),\ |z|_{\mathcal{I}} \leq \max(\,|x|_{\mathcal{I}}, 1\,),\ \text{ for every prime ideal } \mathcal{I} \text{ such that } |\beta|_{\mathcal{I}} = 1\}.$$

One has

- $x \in \mathcal{S}_x$ since $|x|_{\mathcal{I}} \leq \max(\,|x|_{\mathcal{I}}, 1\,)$ for every prime ideal $\mathcal{I}$.
- $\mathbb{Z} \subset \mathcal{S}_x$ since for every integer N and for every ideal $\mathcal{I}$, one has $|N|_{\mathcal{I}} \leq 1 \leq \max(\,|x|_{\mathcal{I}}, 1\,)$.
- $\beta\mathcal{S}_x \subset \mathcal{S}_x$ and $\beta^{-1}\mathcal{S}_x \subset \mathcal{S}_x$ since $|\beta z|_{\mathcal{I}} = |\beta|_{\mathcal{I}}\,|z|_{\mathcal{I}} = |z|_{\mathcal{I}}$ if $|\beta|_{\mathcal{I}} = 1$, and as well $|\beta^{-1}z|_{\mathcal{I}} = |z|_{\mathcal{I}}$.
- $(\mathcal{S}_x, +)$ is a group since the $\mathcal{I}$-adic valuations are ultrametric.

Hence one deduces that:

- $\mathcal{S}_x$ is stable under the action of T_β; indeed
$$T_\beta\mathcal{S}_x \subset \beta\mathcal{S}_x - \mathbb{Z} \subset \mathcal{S}_x - \mathbb{Z} \subset \mathcal{S}_x - \mathcal{S}_x \subset \mathcal{S}_x.$$
- For every integer N, one has $\beta^{-1}(\mathcal{S}_x + N) \subset \beta^{-1}\mathcal{S}_x \subset \mathcal{S}_x$.

In the original proof of [IR02], the analogous of the set $\mathcal{S}_x$ is $\mathbb{Z}[\beta]/q$ for an integer q which depends on x, this set being no more stable under the multiplication by $1/\beta$ in the non-unit case. The keypoint of the present proof is that $\mathcal{S}_x$ is not only stable under the action of T_β but also under the multiplication by $1/\beta$, even when β is not a unit.

Let us consider now the following subset of $\widetilde{\mathcal{R}_\beta}$ obtained by first embeding the points of $\mathcal{S}_x$ into $\mathbb{K}_\beta \times \mathbb{R}$, and then intersecting it with the bounded set $\widetilde{\mathcal{R}_\beta}$:

$$\mathcal{T}_x = (\delta_\beta, Id)(\mathcal{S}_x) \cap \widetilde{\mathcal{R}_\beta} \subset \mathbb{K}_\beta \times \mathbb{Q}(\beta).$$

Let us first observe that $(\delta_\beta(x), x) \in \mathcal{T}_x$.

- **The set $\mathcal{T}_x$ is finite**. Indeed let us prove that all the absolute values of $\mathbb{Q}(\beta)$ take bounded values on the last coordinate of the elements of $\mathcal{T}_x$. Let $z \in \mathcal{S}_x$ such that $(\delta_\beta(z), z) \in \widetilde{\mathcal{R}_\beta}$. Note first that the usual absolute value of z is bounded. Furthermore
  - if $|\cdot|$ is an Archimedean valuation or if $|\cdot|_{\mathcal{I}}$ is an ultrametric valuation satisfying $|\beta|_{\mathcal{I}} \neq 1$, this valuation appears by construction in $\mathbb{K}_\beta$, and thus $|z|$ is bounded since $(\delta_\beta(x), x)$ belongs to the bounded set $\widetilde{\mathcal{R}_\beta}$.
  - if $|\cdot|_{\mathcal{I}}$ is an ultrametric valuation satisfying $|\beta|_{\mathcal{I}} = 1$, then $|z|_{\mathcal{I}}$ is bounded by definition of $\mathcal{S}_x$.
- **The map $\widetilde{T_\beta}$ is onto over $\mathcal{T}_x$.** Let $(\delta_\beta(z), z) \in \mathcal{T}_x$, with $z \in \mathcal{S}_x$. By definition, there exists $(w, u) = ((w_i)_{i\geq 0}, (u_i)_{i\geq 1}) \in \widetilde{X_\beta}$ such that $\widetilde{\varphi_\beta}(w, u) = (\delta_\beta(z), z)$. Hence $z = \sum_{i\geq 1} u_i\beta^{-i}$ and $\delta_\beta(z) = -\delta_\beta(\sum_{i\geq 0} w_i\beta^i)$. An easy computation shows that

$$\widetilde{\varphi_\beta} \circ S^{-1}(w, u) = \left(\delta_\beta\left(\frac{z + w_0}{\beta}\right), \frac{z + w_0}{\beta}\right). \tag{4.1}$$

  Let us note that this computation works thanks to the introduction of a sign minus before δ_β in the expression of $\widetilde{\varphi_\beta}$.

  Let us first assume that $w_0u_1u_2\cdots \neq d^*_\beta(1)$. One thus deduces according to(2.4), that

$$(\delta_\beta(z), z) = \widetilde{\varphi_\beta}(w, u) = \widetilde{\varphi_\beta} \circ S \circ S^{-1}(w, u) = \widetilde{T_\beta} \circ \widetilde{\varphi_\beta} \circ S^{-1}(w, u) \quad = \widetilde{T_\beta} \circ (\delta_\beta, Id)\left(\frac{z + w_0}{\beta}\right).$$

  On the one hand $(\delta_\beta, Id)\left(\frac{z+w_0}{\beta}\right) = \widetilde{\varphi_\beta} \circ S^{-1}(w, u) \in \widetilde{\mathcal{R}_\beta}$, and on the other hand $\frac{z+w_0}{\beta} \in \frac{\mathcal{S}_x+w_0}{\beta} \subset \beta^{-1}(\mathcal{S}_x + \mathbb{Z}) \subset \mathcal{S}_x$. Hence, $(\delta_\beta(z), z) \in \widetilde{T_\beta}(\mathcal{T}_x)$.

Assume now that $w_0 u_1 w_2 \cdots \neq d_\beta^*(1)$. By definition $w_0 = [\beta]$. Moreover, one checks that $z = \beta - w_0$. One has

$$\widetilde{T_\beta} \circ (\delta_\beta, Id)(1) = (h_\beta(1) - [\beta]\delta_\beta(1), \beta - [\beta]) = (\delta_\beta(\beta) - \delta_\beta([\beta]), \beta - [\beta]) = (\delta_\beta(z), z).$$

Since $\frac{z+w_0}{\beta}$ and from (4.1), one has $(\delta_\beta, Id)(1) \in \widetilde{\mathcal{R}_\beta}$ so that $(\delta_\beta(z), z) \in \widetilde{T_\beta}(\mathcal{T}_x)$.

- **The set $\mathcal{T}_x$ is stable by $\widetilde{T_\beta}$.** Let $(\delta_\beta(z), z) \in \mathcal{T}_x$, with $z \in \mathcal{S}_x$. By definition, there exists $(w, u) = ((w_i)_{i\geq 0}, (u_i)_{i\geq 1}) \in \widetilde{X_\beta}$ such that $\widetilde{\varphi_\beta}(w, u) = (\delta_\beta(z), z)$. One has according to (2.3)

$$\widetilde{T_\beta}(\delta_\beta(z), z) = (\delta_\beta, Id) \circ T_\beta(z).$$

Let us first suppose that $u \neq d_\beta^*(1)$. One deduces from (2.4)

$$\widetilde{T_\beta} \circ \widetilde{\varphi_\beta}(w, u) = \widetilde{\varphi_\beta} \circ S(w, u).$$

Now if $u = d_\beta^*(1)$, then $z = \sum_{i\geq 1} u_i \beta^{-i} = 1$, $T_\beta(z) = \beta - u_1$ and $\delta_\beta(\sum_{i\geq 0} w_i \beta^i = \delta_\beta(1)$ so that

$$\widetilde{\varphi_\beta} \circ S(w, u) = (-\delta_\beta(u_1 + \sum_{i\geq 0} w_i \beta^{i+1}), \sum_{i\geq 2} u_i \beta^{-i+1}) = (-\delta_\beta(u_1) + \delta_\beta(\beta), \beta - u_1) = (\delta_\beta, Id) \circ T_\beta(z).$$

Hence, since $\widetilde{T_\beta}$ is onto over a stable finite set, it is one-to-one over this set $\mathcal{T}_x$. By construction, $(\delta_\beta(x), x) \in \mathcal{T}_x$. Hence there exists an integer n such that

$$(\delta_\beta(x), x) = \widetilde{T_\beta}^n(\delta_\beta(x), x) = (\delta_\beta({T_\beta}^n(x)), {T_\beta}^n(x)).$$

We thus deduce that the expansion of x is purely periodic. ■

5. Conclusion

One can also introduce a notion of generalized Rauzy fractal in the non-Pisot case. Namely, a similar characterization holds in the non-Pisot case by introducing in the representation map $\widetilde{\varphi_\beta}$ the coordinate $\sum_{i\geq 1} u_i \lambda^{-i}$ for the conjugates λ of modulus strictly larger that 1 (see for instance [KV98] for a similar description). Yet we are not able to prove that the union in Equation (3.1) is disjoint up to sets of zero measure and that the Haar measure of the generalized Rauzy fractal is non-zero.

It remains now to use this formalism to study further topological or metrical properties of the sets $\mathcal{R}_\beta$ and $\widetilde{\mathcal{R}_\beta}$, even in the non-Pisot case, which will be the object of a subsequent paper. Our motivations are the following: first the construction of explicit Markov partitions of endomorphisms of the torus as initiated in [Sie00], second, the study of rational numbers having a purely periodic expansion in the flavour of [Aki98, Sch80], and third, the spectral study of β-shifts in the Pisot non-unit case according to [Sie03].

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