# Criterion for universality class independent critical fluctuations: example of the 2D Ising model

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# Criterion for universality class independent critical fluctuations: Example of the 2D Ising model

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Order parameter fluctuations for the two dimensional Ising model in the region of the critical temperature are presented. A locus of temperatures $T^*(\mathsf{L})$ and of magnetic fields $B^*(\mathsf{L})$ are identified, for which the probability density function is similar to that for the 2D-XY model in the spin wave approximation. The characteristics of the fluctuations along these points are largely independent of universality class. We show that the largest range of fluctuations relative to the variance of the distribution occurs along these loci of points, rather than at the critical temperature itself and we discuss this observation in terms of intermittency. Our motivation is the identification of a generic form for fluctuations in correlated systems in accordance with recent experimental and numerical observations. We conclude that a universality class dependent form for the fluctuations is a particularity of critical phenomena related to the change in symmetry at a phase transition.
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## INTRODUCTION

There is ubiquitous interest in systems with extended spatial and temporal correlations, from all areas of physics. Recently, observations of fluctuations in global, or spatially averaged measures of such correlations have provided a possible link between critical phenomena and non-equilibrium systems. That is, the probability density function (PDF) for order parameter fluctuations in the low temperature critical phase of the 2D-XY model, shown in Figures 1 and 2 below, is very similar to PDFs [1] for fluctuating spatially, or temporally averaged quantities in turbulent flow [2, 3, 4], for electro-convection in nematic liquid crystals [5], for numerical models of dissipative systems [6, 7, 8, 9] and of "self-organized criticality" [10], for fluctuations of river heights [11, 12], as well as for other equilibrium systems close to criticality [10, 13, 14, 15]. The simplicity of the 2D-XY model allows a complete understanding of fluctuation phenomena in this case [16, 17, 18]. The contrary is true for non-equilibrium systems; the lack of microscopic theory makes the problems extremely complex. Phenomenological observations and analogy with better understood systems can therefore be extremely useful.

However, for other critical systems, such generic behaviour seems only to be observed under restricted conditions. For example, it has been shown that magnetic fluctuations in the two-dimensional Ising model, at a temperature, $T^*(\mathsf{L})$ (where $\mathsf{L}$ is the system size in units of the lattice constant), below but near the critical temperature $T_c$, are similar to those of the 2D-XY model [10, 13, 14, 15]. Given the important role played by universality classes in critical phenomena, this is rather surprising. In fact, it is well established that critical fluctuations, as measured at $T_c$ [19, 20, 21], depend, in general on the universality class of the model, as well as on the shape [16] of the sample and on the boundary conditions [22]. The apparent similarity of the form of the fluctuations, over and above the universality class of the models under consideration therefore seems rather puzzling [13].

Given the generality of the above observations in more complex systems, it is important to understand this point. In this paper we address it in detail for the two-dimensional Ising model and in doing so pose the following questions: Is the similarity in form quantitative, or only qualitative? Is this PDF really a measure of critical fluctuations in the problem? Finally can we, from this investigation shed any further light on the reason for the apparent "super-universality" observed in the wide range of experimental and numerical systems ? In answer to these questions, we show that the distribution functions for the Ising and XY models at $T^*(\mathsf{L})$ are similar, with the latter representing an excellent fit over almost any accessible window of measurements for experimental systems. However they are not the same functions. Numerical evidence suggests that the difference will remain in the thermodynamic limit. The origin of the difference is the structure of phase space associated with the Ising transition. For large amplitude fluctuations the system, localized in one half of phase space is able to surmount the barrier separating it from the other, symmetric half of phase space. If instead one approaches the critical point by applying a small magnetic field, symmetry remains broken and one finds a field $B^*(\mathsf{L})$ giving excellent quantitative agreement between the PDF for the Ising model and the 2D-XY model.

We confirm that, despite the small value of the correlation length, $\xi$ [13, 15] at $T^*(\mathsf{L})$, the system does show evidence of correlations on all scales up to a length of

the order of the system size. A consequence of this is the development of coherent structures, the clusters of spins, up to this macroscopic scale, that dominate the exponential tail of the distribution.
From the above, we conclude that the dependence of the PDF on the universality class comes from the structure of phase space and the way in which symmetry is restored on passing through the phase transition. If long range correlations develop, without the insuing fluctuations exposing a structured phase space, such as occurs in the Ising transition, then the form of the fluctutations will be largely independent of universality class. This is the case in the 2D-XY model [23] and we propose that it is key to the approximate "super-universality" observed in a large array of correlated systems.

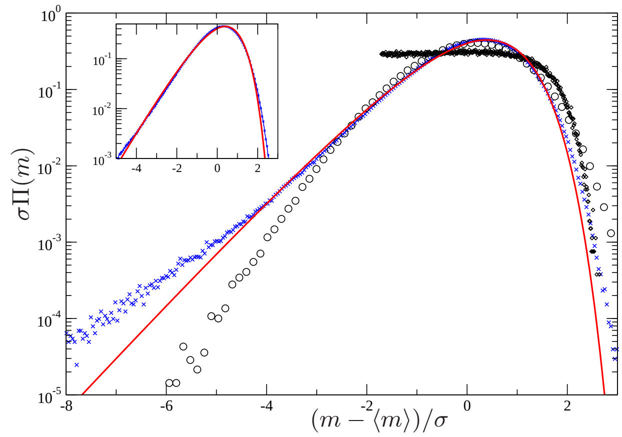

FIG. 1: Order parameter PDF for $T = 2.33J = T_c(\mathsf{L} = 64)$ ($\diamond$)[24], $T^*(\mathsf{L} = 64) = 2.11J$ ($\times$) and $T = 1.54J$ ($\circ$) , with $\mathsf{L} = 64$. The inset is a zoom on $\mu \in [-5; 3]$.

## ORDER PARAMETER FLUCTUATIONS IN THE 2D ISING MODEL

The Hamiltonian for the Ising model is given by:

$$\mathcal{H} = -J \sum_{<i,j>} S_i S_j, \ S_i = \pm 1$$

with $J > 0$, the exchange constant. We study the model on a square lattice of size $N = \mathsf{L} \times \mathsf{L}$ , with periodic boundary conditions and with lattice parameter $a = 1$. The order parameter is defined as the modulus of the magnetization:

$$m = \left| \frac{1}{N} \sum_{i=1}^{N} S_i \right| .$$

Previous studies, by Bramwell *et al.* [10], by Zheng and Trimper [13] and Zheng [15] suggest that there is a temperature, $T^*(\mathsf{L})$ just below the critical temperature, for which the PDF is close to that of the 2D-XY model in the low temperature phase. It was further suggested that $T^*(\mathsf{L})$scales towards the critical temperature as the thermodynamic limit is taken and can thus be interpreted as a critical phenomenon [14]. The first step in our study was to confirm this result. Note that, within the spin wave approximation, the PDF for the 2D-XY model has the form shown in Figures 1 and 2, independently of temperature and hence of critical exponent [16]. The PDF was calculated using the Swendsen-Wang Monte Carlo algorithm [25] for various sizes $\mathsf{L}$ , between 32 and 512. Here we define $T^*(\mathsf{L})$ as the temperature for which the skewness of the distribution, $\gamma(T^*(\mathsf{L}))$ is equal to that for the 2D-XY model, $\gamma_{\mathrm{XY}}$, for data restricted to the window $\mu = (m - \langle m \rangle)/\sigma \in [-6; 3]$. We are able to establish this criterion with a numerical precision $\gamma(T^*(\mathsf{L})) = \gamma_{\mathrm{XY}} \pm 0.01$, giving an error for $T^*(\mathsf{L})$ of less than 2%. This criterion, though precise, is arbitrary and we could choose others. However, as it is only

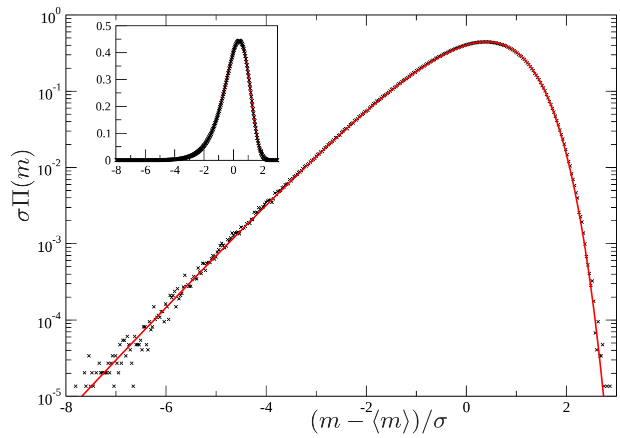

FIG. 2: PDF for a subsystem $\mathsf{L}$ =32, in system of size $\mathsf{L}_0 = 128$, for $B = B^*(\mathsf{L} = 32) = 0.0035J$ at $T_c(\mathsf{L} = 32)$. The plain curve is the PDF for the 2D-XY model.

an approximate agreement between the two sets of data, making alternative definitions does not alter the results discussed below. Figure 1 shows $\log(\sigma\Pi(m))$ against $\mu$ for various temperatures and for $\mathsf{L}$ =64. The solid line is the PDF for the 2D-XY model [16]. The best fit is for $T^*(\mathsf{L} = 64) = 2.11J$, while the critical temperature for the infinite system, as given by Onsager's exact solution is $T_c = 2.25J$. One can see that $T^*(\mathsf{L})$ is significantly shifted below $T_c$. At higher temperatures, the probability for fluctuations below the mean is a concave function of $|\mu|$ and there is no region approximating to an exponential tail. The PDF is cut off at a finite value of $\mu$ corresponding to the constraint $m = 0$. As one expects, this corresponds to a turning point in probability and reflects the access of the finite size system to the complete and symmetric phase space.

A distribution with an exponential or quasi-exponential tail cannot correctly describe this minimum and therefore for symmetry reasons the distribution for the 2D-XY model cannot exactly describe the data for the Ising model in zero field, for any temperature below $T_c(\mathsf{L})$ [24]. However, at $T^*(\mathsf{L})$ the fit is good for fluctuations out to $-5\sigma$ below the mean magnetization, corresponding to a probability density of $10^{-3}$ [15]. From an experimental point of view, reliable data for $\mu < -6$ would be exceptional [2, 5, 9], and in this sense the agreement between the Ising and the XY model data is very satisfactory (see inset Figure 1). Here we have good statistics for fluctuations out as far as $-8\sigma$ from the mean, from which an upturn away from the exponential tail is evident. This is a consequence of the extremum in PDF at $m = 0$. The effect is independent of system size for the values studied and within the numerical error obtained, although we cannot exclude the possibility of corrections to scaling that disappear slowly on the scale of the sizes studied. For temperatures below $T^*(\mathsf{L})$ the turning point at $m = 0$ is moved outside the accessible window of measurement, but the PDF is not sufficiently skewed to give a good fit to the XY model data. For temperatures between $T^*(\mathsf{L})$ and $T = 1.54J$ the the statistics are poor for large fluctuations, which is consistent with having just a few rare events taking the magnetization out towards the constraint $m = 0$. At lower temperature still, the distribution crosses over to a Gaussian, as expected for an uncorrelated system. The relevance of the turning point in the PDF in the critical region depends on the universality class and it is one of the ways in which universality class dependent critical fluctuations appear.

## BOUNDARY CONDITIONS AND MAGNETIC FIELD

It is clear that the quality of the comparison with the 2D-XY model would be improved if the turning point in the probability density at $m = 0$ was either displaced or removed. Two obvious ways in which one might do this are firstly in changing the boundary condition and secondly in adding a magnetic field. Changing from periodic to fixed, or window boundaries one can expect to observe small changes in the form of the universal scaling function [16, 22]. One might think that these boundaries would make a finite size system more rigid with respect to a global spin flip, thus reducing the probability of a microstate with $m = 0$ and improving the quality of the fit. We have studied the distribution for a window of size $\mathsf{L}$ embedded in a larger system of size $\mathsf{L}_0$. The fits to the PDF for the 2D-XY model are qualitatively better, but the same upturn away from the exponential tail is observed for fluctuations of more than about $5\sigma$ from the mean. The situation is not quantitatively changed compared with periodic boundaries. The data are not show here.

A real quantitative improvement is found however, for the study of fluctuations in a magnetic field. Approaching the critical point by fixing $T = T_c(\mathsf{L})$ and applying a small field $B$, a field $B^*(\mathsf{L})$ is defined for which the PDF of magnetic fluctuations gives the best fit to the 2D-XY data. The data for $B^*(\mathsf{L}) = 0.0035J$ for window boundaries, with $\mathsf{L} = 64$ and $\mathsf{L}_0 = 128$, are shown in Figure 2. To the eye the quality of the fit is excellent. This is confirmed quantitatively by measuring the skewness, $\gamma$, and kurtosis $\kappa$, of the distribution. We find $\gamma = 0.890 \pm 0.01$, $\kappa = 4.495 \pm 0.01$, which are in excellent agreement with those of the 2D-XY model ($\gamma_{\mathrm{XY}} = 0.890$ and $\kappa_{\mathrm{XY}} = 4.489$). The application of a magnetic field breaks the symmetry and removes the minimum from the PDF for the order parameter. This can be seen in Figure 3, where we show the evolution of the PDF, for a window of size $\mathsf{L} = 32$ in a larger system of size $\mathsf{L}_0 = 128$ for field varying between zero and $B^*(\mathsf{L} = 32)$. For zero field the minimum in probability at $m = 0$ is clearly visible and the fluctuations below the mean are cut off by the constraint $m \geqslant 0$. As the field increases the mean magnetization increases and the variance, $\sigma$ reduces, pushing the constraint $m \geqslant 0$ out to larger negative values of $\mu$. At the same time the end point of the distribution

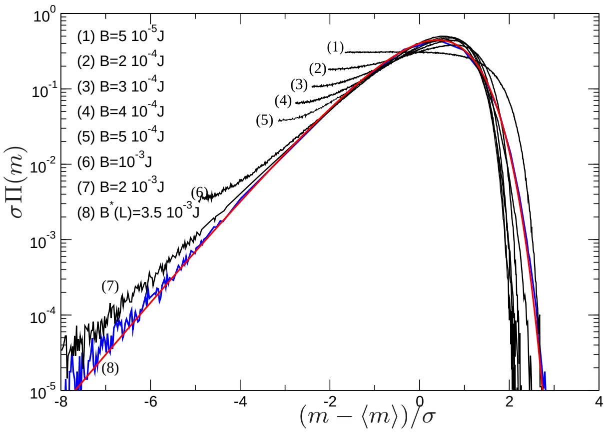


FIG. 3: PDF for a subsystem $\mathsf{L}$ =32, in system of size $\mathsf{L}_0$=128, for various values of the magnetic field $B$ at $T_c(\mathsf{L} = 128)$. The plain curve is the PDF for the 2D-XY model.

is no longer a minimum, that is, the PDF terminates with a finite slope. Within the window of observation, one clearly sees an exponential tail developing which approaches asymptotically that of the 2D-XY model. For larger fields the asymmetry reduces and the PDF crosses over to a Gaussian. The curve for the 2D-XY model therefore seems to define an envelope giving the maximum possible asymmetry as the field varies in the region of the critical point. Behaviour for periodic boundary conditions is similar. One again sees the development of an exponential tail with the same slope as the 2D-XY model, but for the best fit the cut off corresponding to $m = 0$ remains within the window $-8 < \mu < 3$. The data are not shown here.

## CORRELATION LENGTH AT $T^*(\mathsf{L})$

In this section we concentrate on the critical properties of $T^*(\mathsf{L})$. Figure 4 shows how $T^*(\mathsf{L})$ scales with system size. As noted in references [10, 13], $T^*(\mathsf{L})$ scales with $\mathsf{L}$ as : $T_c - T^*(\mathsf{L}) \propto \mathsf{L}^{-1/\nu}$, with $\nu \simeq 1$, the expected value conforming to the scaling hypothesis for the 2d Ising model. These finite size scaling results imply that

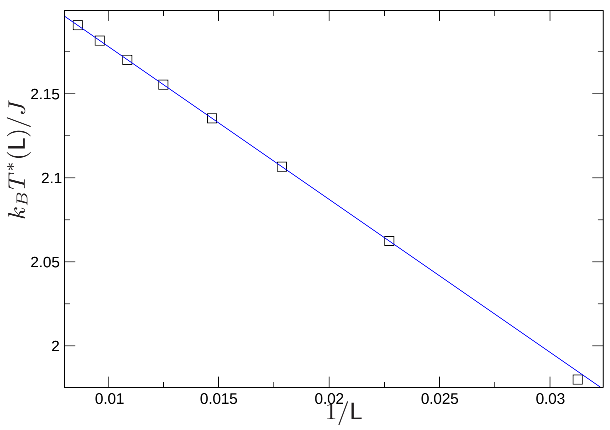


FIG. 4: Evolution of $T^*(\mathsf{L})$ with $\mathsf{L}$ . The solid line is the best linear fit.

the magnetic correlation length $\xi(\mathsf{L})$ is diverging with the system size at $T^*(\mathsf{L})$, such that $\xi(\mathsf{L})/\mathsf{L}$ = constant. This is confirmed by the observed universality of $\Pi(\mu)$ along this locus of points [19]. The correlation length can be computated from the spin-spin connected correlation function:

$$\mathrm{G}_{\mathsf{L}}(|\boldsymbol{r}_i - \boldsymbol{r}_j|) = \langle S(\boldsymbol{r}_i).S(\boldsymbol{r}_j)\rangle - \langle S(\boldsymbol{r}_i)\rangle . \langle S(\boldsymbol{r}_j)\rangle .$$

Different curves, obtained for $\mathsf{L}$ =128 and various temperatures are plotted on Figure 5.

The numerical data is fitted well by the expression

$$\mathrm{G}_{\mathsf{L}}(r, \mathrm{T}) = \frac{1}{r^{\eta}} e^{-\frac{r}{\xi_{\mathsf{L}}(\mathrm{T})}},$$

with $\eta = 0.24 \pm 0.01$, in good agreement with the theoretical value $\eta = \frac{1}{4}$. Values of $\xi(\mathsf{L})$ and $\xi(\mathsf{L})/\mathsf{L}$ are shown in table I. The main criticism of Zheng and Trimper [13],

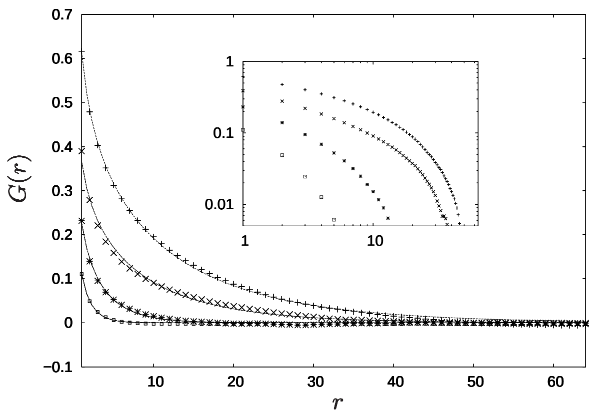


FIG. 5: Correlation functions for $\mathsf{L}$ = 128 and various temperatures $T = 2.30J(+)$, $2.27J(\times)$, $2.24J(*)$, $2.17J(\Box)$. The plain curves are the best fits.

that $\xi(\mathsf{L})$ is small, is immediately apparent, however the ratio $\xi(\mathsf{L})/\mathsf{L} \simeq 0.03$ is indeed constant, to a good approximation, confirming that the correlation length is diverging with system size, as conjectured. In this respect we are indeed dealing with a critical phenomena despite the small values measured for the system sizes considered. This "small, yet diverging" quantity plays an important role in the approximate universality observed between disparate systems.

## CLUSTER SIZES AND DISTRIBUTION

Definitions of clusters in the Swendsen-Wang algorithm is a good way to study structures in the 2d Ising model [26]. A cluster is defined as a connected graph of spins in the sense of reference [26], with magnetization opposed to the spatially average value over all the lattice. In concrete terms, a cluster is a white object in the snapshots shown in Figures 6, 7 and 8. Cluster sizes are calculated for each generated spin configuration and averaged over many realizations.

| $\mathsf{L}$ | 32 | 36 | 40 | 44 | 52 | 56 | 64 | 128 |
|---|---|---|---|---|---|---|---|---|
| $\xi(\mathsf{L})$ | 0.83 | 1.0 | 1.1 | 1.25 | 1.45 | 1.42 | 1.7 | 2.9 |
| $\xi(\mathsf{L})/\mathsf{L}$ | 0.026 | 0.028 | 0.027 | 0.028 | 0.028 | 0.025 | 0.026 | 0.026 |

TABLE I: $\xi$ and $\xi/\mathsf{L}$ for various $\mathsf{L}$ .

Given the small value for the correlation length at $T^*(\mathsf{L})$ one might expect the range of cluster sizes to be extremely limited. One of the surprises of this study is that this is not the case. One can get a feeling for this by first studying snap shots. In Figures 6, 7 and 8, we show three configurations, the first has magnetization $m$ close to the mean value. One can clearly see a range of cluster sizes up to a characteristic size that is small compared with $\mathsf{L}$. The second snap shot shows a configuration with magnetization four standard deviations below the mean, $m = \langle m \rangle - 4\sigma$. A much bigger cluster is present. The large fluctuation is due to the presence of this large coherent structure. This is very different from what one would expect for fluctuations of an uncorrelated system: in this case, a large deviation would correspond to a configuration with many, small and uncorrelated clusters appearing spontaneously. This scenario is extremely unlikely, which is why fluctuations away from the critical point are Gaussian. Here it is clear that, while the correlation length fixes the size of typical clusters, much bigger clusters are not excluded. They are rare events, but not so rare as to be experimentally irrelevant. This can be compared with the configuration taken at $T_c(\mathsf{L})$, shown in Figure 8. Here, coherent structures spanning the entire system are not rare. In this situation the PDF depends strongly on the universality class and generic behaviour is not expected. Moving along the locus of points $T^*(\mathsf{L})$, the size of typical clusters scales with $\mathsf{L}$ through the scaling of $\xi(\mathsf{L})$ and we expect the size of rare clusters to scale in the same way. In this sense the only length scale in the problem along $T^*(\mathsf{L})$ is $\mathsf{L}$. Apart from this, the system is scale free, despite the small values of $\xi(\mathsf{L})$ extracted numerically.

From the mapping of the Ising model at criticality onto a percolation problem [26, 27] one expects the clusters to have a power law distribution of sizes. Results for the distribution of clusters sizes $P(s)$, obtained for $\mathsf{L} = 128$ for various temperatures below $T_c(\mathsf{L})$ are presented in Figure 9. As the temperature is below $T_c(\mathsf{L})$ the system possesses a spanning cluster whose spin direction defines the global magnetization direction. Note that, while for standard Metropolis Monte Carlo the magnetization direction of the spanning cluster changes extremely rarely, for Swendson-Wang Monte Carlo symmetry is not broken and the direction oscillates between "spin up" and "spin down". The spanning cluster is included in the statistics shown in Figure 9. As expected, as the temperature approaches $T_c(\mathsf{L})$ the probability of finding large clusters increases and the distribution approaches a power law. However, for temperatures below $T^*(\mathsf{L})$ there is a separation of scales, with a gap in probability between the largest non-spanning cluster and the spanning cluster. The gap closes at about $T^*(\mathsf{L})$ and above this temperature the statistics of the spanning and the secondary clusters are mixed [28]. When this is the case the largest clusters of up spins and down spins will be of the same

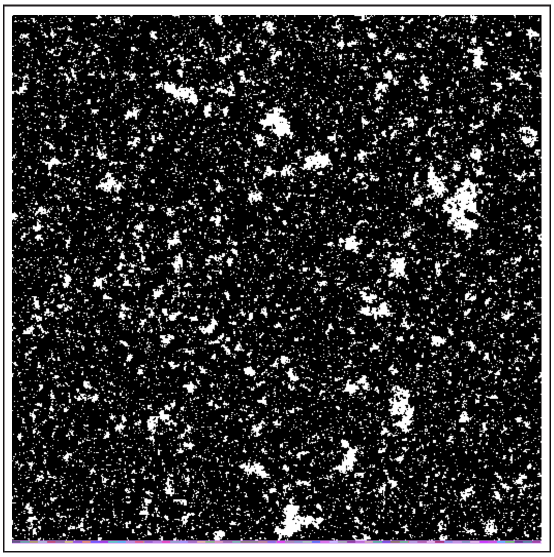

FIG. 6: Typical configuration at $T^*(\mathsf{L})$.

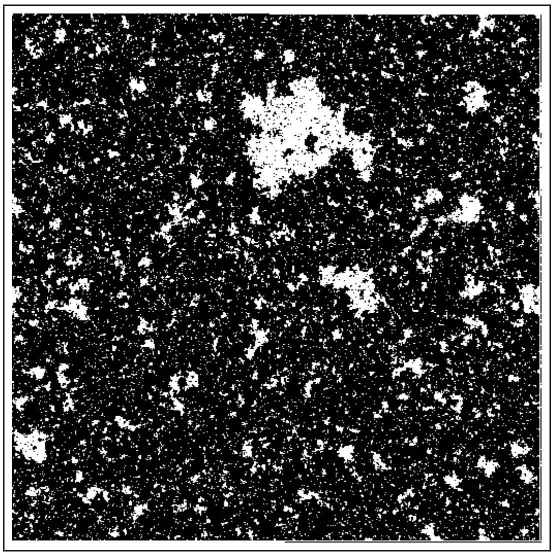

FIG. 7: Rare event at $T^*(\mathsf{L})$.

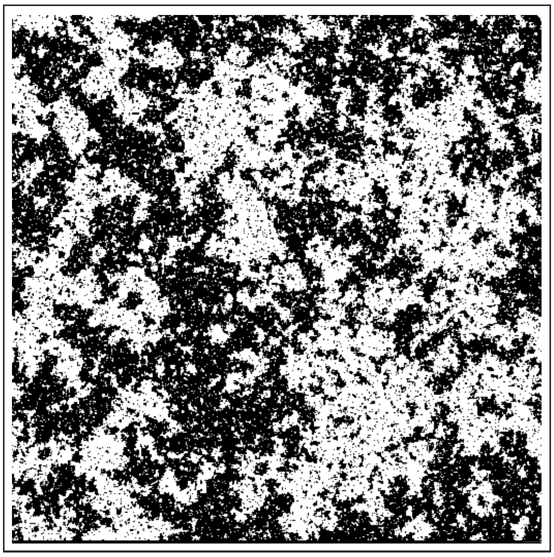

FIG. 8: Typical configuration at $T_c$.

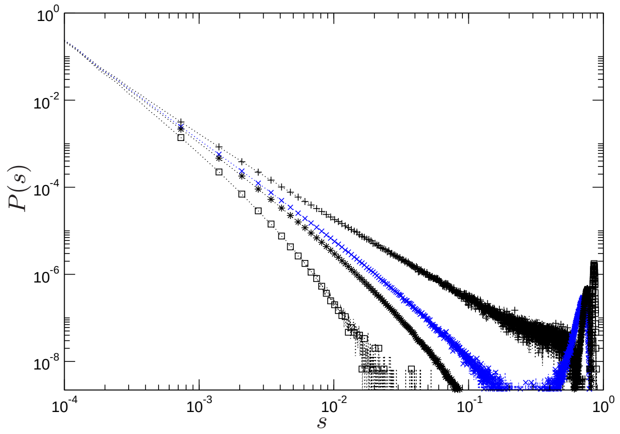


FIG. 9: Clusters size distribution $P(s)$ for $\mathsf{L}$ =128 and $T = 2.29J(+)$, $T^*(\mathsf{L} = 128) = 2.25J(\times)$, $T = 2.22J(*)$ and $T = 2.13J(\Box)$ ($s = N_{\text{cluster}}/N$).

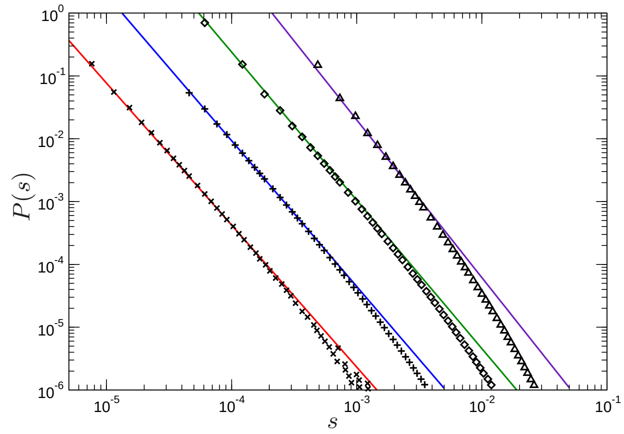


FIG. 10: Clusters size distribution at $T^*(\mathsf{L})$ for $\mathsf{L}$ =64($\triangle$), 128($\diamond$), 256(+) and 512($\times$). The solid lines are the best power laws obtained.

size and there will be a non-negligible probability of having zero magnetization. The mixing of statistics of the spanning and secondary clusters is therefore perfectly consistent with the observation that the turning point of the PDF, at $m = 0$, moves into the window of numerical or experimental observations for $T$ above $T^*(\mathsf{L})$ and that the tail of the distribution $\Pi(\mu)$ is no longer well approximated by an exponential.

At $T^*(\mathsf{L})$ the cluster distribution is well represented by a power law out to cluster size $s = \frac{N_{\text{cluster}}}{N} < \frac{2}{100}$. This is just about the size of the large cluster in the second snap shot of Figure 7. Above this size, corrections to scaling are manifest as one might expect, but it is worth noting that the limit of power law behaviour corresponds to clusters of sizes well in excess of the correlation length extracted from the autocorrelation function. As shown in Figure 10 the range of application of a power law distribution increases with system size. For $\mathsf{L} = 512$ it extends over almost six orders of magnitude of probability and three of cluster size. The fitted exponent decreases with $\mathsf{L}$, as shown in table II. For $\mathsf{L} = 512$ the best fit is for $P(s) \sim s^{-\tau(\mathsf{L})}$ with $\tau(\mathsf{L}) = 2.2 \pm 0.1$. This seems to correspond to the estimated exponent for percolating clusters [29, 30], $\tau = 2 + \frac{\beta}{\nu D_f} \simeq 2.1$, where $D_f = 187/96$ is the fractal dimension for the 2D Ising model [31, 32],

| $\mathsf{L}$ | 16 | 32 | 64 | 128 | 256 | 512 |
|---|---|---|---|---|---|---|
| $\tau(\mathsf{L})$ | 3.6 | 3.6 | 2.5 | 2.4 | 2.3 | 2.2 |

TABLE II: Exponent $\tau(\mathsf{L})$ of the power laws obtained for the cluster size distributions.

confirmation of the critical nature of our observations.

In this section we have shown that clusters at $T^*(\mathsf{L})$ show, rather remarkably, both scale free behaviour, and a separation of scales: Typical cluster size, maximum secondary cluster size and spanning cluster size are all fixed by $\mathsf{L}$, however their amplitudes are sufficiently different to ensure a minimum of interference between the three scales. In the low temperature phase of the 2D-XY model, the ratio of variance to mean magnetization is, $\sigma/\langle m\rangle = AT/J$ [23], where $A \approx 0.04$. This is a critical phenomenon, in that the ratio is system size independent. However, it goes to zero at zero temperature, meaning that the critical fluctuations have zero amplitude and despite their singular nature, the system visits an infinitesimal part of phase space near the mean value of the magnetization. The separation of scales for cluster formation in the Ising case is analogous to this. It corresponds to a situation where the fluctuations are critical, but where the limits of phase space ($m = 1$, $m = 0$, or $m = -1$) are not approached. We propose here that observed generic behaviour and universality classes independance must have its origin in this point.

## EXTREME CLUSTERS DISTRIBUTIONS

The form of the distribution in Figure 2 bares a strong resemblance with Gumbel's first asymptotic solution for extreme values [10, 16]. Indeed there have been a series of recent papers searching for a connection between this form for global fluctuation and extreme statistics [22, 33]. Such a connection has failed to emerge in Gaussian interface models, related to the 2D-XY model [18], but as there is here direct access to obvious real space objects, the clusters, it seems natural to investigate extreme statistics for the clusters. One relevant question is: does the largest cluster dominate $\Pi(\mu)$ rendering the problem of non-Gaussian fluctuations an extreme value problem for the clusters ? The answer is no! While the distribution for the largest cluster is skewed in the right direction and looks qualitatively quite similar to $\Pi(\mu)$, it is not sufficient to reproduce the global fluctuations quantitatively. This conclusion has been tested in detail: We express the magnetization as

$$m = 1 - 2\sum_{j=1}^{\infty} \frac{n_j}{\mathrm{N}},$$

with $n_j = 0, \forall j > j_{\max}$. Approximate order parameters $m_k$ are constructed:

$$m_k = 1 - 2\sum_{j=1}^{k} \frac{n_j}{\mathrm{N}}.$$

If extreme values statistics are relevant for the complete order parameter PDF, then starting from $k = 1$ $\Pi(\mu_k)$, with $\mu_k = (m_k - \langle m_k \rangle)/\sigma$ should converge to $\Pi(\mu)$ for just a few values of $k$. Results are shown in Figure 11 for $k = 1, 5, 10$ and compared with the complete order parameter PDF. The convergence is slow and $k = 10$ is not sufficient to reproduce well the global PDF. We conclude that all the clusters are required to reproduce the global fluctuations. This is not then an extreme value problem, at least in terms of clusters.
This result is rather similar to that obtained for the 2D-XY model [17] and for related Gaussian interface models [18], although in the latter more detailed information on the microscopic distributions is available. The non-Gaussian fluctuations occur because the largest cluster makes a macroscopic contribution to the many body sum. However, this contribution does not dominate, it is the same order as the sum over all the other clusters. In this sense critical fluctuations are a marginal case between statistics dominated by the majority, leading to the central limit, and statistics dominated by a single event, for example Lévy statistics.

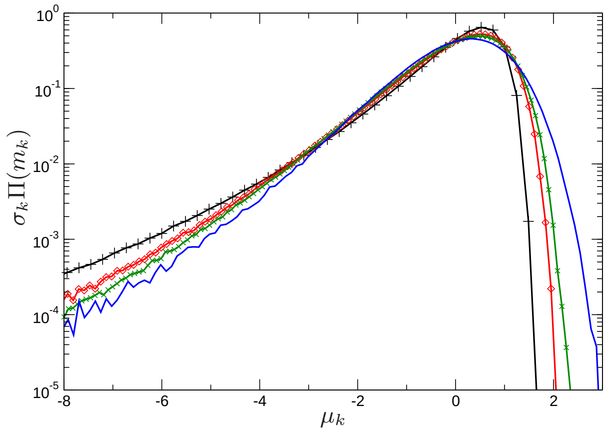


FIG. 11: PDF of $m_k$, for $k = 1(+), 5(\diamond), 10(\times)$, and non approximate order parameter (solid line), for $\mathsf{L} = 64$ at $T^*(\mathsf{L} = 64)$.

## INTERMITTENCY AND CRITICAL FLUCTUATIONS

Figure 12 shows a sequence of magnetization values for configurations generated by a Metropolis Monte Carlo algorithm at temperature $T^*(\mathsf{L})$. The asymmetric nature of the distribution is evident from the stochastic time series. We remark that the data bare a striking resemblance to the time series of injected power into a closed turbulent flow at fixed Reynolds number [3]. The two systems share the characteristic of making large deviations from the mean value, on a scale set by the variance of the distribution $\sigma$. This is the so called "intermittency" which is so important in studies of turbulence. It is not the purpose of this paper to discuss the detailed properties of intermittency, we simply make a comment concerning the scale of intermittency in the 2D Ising model in the region of the critical point. In Figure 13, a similar time series is shown for a simulation at $T_c(\mathsf{L})$. One can clearly see that the range of fluctuations, on the scale of $\sigma$ is smaller at $T_c(\mathsf{L})$ than at $T^*(\mathsf{L})$. This is an unconventional way of characterizing fluctuations in the context of critical phenomena. Usually one discusses the susceptibility, $\chi = (1/T)\sigma^2$, which is a diverging quantity at a critical point. For a finite size system, $T_c(\mathsf{L})$ is generally defined at the temperature where $\chi$ and hence $\sigma$ is a maximum [34]. What we observe here is that, although the fluctuations on an absolute scale are maximized at $T_c(\mathsf{L})$, on a relative scale, as fixed by $\sigma$ they are maximized at $T^*(\mathsf{L})$. Between $T^*(\mathsf{L})$ and $T_c(\mathsf{L})$, large deviations from the mean value are cut off because of the constraints on the phase space in which the fluctuations can occur. Specifically, the free energy barrier between "spin up" configurations and "spin down" configurations is surmountable with the consequence that the limits $0 < \langle m \rangle < 1$ become the relevant limits for fluctuations and the variance and the mean order parameter values become of the same size. In this way the magnetic symmetry, which is broken in the ordered phase, is re-established on passing through the phase transition. For temperatures up to $T^*(\mathsf{L})$, despite the growing fluctuations the system behaves *almost* as if only a single phase existed, for which the order parameter could vary

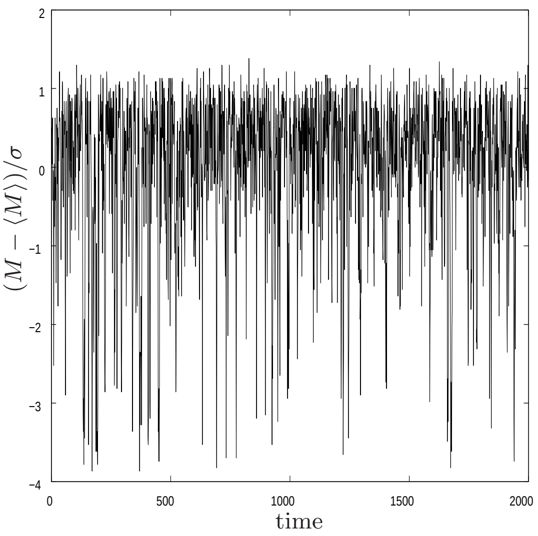


FIG. 12: Metropolis evolution of $m$ at $T^*(\mathsf{L})$.

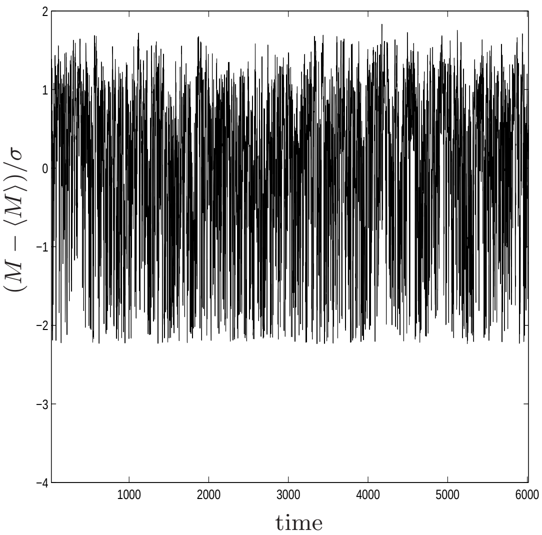


FIG. 13: Metropolis evolution of $m$ at $T_c(\mathsf{L})$.

within the limits $-\infty < m < \infty$. The distribution for the 2D-XY model corresponds to the case where this is a limitingly good approximation [23]. If, therefore we make a rather loose definition of intermittency, as the tendency to make fluctuations away from the mean, on the scale of $\sigma$ with a probability that differs from that for a Gaussian function, then maximum intermittency occurs at $T^*(\mathsf{L})$ and not at $T_c(\mathsf{L})$.

## CONCLUSION

In this article we have compared the order parameter fluctuations of a finite size 2D Ising model, in the region of the critical temperature, with those of the 2D-XY in the low temperature phase. Our motivation for doing this is the observation that fluctuations of global quantities in a wide range of different correlated systems are of similar form to the 2D-XY model, or equivalent Gaussian interface models [22]. Approaching the critical temperature $T_c$, either along the zero field axis, or applying a small field, $B$, at the critical temperature one can identify a temperature $T^*(\mathsf{L})(B = 0)$ and a field $B^*(\mathsf{L})$, close to the critical point for which the order parameter fluctuations are similar to those of the 2D-XY model. We have established the critical scaling behaviour of the locus of temperatures $T^*(\mathsf{L})$. The correlation length $\xi$ diverges with system size along the locus of temperatures $T^*(\mathsf{L})$ , as one might expect for critical scaling. However, the ratio $\xi/\mathsf{L} \sim 0.03$ is a small constant. This is a key point: Critical fluctuations are usually associated with a phase transition and a change in phase space symmetry. In the 2D Ising model the symmetry between the two competing phases is reflected in the form of the PDF, imposing that $m = 0$ corresponds to a turning point in the probability density. However the 2D-XY model is an exception, in that there is a continuous line of critical points in the low temperature phase, but no phase transition or associated change in symmetry [35]. To an excellent approximation the critical fluctuations occur in an unconstrained phase space [23]. Hence, criticality in the Ising model can only resemble that in the 2D-XY model if the change in symmetry is not apparent. This corresponds to the condition that the ratio $\xi/\mathsf{L}$ is small. In the case of $B^*(\mathsf{L})$, the agreement between the PDFs for order parameter fluctuations in the two systems is exceptionally good: High quality numerical data for the Ising model are indistinguishable from the analytical results for the 2D-XY model. However, differences are observable for data at $T^*(\mathsf{L})$. The agreement is better in the former case, as the field breaks the symmetry between the two competing ordered phases, thus eliminating the turning point, or the minimum value from the PDF. In this case, the phase space available for fluctuations strongly resembles that of the 2D-XY model.

Driven, non-equilibrium systems, showing strong correlations, such as turbulent flow [2, 4], resistance networks [9], self-organized critical systems [10], or growing interfaces, resemble the 2D-XY model, in that there is no associated phase transition and no sudden change of limits that constrain the fluctuations as the correlations build up. We propose that observation of a non-Gaussian PDF with finite skewness and an exponential tail, for fluctuations in a global quantity, is a characteristic of correlated fluctuations in an effectively unbounded phase space. The example of the 2D Ising model serves to show that this is not the case for critical phenomena associated with a $2^{\text{nd}}$ order phase transition. However, this is a detail specific to critical phenomena, related to the fact that the fluctuations become so large that they allow the system to explore the whole allowed phase space of order parameter values. From this analysis one could argue that fluctuations at $T^*(\mathsf{L})$ are not strong. This is true on

an absolute scale: For example the susceptibility, which is a measure of the variance of the distribution, is small at $T^*(\mathsf{L})$ compared with the maximum value from which one defines $T_c(\mathsf{L})$. However at this temperature extreme fluctuations compared with the variance are capped by the constraint, $0 < m < 1$. Parameterizing in terms of the reduced variable $\mu = (m - \langle m \rangle)/\sigma$ changes this conclusion: Fluctuations in $\mu$ are essentially unbounded at $T^*(\mathsf{L})$ while they are constrained at $T_c(\mathsf{L})$. Hence the largest fluctuations in $\mu$ occur at $T^*(\mathsf{L})$ and not at $T_c(\mathsf{L})$. It is the variance of the distribution which defines the scale of the fluctuations that one might observe experimentally [2] or numerically, and so in this sense $T^*(\mathsf{L})$ is highly relevant. Physically, this means that at $T^*(\mathsf{L})$ large fluctuations are rare enough not to modify $\sigma$, but not too rare to be observed. As long as the constraints relevant to the phase transition remain unimportant, increasing the level of fluctuations, that is increasing the ratio $\langle m \rangle / \sigma$ there will be little, if any evolution of the PDF. This is exactly what is observed for the 2D-XY model: The PDF has the universal form discussed above, independently of the critical exponent along the critical line [16, 23]. This seems consistent with the observed generic behaviour in disparate systems and also appears to be compatible with recent renormalization group calculations on Gaussian interface models with quenched disorder [36]. In the latter a study is made of the PDF for Gaussian interfaces (of which the low temperature phase of the 2D-XY model is an example) in the presence of disorder. This is shown to be highly irrelevant, with the PDF being unchanged from that of the underlying Gaussian model.
It remains to quantify what we mean by "little evolution of the PDF". In this paper we have shown that at $T^*(\mathsf{L})$, or $B^*(\mathsf{L})$, dependence on universality class is largely absent. It has been further shown that, while the boundary conditions are important for quantitative comparison, there effects are not very significant. However, we have remained firmly in two-dimensions. Moving to three dimensions will no doubt lead to variations and this will prove an interesting test for our explanation of the observed approximate universality for global fluctuations in correlated systems.

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