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"On se persuade mieux, pour l'ordinaire , par les raisons qu'on a soi-même trouvées que par celles qui sont venues dans l'esprit des autres"(Pascal).

Abstract : In [AO] the authors study Steiner bundles via their unstable hyperplanes and proved that (see [AO], Tmm 5.9) :
A rank n Steiner bundle on $\mathbb{P}^n$ which is $SL(2,\mathbb{C})$ invariant is a Schwarzenberger bundle.
In this note we give a very short proof of this result based on Clebsch-Gordon problem for $SL(2,\mathbb{C})$-modules.

1 Introduction

Let A, B, and C three vector spaces over $\mathbb{C}$ and $\phi : A\otimes B \to C^*$ a linear surjective map. We consider the sheaf $\mathcal{S}_\phi$ on $\mathbb{P}(A)\times\mathbb{P}(B)$ defined by,

$$0 \longrightarrow \mathcal{S}_\phi \longrightarrow C\otimes O_{\mathbb{P}(A)\times\mathbb{P}(B)} \xrightarrow{\ {}^t\phi\ } O_{\mathbb{P}(A)\times\mathbb{P}(B)}(1,1)$$

with fibers $\mathcal{S}_\phi(a\otimes b) = \{c\in C \mid \phi(a\otimes b)(c) = 0\}$.

Remark 1. If $\dim_{\mathbb{C}}C < \dim_{\mathbb{C}}A + \dim_{\mathbb{C}}B - 1$ then $\ker(\phi)$ meets the set of decomposable tensors, so there exist $a\in A$, $a\neq 0$ and $b\in B$, $b\neq 0$ such that $\phi(a\otimes b) = 0$.

Remark 2. When $\mathcal{S}_\phi$ is a vector bundle it gives two "associated" Steiner bundles S_A on $\mathbb{P}(A)$ and S_B on $\mathbb{P}(B)$ after projections (see [DK], prop. 3.20).

We denote by $(\mathbb{P}(A)\times\mathbb{P}(B)\times\mathbb{P}(C))^\vee$ the variety of hyperplanes tangent to the Segre $\mathbb{P}(A)\times\mathbb{P}(B)\times\mathbb{P}(C)$ and by $(\{a\}\times\{b\}\times\mathbb{P}(C))^\vee$ the set of hyperplanes in $\mathbb{P}(A\otimes B\otimes C)$ containing $\{a\}\times\{b\}\times\mathbb{P}(C)$.

The next proposition is a reformulation of many results from, [GKZ] (see for instance Thm 3.1' page 458, prop 1.1 page 445), [AO] (see thm page 1) and [DO] (see cor.3.3).

Proposition 1.1. *Let A, B, and C three vector spaces over $\mathbb{C}$ with $\dim_{\mathbb{C}}A = n+1$, $\dim_{\mathbb{C}}B = m+1$ and $\dim_{\mathbb{C}}C \geq n+m+1$ and $\phi : A\otimes B\to C^*$ a surjective linear map. Then the following propositions are equivalent :*
1) $\mathcal{S}_\phi$ is a vector bundle over $\mathbb{P}(A)\times\mathbb{P}(B)$.
2) $\phi(a\otimes b)\neq 0$ for all $a\in A$, $a\neq 0$ and $b\in B$, $b\neq 0$.
3) $\phi\notin(\{a\}\times\{b\}\times\mathbb{P}(C))^\vee$ for all $a\in A$, $a\neq 0$ and $b\in B$, $b\neq 0$.
4) $\phi\notin(\mathbb{P}(A)\times\mathbb{P}(B)\times\mathbb{P}(C))^\vee$

Remark 3. When $\dim_{\mathbb{C}}C = \dim_{\mathbb{C}}A + \dim_{\mathbb{C}}B - 1$ the variety $(\mathbb{P}(A)\times\mathbb{P}(B)\times\mathbb{P}(C))^\vee$ is an hypersurface in $\mathbb{P}((A\otimes B\otimes C)^\vee)$. This hypersurface is defined by the vanishing of the hyperdeterminant, say $Det(\Phi)$ where Φ is the generic tridimensional matrix (see [GKZ], chapter 1 and 14).

Proof. It is clear that 1) 2) and 3) are equivalent. It remains to show that 3) and 4) are equivalent too.
Since $\partial(abc) = (\partial(a))bc + a(\partial(b))c + ab(\partial(c))$ an hyperplane H is tangent to the Segre in a point (a, b, c) if and only if it contains $\mathbb{P}(A) \times \{b\} \times \{c\}$ and $\{a\} \times \mathbb{P}(B) \times \{c\}$ and $\{a\} \times \{b\} \times \mathbb{P}(C)$. We prove here that the third condition implies the two others. Let H an hyperplane containing $\{a\} \times \{b\} \times \mathbb{P}(C)$, we show that there exists $c \in C$ such that H contains $\mathbb{P}(A) \times \{b\} \times \{c\}$ and $\{a\} \times \mathbb{P}(B) \times \{c\}$. Let ϕ the trilinear application corresponding to H. Since $\phi(a \otimes b)(C) = 0$ we have a $\dim_{\mathbb{C}} C$-dimensional family of bilinear forms vanishing on (a, b). Now finding a bilinear form of the above family (i.e. finding $c \in C$) which verify $\phi(A \otimes b)(c) = 0$ and $\phi(a \otimes B)(c) = 0$ imposes at most $n+m$ conditions. Since $\dim_{\mathbb{C}} C \geq n + m + 1$, this point c exists. $\square$

2 Invariant tridimensional matrix under $SL(2, \mathbb{C})$-action

In the second part of this note we will consider the boundary case

$$\dim_{\mathbb{C}} C = \dim_{\mathbb{C}} A + \dim_{\mathbb{C}} B - 1$$

Then, instead of writing ϕ induces a vector bundle or $\phi \notin (\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))^{\vee}$ we will write equivalently $Det(\phi) \neq 0$.
We denote by S_i the irreducible $SL(2, \mathbb{C})$-representations of degree i and by $(x^{i-k}y^k)_{k=0,\cdots,i}$ a basis of S_i.

Theorem 2.1. *Let A, B and C be three non trivial $SL(2, \mathbb{C})$-modules with dimension $n+1$, $m+1$ and $n+m+1$ and $\phi \in \mathbb{P}(A \otimes B \otimes C)$ an invariant hyperplane under $SL(2, \mathbb{C})$. Then,*

$$Det(\phi) \neq 0 \Leftrightarrow \phi \text{ is the multiplication } S_n \otimes S_m \rightarrow S_{n+m}$$

Proof. When $\phi \in \mathbb{P}(S_n \otimes S_m \otimes S_{n+m})$ is just the multiplication $S_n \otimes S_m \rightarrow S_{n+m}$ it is well known that it corresponds to Schwarzenberger bundles (see [DK], prop 6.3).
Conversely, let $A = \oplus_{i \in I} S_i \otimes U_i$, $B = \oplus_{j \in J} S_j \otimes V_j$ where U_i, V_j are trivial $SL(2, \mathbb{C})$-representations of dimension n_i and m_j. Let $x^i \in S_i$, $x^j \in S_j$ be two highest weight vectors and $u \in U_i$, $v \in V_j$. Since $Det(\phi) \neq 0$, $\phi((x^i \otimes u) \otimes (x^j \otimes v)) \neq 0$ and by $SL(2, \mathbb{C})$-invariance $\phi((x^i \otimes u) \otimes (x^j \otimes v)) = x^{i+j} \phi(u \otimes v) \in S_{i+j} \otimes W_{i+j}$. By hypothesis $\phi(u \otimes v) \neq 0$ for all $u \in U_i$ and $v \in V_j$ so, by the Remark 1, it implies that $\dim W_{i+j} \geq n_i + m_j - 1$, and $S_{i+j}^{n_i+m_j-1} \subset C^*$.
Assume now that B contains at least two distinct irreducible representations. Let i_0 and j_0 the greatest integers in I and J. We consider the submodule B_1 such that $B_1 \oplus S_{j_0}^{m_{j_0}} = B$. Then the restricted map $A \otimes B_1 \rightarrow C^*$ is not surjective because the image is concentrated in the submodule C_1^* of C^* defined by $C_1^* \oplus S_{i_0+j_0}^{n_{i_0}+m_{j_0}-1} = C^*$. Now since

$$\dim_{\mathbb{C}}(C_1) < \dim_{\mathbb{C}}(A) + \dim_{\mathbb{C}}(B_1) - 1$$

there exist $a \in A$, $b \in B_1 \subset B$ such that $\phi(a \otimes b) = 0$. A contradiction with the hypothesis $Det(\phi) \neq 0$.

So $A = S_i^{n_i}$, $B = S_j^{m_j}$ and $S_{i+j}^{n_i+m_j-1} \subset C^*$. Since $\dim_{\mathbb{C}} C = \dim_{\mathbb{C}} A + \dim_{\mathbb{C}} B - 1$, we have $(i+1)n_i + (j+1)m_j - 1 = \dim_{\mathbb{C}} C \geq (i+j+1)(n_i + m_j - 1)$ which is possible if and only if $n_i = m_j = 1$ and $C = S_{i+j}$. $\square$

Corollary 2.2. *A rank n Steiner bundle on $\mathbb{P}^n$ which is $SL(2,\mathbb{C})$ invariant is a Schwarzenberger bundle.*

Proof. Let S a rank n Steiner bundle on $\mathbb{P}^n$, i.e S appears in an exact sequence

$$0 \longrightarrow S \longrightarrow C \otimes O_{\mathbb{P}(A)} \longrightarrow B^* \otimes O_{\mathbb{P}(A)}(1) \longrightarrow 0$$

where $\mathbb{P}(A) = \mathbb{P}^n$, $\mathbb{P}(B) = \mathbb{P}^m$ and $\mathbb{P}(C) = \mathbb{P}^{n+m}$. If $SL(2,\mathbb{C})$ acts on S the vector spaces A, B and C are $SL(2,\mathbb{C})$-modules since A is the basis, $B^* = H^1 S(-1)$ and $C^* = H^0(S^*)$. If S is $SL(2,\mathbb{C})$-invariant the linear surjective map

$$A \otimes (H^1 S(-1))^* \to H^0(S^*)$$

is $SL(2,\mathbb{C})$-invariant too. $\square$

Remark. The proofs of the theorem and the proposition, given in this paper, are still valid for more than three vector spaces when the format is the boundary format.

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