Analysis and optimisation of the tuning of the twelfths for a clarinet resonator

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Abstract

Even if the tuning between the first and second register of a clarinet has been optimized by instrument makers, the lowest twelfths remain slightly too large (inharmonicity). In this article, we study the problem from two different points of view. First, we systematically review various physical reasons why this inharmonicity may take place, and the effect of different bore perturbations inserted in cylindrical instruments such as bore flare, open and closed holes, taper, temperature gradients, visco-thermal effects, etc. Applications to a real clarinet resonator and comparisons with impedance measurements are then presented. A commonly accepted idea is that the register hole is the dominant cause for this inharmonicity: it is natural to expect that opening this hole will shift the position of the resonances of the instrument to higher frequencies, except of course for the note for which the hole is exactly at the pressure node. We show that the real situation is actually more complicated because other effects, such as open holes or bore taper and bell, introduce resonance shifts that are comparable but with opposite sign, so that a relatively good overall compensation takes place. This is checked by experimental and theoretical studies of the acoustical impedance curves of a clarinet. The origin of the observed inharmonicity in playing frequencies is therefore different, maybe related to the reed or the vocal tract. In a second part, we use an elementary model of the clarinet in order to isolate the effect of the register hole: a perfect cylindrical tube without closed holes. Optimization techniques are then used to calculate an optimum location for the register hole (without taking into account the use of the register hole as a B flat tone hole); the result turns out to be close to the location chosen by clarinet makers. Finally, attempts are made numerically to improve the situation by introducing small perturbations in the higher part of the cylindrical resonator, but no satisfactory improvement is obtained.

Key words: clarinet, inter-register inharmonicity, register hole, length corrections.

1 Introduction

Many contributions, either experimental or theoretical, have been made in the past to improve our understanding of single reed woodwind instruments (see e.g.[1,2,3,4]). Our knowledge of the linear behaviour of the resonator is now satisfactory, which probably explains why most of the recent literature deals with the understanding of the sound production and oscillation regimes, an inherently non-linear problem. This does not mean that all interesting questions concerning the linear behaviour of the resonator have been solved. For instance, Benade [5] proposed in the seventies some basic ideas and methods allowing the qualities of a wind instrument to be characterised (impedance peak alignment), but more detailed explorations along this line would be useful; Meynial and Kergomard [6] proposed relatively recently simple acoustical systems that shift the scale of a woodwind by a given micro-interval. Concerning the relation between linear properties and non-linear oscillations, ref.[7] discusses in general how it is possible to predict the emission frequencies, and even some aspects of the clean intonation and tone colour, which naturally leads to the design of modified instruments as soon as appropriate optimization criteria are defined. Recently, optimization techniques have been used in order to define longitudinal profiles of brass instruments [8].

An example of an unsolved problem is given by the clarinet. While many musicians agree that it is now a well tuned instrument, the twelfths corresponding to the four lowest tones remain slightly too large, by 20 or 30 cents [9,10,11]. One could imagine many reasons for this problem: deviation of the bore from purely cylindrical shape, existence of open or closed side holes (cavities), temperature gradient, etc. A generally accepted idea states nevertheless that the main reason resides in the register hole, because it cannot be ideally placed for every fingering.

The first purpose of the present article is to analyze the origin of this tuning problem in detail. We will actually see that the occurrence of wide twelfths at either end of the clarinet register is not an obvious problem in terms of linear impedance theory, in contradiction with common wisdom. In real clarinets, various effects of opposite signs tend to compensate each other, at least partially (role of the flaring bell for instance). The second purpose of the article is to investigate whether or not it is possible to design a register hole allowing the first two complete registers to be perfectly well tuned (the definition of the registers will be given in the next section), ignoring all the other sources

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for inharmonicity except the register hole. We will then reason on a simplified shape, cylindrical and without tone holes, the different values of the tones being adjusted simply by choosing different lengths for the tube. Two questions will be discussed :

i) what is the optimum location of the register hole for the tuning of the twelfths intervals between the first two registers?
ii) is it possible to propose some simple system, located upstream from the highest tone hole, to provide a good correction to the residual tuning defaults?

The outline of this paper is as follows: the second section recalls some general features of the clarinet. The third section is devoted to the first study: it analyzes the length corrections and their variation with frequency for the main kinds of discontinuities encountered in wind instruments. It ends with an application to a real clarinet resonator in order to find whether the register hole is the main culprit of the clarinet tuning defect; calculations are compared with measurements. A validation of the calculations to first order in the ratio of the length correction to the wavelength is also given as well as a complete analysis of the effects of an open and closed tone holes. Then, sections 4 and 5 are concerned with the second part of our study and answer questions i) and ii) respectively. Some useful formulae are given in appendix A.

2   Generalities on the clarinet

2.1   Definitions

For a given fingering, several resonance frequencies of the resonator occur, and several oscillation regimes can be obtained. The term register is used to describe the set of tones obtained for the same regime: the *first register* involves the tones corresponding to the first mode (i.e. the first resonance frequency) of the resonator, the frequencies being denoted f_1 , and the *second register* involves the tones corresponding to the second mode, the frequencies being denoted f_2 (where f_2 is approximately $3f_1$). The opening of the register hole allows the musician to "jump" from a given tone of the first register to the corresponding tone of the second register, one octave and a fifth above, the first two registers covering ideally more than 3 octaves [1] . For the fingerings

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[1]   We notice that while this division into two registers seems obvious to the scientist, the analysis of the characteristics of the tone colour is not so clear, and musicians divide this musical compass in four registers: chalumeau, throat, clarinet and extrem [9].

corresponding to the higher tones of the first register, i.e. for fingerings from $f'\#$ to $a'\#^2$, the opening of the register hole is not sufficient in order to get the proper twelfth, and it is necessary to modify slightly the fingering in order to ensure the good tuning. Thus, in order to play the corresponding twelfths (i.e tones from $c'''\#$), the third resonance, corresponding to an interval of a 17th, is used, corresponding to that the physicist can call the third register (see figure 1).
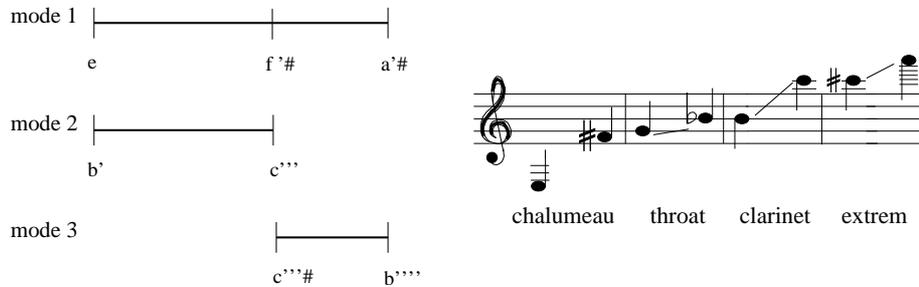

Fig. 1. *Compass and operative resonance frequency for a clarinet.*

The most commonly used clarinet is the B-flat clarinet which sounds one tone below the written note. Five parts can be distinguished on a clarinet: the mouthpiece, the barrel, the upper and lower joints and the bell. To first approximation, the clarinet is a cylindrical instrument, but a detailed determination of the geometrical dimensions of the instrument reveals deviations from this regular, cylindrical shape (see figure 2) over parts which are short with respect to the wavelength. Moreover, the existence of tone and register holes, bore irregularities, thermal gradient and dispersion due to visco-thermal effects alter the resonance frequencies and their harmonicity; they must be included for a detailed study of the tuning of the instrument.

2.2   Inharmonicity and sound production

The sound of the clarinet is produced by self-sustained oscillations. Theory shows [13] that, at low sound intensity, the playing frequencies are determined by the zero of the imaginary part of the input admittance of the resonator (they are therefore very close to the frequencies of the maxima of the input impedance modulus). In addition, the reed has also a small influence on the playing frequency, for two different reasons: the volume velocity created by the motion of the reed movement, which adds to that produced by the pressure difference across the reed opening ; the frequency pulling effect of the resonance of the reed (at higher frequencies) combined with its damping. It can

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[2] The notation system adopted in this paper is the same as that used by Baines [12].
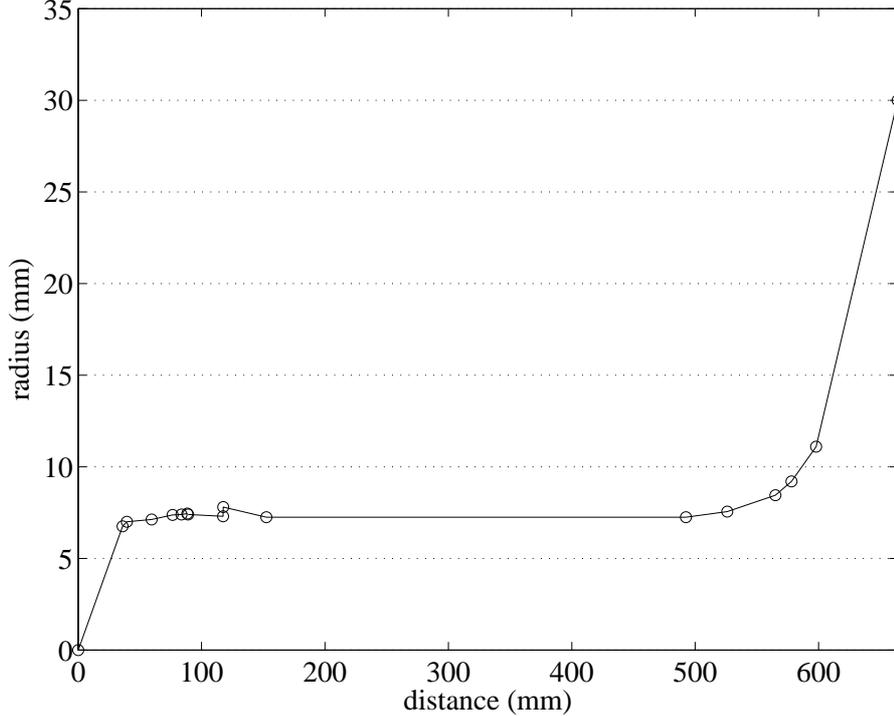
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Fig. 2. *Radius of a clarinet as a function of the distance from the reed tip (see table 3).*

be shown that these two effects can be taken into account by introducing appropriate length corrections to the instrument, which are almost independent of the frequency [1,7]. At higher sound intensities, the playing frequency also depends on other impedance peaks; it can slightly change under the effect of the inharmonicity of the resonances of the resonator. A proper harmonicity of the first two resonances is therefore important (even for a single note); then, the sound frequency remains independent of the playing level, even if the first two resonances have similar magnitude, and good "impedance peak cooperation" [14] ensures a stable sound emission. This question is actually intricate because it involves the influence of many parameters. Here, for the sake of simplicity we limit ourselve to low intensity sounds; in other words, our goal will just be to achieve a perfect tuning at least for pianissimo levels.

In this article, we will distinguish between *inter-resonance* inharmonicity and *inter-register* inharmonicity. The former refers to the inharmonicity between the first two resonance frequencies of the resonator for a given fingering; the latter defines the ratio between the resonance frequencies of the first impedance peak when the register hole is closed and the second one when the register hole is open [3]

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[3] The targeted interval is a pure twelfth, i.e. almost a ratio of 3. Actually for a tempered scale, the ratio is slightly different, because the tempered intervals are different from the natural ones: the exact value is $2^{19/12} = 2.9966$. The relative difference is 0.11%, i.e. 2 cents (it is the difference between the tempered fifth and

The shape of a clarinet is very close to a pure cylinder. For a given fingering of the chalumeau, with first resonance frequency f_1, an effective length ℓ_{eff} can be defined as

$$\ell_{eff} = c/4f_1 \ ,$$

where c is the speed of sound. A first approximation of this length is given by the physical length between the input of the instrument (taking into account a length correction for the above mentionned reed effects) and the first open tone hole position, where the tube is assumed to be cut. All the perturbations from the cylindrical shape, such as enlargments or contractions, closed tone holes, as well as the precise effects of the open tone hole (depending on the downstream geometry and radiation property), can be regarded as small corrections to this first approximation leading to the correct value, ℓ_{eff}. For the lowest tone, for which all holes are closed, an equivalent length of the flaring horn is used as in [1]. If the resonator was purely cylindrical and lossless, the second resonance frequency would be exactly $f_2 = 3f_1$. However all the perturbations produce corrections slightly different from those on the first resonance: this creates the inter-resonance inharmonicity (for inter-register inharmonicity, the opening of the register hole also needs to be taken into account). Here, we are actually concerned not by the effect itself of the perturbations from the cylindrical shape, but by the variation of this effect between the two registers.

The concept of frequency dependent length corrections is an adequate tool for this study. As in reference [6] the corrections upstream of the first open hole can be calculated separately for each kind of perturbation, with respect to the effective input of the tube (taking into account the reed effects) where the imaginary part of the admittance must be zero for self-sustained oscillations. The frequency dependence of the length corrections for all fingerings provides the inharmonicity. Only the effect of the part that is upstream the first open hole needs to be studied for each fingering.

All calculations are carried out by ignoring the different kinds of dissipation (due to visco-thermal effects in the boundary layers, to radiation, etc...): weak dissipation is known to have a negligible effect on the resonance frequencies [16]. In Appendix B the effect of the resistance of a small hole is discussed, and even if nonlinear effects are taken into account, it is shown to be negligible. A consequence is the systematic use of purely imaginary impedances, which means for the resonance condition, that the input impedance is infinite. Perturbation to the planar mode theory is classically taken into account using

the harmonic one, called the "skhisma"). Nevertheless this difference is very small and for simplicity in what follows we will ignore it (2 cents are almost inaudible, and within the "tunability range", especially for higher notes [15]).

lumped elements representing the effects of higher order, evanescent modes of the tubes.

3 Length correction and inharmonicity produced by a small perturbation: analysis of the different effects for a real clarinet resonator

In this section we study the analytic expressions of length corrections and inharmonicity of the first two resonance frequencies associated with small perturbations to a purely cylindrical shape. This formulation gives a good idea of the inharmonicity encountered on a real clarinet resonator.

3.1 Length corrections: definition

The effect of a geometrical perturbation on a cylindrical resonator may be expressed conveniently in terms of a length correction to the main tube, denoted $\Delta\ell$. Using an exact formulation of $\Delta\ell$ is possible but in this study, we use an approximation of length corrections to first order in the ratio of the length correction to the wavelength: this gives a sufficiently accurate determination of the resonance frequencies. Moreover, the length corrections associated with different perturbations can be simply added as we will see in a particular example. The perturbation is located at a distance ℓ from the effective input, which is the origin of the coordinates once the clarinet embouchure has been replaced by an equivalent cylinder and the reed effects have been taken into account (see 3.5.1).

3.2 Inharmonicity of the resonance frequencies

Inharmonicity can be defined as the relative difference between the resonance frequency f_n and n times the first resonance f_1, as follows:

$$IH = \frac{f_n - nf_1}{nf_1} = \frac{\ell_{eff} + \Delta\ell_1}{\ell_{eff} + \Delta\ell_n} - 1 = -\frac{\Delta\ell_n - \Delta\ell_1}{\ell_{eff}} + o(\frac{\Delta\ell}{\ell_{eff}}) \,, \qquad (1)$$

where ℓ_{eff} is the acoustic length of the unperturbed system and $\Delta\ell_n$ is the length correction associated to the nth resonance.
In the case $IH > 0$ the basic intervals are enlarged; when $IH < 0$ the intervals are reduced. In the present paper, results are given in *cent*; the cent is the

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micro-interval equal to one hundredth of a tempered semi-tone:

$$1 \ cent = 5.78 \ 10^{-4} \quad \text{and} \quad IH)_{cents} = \frac{IH}{5.78 \ 10^{-4}} \ .$$

We notice that the smallest frequency deviation perceptible by the human ear, estimated to be 4 cents, corresponds to $IH \simeq 0.25\%$.

3.3 Length corrections and inharmonicity formulae for acoustic basic systems

This section contains analysis of inharmonicity associated with the simple acoustical systems usually encountered in a clarinet, and depicted in table 1. From the calculation of first order length corrections[4], analytical expressions of inharmonicity between the first and second resonance frequencies are derived. For exact formulations of length corrections, the reader is referred to appendix A.

3.3.1 Insertion of a side hole

This section uses results obtained by Meynial and Kergomard [6] concerning the effects of the insertion of an admittance branched in parallel and an impedance branched in series, on a cylindrical tube. Starting from the tone hole model proposed by Dubos *et al.* in [17], we propose a new calculation of an equivalent length that includes the effects of both the series and shunt impedances. Consequences for the cases of a register hole, an open side hole and a closed hole are subsequently discussed.

General formulation for an admittance in parallel and an impedance in series We consider a side hole, or any discontinuity inserted in parallel, branched on a straight cylindrical tube (a typical example is the register hole shown in table 1). Y is the admittance of the inserted discontinuity [5] and Y_{up} and Y_{down} are the main tube admittances upstream and downstream the discontinuity respectively. The following equation can be written at the location of the discontinuity $(x = \ell)$,

$$Y_{up} = Y + Y_{down} \ , \tag{2}$$

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[4] The expression *first order length corrections* refers to length corrections expanded to first order in the ratio of the length correction to the wavelength.
[5] Throughout the paper the admittance is defined as a ratio of an acoustic volume velocity to an acoustic pressure.

register hole	ℓ → $\ell + \Delta\ell$
tone hole	ℓ_d → $\ell_d + \Delta\ell$
closed-hole	ℓ → $\ell + \Delta\ell$
abrupt change in cross section in the upper part of the instrument	S' ℓ S → $\ell + \Delta\ell$ S
localized enlargement/contraction in the upper part of the instrument	ℓ' S' ℓ S → $\ell + \ell' + \Delta\ell$ S
truncated cone in the upper part of the instrument	s' ℓ ℓ_c S → $\ell + \ell_c + \Delta\ell$ S

Table 1

Acoustic basic systems and associated length corrections.

Looking backwards to the top-end of the resonator and writing $Y_{down} = -jY_c \tan k(\ell + \Delta\ell)$, we get from equation (2):

$$-jY_c \tan k\ell = Y - jY_c \tan k(\ell + \Delta\ell) \, , \qquad (3)$$

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where $\Delta\ell$ is the length correction, $Y_c = S/\rho c$ is the characteristic admittance of the main tube (ρ is the density of air and S the cross section area of the main tube), $k = 2\pi f/c$ is the wavenumber and $j = \sqrt{-1}$. After some algebra, the following result is obtained:

$$k\Delta\ell = \text{Arctan}\left(\frac{-j\frac{Y}{Y_c}\cos^2 k\ell}{1 - \frac{1}{2}j\frac{Y}{Y_c}\sin 2k\ell}\right) \quad , \tag{4}$$

which can be approximated in the limit of small Y/Y_c by:

$$k\Delta\ell \simeq -j\frac{Y}{Y_c}\cos^2 k\ell \quad . \tag{5}$$

For the case of a series impedance Z branched on a cylindrical tube, similar expressions are found to be, respectively:

$$k\Delta\ell = \text{Arctan}\left(\frac{-j\frac{Z}{Z_c}\sin^2 k\ell}{1 + \frac{1}{2}j\frac{Z}{Z_c}\sin 2k\ell}\right) \quad , \tag{6}$$

and

$$k\Delta\ell \simeq -j\frac{Z}{Z_c}\sin^2 k\ell \quad . \tag{7}$$

Formulae (5) and (7) can also be deduced from Rayleigh's variational principle [18]. They are useful for different kinds of small discontinuities (see reference [6]), but for some particular cases, other formulae need to be derived as we see below.

Side hole Referring to [17] for the tone hole model, the acoustic pressure and the acoustic volume velocity at both sides of the hole can be related by the following transfer matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{1 - Y_s Z_a/4}\begin{pmatrix} 1 + Y_s Z_a/4 & Z_a \\ Y_s & 1 + Y_s Z_a/4 \end{pmatrix} \quad , \tag{8}$$

where Z_a and Y_s are the series impedance and shunt admittance respectively. This formulation emphasizes the dual role of the two terms Z_a and Y_s, as opposed to formulations based on the classical T or Π equivalent circuits. Manipulation of equation (8) gives the length correction:

$$-j\,Y_c\,\tan k(\ell + \Delta\ell) = \frac{-Y_s + DY_\ell}{A - Z_a Y_\ell} \quad , \tag{9}$$

where $Y_\ell = -j\,Y_c\,\tan k\ell$. Finally:

$$k\Delta\ell = \text{Arctan}\left(\frac{-jY_s/Y_c\cos^2 k\ell - jZ_a/Z_c\sin^2 k\ell}{1 + Y_s Z_a/4 + j(Z_a/Z_c - Y_s/Y_c)\sin k\ell\cos k\ell}\right) \quad . \tag{10}$$

As a result, equation (10) is a general expression for the length correction of a tube branched off a cylindrical tube. It gives the influence of both the series and shunt impedances in the determination of the equivalent length. No assumption has been made concerning the dimensions and shape of the hole: it is therefore possible to derive first order approximations for special cases as tone hole, register hole or closed hole from the relative influence of Z_a and Y_s. For instance, in the case of an open hole, both Z_a and Y_s are inductive. With the low frequencies approximation, equation (10) shows that $\Delta\ell$ is mostly determined by the shunt admittance. If $Y_s/Y_c \ll 1$, the length correction is small and equation (10) gives equation (5); this is the case of the register hole.

Open hole: register hole The register hole is a side chimney with small diameter but large height located a distance $\ell \simeq 140\ mm$ down the effective input (typically $r = 1.5\ mm$, $h = 13\ mm$). Using [17] to calculate the elements of the model, we obtain the series impedance as:

$$Z_a = j\omega\rho\ell_a^{(o)}/S \ , \tag{11}$$

where $\ell_a^{(o)}$ is the series length correction for an open hole. The shunt impedance is:

$$Z_s = Y_s^{-1} = j\frac{\rho c}{S_h}\left(\tan[k(h + h_m + h_r) + jkh_s]\right) \ , \tag{12}$$

where S_h is the cross-sectional area of the hole, and h_m, h_s and h_r are height corrections associated with the matching volume, the higher order modes impedance and radiation, respectively. Expressions for these lengths can be found in [17,19] (see also [20] for a correction to [17] for h_s) and are given hereafter:

$$\ell_a^{(o)} = -r\delta^2[1.78\tanh(1.84(h + h_m)/r) + 0.940 + 0.540\delta + 0.285\delta^2]^{-1} \tag{13}$$

$$h_m = \frac{r\delta}{8}(1 + 0.207\delta^3) \tag{14}$$

$$h_s = r(0.82 - 0.193\delta - 1.09\delta^2 + 1.27\delta^3 - 0.71\delta^4) \tag{15}$$

$$h_r = 0.82r\left(\frac{1 + (0.77kr)^2}{1 + 0.77kr}\right)^{-1} \tag{16}$$

where $\delta = r/R$, k is the wavenumber of the played tone and the subscript o is used to refer to the open hole case. Since the low frequency limit is valid, the shunt impedance is well approximated by:

$$Z_s = j\frac{\rho c}{S_h}kh' \ , \tag{17}$$

where:

$$h' = h + h_m + h_s + h_r \ . \tag{18}$$

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This equation allows us to discuss the relative influence of the series and shunt impedances. Calculation gives $\mid Y_s Z_a \mid \simeq 0.0045\%$ which confirms that the series impedance can be ignored in the case of a long chimney with a small diameter. Finally, equations (10) is identical to (5), which provides:

$$k\Delta\ell \simeq -\frac{S_h\ell}{Sh'}\frac{\cos^2 k\ell}{k\ell} \ , \tag{19}$$

where ℓ is the distance of the hole from the effective input. Equation (19) shows that a small open side hole always introduces a negative length correction, i.e. an increase in the resonance frequencies. It also shows that $\Delta\ell$ decreases as the inverse of the square of frequency. Therefore, the register hole opening affects the first resonance frequencies much more than the second one, which is an important requirement for the clarinet to overblow correctly [21].

From equation (19), since the register hole has no effect on tones of the first register for which it is closed, the frequency shift of the second resonance frequency due to the register hole opening can be derived. We then get:

$$IH \simeq -\frac{\Delta\ell_2}{\ell_{eff}} \simeq \frac{2}{3\pi}\frac{S_h}{Sh'}\frac{cos^2 k_2\ell}{k_2} \ , \tag{20}$$

where $k_2 = 3k_1$, k_1 being the wavenumber of the fundamental frequency of the played tone. This expression shows that the register hole opening generates positive inharmonicity by pulling the second vibration mode upward in frequency at both ends of the register scale and has no effect at a pressure node as it is well known (see figure 3 where $r = 1.4 \ mm$, $h = 13 \ mm$ and $\ell = 140 \ mm$). It also appears that the two geometrical parameters which control inter-resonance inharmonicity are Sh'/S_h and ℓ.

Open hole: tone hole   In the limit of zero frequency, the shunt admittance of an open hole increases to infinity (see equation (5)) and the main tube behaves as if it was cut at the location of the hole. The ratio Y_s/Y_c becomes large for tone holes and equation (10) can no longer be used since the perturbation cannot be considered as small anymore. As a consequence, $\Delta\ell$ is no longer seen as an extension of the entire air column of length ℓ_{eff} but as an additional length to ℓ the distance from the effective input to the hole location. For an open hole of height h' (see eq.(18)), radius r at distance ℓ_d from the open end, the length correction is now obtained from:

$$-j\,Y_c \cotan k\Delta\ell = \frac{Y_s + DY_\ell}{A + Z_aY_\ell} \ , \tag{21}$$

where now $Y_\ell = -j\,Y_c\,\cot k\ell_d$. Thus:

$$k\Delta\ell = \text{Arctan}\left(\frac{-jY_c[(1 + Z_aY_s/4) - jZ_aY_c\cot k\ell_d]}{Y_s - (1 + Z_aY_s/4)jY_c\cot k\ell_d}\right) \ . \tag{22}$$
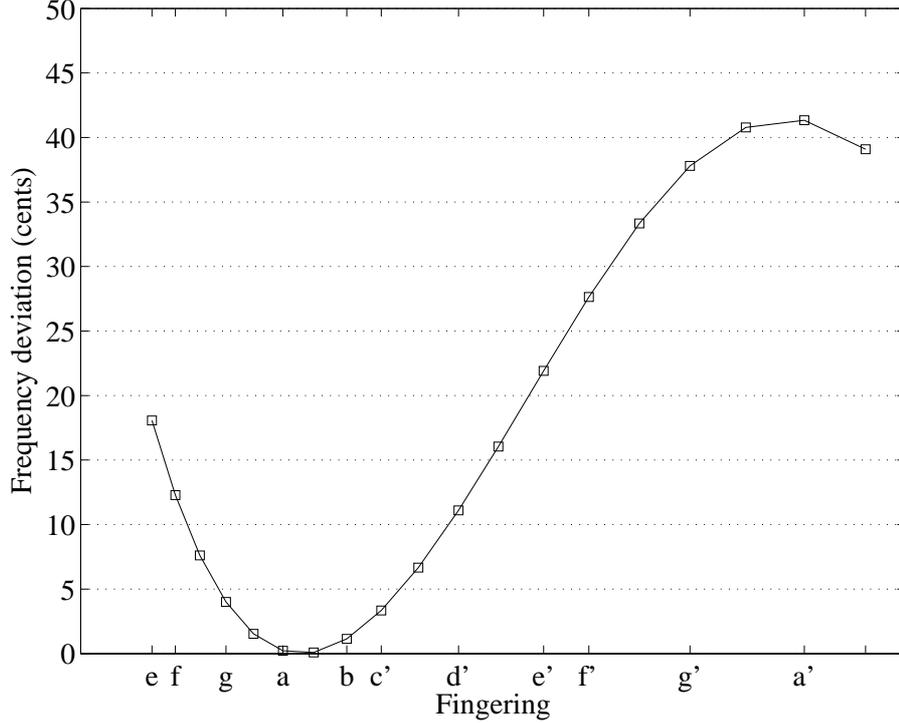
Fig. 3. *Shift of the second resonance frequency due to a register hole of radius r = 1.4 mm, height h = 13 mm, located a distance ℓ = 140 mm from the effective input. For the b flat fingering, the register hole does not alter the second vibration mode since, in this case, the register hole is located at one third the effective length.*

For a real clarinet using the expressions for Z_a, Y_s and Y_c for an open hole, we see that the quantity $\mid Z_a Y_s/4 \mid$ is small compared to unity, except for the four bottom tones for which the order of magnitude is to 0.02 . In the same way, the term in bracket at the numerator is close to unity. This leads us to assume that Z_a has a negligeable effect in this study for open holes especially since we focus on inharmonicity. Setting the term in bracket to unity in equation (22) and using Taylor's formula to the third order in $k\ell_d$ and to the first order in $k\Delta\ell$, the tone hole equivalent length becomes:

$$\Delta\ell \simeq \frac{1}{q + \frac{1}{\ell_d(1+(k\ell_d)^2/3)}} \ , \tag{23}$$

where $q = r^2/(R^2 h')$ is relative to the geometry of the tone hole. From equation (23),the length correction becomes a constant value in the lower frequency limit given by $\Delta\ell = \ell_d\ell_{hole}/(\ell_d+\ell_{hole})$,where $\ell_{hole} = h'S/S_h$. When frequency increases, the tone hole length correction increases too. Then, any upper resonance frequency relative to the lowest one is flattened, and negative inharmonicity is generated as it can be seen in figure 4 (for a tone hole geometry given by $r = 4.5$ *mm*, $h = 4$ *mm* located, for each length of tube, at a distance $\ell_d = 30$ *mm* from the open end).
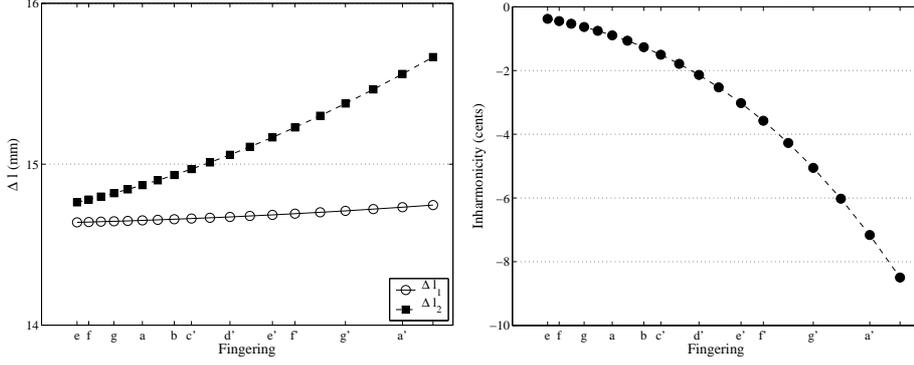
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Fig. 4. *Length corrections (left) and corresponding inter-resonance inharmonicity (right) for a tone hole of radius $r = 4.5$ mm, height $h = 4$ mm, located at a distance $\ell_d = 30$ mm from the open end for each tube length.*

Concerning the inter-resonance inharmonicity, equation (23) leads to the expression

$$IH \simeq -\frac{16}{3\pi} \frac{k_1^3 \ell_d^3}{(1 + q\ell_d)^2} \; . \tag{24}$$

Discrepancy between equation (24) and numerical calculations from equation (22) gives less than a 4-cent difference for all fingerings except for three tones, $c'\#$, f' and g' for which results differ of an amount to about 8 cents. This appears to be due to the use of the Taylor expansion of the *cotan* function to obtain equation (23). Finally, equation (24) is a simple analytic formula which also appears reliable, and will be used in the next. It also gives an interesting feature: the freedom it gives the designer to control inharmonicity due to open hole in the choice of the hole size. Large tone holes would be prefered to reduce inharmonicity of the first two resonance frequencies.

Closed hole (or cavity) According to [17], the series and shunt impedances for a closed hole are

$$Z_s = \frac{\rho c}{S_h}[-j \cot k(h + h_m) + jkh_s] \; , \tag{25}$$

and

$$Z_a = j\omega\rho\ell_a^{(c)}/S \; , \tag{26}$$

respectively, where $\ell_a^{(c)}$ is the series equivalent length for a closed tube given by

$$\ell_a^{(c)} = -r\delta^2[1.78\mathrm{cotanh}(1.84(h + h_m)/r) + 0.940 + 0.540\delta + 0.285\delta^2]^{-1} \; . \tag{27}$$

In the low frequency approximation, the shunt impedance becomes

$$Z_s = \frac{\rho c}{S_h} \frac{-j}{k(h + h_m)} \; ,$$

and shows that it behaves as a shunt acoustic compliance $C = v/\rho c^2$ determined by its volume $v = S_h(h + h_m)$. Substituting Z_a and Z_s in equation (10) and denoting $X = v/S\ell$, the length correction to first order is found to be:

$$\Delta\ell \simeq X\ell \ \cos^2 k\ell + \ell_a^{(c)} \sin^2 k\ell \ , \qquad (28)$$

which can be rewritten in

$$\Delta\ell \simeq (X\ell - \ell_a^{(c)}) \cos^2 k\ell + \ell_a^{(c)} \ . \qquad (29)$$

This expression shows that the effect of a closed hole inserted on a cylindrical tube depends mainly on its size and position. It also shows that the change in inertance described by $\ell_a^{(c)}$ appears in the determination of the series closed-hole length correction. The pratical magnitude of this term on the total closed-hole effect is to an amount up to 30% for the bottom holes of a clarinet; this cannot be neglected as it could be found in several papers [22,6]. This term must be taken into account especially when investigating inharmonicity of a real instrument for which many cavities can act for a given fingering. Since $k\ell \ll 1$, the effect of a cavity is proportional to the ratio of the inserted volume to the volume of air included between the reed tip and the closed hole. As a consequence, the virtual volume of air corresponding to the flow induced by the reed movement introduced by Nederveen, does not produce inharmonicity. With the use of equation (29), inter-resonance inharmonicity becomes:

$$\begin{aligned}\Delta\ell_1 - \Delta\ell_2 &= -(\frac{v}{S} - \ell_a^{(c)})(\cos^2 3k_1\ell - \cos^2 k_1\ell) \\ &= 2(\frac{v}{S} - \ell_a^{(c)}) \sin^2 2k_1\ell \cos 2k_1\ell \ ,\end{aligned}$$

so that:

$$IH \simeq \frac{4}{\pi}(\frac{v}{S} - \ell_a^{(c)}) \, k_1 \ \sin^2 2k_1\ell \cos 2k_1\ell \ . \qquad (30)$$

The result depends on two geometrical parameters, the location ℓ and the ratio v/S. Dealing with the case of real clarinet resonator and ignoring the effect of $\ell_a^{(c)}$ can undersestimate the effect of cavities to about 10 cents for the lowest fingerings since each cavity contribution is added. This confirms the necessity to take into account the series closed-hole effect. Besides, as a consequence of the term $\cos 2k_1\ell$, either negative or positive inharmonicity is associated with a closed hole (see figure 5 for a hole volume equal to $v = 0.3 \ cm^3$). Finally, since the magnitude is proportional to the wavenumber k_1, inharmonicity associated with a hole of fixed size and position increases with the fundamental frequency of the played tone.
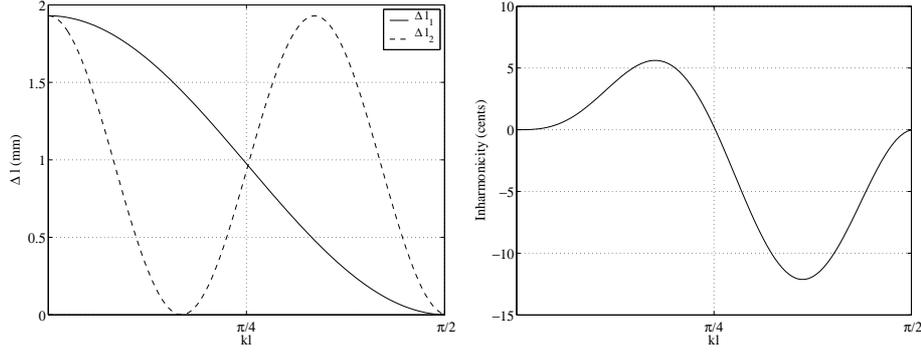
15

Fig. 5. *Length corrections (left) and corresponding inter-resonance inharmonicity (right) for a closed hole whose volume $v = 0.3\ cm^3$ as a function of $k_1\ell$.*

3.3.2 Localized enlargement/contraction

Consider a localized enlargement (or contraction) of length ℓ' located at distance ℓ from the effective input in a cylindrical air column and let $\alpha = S'/S$ be the ratio of the cross section area of the enlargement (or contraction) to the one of the main tube. Assuming α to be close to the unity (only small bore changes are considered), we obtain the change in resonance frequencies expressed through length correction according to (see appendix A eq.(A.2)):

$$k\Delta\ell = (\alpha - 1)\sin k\ell' \cos 2k(\ell + \ell') \ . \tag{31}$$

Then, the relationship between first and second resonance frequencies is:

$$IH \simeq \frac{4}{3\pi}(\alpha - 1)(\sin^3 2k_1(\ell + \ell') - \sin^3 2k_1\ell) \ , \tag{32}$$

which states that the two diameter discontinuities located at points ℓ and $\ell + \ell'$ create either positive or negative inharmonicity. This is shown in figure 6 in the case of an enlargement ($\alpha = 1.04$) with a 10 cm-long inserted tube.
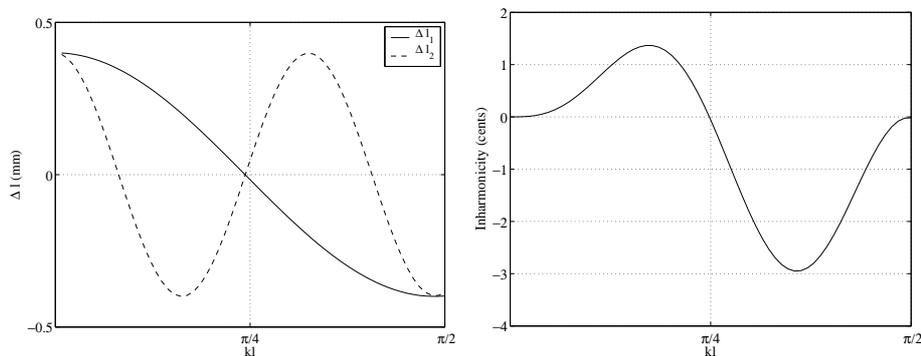

Fig. 6. *Length corrections (left) and corresponding inter-resonance inharmonicity (right) for a localized enlargement as a function of $k_1\ell$ ($\ell' = 10\ mm$, $S' = 1.04\,S$).*

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Substituting $\ell = 0$ in equations (31) and (32), the effects due to a change in cross section in part of a cylindrical tube are calculated. The length correction becomes

$$k\Delta\ell \simeq \frac{1}{2}(\alpha - 1)\sin 2k\ell ,\qquad(33)$$

and the inharmonicity can be written

$$IH \simeq \frac{4}{3\pi}(\alpha - 1)\sin^3 2k_1\ell ,\qquad(34)$$

where the term $\sin 2k_1\ell$ still remains positive since $k_1\ell \in [0 , \pi/2]$. As a consequence, the sign of inharmonicity associated with a discontinuity depends only on the value of α as shown in figure 7 for a bore widening and contraction of 2%.
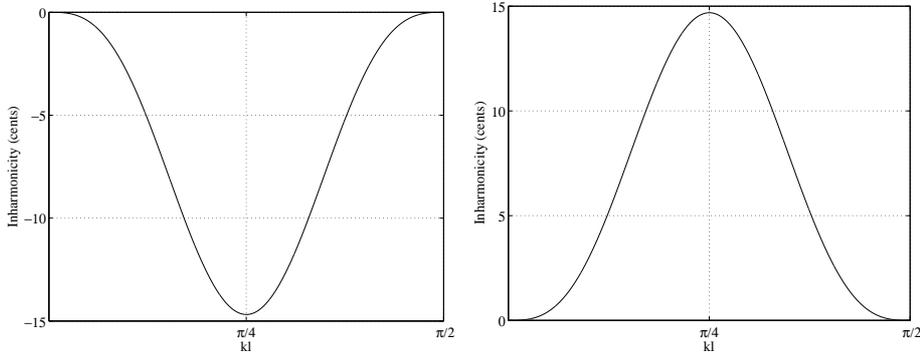

Fig. 7. *Inter-resonance inharmonicity due to an abrupt change in cross section area as a function of $k_1\ell$: contraction $S' = 0.98\,S$ (left) and enlargement $S' = 1.02\,S$ (right).*

3.3.3 Change in taper close to the tube input

The acoustical behaviour of a change in taper over a length ℓ_c can be represented with an equivalent electrical circuit including two inductances of opposite sign and the elements of a cylindrical tube of length ℓ_c [23]. Writing $X_1 = \ell/x_1$ and $X_2 = (\ell + \ell_c)/x_2$, we obtain the length correction calculated to first order in X_i $(i = 1, 2)$ (see appendix A eq.(A.3))

$$k\Delta\ell \simeq X_2 \frac{\cos^2 k(\ell + \ell_c)}{k(\ell + \ell_c)} - X_1 \frac{\cos^2 k\ell}{k\ell} .\qquad(35)$$

Equation (35) states that a single taper change is equivalent to two open side holes with a positive and a negative inertance respectively. It is valid either for a positive or negative taper change, the difference being in the sign of the x_i which are positive for a diverging cone and negative for a converging cone.
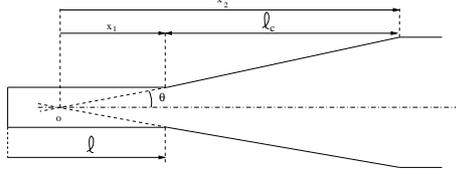
17

Fig. 8. *Geometry and symbol used for the case of a truncated cone.*

In order to evaluate the inharmonicity generated by a small truncated cone, it is convenient to reformulate the expression of the length correction in terms of two control parameters by rewriting equation (35). Since approximations $k\ell_c \ll 1$ and $\ell_c/x_1 \ll 1$ are still valid, it is possible to write:

$$\cos^2 k(\ell + \ell_c) = \cos^2 k\ell - k\ell_c \sin 2k\ell + o(k\ell_c) \ ,$$

and

$$\frac{1}{kx_2} = \frac{1}{kx_1(1 + \ell_c/x_1)} \simeq \frac{1}{kx_1} \ .$$

Therefore a simplified expression for the length correction is derived

$$k\Delta\ell \simeq -\frac{\ell_c}{x_1} \sin 2k\ell \ , \tag{36}$$

where ℓ_c/x_1 and ℓ are the two parameters. Under these conditions and with the use of equation (36), inharmonicity is given by:

$$IH \simeq -\frac{8}{3\pi} \frac{\ell_c}{x_1} \sin^3 2k_1\ell \ . \tag{37}$$

An example of diverging cone of length $\ell_c = 5$ *mm*, large-end radius $R = 7.5$ *mm* and half-angle $\theta = 1.7°$ is shown in figure 9. Finally, looking at equation (36), the equivalence between a positive truncated cone and an abrupt change in cross section for the case $S' < S$ can be noticed.

3.4 Other effects: radiation, dispersion and temperature

Because of dispersion due to visco-thermal effects, the eigenfrequencies of the cylindrical air column cannot be exactly harmonically related. From the well known expression of the speed of sound with respect to frequency [24], it can be shown that dispersion introduces a positive inter-resonance inharmonicity given by

$$IH = \frac{\Gamma_1 - \Gamma_3}{1 - \Gamma_1} \ , \tag{38}$$

where $\Gamma_n = \frac{1}{R\sqrt{2k_n}}(\sqrt{\ell_v} + (\gamma - 1)\sqrt{\ell_h})$ is the dispersion factor associated to the nth eigenfrequencies, ℓ_v and ℓ_h are the viscous and thermal characteristic lengths and γ the ratio of specific heats. The order of magnitude is given in
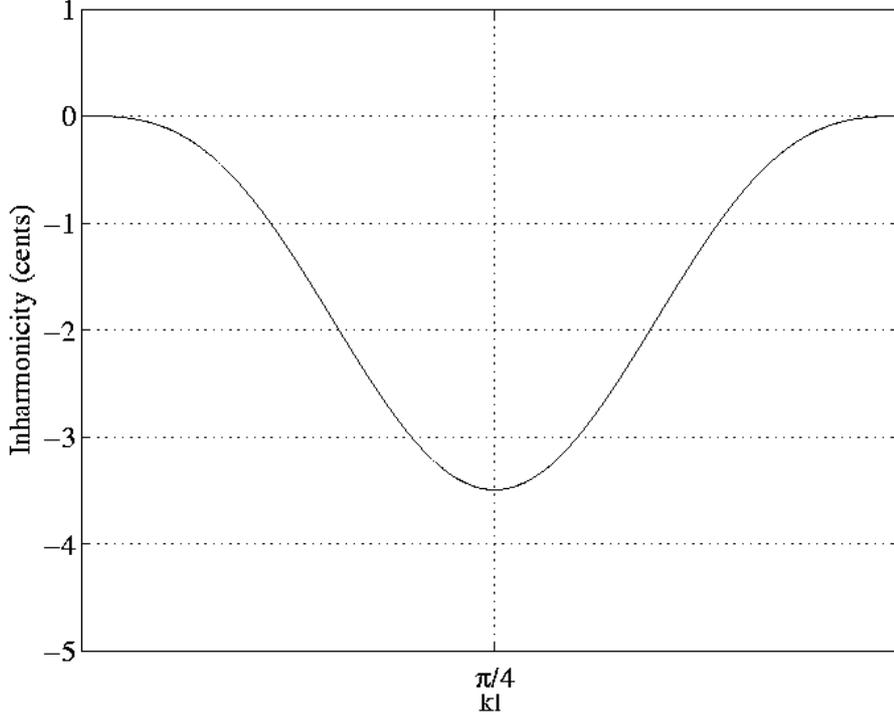
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Fig. 9. *Inter-resonance inharmonicity for a positive truncated cone as a function of $k_1\ell$ ($R = 7.5$ mm, $\theta = 1.7°$, $\ell_c = 5$ mm).*

the next section.

Another effect which affects the relationship between the resonance frequencies is the axial temperature drop. With the relation $\frac{\Delta T}{T} = -\frac{\Delta \rho}{\rho}$, the temperature gradient can be seen as a small perturbation in series which modifies the air density. Looking at a location x, the infinitesimal length correction is

$$\delta x = -jZ/Z_c \sin^2 kx , \qquad (39)$$

where $Z = j\omega\Delta\rho dx/S$. The total length correction is obtained by an integration over the length of the perturbed part of the tube. Assuming arbitrarily a linear temperature profile over the upper third of the length ℓ_{eff} of the instrument $T(x) = 3\frac{T_{out}-T_{in}}{\ell_{eff}}x + T_{in}$, where T_{in} is the temperature at the input ($x = 0$) and T_{out} is the external temperature, the effect of the thermal gradient is evaluated as follows:

$$\Delta\ell = -3\frac{T_{out} - T_{in}}{T_{out}} \int_0^{x_0} x\sin^2 kx \, dx + \frac{T_{in} - T_{out}}{T_{out}} \int_0^{x_0} \sin^2 kx \, dx , \qquad (40)$$

where $x_0 = \ell_{eff}/3$. Expression (40) can be calculated analytically since the lower and upper bounds are given by $k_1\ell = \pi/6$ and $k_2\ell = \pi/2$ and approximation for inharmonicity is given by

$$IH = -0.0422 \frac{T_{out} - T_{in}}{T_{out}} . \qquad (41)$$

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With these assumptions, the thermal gradient appears to produce a constant 5-cent positive inharmonicity across the complete register. Applying a linear thermal gradient over the entire instrument instead of the third of its effective length, would lead to inharmonicity of +10 cents . Finally, with an arbitrary linear temperature profile which is surely not very realistic, it can be said that the thermal gradient have a significant effect on inharmonicity but no more than 5 and 10 cents over the entire scale.

Since radiation of wind instruments depends on frequency, radiation is also a cause of inharmonicity but its effect is very small compared to the previous effects (less than 1 cent).

3.5 Theoretical analysis of a clarinet resonator

Our major results concerning inharmonicity are summarized in table 2. To

Basic perturbation	Sign of IH	Effective parameters
tone hole	< 0	Sh'/S_h and ℓ_d
closed-hole	$k\ell \lesssim \pi/4 \Rightarrow IH > 0$ $k\ell \gtrsim \pi/4 \Rightarrow IH < 0$	v/S and ℓ
abrupt change in cross section in the upper part of the instrument	$S' > S \Rightarrow IH > 0$ $S' < S \Rightarrow IH < 0$	S'/S and ℓ
localized enlargement/contraction in the upper part of the instrument	> 0 or < 0	S'/S and ℓ
diverging truncated cone in the upper part of the instrument	< 0	ℓ/x_1 and ℓ
converging truncated cone in the upper part of the instrument	> 0	ℓ/x_1 and ℓ
register hole	> 0	Sh'/S_h and ℓ

Table 2

Sign of inharmonicity associated with basic acoustic perturbations close to the tube input.

compare them with a real clarinet, a Buffet Crampon clarinet has been investigated theoretically and experimentally. From the geometrical dimensions of the instrument, given in table 3, and with the mathematical formulations of inharmonicity between the first two resonance frequencies, both inter-resonance and inter-register inharmonicities were predicted directly for each fingering,

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simply adding inharmonicities associated to each perturbation. In order to validate these calculations, computations of the input impedance of the clarinet were carried out using a transmission-line model and transfer matrices. Multiplying sequentially the transfer matrix of each element from the open end to the mouthpiece results in the input response of the instrument. From the obtained impedance curves, inharmonicities between the resonance frequencies were determined for all fingerings.

Calculations with both transfer matrix and length correction approximate formulae coincide satisfactorily for the total inter-register inharmonicity as shown in figure 10. Focusing on the influence of a single perturbation, a good agreement is also obtained even if the two methods do not coincide for few tones. For a detailed validation of length correction calculations, readers are referred [25]. Since the two methods give similar results, our method based on adding corrections to first order seems appropriate.
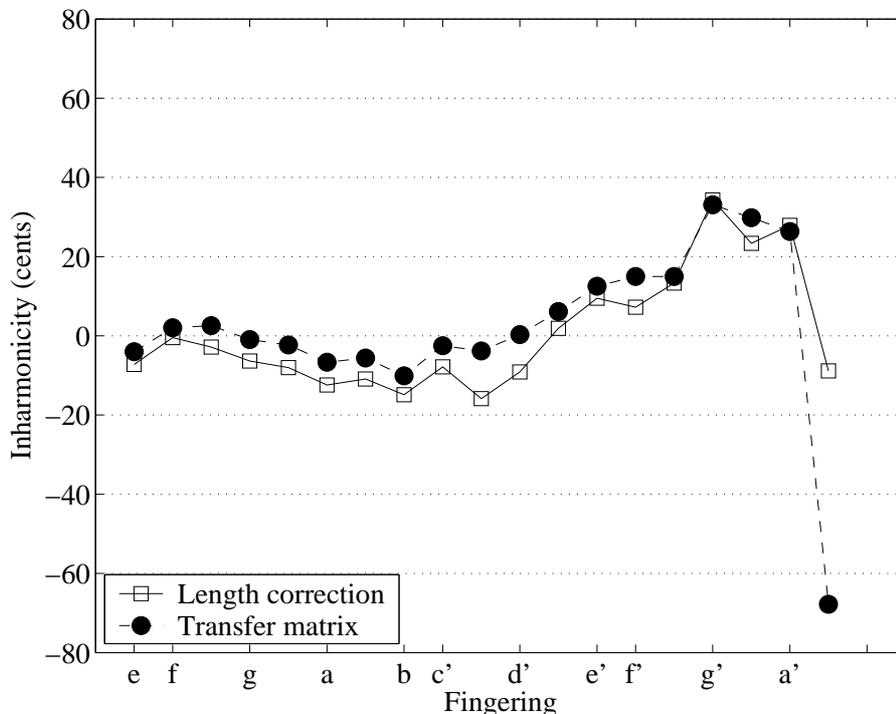

Fig. 10. *Total inter-register inharmonicity for a Buffet Crampon clarinet. Length correction calculations (□) and transfer matrix calculations (•) (the temperature gradient has been ignored).*

3.5.1 Calculation procedure and results

The influence on inharmonicity of each bore perturbation found on a clarinet is now discussed and results are plotted in figures 11 and 12. Results for the register hole have been shown in figure 3. The principles of the length correction calculations are given for every kind of perturbations.

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no	ℓ (mm)	r (mm)	h (mm)
24	152.7	1.4	13.0
23	166.9	2.35	7.1
22	193.4	2.8	7.1
21	202.9	2.7	7.1
20	213.2	2.55	7.2
19	230.4	2.4	6.4
18	238.9	3.9	10.8
17	241.3	2.35	6.8
16	252.2	2.45	8.8
15	270.5	2.3	6.9
14	284.3	3.3	8.8
13	287.1	2.7	6.8
12	289.3	2.85	6.9
11	308.2	3.95	7.4
10	318.8	2.4	7.2
9	348.9	3.55	6.5
8	364.8	4.05	8.9
7	369.5	3.95	6.1
6	391.0	3.85	8.9
5	413.1	4.85	8.9
4	445.6	5.4	6.0
3	472.9	6.1	5.0
2	504.9	5.6	5.2
1	543.7	5.8	4.4

position (mm)	diameter (mm)
0	0
89	14.8
117.5	14.6
117.5	15.6
152.5	14.5
492.5	14.5
526	15.1
565	16.9
578.2	18.4
598	22.2
664	60

Table 3

Numerical data concerning the Buffet Crampon clarinet investigated. Tones holes are numbered from 1 to 24 in order of decreasing distance from mouthpiece, the 24th being the register hole. Distances are given from the reed tip. For the input impedance measurements the mouthpiece is replaced by a cylindrical tube of length 77 mm. For the comparison with computations, a position correction of about -12 mm is introduced for the tone hole locations.
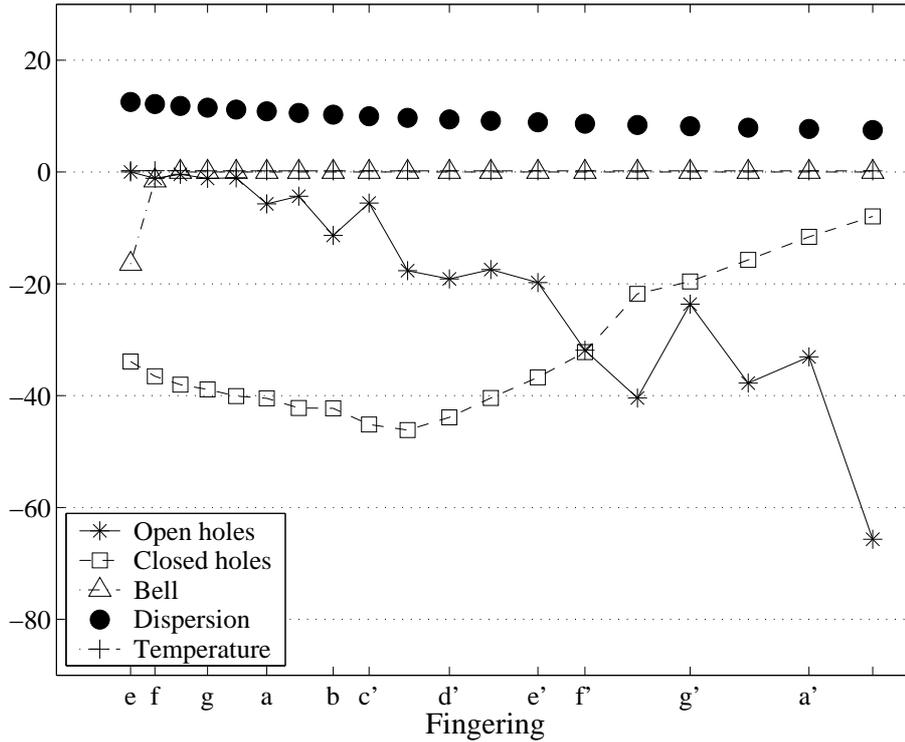
Fig. 11. *Inter-resonance inharmonicities due to open holes (∗), closed holes (□), dispersion (●), temperature (+) and the flaring bell (△).*

Mouthpiece The acoustical top of the instrument (i.e the effective input) for measurements is defined assuming a cylinder of length $\ell_G = 77\ mm$. Even if this can be seen as the sum of an equivalent embouchure of volume $v = 11\ cm^3$ plus a length correction due to reed effects equal to 10 mm as measured in reference [7] by comparing the resonance and playing frequencies, it must be noticed that this equivalent length is imposed by the experimental setup and ignores the variation of reed effects with frequency. This characterisation of the mouthpiece does not take into account the geometry of the input volume which influences the functioning. Actually, the theoretical analysis is made in two steps:

i) the analysis of the input impedance with the equivalent embouchure (present section 3.5.1);

ii) after a comparison of the results of the present section with experiments, a discussion of the effect of the conicity of the mouthpiece is given in the next section (see 3.6).

Barrel and top part of the upper joint Both the barrel and the top part of the upper joint are tapered towards the bottom with a step discontinuity between them. Inharmonicity due to change in cross-section is evaluated with equation (34) and since the assumption $X \ll 1$ is not true for both changes in
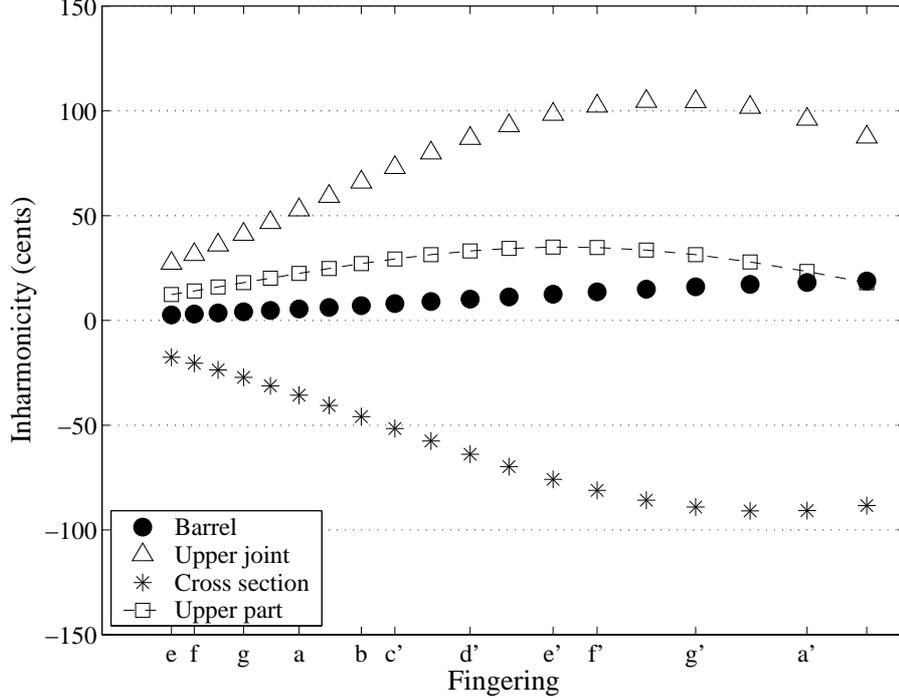
Fig. 12. *Inter-resonance inharmonicities due to bore perturbations of the upper part: barrel (•), change in cross sections (∗), upper joint (△) and upper part (sum of the three curves) (□).*

taper, inharmonicities are derived using exact formulae for length corrections (see appendix A eq.(A.3)) and equation (1).

It appears that the barrel entails positive inharmonicity and affects much more the highest part of the register than the lower part. Besides, even if the change in cross section tends to balance the effect of the inverse conical upper joint (see figure 12), the total inharmonicity associated with bore perturbations at the input is positive and appears to be essential to correct the damage caused by open holes, cavities, dispersion and the flaring horn as it will be shown below.

An important point to notice is that since length corrections due to taper and discontinuity in cross section are frequency dependent, a correction is needed for the location of closed tone holes, depending on the considered fingering.

Open holes   As apparent with equation (24), inharmonicity associated with open holes depends strongly on the length ℓ_d of the tube-part below the first open hole. To allow realistic calculations, a good estimation of this parameter is nedeed. Instead of approximating ℓ_d as the distance between the first two open tone holes, the following method is used. Starting from fingering f with hole 1 open, calculation of ℓ_d is self-evident: an equivalent length $\Delta\ell$ of the two removed admittances (Y and Y_{down}) can be deduced with equation (23). This gives the substitution tube of length $\ell + \Delta\ell$ to produce tone f. Deal-

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ing with the next tone $f\#$, ℓ_d is now approximated by the distance between the first two open holes increased by the tube-piece of length $\Delta\ell$ calculated previously. This allows to take into account, up to a certain extent, the effect of the tone hole lattice (only two or three open holes may play a role in the determination of the playing frequencies [26]). Then, inharmonicity is deduced for this fingering, a new $\Delta\ell$ is calculated and so on. In practice, the method is very convenient but fails for few situations such as cross-fingering, because of the proximity effects between two open holes.

As it is expected, examination of inharmonicity due to open side holes shows the tendency to produce negative inharmonicity across the entire scale (see figure 11). Besides, it appears that the highest tones of the register are more affected than the bottom tones because of the strong dependence of inharmonicity with the wavenumber k_1 (see equation (24)).

Closed holes   Inharmonicity due to a closed side-hole located above the first open hole is calculated with equation (30) assuming that every hole is independent of the others. Adding each closed hole contribution, the total closed-side hole inharmonicity is determined.

Figure 11 shows the important role of the closed holes. Inharmonicity is negative and the shape of this curve can be interpreted by studying the combined action of cavities located at both extremities of the tube of length ℓ_{eff}. It must be reminded that inharmonicity due to a closed hole is mainly proportional to the hole volume, the hole volumes being larger for cavities near the open-end than those located at the input of the instrument. On one hand, for tones from e to $c'\#$, cavities located near the open-end have a larger (negative) effect than those located near the mouthpiece which produce small positive inharmonicity. As a consequence, harmonicity is more and more altered. On the other hand, for tones from $c'\#$ to $a'\#$, inharmonicity still remains negative but decreases when the chromatic scale is played which is mainly due to the decreasing of the number of closed holes.

Flaring horn   The expanding part at the open-end of the clarinet has been modeled as a catenoidal horn with a horn constant $h = 0.085$. Using the procedure given in [1] and restricting our investigation to the two lowest tones only, a length correction referring to the geometrical length has been assigned to the flaring bell according to

$$\Delta\ell_h = -\ell_h + 1/k\,\text{Arctan}(kh'\,\tan(L_h/h')), \tag{42}$$

where L_h is the effective length including the classical end-correction $0.6R_f$ (R_f being the end-radius of the flare) and $h' = h/\sqrt{1 - k^2 h^2}$. Besides, a position correction has been given to the two holes located in the expanding part of the bell as Nederveen did. As expected, the flaring horn produces negative

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inharmonicity for the lowest tone to about -15 cents and has small influence on fingering f.

Dispersion and temperature Calculations of the effects of dispersion is obtained with equations (38) and results are plotted in figure 11. Dispersion effect is continuous over the entire scale of an amount to about +10 cents. Concerning the effect of the thermal gradient, it has been discussed in section 3.4.

3.6 Comparison between theory and measurements

To correlate theoretical results and measurements, an input impedance measurement device is used (see reference [27]). It gives the acoustic linear response of the passive instrument (i.e without reed effects) to an harmonic excitation. Measurements are made in a well-insulated room and temperature is 18.5^oC [6] . The resonance frequencies are obtained by interpolating the imaginary part of the admittance in the vicinity of each resonance, the oscillating frequency satisfying $\mathcal{I}m(Y) = 0$. The absolute uncertainty on the measurements are determined to be ± 5 cents.

Figures 13 and 14 show that experiment and theory are in a good agreement, except for few tones for which discrepancies can be explained by an inconvenient determination of the length ℓ_d when investigating the open hole effect. Figure 14 also shows the inter-register inharmonicity with conditions close to the normal use of a clarinet, i.e when both the thermal gradient and the conicity of the embouchure at the end of the mouthpiece are taken into account. With the use of the geometry of a classical embouchure (see reference [28]), the effect of the conical part at the end of the embouchure has been evaluated with the exact formulae of length corrections given in appendix A (see eq.(A.3)). It appears that the embouchure conicity narrows the inharmonicity of the highest tones of the register.

Otherwise, two clarinets were investigated experimentally. It appears that both clarinets have the same general tendencies in inharmonicity as figure 15 shows, even if small differences can be noticed for few tones. This fact has previously been reported in [7] and is linked to differences in bore profile of the upper part of the instrument. In the same reference, figure 11 shows results for two different clarinets, very similar to the results of figure 15 for the inter-resonance inharmonicity: nevertheless a tendency for the results of the present study is to be 5 cents lower than the results for the clarinets measured in [7]. Moreover,

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[6] Two other changes in cross section occur in the experimental setup at the input of the instrument but their effect on the resonance frequencies is substracted algebraically.
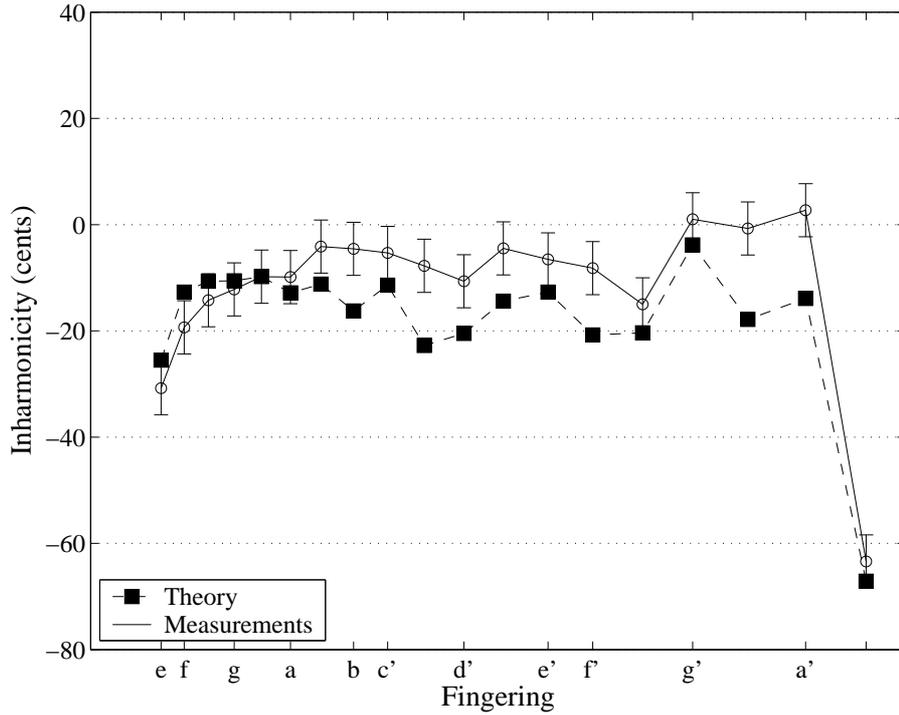
Fig. 13. *Inter-resonance inharmonicity. Comparison between measurements (error bars) and length correction calculations (■) (uniform temperature).*
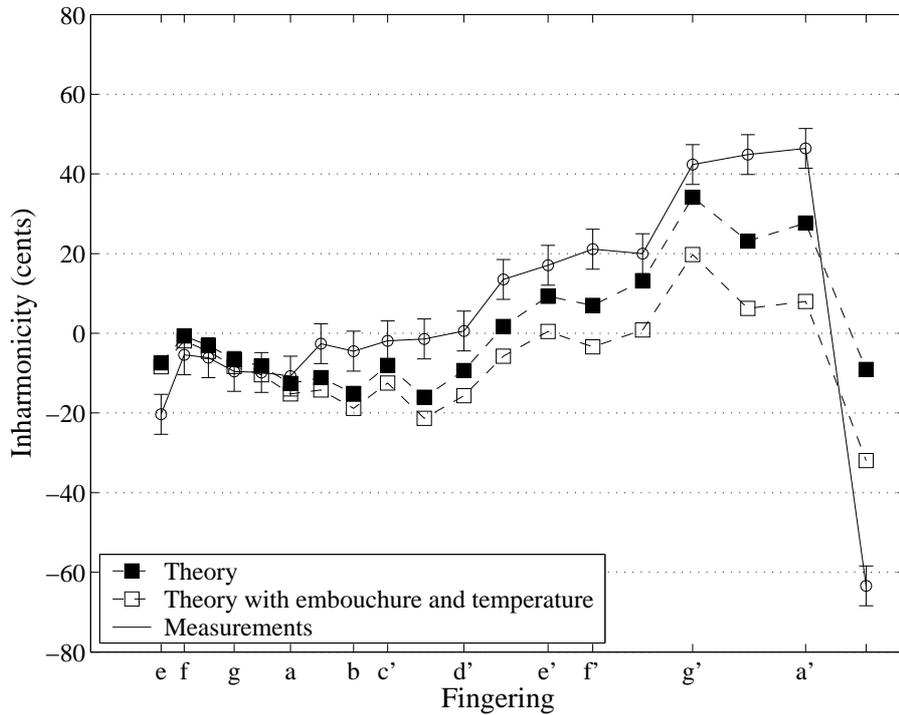

Fig. 14. *Inter-register inharmonicity. Comparison between measurements (error bars) and length correction calculations (■). Adding the effects of the thermal gradient and the conicity of the embouchure to (■) gives (□).*

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adding the results for the inter-resonance inharmonicity and for the inharmonicity due to register hole leads to very similar inter-register inharmonicity results for the two studies.
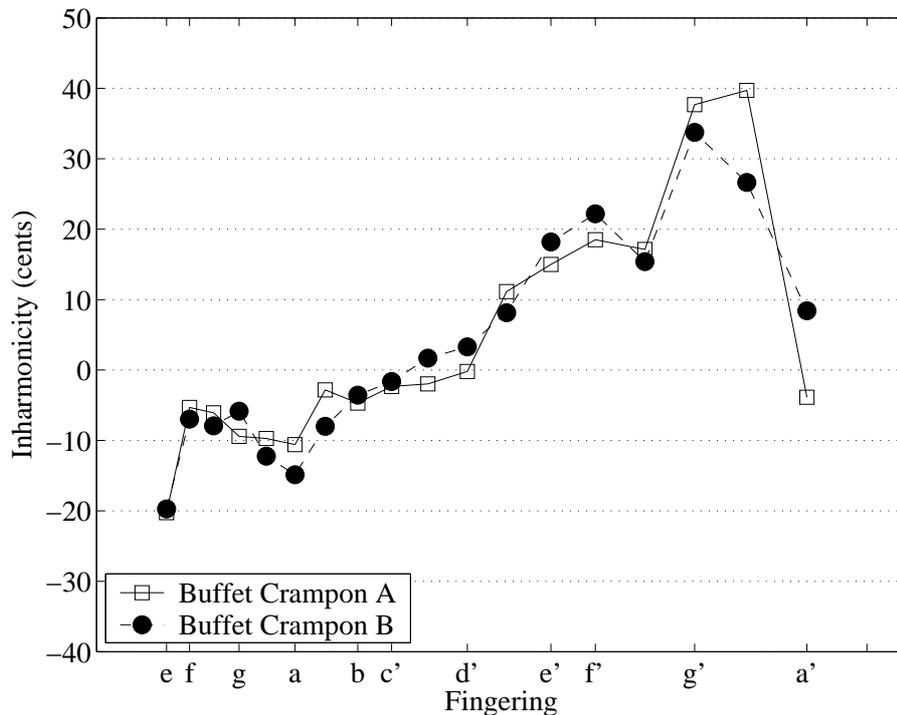

Fig. 15. *Measured inter-register inharmonicities for two Buffet Crampon clarinet models.*

3.7 Discussion and conclusions

i) All our calculations deals with resonance frequencies and no excitation mechanism has been introduced. As shown by figures 11 and 16, our analysis leads to the following statements concerning the origin of the general tendency of clarinets to have tuning deficiencies for the lowest twelfths:

- the used linear theory does not reveal a tuning problem for the lowest tones,
- certain dimensions of the tone holes have been chosen by makers. When their effects are considered as imposed, the remaining parameters to achieve a correct tuning are mainly in the bore profile and the location of the register hole,
- a good compromise between the bell profile and the register hole location and size, is needed to control the tuning of the lowest twelfth (tone *e*),
- the barrel and bore profile at the top end of the upper joint are of great importance in order to bring the lowest and middle twelfths in tune. This effect tends to balance the effect of the closed holes over the entire scale.
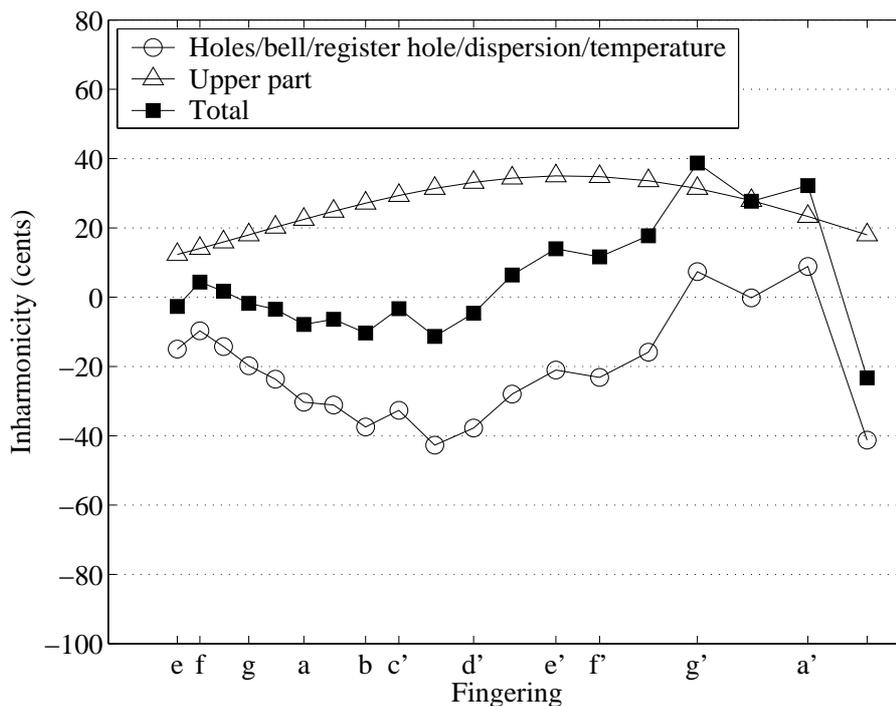
Fig. 16. *Inharmonicities due to perturbations found in the upper part (△), due to hole/bell/register hole/temperature/dispersion (○), and resulting total inter-register inharmonicity (sum of the two curves) (■).*

Finally, if positive inharmonicity of the resonator is observed for the lowest twelfths, the origin of the problem may lie in combination of the combined action of register hole opening, cavities, bell and inappropriate bore perturbations in the upper part. Nederveen [1] has raised the question of the necessity of bore perturbations in musical instruments. In the case of the clarinet, calculations reveal the need to resort to bore irregularities close to the input to compensate for the negative inharmonicity caused by open holes, cavities, flaring bell and dispersion. Reaming the upper bore profile appears to be a difficult task and must be done carrefully, since this part seems to control the accuracy of the twelfths.

ii) As discussed in section 2, predicting playing frequencies from resonance curves is not an easy task. Reference [7] shows that the measured length correction due to the reed effects (volume velocity of the reed and reed damping) is rather constant over the entire first register and the first half of the second one, then increases strongly. This allows compensation for the large increase of inter-register inharmonicity shown e.g. in figure 15. Nevertheless this kind of measurement, done using an artificial mouth, is rather difficult. It is also difficult to compare directly to the results obtained at different levels by an instrumentalist. An example of this difficulty is given in [29]. From our results for the two clarinets of figure 15, it is not possible to explain why the players

are obtaining too large twelfths for the first tones (e to g). It is a way for future investigation: maybe the analysis could be made using also an artificial mouth, but the fact is that the players often complain about too large twelfths, and this fact seems to be quite common: as an example, recently Buffet Crampon has designed a new clarinet, called *Tosca*, correcting the tone f. For these low tones, what is the optimum inharmonicity of the resonator for having well tuned twelfth? Is it zero? The answer is not obvious, depending also on the excitation level, and certainly on the "tunability range" [15,11], due to several factors, including the modification of both the embouchure (reed opening and reed damping) and the vocal tract. These musician-control parameters have not been extensively investigated yet. Moreover the "tunability range" increases with the pitch of the played tone: it is larger for the higher register than for the lower one, particularly due to the low level of the second impedance peak (for the second register, the second peak corresponds to the sixth peak of the first register).

iii) In the next sections, we continue our study in the same direction but in a different perspective. First, the register hole is searched to allow a register jump for the 19-tone complete register. This is an alternative to what makers do since clarinettists use different fingerings to play tones from $f'\#$ to $a'\#$ and their associated twelfths. Second, instead of working with a real instrument, we consider a perfect cylindrical tube (without tone holes and flare) with only one register hole. The next sections are devoted to answer to the following general question: is it possible to find a register hole location on a cylindrical tube, combined with a simple perturbation in the upper part, which allows for the first two complete registers to be perfectly well tuned? In other words, we examine if it is possible to design a well-tuned instrument which is provided with additionnal bore perturbations in the upper part which exactly compensate for inharmonicities due to tone holes and other effects.

4 Optimization of the location of the register hole and tuning corrections : statement of the problem

4.1 Assumptions

Our goal now is to achieve harmonicity of the first two resonance frequencies f_1 and f_2 , for the effective lengths corresponding to each hole, i.e. between two extreme values of ℓ_{eff}, written $\ell_{\min}$ and $\ell_{\max}$. The precise intermediate values of the length is without importance for the present objective. As a consequence, the absolute effect of small discontinuities, e.g. cavities or tapers, is not important as well, only their relative effect between these two registers

being important. It is therefore convenient to consider a continuous variation of the length between the two extreme values. After the optimization of the intervals between the two registers between these two values, it will be possible to find the precise location of the tone holes achieving the desired scale.

How can harmonicity between the frequency f_1 of the tone of the first register and the frequency f_2 of the tone when the register hole is open, be achieved for all lengths between ℓ_{min} and ℓ_{max}? First we will find an optimal location for the register hole, located upstream of ℓ_{min}, secondly we will study if a correction system can compensate the residual defaults of this register hole. The dimensions of the hole are considered to be optimized by the practice of makers, and therefore are regarded as imposed. It is actually a difficult question, related to nonlinear effects as well as water effects, the important fact being that the linear behaviour, at low level, is well known [30]. We do not take into account the use of the register hole as a B flat tone hole. The correction systems are sought in order to be without manipulation by the instrumentalist, and therefore to act on both the first and the second registers, contrary to the register hole itself.

4.2   Formulation of the optimization problem

Optimization techniques are used to find a set of design parameters $x_\star = \{x_\star^{(1)}, \ldots, x_\star^{(n)}\}$ that can in some way be defined as optimal. These parameters are obtained by minimizing (or maximizing) a criterion function $\mathscr{F}$ which may be subject to constraints and/or parameter boundaries. Thus, the design parameters are subjected to the following requirements:

- small changes location must be less than the distance between the reed tip and the first tone hole;
- dimensions of the acoustical systems must be reasonable for the realization;
- geometrical dimensions are positive.

Our optimization problem is formulated as follows:

$$
\begin{cases}
\min\limits_{x \in I\!\!R^n} \quad \mathscr{F}(x) \quad \text{where } \mathscr{F} \text{ is the criterion function} \\
l_{b_i} \leq x_i \leq u_{b_i} \quad \text{i=1,\dots, n}
\end{cases}
\tag{43}
$$

where x is the vector of design parameters ($x \in I\!\!R^n$) and l_{b_i} and u_{b_i} are the lower and upper parameter boundaries respectively, for the design parameter x_i.

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4.3 Criterion functions

4.3.1 Location of the register hole

As previously shown, the presence of the register hole means that it is probably impossible to built a clarinet in a way that the two registers are played accurately in tune with the same fingerings. Thus, the objective is to find the register hole location that entails the smallest frequency shift of the second resonance frequency for all fingerings. The first optimization is performed in order to play a 19-tone compass i.e a complete register. Then, a second optimization is performed restricting attention to fingerings from e to f' (above this fingering, clarinettists do not use the same fingering to play the fundamental and the associated twelfth). The distance ℓ of the hole from the effective input is the only design parameter. The optimization problem deals with equation (20) which predicts theoretically the frequency deviation associated with the opening of the register hole. It can be formulated with one of the two following criterions:

- criterion 1: to minimize the maximum of the frequency deviation;
- criterion 2: to minimize the mean of the square of the derivative with respect to k of equation (20).

The first criterion consists in limiting the most important tuning default of the instrument. This criterion, which is very simple and intuitive for anyone who is interested in instrument design, can be written:

$$\textbf{criterion 1}: \quad \mathscr{F}_1 = \sup\left(\frac{2}{3\pi}\frac{s}{Sh'}\frac{cos^2 k_2\ell}{k_2}\right) . \qquad (44)$$

An interesting point about criterion 1 is that the solution can be approximated analytically as it is shown in section 5.1.1. On the contrary, once the maximum of the deviation is achieved for a fixed register hole location, the frequency deviations associated with other fingerings are not taken into account for the evaluation of the function.

Concerning criterion 2, the criterion function to minimize is formulated as

$$\textbf{criterion 2}: \quad \mathscr{F}_2 = \frac{1}{2}\int_{k_{min}}^{k_{max}}\left[\frac{\partial}{\partial k_2}\left(\frac{2}{3\pi}\frac{s}{Sh'}\frac{cos^2 k_2\ell}{k_2}\right)\right]^2 dk_2 , \qquad (45)$$

where $\partial/\partial k_2$ is the derivative with respect to the wavenumber k_2, k_{min} and k_{max} being related to the playing frequency via the effective length by $k_{min} = 3\pi/2\ell_{max}$ and $k_{max} = 3\pi/2\ell_{min}$ respectively. Contrary to criterion 1, this criterion function takes into account the deviation associated with all fingerings for its evaluation. The global minimum of this criterion function is achieved when the integrand is zero, i.e when the frequency deviation is constant for all

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fingerings. Nevertheless, noting that equation (20) is necessarily equal to zero for a particular frequency, $k_2\ell = 3\pi/2$, it appears that the register hole location given by this criterion function is the one that minimizes inharmonicity variations around zero and may lead to a more homogeneous register jump across the entire register.

4.3.2 Correction of the register hole effect

The second aim of this optimization is to give suggestions on how to compensate for the register hole effect. We are looking for geometrical dimensions of acoustical systems whose effects alter the resonance frequencies in order to restore the original f_2/f_1-mode frequency ratio. Similarly to the study of the location of the register hole and denoting IH_{reg} and IH_{pert} the inharmonicities associated with the register hole and the perturbation respectively, a criterion function can be written as

$$\textbf{criterion 1}: \quad \mathscr{F}_1 = \sup\left(|IH_{reg} + IH_{pert}|\right) \ , \tag{46}$$

which deals with the maximum of the total inharmonicity. Moreover, a second criterion function has also been investigated by minimizing the function defined as

$$\textbf{criterion 3}: \quad \mathscr{F}_3 = \frac{1}{2}\int_{k_{min}}^{k_{max}} \left[IH_{reg} + IH_{pert}\right]^2 dk \ . \tag{47}$$

for which the minimum is reached when inharmonicity associated with the inserted perturbation is of the same magnitude as the register hole deviation and in the opposite direction.

5 Results and discussions

For the optimization, the radius of the main tube is taken as $R = 7.5 \ mm$. The height of the register hole is $12.5 \ mm$ and its radius $1.55 \ mm$. The upper bound for the acoustical system location, is set to $154 \ mm$ which is the first tone hole location according to table 3. The other parameter boundaries for all perturbations have been chosen in order to make the realization possible.

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5.1 Register hole location

5.1.1 Criterion 1: maximum of the frequency deviation

As mentioned earlier, an optimum of criterion 1 defined by equation (44) can be derived analytically. In order to achieve the solution, we rewrite the frequency shift as follows

$$IH = \frac{2}{3\pi} \frac{s}{Sh} \ell \frac{cos^2 k_2 \ell}{k_2 \ell} \ , \tag{48}$$

and note that the magnitude of the deviation is proportional to ℓ and varies as the function $F(x) = \cos^2 x / x$ where $x = k_2 \ell$. We will first proof that the maximum of $F(x)$ does not depend on ℓ for a certain interval of values of $k_2 \ell$ and therefore the optimum value of IH is obtained for the minimum of ℓ on this interval.

The effective length of the tube varies between $\ell_{max} = L = 585\ mm$, the total length of the instrument including reed and radiation length corrections, and ℓ_{min}. If the compass of a register is one twelfth minus one semi-tone, the two extreme values of the effective length are related by $\ell_{min} = \ell_{max}/2^{18/12}$ which for simplicity we assume first to be $\ell_{min} = \ell_{max}/3$. The location of the register hole, defined by ℓ, is in the upper part of the instrument and satisfies $\ell < \ell_{min}$.

Then, for a given effective length, ℓ_{eff} lying between ℓ_{min} and ℓ_{max}, the wavenumber is defined as $k_2 = 3\pi/2\ell_{eff}$, and therefore the argument of $F(x)$ varies follows :

$$\frac{3\pi\ell}{2\ell_{max}} < k_2 \ell < \frac{3\pi\ell}{2\ell_{min}} \ , \tag{49}$$

thus

$$\frac{3\pi}{2} \frac{\ell}{L} < k_2 \ell < \frac{9\pi}{2} \frac{\ell}{L} \ . \tag{50}$$

Because the ratio ℓ/L is less than $1/3$, the upper bound for $x = k_2 \ell$ is therefore $3\pi/2$ for which $F(x) = 0$. Figure 17 shows the variation of $F(x)$. From $\ell = L/3$, for the interval defined by inequalities (50), $F(x)$ varies between 0 and 0, with a maximum value equal to 0.327, for $x_0 = 2.975$. When ℓ decreases from $L/3$, figure 17 shows that the maximum values remains constant, equal to $F(x_0)$, except if ℓ becomes so small that the value of $F(x)$ for the minimum value of $k_2 \ell$ reaches the same value 0.327. Using the subscripts $\star$ for referring to the optimal value and (i) $(i = 1, 2, 3)$ for referring to the studied criterion, this corresponds to $\ell_\star^{(1)} = 0.205\ L$. Below this value, the maximum of $F(x)$ grows rapidly. As a consequence, the optimum of IH for the criterion 1 is finally the value $\ell_\star^{(1)} = 0.205\ L$. Nevertheless, this value is not so critical, because the criterion function $\mathscr{F}_1$ for criterion 1 is linear with ℓ, over the interval
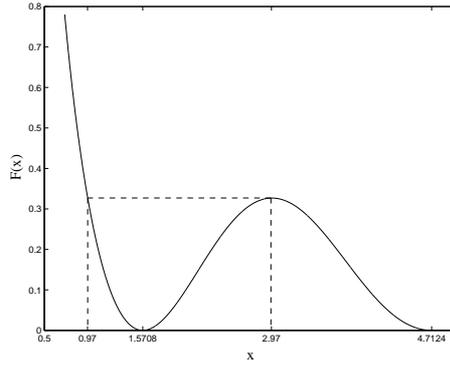
Fig. 17. *Perturbation function* $F(x) = \cos^2 x/x$. *The same local maximum is reached for* $x_0 = 2.97$ *and* $x_\star = 0.97$.

$[0.205\, L, L/3]$, thus the variation is not strong. Moreover, it appears clearly that any other criterion relative to the maximum of the frequency deviation will lead to a value within the interval, because the maximum is reached two times instead of one only.

Finally, taking into account that the interval $[\ell_{min}, \ell_{max}]$ corresponds actually to a slightly smaller interval than a twelfth i.e $2^{18/12}$, a numerical study leads to the optimal value for criterion 1 slightly smaller than $\ell = 0.205\, L$, i.e $\ell_\star^{(1)} = 0.2041\, L$. Figure 18 confirms that for criterion 1, the function $\mathscr{F}_1$ increases linearly above this optimum value but increases strongly when ℓ decreases below it.
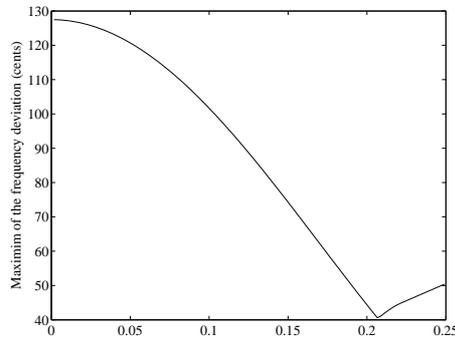

Fig. 18. *Maximum of the frequency deviation as a function of* ℓ/L. *Above* $\ell/L = 0.2041$, *the maximum value of the frequency shift increases linearly with the register hole location.*

The register hole deviation obtained with this "optimal" position is plotted on figure 19. As expected, the maximum of deviation appears at the beginning of the register and is almost equal to a quarter tone. Moreover, this result shows that the register hole is located close to its ideal position to produce the c'/g'' transition: the register hole is located at one third of the effective length for this fingering so that the frequency shift is zero.

5.1.2 Criterion 2: mean of the square of the derivative of the deviation

The register hole location achieved with the minimization of (45) is found to be

$$\frac{\ell_\star^{(2)}}{L} = 0.250 \; , \tag{51}$$

which is not only larger than the previous result but also larger than the one found with numerical data given by Nederveen ($\ell/L = 0.230$). For the investigated clarinet (see table 3), this ratio gives $\ell/L = 0.240$. The frequency deviations obtained with the optimal position defined by (51) and the one found in literature [7,21] are very similar: the a-e'' transition is correct and the frequency shift at the beginning of the register is about 20 cents (see figure 19).
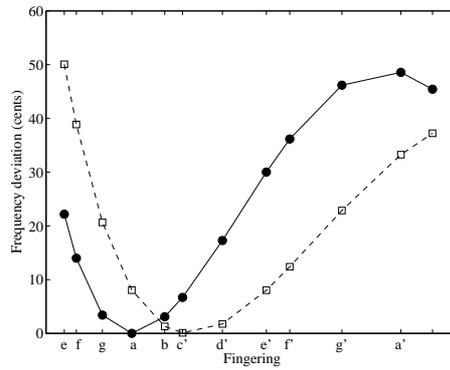

Fig. 19. *Frequency deviations associated with the register hole. The register hole is found to be located at $\ell_\star^{(1)} = 120.45$ mm from the top end for criterion 1 ($\square$) and $\ell_\star^{(2)} = 146.2$ mm for criterion 2 ($\bullet$).*

5.1.3 Conclusion

Performing optimization by minimizing the criterion functions defined by (44) and (45) gives different results and indicates that the final location of the register hole is the result of a compromise. However, as expected with equation (48), the magnitude of the maximum of deviation increases with the distance ℓ but very slightly for location larger than $\ell/L = 0.2041$. As a consequence, above this critical value, the location of the register hole is not so essential on this point of view and this can explained the difference in the register hole location observed between Nederveen's and the investigated clarinets for instance. Finally, it appears that a register hole location far from the effective input and above the critical value should be an interesting compromise in order to accurate the first twelfths of the register. Hence, we choose the location given by the optimization of criterion 2 that is $\ell_\star^{(2)} = 146.2$ mm.

Restricting now attention to fingerings from e to f' (and not up to $a'\#$) may be interesting in order to understand what instrument makers do. The result is that the optimization process converges to a position for the register

hole between the two previous extreme values for the two criterion functions. Moreover, for the criterion 1, the distance from the clarinet effective input is found to be $\ell_\star^{(1)} = 133.7 \ mm$. This result corresponds very well to the location used by Nederveen in its calculation [1] (i.e $\ell = 135 \ mm$). For that case, the twelfths at the bottom of the scale are still very large but the maximum of the frequency deviation, which is also obtained for the lowest tone, has noticeably fallen to 35 cents (see figure 20). Concerning criterion 2, the location is very close from the one found for the large compass.
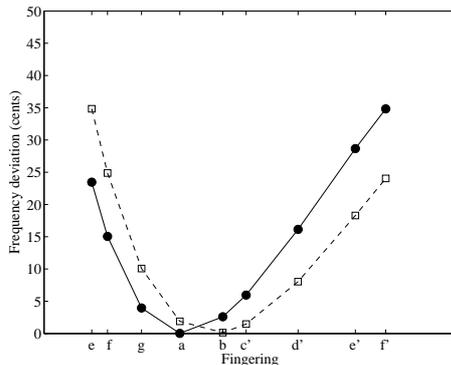

Fig. 20. *Frequency deviation associated with the register hole in the case of a real clarinet (i.e restricting our optimization to fingering from e to f': above f', fingerings are modified to ensure the well tuning and the third resonance is used). Optimal positions are $\ell_\star^{(1)} = 133.7 \ mm \ (\square)$ and $\ell_\star^{(2)} = 145.0 \ mm \ (\bullet)$*

5.2 Adjustements of natural frequencies by means of small changes of the bore

5.2.1 Overview of the possibilities

As mentioned earlier, the criterion is now to compensate for the frequency deviation due to the register hole by means of small changes of the bore. We are looking for a solution localized in the upper part of the instrument, i.e a solution acting for all fingerings. Looking at table 2, only three systems give inharmonicity in the right direction: an abrupt change in cross section area with $S' < S$, a change of conicity at the top end, and a localized enlargement or contraction. Concerning the case of a closed cavity, it has been shown (see figure 5) that both positive and negative inharmonicity can be generated: in order to produce negative inharmonicity, the condition $k_1 \ell > \pi/4$ which corresponds to playing frequencies larger than $f_1 = c/8\ell_{max}$ must be valid. Thus, the accuracy of the twelfths at the end of the second-register scale (from e' fingering) would be improved only.

5.2.2 Abrupt change in cross section area: $S' < S$

Fixing the radius of the tube downstream the discontinuity equal to 7.5 mm, the upstream tube radius R' and the location ℓ of the discontinuity are used as optimization variables. Performing optimization leads to the following results :

- criterion 1 : $\ell_\star^{(1)} = 76.9\ mm$ and $R_\star^{\prime(1)} = 7.1\ mm$.
- criterion 3 : $\ell_\star^{(3)} = 76.4\ mm$ and $R_\star^{\prime(3)} = 7.1\ mm$.

The deviation from the tempered scale of the first resonance frequencies and inter-register for each fingering are plotted on figure 21. Results show that even if "optimal" parameters are quite different, the shape and the magnitude of final inharmonicities obtained with the two criterion functions are very similar. The f_2/f_1 frequency ratio is noticeably improved for all fingerings except for the twelfth associated with the lowest tone which is still 20-cent large. An interesting result is that the discontinuity in the diameter is found to be located near the barrel joint which is at a distance of $60\pm4\ mm$ from the closed-end on a clarinet. This result agrees with the practice of many makers: Nederveen noticed that change in cross section area are mostly found near the embouchure for reed instruments.
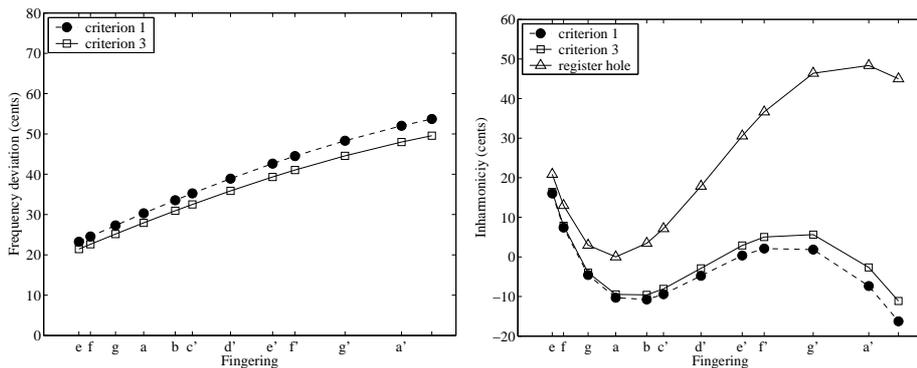

Fig. 21. *Frequency deviation for the first register (left) and inter-register inharmonicity (right): abrupt change in cross section area.*

5.2.3 Localized enlargement/contraction

The optimization variables used in this case are the location ℓ of the perturbation, its length ℓ' and the radius R' as indicated on table 1. Performing optimization for the case of a localized enlargement leads to the conclusion that small enlargement does not improve the relationship between first and second resonance frequencies. Even if optimization process converges with both criterion functions, final inharmonicities are not satisfactory. Figure 22 shows the tendency of the bottom notes to widen the f_2/f_1 ratio which is musically

unacceptable. On the contrary, for the considered particular case, contracting
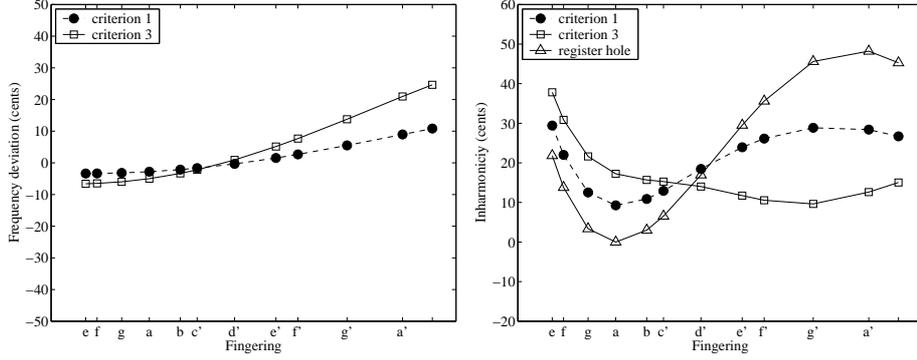

Fig. 22. *Frequency deviation for the first register (left) and inter-register inharmonicity (right): localized enlargement.*

the air column causes negative inharmonicity across the entire register and tends to improve the inharmonicity (see figure 23). The optimization process converges to the following parameters :

$$
\text{criterion 1} \begin{cases} \ell_\star^{(1)} = 62.74 \ mm \\ \ell'^{(1)}_\star = 36.76 \ mm \\ R'^{(1)}_\star = 7.17 \ mm \end{cases} \quad \text{criterion 3} \begin{cases} \ell_\star^{(3)} = 60.0 \ mm \\ \ell'^{(3)}_\star = 22.41 \ mm \\ R'^{(3)}_\star = 7.0 \ mm \end{cases}
$$

When minimizing the criterion function defined by equation (46), an interesting result is that the design parameters given by the optimization process are very similar to those of a clarinet barrel. As for an abrupt change in cross section, the contraction is located at a point near the barrel joint. In addition, the acoustic length of the contraction is the same as a common barrel ($\ell_{barrel} \sim 40 \ mm$). When looking at what makers do, their practice varies considerably concerning the geometry of the barrel. However, some makers insert narrower barrel than the cylindrical portion of the entire instrument in order to get accurate twelfths in the upper part of the scale [31]. While the results given by the minimization of (47) generate negative inharmonicity (see figure 23), the optimization process converges to two lower boundaries ($R'^{(3)}_\star = R'_{max}$ and $\ell_\star^{(3)} = \ell_{min}$). When enlarging the domain where optimization variables are looking for, a new minimum (lower than the one found) is achieved. Thus, we do not consider this solution in the next. Finally, the mean value of inharmonicity is about -0.6 cents and the standard deviation of the mean value is equal to 11.9 cents.
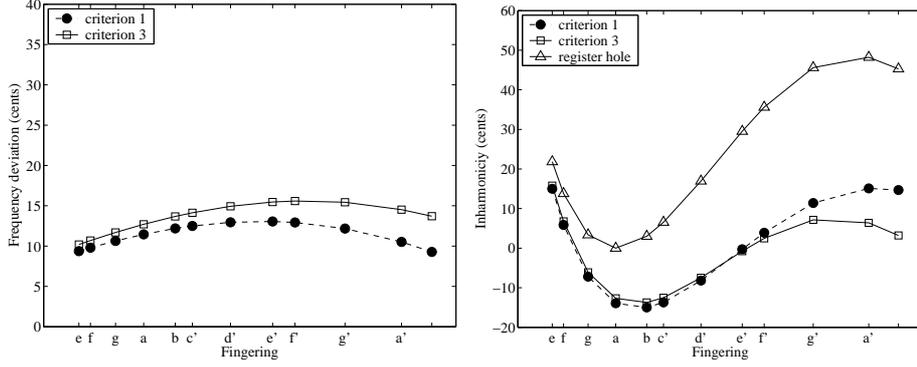
Fig. 23. *Frequency deviation for the first register (left) and inter-register inharmonicity (right): localized contraction.*

5.2.4 Change in conicity at the input

The optimization variables used for the case of tapered perturbation (see table 1) are the radius of the upper end of our model R', the top angle θ and the location point ℓ. For each criterion function, the optimization process converges to the following design paremeters :

$$\text{criterion 1} \begin{cases} \ell_\star^{(1)} = 98.5\ mm \\ \theta_\star^{(1)} = 0.10° \\ R'^{(1)}_\star = 7.46\ mm \end{cases} \qquad \text{criterion 3} \begin{cases} \ell_\star^{(3)} = 73.05\ mm \\ \theta_\star^{(3)} = 0.22° \\ R'^{(3)}_\star = 7.37\ mm \end{cases}$$

The lengths of the tapered perturbations corresponding to criterion 3 and criterion 1 are about 44.5 mm and 46.5 mm which are both larger than the length of a classic barrel. Concerning the location point, it is near the barrel joint for criterion 3 and slightly below this point for criterion 1. An important thing to note is that the location achieved by the optimization process is always close to the barrel joint whatever small change is used.

Figure 24 shows final inharmonicities obtained with the minimization of the two criterion functions. The f_2/f_1 frequency ratio is improved globally across the entire scale for both criterion functions except for the first fingering which is deteriorated slightly for criterion 3. As a consequence, only the design parameters given by the minimization of criterion 1 are considered. The mean value of the total inharmonicity is found to be 3.6 cents and the standard deviation of the mean value is about 11 cents.

Finally, results indicate that the 20-cent deficiency still remains for the notes at the bottom of the scale whatever acoustic system is used in order to alter the resonance frequencies. Thus, no improvement can be done to compensate for the effect of the register hole except for tones at the end of the scale.
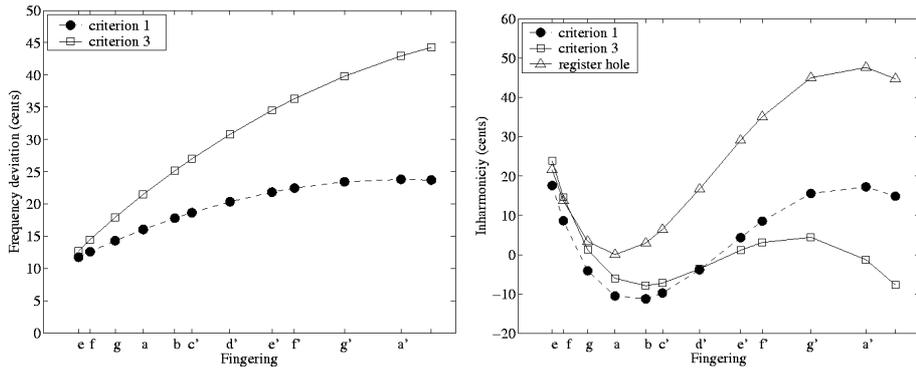
Fig. 24. *Frequency deviation for the first register (left) and inter-register inharmonicity (right): positive truncated cone.*

6 Conclusion

Many deviations from the standard cylindrical tube can be observed on a woodwind instrument. The present work provides some theoretical results concerning the effects of these bore irregularities in the perspective of the relationship between the first two resonance frequencies for the two first registers. The calculations using small perturbations are classical but the case of a side hole has been treated precisely, taking into account both the series and shunt impedances.

Calculations of inharmonicity associated with small perturbations and application to a real case give access to the relative influence on the first two resonance frequencies of each pertubating system found on a woodwind and reveal, in the case of the clarinet, the necessity to resort to bore perturbations to compensate for the damage on the resonance frequencies caused mainly by the presence of closed holes and the register hole.

In addition, the use of simple optimization techniques has been shown to be useful in order to give a more general view of the problem. However, optimization is very sensitive to the definitions of the criteria; for our purpose, performing optimization on the accuracy of the twelfths does not impose a unique possible criterion. For instance, the use of optimization techniques with weight for the lowest tones is of course possible and would probably give other results.

The main results of the present work can be summarized as follows:

- the origin of the occurence of wide twelfths for the lowest tones of a clarinet is not to be found in shifts of the resonance frequencies of the resonator, as opposed to common wisdom. A way for future investigations may be the influences of the blowing parameters, reed parameters and the vocal tract since players can modify both of them and claim to do it.
- with a single corrective system, designing a new instrument having for the two registers a compass of a twelfth with the same fingering except the

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register hole opening, is impossible without tuning deficiencies;
- if the aim is the improvement of a common clarinet, the use of contractions near the barrel can help to some extent; nevertheless, no simple solution located in the upper part of the instrument can ensure the perfect tuning of the twelfths corresponding to the first lowest tones.

Finally, combining global and local solutions, i.e solutions acting mainly on one particular tone, is necessary in order to achieve the best tuning.

Acknowledgements

Appendix

A   Exact calculation of length corrections

The aim of the appendix A is to give some precise definitions and exact formulae for some elements often encountered on wind instruments. Exact formulae can be useful for the computation of resonance frequencies by using an iterative procedure. The geometries of the different elements are shown in table 1.

Abrupt change in cross section

Looking for a equivalent system of section S, one can write at the discontinuity

$$-jY'_c \tan k\ell = -jY_c \tan k(\ell + \Delta\ell)$$

where $Y'_c = S'/\rho c$ and $Y_c = S/\rho c$. Manipulating the above expression gives result in

$$k\Delta\ell = \text{Arctan}\left(\frac{\frac{1}{2}(\alpha - 1)\sin 2k\ell}{1 + (\alpha - 1)\sin^2 k\ell}\right) , \tag{A.1}$$

where $\alpha = S'/S$ is the ratio of cross sections of the tubes located upstream and downstream the discontinuity.

Localized enlargement/contraction at the input of the instrument

Considering a localized enlargement or contraction of length ℓ' and cross section area S' located at point ℓ and denoting $\alpha = S'/S$ the change in diameter where S is the cross section area of the main tube, it can be written:

$$\begin{cases} Y_{up_1} = -jY_c \tan k\ell & \text{at point } \ell \\ Y_{down_2} = -jY_c \tan k(\ell + \ell' + \Delta\ell) & \text{at point } \ell + \ell' \, . \end{cases}$$

which results in after some algebra

$$k\Delta\ell = \text{Arctan}\left(\tan k\ell' \, \frac{TT_2 - \alpha(1 + TT_2) + \alpha^2}{-T \tan k\ell' + \alpha(1 + TT_2) + \alpha^2 \tan k\ell' T_2} \right) ,$$

where $T = \tan k\ell$ and $T_2 = \tan k(\ell + \ell')$.
Denoting $\alpha = 1 + X$, the expression of the length correction becomes

$$k\Delta\ell = \text{Arctan}\left(X \sin k\ell' \frac{\cos k(2\ell + \ell') + X \cos k\ell \cos k(\ell + \ell')}{1 + X \cos^2 k\ell' + X(X + 2)\sin k\ell' \cos k\ell \sin k(\ell + \ell')} \right) .$$
$$\text{(A.2)}$$

In the limit that α is close to the unity, an approximation of equation (A.2) is given by equation (31).

Truncated cone at the input

This paragraph deals with the case of a perturbation in taper of length ℓ_c located at distance ℓ from the reed tip. Calculations for a positive and a negative taper are identical. The unique difference is that the sign of both quantities X and X' has to be inverted.
Assuming the quantities x_i to be positive and $x_2 > x_1$, and noting that the two inductances located at points ℓ and $\ell+\ell_c$ are positive and negative respectively, the following system is derived :

$$\begin{cases} Y_{up_2} = Y_2 + Y_{down2} & \text{at point } \ell + \ell_c \ (x_2 > 0) \text{ and } L > 0 \, , \\ Y_{up_1} = -Y_1 + Y_{down} & \text{at point } \ell \ (x_1 > 0) \text{ and } L < 0 \, , \\ Y_{down1} = -jY_c \tan k(\ell + \ell_c + \Delta\ell) \, , \end{cases}$$

where $\Delta\ell$ is the global length correction.
Denoting $X = \dfrac{\ell}{x_1}$ and $X' = \dfrac{\ell + \ell_c}{x_2}$, the result is

$$k\Delta\ell = \text{Arctan} \, \Lambda \, , \tag{A.3}$$

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where

$$\Lambda = \left(\frac{X\frac{\cos^2 k(\ell+\ell_c)}{k(\ell+\ell_c)} - X'\frac{\cos^2 k\ell}{k\ell} + XX'\frac{\cos k\ell}{k\ell}\frac{\cos k(\ell+\ell_c)}{k(\ell+\ell_c)}\sin k\ell_c}{1 - \frac{1}{2}X'\frac{\sin 2k\ell}{k\ell} + \frac{1}{2}X\frac{\sin 2k(\ell+\ell_c)}{k(\ell+\ell_c)} + XX'\frac{\cos k\ell}{k\ell}\frac{\sin k(\ell+\ell_c)}{k(\ell+\ell_c)}\tan k(\ell+\ell_c)\sin k\ell_c} \right) .$$

In the limit of small perturbation, the length correction for a truncated cone is the sum of the length correction associated with both changes in taper.

B Effect of the resistive term in the hole impedance on resonance frequencies

The aim of this appendix is to give an analytic expression of the playing frequency in term of the resistive part R of the hole impedance. This can be interesting especially when resistive effects are large and can occur at high level when nonlinear effects appear, especially for narrow holes, like the register hole. This appendix describes the calculations which lead to the following expression for the frequency of the played tone

$$f = \frac{f_o + Af_c}{1+A} \tag{B.1}$$

where f_o and f_c are the frequencies obtained when the resistance term is zero and infinite respectively (when R is infinite, the effect of the hole disappears as it was closed), and A is a dimensionless factor given by

$$A = \left(\frac{2R}{\pi}\right)^2 \frac{1}{\ell_{hole}}\frac{L\ell}{\ell_d + \Delta\ell} ,$$

with L the total length of the resonator, ℓ the hole location, $\ell_{hole} = h'S/S_h$ and $\Delta\ell$ the equivalent length of the hole. As done in this study for the case of open hole, the series impedances Z_a are assumed to be small (see section 3.3.1).

Using dimensionless impedance quantities, the impedance equivalent Z_{eq} to the tone hole impedance and to the tube downstream the discontinuity can be written at the discontinuity point as

$$Z_{eq} = Z_h \,/\!/\, Z_{down} ,$$

where $Z_h = R + jX$ is the tone hole impedance and Z_{down} the impedance of the tube donwstream the discontinuity. Then, the input impedance Z_{in} of the

system can be derived as follows:

$$Z_{in} = \frac{-t + jZ_hT\left(\dfrac{1+t^2}{T-t}\right)}{Z_h\left(\dfrac{1+t^2}{T-t}\right) + j} \ , \tag{B.2}$$

where

- Z_{in} is the input impedance of the system ;
- $Z_h = R + jX$ is the tone hole impedance ;
- $t = \tan k\ell$;
- $T = \tan kL$.

For the resonance frequencies, the imaginary part of the input impedance vanishes, i.e

$$\mathcal{I}m(Z_{in}) = 0 \Longleftrightarrow \left(\frac{T-t}{1+t^2} + X\right)\left(t\frac{T-t}{1+t^2} + XT\right) + TR^2 = 0 \ . \tag{B.3}$$

Denoting

$$A = \sin k(L - 2\ell) + \sin kL + 2\chi\cos kL \ ,$$
$$B = \cos k(L - 2\ell) - \cos kL + 2\chi\sin kL \ ,$$
$$C = 2R^2\sin 2kL \ ,$$

where $\chi = k\ell_{hole}$, equation (B.3) can be rewritten in the form

$$AB + C = 0 \ . \tag{B.4}$$

By writing $kL = k_1L + \epsilon_1$ where k_1 is the wavenumber of the fundamental frequency of the played tone defined as $A = 0$ (when R vanishes):

$$\sin k_1(L - 2\ell) + \sin k_1L + 2\chi\cos k_1L = 0 \ , \tag{B.5}$$

the use of Taylor's formula to the first order for the A and B quantities results in:

$$A = \epsilon_1[(L - 2\ell)\cos k_1(L - 2\ell) + L\cos k_1L + 2\ell_{hole}\cos k_1L - 2\chi L\sin k_1L] \ , \tag{B.6}$$

$$\begin{aligned} B = &\cos k_1(L - 2\ell) - \cos k_1L + 2\chi\sin k_1L \\ &+ \epsilon_1[L\sin k_1L - (L - 2\ell)\sin k_1(L - 2\ell) + 2\ell_{hole}\sin k_1L \\ &+ 2\chi L\cos k_1L] \ . \tag{B.7} \end{aligned}$$

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Neglecting terms of second order in ϵ_1 and using equation (B.5), the quantity AB becomes:

$$AB = \left(\frac{1}{\cos k_1 \ell_{hole}}[L(1 + \cos 2k_1\ell) + 2\ell_{hole}\cos^2 kL - 2\ell\cos k(L - 2\ell)\cos kL\right)$$

$$\left(\frac{\cos 2k_1\ell_d - 1}{\cos k_1 L}\right). \quad \text{(B.8)}$$

In the limit of $k_1\ell_d \ll 1$ (i.e $\frac{\ell_d}{\ell}$ and $\frac{\Delta\ell}{\ell}$ are smaller than unity) and noting $k_1 = \frac{\pi}{2(\ell + \Delta\ell)}$, we have:

$$\cos k_1 L = -\frac{\pi}{2}\frac{\ell_d{}^2}{\ell(\ell_{hole} + \ell)} + o(\frac{\ell_d}{\ell})$$

$$\cos 2k_1\ell_d = 1 + o(\frac{\ell_d}{\ell})$$

$$\cos k_1(L - 2\ell) = \sin\frac{\pi}{2}\frac{\Delta\ell + \ell_d}{2} = \frac{\pi}{2}\frac{\Delta\ell + \ell_d}{2} + o(\frac{\ell_d}{\ell})$$

$$\cos 2k_1\ell = -1 + o(\frac{\ell_d}{\ell}) .$$

Substituting these results in equation (B.4) leads to the following equation

$$\left(\pi^2\ell_{hole}\frac{\ell_d + \Delta\ell}{\ell}\right)\epsilon_1 + 2R^2\sin 2kL = 0 .$$

Noting $kL = k_cL + \epsilon_2$ where $k_cL = \frac{\pi}{2}$ (k_c is the wavenumber when the hole is closed), it can be written :

$$\pi^2\ell_{hole}\frac{\ell_d + \Delta\ell}{\ell}(k - k_1) + 4R^2L(k - k_c) = 0 , \quad \text{(B.9)}$$

therefore,

$$f = \frac{f_o + Af_c}{1 + A} , \quad \text{(B.10)}$$

where

$$A = \left(\frac{2R}{\pi}\right)^2\frac{1}{\ell_{hole}}\frac{L\ell}{\ell_d + \Delta\ell} , \quad \text{(B.11)}$$

f_o is the frequency of the tone with no resistance, and f_c is the frequency of the tone with an infinite resistance term.

A careful limiting process shows that all is in order in the two extreme cases i.e when $R \to 0$ and $R \to \infty$. In the limit that R goes to zero, equation (B.10) gives the frequency to tend to f_o, the frequency of the tone when the hole is open. In the opposite limit when R tends to infinity, equation (B.1) gives the frequency to be f_c. Figure B.1 shows that the playing frequencies given by

equation (B.10) coincide quite well with the playing frequencies obtained by the zero values of the imaginary part of the input admittance. Moreover, in
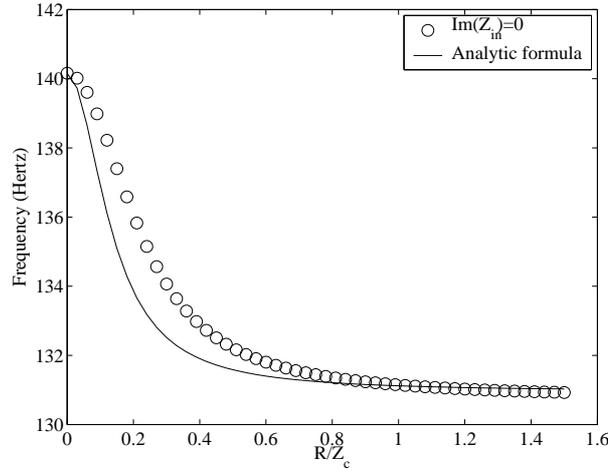

Fig. B.1. *Playing frequency as a function of the real part of the tone hole impedance ($R = 10$ mm, $r = 7$ mm,$h = 9$ mm). In the limit that $R \to 0$, the frequency of the played tone tends to f_o, the frequency obtained when the hole is open. In the limit that $R \to \infty$, the frequency of the played tone approaches f_c, the frequency when the hole being closed.*

the limit that R is much smaller than unity, it follows

$$f = f_o + A(f_f - f_o) \, . \tag{B.12}$$

With the use of experimental values obtained by Dalmont et al. [20] about the non linear behavior of an open side hole, it appears that the resistive term of the shunt impedance has small effects in the determination of the playing frequency. For instance, a 5-cent difference is obtained with $R/Z_c = 0$ and $R/Z_c = 0.05$.

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