The hyperbolic, the arithmetic and the quantum phase

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Abstract. We develop a new approach of the quantum phase in an Hilbert space of finite dimension which is based on the relation between the physical concept of phase locking and mathematical concepts such as cyclotomy and the Ramanujan sums. As a result, phase variability looks quite similar to its classical counterpart, having peaks at dimensions equal to a power of a prime number. Squeezing of the phase noise is allowed for specific quantum states. The concept of phase entanglement for Kloosterman pairs of phase-locked states is introduced.

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1. Introduction

Time and phase are not well defined concepts at the quantum level. The present work belongs to a longstanding effort to model phase noise and phase-locking effects that are found in highly stable oscillators. It was unambiguously demonstrated that the observed variability, (i.e., the 1/f frequency noise of such oscillators) is related to the finite dynamics of states during the measurement process and to the precise filtering rules that involve continued fraction expansions, prime number decomposition, and hyperbolic geometry [1]-[4].

We add here a quantum counterpart to these effects by studying the notions of quantum phase-locking and quantum phase entanglement. The problem of defining quantum phase operators was initiated by Dirac in 1927 [5]. For excellent reviews, see [6]. We use the Pegg and Barnett quantum phase formalism [7] where the calculations are performed in an Hilbert space H_q of finite dimension q. The phase states are defined as superpositions of number states from the so-called quantum Fourier transform (or QFT)

$$|\theta_p\rangle = q^{-1/2}\sum_{n=0}^{q-1}\exp(\frac{2i\pi pn}{q})|n\rangle \ . \tag{1}$$

in which $i^2=-1$. The states $|\theta_p\rangle$ form an orthonormal set and in addition the projector over the subspace of phase states is $\sum_{p=0}^{q-1}|\theta_p\rangle\langle\theta_p| = 1_q$ where 1_q is the identity operator in H_q. The inverse quantum Fourier transform follows as $|n\rangle = q^{-1/2}\sum_{p=0}^{q-1}\exp(-\frac{2i\pi pn}{q})|\theta_p\rangle$. As the set of number states $|n\rangle$, the set of phase states $|\theta_p\rangle$ is a complete set spanning H_q. In addition the QFT operator is a q by q unitary matrix with matrix elements $\kappa_{pn}^{(q)} = \frac{1}{\sqrt{q}}\exp(2i\pi\frac{pn}{q})$.

From now we emphasize phase states $|\theta'_p\rangle$ satisfying phase-locking properties. The quantum phase-locking operator is defined in (19). We first impose the coprimality condition

$$(p,q)=1, \tag{2}$$

where (p,q) is the greatest common divisor of p and q. Differently from the phase states (1), the $|\theta'_p\rangle$ form an orthonormal base of a Hilbert space whose dimension is lower, and equals the number of irreducible fractions p/q, which is given by the Euler totient function $\phi(q)$. These states were studied in our recent publication [3]‖.

Guided by the analogy with the classical situation [2], we call these irreducible states the phase-locked quantum states. They generate a cyclotomic lattice L [8] with generator matrix M of matrix elements $\kappa_{pn}^{'(q)}$, (p,q)=1 and of size $\phi(q)$. The corresponding Gram matrix $H=M^\dagger M$ shows matrix elements $h_{n,l}^{(q)} = c_q(n-l)$ which

‖ Some errors or misunderstandings are present in that earlier report. The summation in (3),(5),(7) and (9) should be (this is implicit) from 0 to $\phi(q)$. The expectation value $\langle\theta_q^{\mathrm{lock}}\rangle$ in (8) should be squared. There are also slight changes in the plots.

are Ramanujan sums

$$c_q(n) = \sum_p \exp(2i\pi \frac{p}{q} n) = \frac{\mu(q_1)\phi(q)}{\phi(q_1)}, \quad \text{with } \mathrm{q}_1 = \mathrm{q}/(\mathrm{q},\mathrm{n}). \tag{3}$$

where the index p means summation from 0 to $q-1$, and $(p,q)=1$. Ramanujan sums are thus defined as the sums over the primitive characters $\exp(2i\pi\frac{pn}{q})$, (p,q)=1, of the group $Z_q = Z/qZ$. In the equation above $\mu(q)$ is the Möbius function, which is 0 if the prime number decomposition of q contains a square, 1 if $q=1$, and $(-1)^k$ if q is the product of k distinct primes [9]. Ramanujan sums are relative integers which are quasi-periodic versus n with quasi-period $\phi(q)$ and aperiodic versus q with a type of variability imposed by the Möbius function. Ramanujan sums were introduced by Ramanujan in the context of Goldbach conjecture [9].

They are also useful in the context of signal processing as an arithmetical alternative to the discrete Fourier transform [10]. In the discrete Fourier transform the signal processing is performed by using all roots of unity of the form $\exp(2i\pi p/q)$ with p from 1 to q and taking their nth powers $e_p(n)$ as basis function. We generalized the classical Fourier analysis by using Ramanujan sums $c_q(n)$ as in (3) instead of $e_p(n)$. This type of signal processing is more appropriate for arithmetical functions than is the ordinary discrete Fourier transform, while still preserving the metric and orthogonal properties of the latter. Notable results relating arithmetical functions to each other can be obtained using Ramanujan sums expansion while the discrete Fourier transform would show instead the low frequency tails in the power spectrum.

In this paper we are also interested in pairs of phase-locked states which satisfy the two conditions

$$(p,q) = 1 \;\; \mathit{and} \;\; p\bar{p} = -1(\mathit{mod}\; q). \tag{4}$$

Whenever it exists $\bar{p}$ is uniquely defined from minus the inverse of p modulo q. Geometrically the two fractions p/q and $\bar{p}/q$ are the ones selected from the partition of the half plane by Ford circles. Ford circles are defined as the set of the images of the horizontal line z=x+i, x real, under all modular transformations in the group of 2×2 matrices $SL(2,Z)$ [4]. Ford circles are tangent to the real axis at a Farey fraction p/q, and are tangent to each other. They have been introduced by Rademacher as an optimum integration path to compute the number of partitions by means of the so-called Ramanujan's circle method. In that method two circles of indices p/q and $\bar{p}/q$, of the same radius $\frac{1}{2q^2}$ are dual to each other on the integration path [see also Part 2 and Fig. 3].

2. Hints to the hyperbolic geometry of phase noise

2.1. The phase-locked loop, low pass filtering and $1/f$ noise

A newly discovered clue for $1/f$ noise was found from the concept of a phase locked loop (or PLL) [2]. In essence two interacting oscillators, whatever their origin, attempt to cooperate by locking their frequency and their phase. They can do it by exchanging continuously tiny amounts of energy, so that both the coupling coefficient and the beat frequency should fluctuate in time around their average value. Correlations between amplitude and frequency noises were observed [11].

One can get a good level of understanding of phase locking by considering the case of quartz crystal oscillators used in high frequency synthesizers, ultrastable clocks and communication engineering (e.g., mobile phones). The PLL used in a FM radio receiver is a genuine generator of $1/f$ noise. Close to phase locking the level of $1/f$ noise scales approximately as $\tilde{\sigma}^2$, where $\tilde{\sigma} = \sigma K/\tilde{\omega_B}$ is the ratio between the open loop gain K and the beat frequency $\tilde{\omega_B}$ times a coefficient σ whose origin has to be explained. The relation above is explained from a simple non linear model of the PLL known as Adler's equation

$$\dot{\theta}(t) + KH(P)\sin\theta(t) = \omega_B, \tag{5}$$

where at this stage $H(P) = 1$, $\omega_B = \omega(t) - \omega_0$ is the angular frequency shift between the two quartz oscillators at the input of the non linear device (a Schottky diode double balanced mixer), and $\theta(t)$ is the phase shift of the two oscillators versus time t. Solving (5) and differentiating one gets the observed noise level $\tilde{\sigma}$ versus the bare one $\sigma = \delta\omega_B/\tilde{\omega_B}$. Thus the model doesn't explain the existence of $1/f$ noise but correctly predicts its dependence on the physical parameters of the loop [2].

Besides one can get detailed knowledge of harmonic conversions in the PLL by accounting for the transfer function $H(P)$, where $P = \frac{d}{dt}$ is the Laplace operator. If $H(P)$ is a low pass filtering function with cut-off frequency f_c, the frequency at the output of the mixer + filter stage is such that

$$\mu = f_B(t)/f_0 = q_i|\nu - p_i/q_i| \leq f_c/f_0, \quad p_i \text{ and } q_i \text{ integers.} \tag{6}$$

The beat frequency $f_B(t)$ results from the continued fraction expansion of the input frequency ratio

$$\nu = f(t)/f_0 = [a_0; a_1, a_2, \ldots a_i, a, \ldots], \tag{7}$$

where the brackets mean expansions $a_0+1/(a_1+1/(a_2+1/\ldots+1/(a_i+1/(a+\ldots))))$. The truncation at the integer part $a = [\frac{f_0}{f_c q_i}]$ defines the edges of the basin for the resonance p_i/q_i; they are located at $\nu_1 = [a_0; a_1, a_2, \ldots, a_i, a]$ and $\nu_2 = [a_0; a_1, a_2, \ldots a_i - 1, 1, a]$ [1]. The two expansions in ν_1 and ν_2, prior to the last filtering partial quotient a, are the two allowed ones for a rational number. The convergents p_i/q_i at level i are obtained using the matrix products

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}\cdots\begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_i & p_{i-1} \\ q_i & q_{i-1} \end{bmatrix}. \tag{8}$$

Using (8), one can get the fractions ν_1 and ν_2 as $\nu_1 = \frac{p_a}{q_a}$ and $\nu_2 = \frac{p_i(2a+1)-p_a}{q_i(2a+1)-q_a}$, so that with the relation relating convergents $(p_i q_{i-1} - p_{i-1}q_i) = (-1)^{i-1}$, the width of the basin of index i is $|\nu_1 - \nu_2| = \frac{2a+1}{q_a(q_a+(2a+1)q_i)} \simeq \frac{1}{q_a q_i}$ whenever $a > 1$.

In previous publications of one of the authors a phenomenological model for $1/f$ noise in the PLL was proposed, based on an arithmetical function which is a logarithmic coding for prime numbers [1],[2]. If one accepts a coupling coefficient evolving discontinuously versus the time n as $K = K_0\Lambda(n)$, with $\Lambda(n)$ the Mangoldt function which is $\ln(p)$ if n is the power of a prime number p and 0 otherwise, then the average coupling coefficient is K_0 and there is an arithmetical fluctuation $\epsilon(t)$

$$\psi(t) = \sum_{n=1}^{t} \Lambda(n) = t(1+\epsilon(t)),$$
$$t\epsilon(t) = -\ln(2\pi) - \frac{1}{2}\ln(1-t^{-2}) - \sum_{\rho} \frac{t^{\rho}}{\rho}. \quad (9)$$

The three terms at the right hand side of $t\epsilon(t)$ come from the singularities of the Riemann zeta function $\zeta(s)$, that are the pole at $s = 1$, the trivial zeros at $s = -2l$, l integer, and the zeros on the critical line $\Re(s) = \frac{1}{2}$[1]. Moreover the power spectral density roughly shows a $1/f$ dependance versus the Fourier frequency f. This is the proposed relation between Riemann zeros (the still unproved Riemann hypothesis is that all zeros should lie on the critical line) and $1/f$ noise.

We improved the model by replacing the Mangoldt function by its modified form $b(n) = \Lambda(n)\phi(n)/n$, with $\phi(n)$ the Euler (totient) function [10]. This seemingly insignificant change was introduced by Hardy [9] in the context of Ramanujan sums for the Goldbach conjecture and resurrected by Gadiyar and Padma in their recent analysis of the distribution of pairs of prime numbers [12]. Then by defining the error term $\epsilon_B(t)$ from the cumulative modified Mangoldt function

$$B(t) = \sum_{n=1}^{t} b(n) = t(1+\epsilon_B(t)), \quad (10)$$

its power spectral density $S_B(f) \simeq \frac{1}{f^{2\alpha}}$ exhibits a slope close to the Golden ratio $\alpha \simeq (\sqrt{5}-1)/2 \simeq 0.618$ (see Fig. 1).

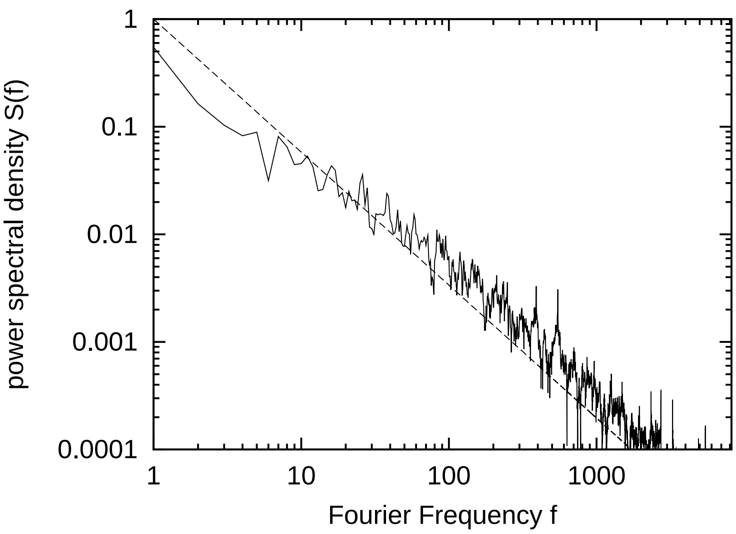

Figure 1. Power spectral density of the error term in the modified Mangoldt function b(n) in comparison to the power law $1/f^{2\alpha}$, with the Golden ratio $\alpha = (\sqrt{5}-1)/2$.

The modified Mangoldt function occurs in a natural way from the logarithmic derivative of the following quotient

$$Z(s) = \frac{\zeta(s)}{\zeta(s+1)} = \sum_{n\geq 1} \frac{\phi(n)}{n^{s+1}}, \tag{11}$$

since $-\frac{Z'(s)}{Z(s)} = \sum_{n\geq 1} \frac{b(n)}{n^s}$. This replaces the similar relation from the Riemann zeta function where $-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}$.

In the studies of $1/f$ noise, the fast Fourier transform (FFT)plays a central role. But the FFT refers to the fast calculation of the discrete Fourier transform (DFT) with a finite period $q = 2^l$, l a positive integer. In the DFT one starts with all q^{th} roots of the unity $\exp(2i\pi p/q)$, $p = 1 \ldots q$ and the signal analysis of the arithmetical sequence $x(n)$ is performed by projecting onto the nth powers (or characters of $Z/\text{q}Z$) with well known formulas.

The signal analysis based on the DFT is not well suited to aperiodic sequences with many resonances (naturally a resonance is a primitive root of the unity: $(p, q) = 1$), and the FFT may fail to discover the underlying structure in the spectrum. We recently introduced a new method based on the Ramanujan sums defined in (3) [10].

Mangoldt function is related to Möbius function thanks to the Ramanujan sums expansion found by Hardy [12]

$$b(n) = \frac{\phi(n)}{n}\Lambda(n) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n). \tag{12}$$

We call such a type of Fourier expansion a Ramanujan-Fourier transform (RFT). General formulas are given in our recent publication [10] and in the paper by Gadiyar [12]. This author also reports on a stimulating conjecture relating the autocorrelation function of $b(n)$ and the problem of pairs of prime numbers. In the special case (12), it is clear that $\mu(q)/\phi(q)$ is the RFT of the modified Mangoldt sequence $b(n)$.

Using Ramanujan-Fourier analysis the $1/f^{2\alpha}$ power spectrum gets replaced by a new signature shown on Fig. 2, not very different of $\mu(q)/\phi(q)$ (up to a scaling factor).

2.2. The hyperbolic geometry of phase noise and $1/f$ frequency noise

The whole theory can be justified by studying the noise in the half plane $H = \{z = \nu + iy,\ i^2 = -1¶,\ y > 0\}$ of coordinates $\nu = \frac{f}{f_0}$ and $y = \frac{f_B}{f_c} > 0$ and by introducing the modular transformations

$$z \to \gamma(z) = z' = \frac{p_i z + p_i'}{q_i z + q_i'}, \quad p_i q_i' - p_i' q_i = 1. \tag{13}$$

The set of images of the filtering line $z = \nu + i$ under all modular transformations can be written as

$$|z' - (\frac{p_i}{q_i} + \frac{i}{2q_i^2})| = \frac{1}{2q_i^2}. \tag{14}$$

¶ The imaginary symbol i should not be confused with the index i in integers p_i, q_i and in related integers.

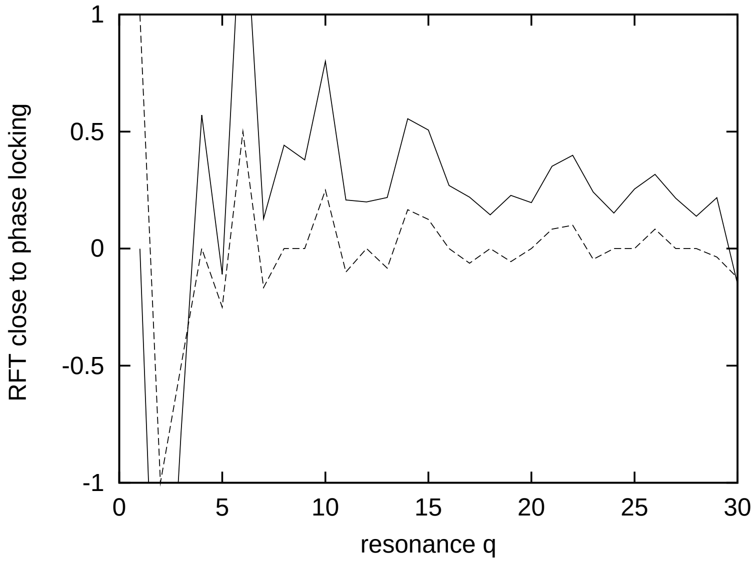

Figure 2. Ramanujan-Fourier transform (RFT) of the error term (upper curve) of modified Mangoldt function $b(n)$ in comparison to the function $\mu(q)/\phi(q)$(lower curve).

Equation (14) defines Ford circles (see Fig. 3) centered at points $z = \frac{p_i}{q_i} + \frac{i}{2q_i^2}$ with radius $\frac{1}{2q_i^2}$ [13]. To each $\frac{p_i}{q_i}$ a Ford circle in the upper half plane can be attached, which is tangent to the real axis at $\nu = \frac{p_i}{q_i}$. Ford circles never intersect: they are tangent to each other if and only if they belong to fractions which are adjacent in the Farey sequence $\frac{0}{1} < \cdots \frac{p_1}{q_1} < \frac{p_1+p_2}{q_1+q_2} < \frac{q_1}{q_2} \cdots < \frac{1}{1}$[13]. The half plane H is the model of Poincaré

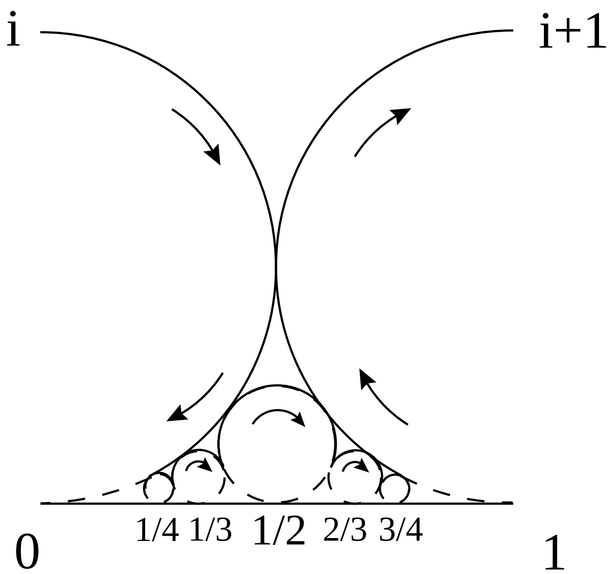

Figure 3. Ford circles: the mapping of the filtering line under modular transformations (13). The arrows indicates that Ford circles were used as an integration path by Rademacher to compute the partition function $p(n)$ [13].

hyperbolic geometry. A basic fact about the modular transformations (13) is that they form a discontinuous group $\Gamma \simeq SL(2,Z)/\{\pm 1\}$, which is called the modular group. The action of Γ on the half-plane H looks like the one generated by two independent linear translations on the Euclidean plane, which is equivalent to a tesselation the complex plane C with congruent parallelograms. One introduces the fundamental domain of Γ (or modular surface) $F = \{z \in H: \ |z| \geq 1, \ |\nu| \leq \frac{1}{2}\}$, and the family of domains $\{\gamma(F), \gamma \in \Gamma\}$ induces a tesselation of H [14].

It can be shown [4],[14] that the noise amplitude is a particular type of solution of the eigenvalue problem with the non-Euclidean Laplacian $\Delta = y^2(\frac{\partial^2}{\partial\nu^2} + \frac{\partial^2}{\partial y^2})$. The solution corresponds to the scattering of waves in the fundamental domain F. It can be approximated as a superposition of three contributions. The first one is an horizontal wave and is of the power law form y^s, the second one is also horizontal wave of the form $S(s)y^{1-s}$ and corresponds to a reflected wave with a scattering coefficient of modulus $|S(s)| = 1$, whereas the remaining part $T(y,\nu)$ is a complex superposition of waves depending of y and the harmonics of $\exp(2i\pi\nu)$, but going to zero for $y \to \infty$. Extracting the smooth part in $S(s)$ one is left with a random factor which is precisely equal to the function $Z(2s-1)$ defined in (11) as the quotient of two Riemann zeta functions at $2s-1$ and $2s$, respectively. An interesting case is when s is on the critical line, i.e. $s = \frac{1}{2} + ik$ in which case the superposition $T(y,\nu)$ vanishes and the reflexion coefficient is

$$S(k) = \exp[2i\theta(k)], \quad \text{with } \theta'(k) = \frac{d\ln S(s)}{ds} \ \text{ at } s = \frac{1}{2} + ik. \tag{15}$$

The scattering of waves from the modular surface is thus similar to the phase-locking model plotted in the set of equations (5)-(11). It explains the relationship between the hyperbolic phase and the $1/f$ noise found in the counting function $\theta'(k)$. The phase factor $\theta(k)$ is represented in Fig. 4.

3. Quantum phase-locking

3.1. The quantum phase operators

Going back to the quantum definition of phase states announced in the introduction one calculates the projection operator over the subset of phase-locked quantum states $|\theta'_p\rangle$ as

$$P_q^{\mathrm{lock}} = \sum_p |\theta'_p\rangle\langle\theta'_p| = \frac{1}{q}\sum_{n,l} c_q(n-l)|n\rangle\langle l|, \tag{16}$$

where the range of values of n, l is from 0 to $\phi(q)$. Thus the matrix elements of the projection are $q\langle n|P_q|l\rangle = c_q(n-l)$. This sheds light on the equivalence between cyclotomic lattices of algebraic number theory and the quantum theory of phase-locked states.

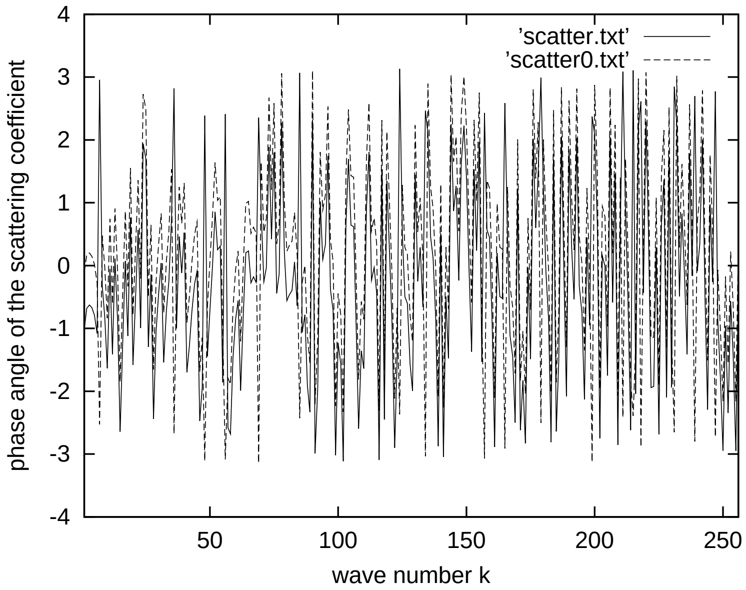

Figure 4. The phase angle $\theta(k)$ for the scattering of noise waves on the modular surface. Plain lines: Exact phase factor. Dotted lines: Approximation based on the quotient of two Riemann zeta functions[4] .

The projection operator over the subset of pairs of phase-locked quantum states $|\theta'_p\rangle$ is calculated as

$$P_q^{\mathrm{pairs}} = \sum_{p,\bar{p}} |\theta'_p\rangle\langle\theta'_{\bar{p}}| = \frac{1}{q}\sum_{n,l} k_q(n,l)|n\rangle\langle l|, \tag{17}$$

where the notation $p, \bar{p}$ means that the summation is applied to such pairs of states satisfying (4). The matrix elements of the projection are $q\langle n|P_q^{\mathrm{pairs}}|l\rangle = k_q(n,l)$, which are in the form of so-called Kloosterman sums [15]

$$k_q(n,l) = \sum_{p,\bar{p}} \exp[\frac{2i\pi}{q}(pn - \bar{p}l)]. \tag{18}$$

Kloosterman sums $k_q(n,l)$ as well as Ramanujan sums $c_q(n-l)$ are relative integers. They are given below for the Hilbert dimensions $q = 5(\phi(5) = 4)$ and $q = 6(\phi(6) = 2)$.

$$q = 5: \quad c_5 = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}, \quad k_5 = \begin{bmatrix} -1 & -1 & -1 & 4 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ 4 & -1 & -1 & -1 \end{bmatrix},$$

$$q = 6: \quad c_6 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad k_6 = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}.$$

One defines the quantum phase-locking operator as

$$\Theta_q^{\mathrm{lock}} = \sum_p \theta_p |\theta'_p\rangle\langle\theta'_p| = \pi P_q^{\mathrm{lock}} \quad \text{with } \theta_{\mathrm{p}} = 2\pi\frac{\mathrm{p}}{\mathrm{q}}. \tag{19}$$

The Pegg and Barnett operator [7] is obtained by removing the coprimality condition. It is Hermitian with eigenvalues θ_p. Using the number operator $N_q = \sum_{n=0}^{q-1} n|n\rangle\langle n|$ a generalization of Dirac's commutator $[\Theta_q, N_q] = -i$ has been obtained.

Similarly one defines the quantum phase operator for Kloosterman pairs as

$$\Theta_q^{\mathrm{pairs}} = \sum_{p,\bar{p}} \theta_p |\theta'_p\rangle\langle\theta'_{\bar{p}}| = \pi P_q^{\mathrm{pairs}} \text{ with } \theta_{\mathrm{p}} = 2\pi\frac{\mathrm{p}}{\mathrm{q}}. \tag{20}$$

The phase number commutator for phase-locked states calculated from (19) is

$$C_q^{\mathrm{lock}} = [\Theta_q^{\mathrm{lock}}, N_q] = \frac{\pi}{q}\sum_{n,l}(l-n)c_q(n-l)|n\rangle\langle l|, \tag{21}$$

with antisymmetric matrix elements $\langle l|C_q^{\mathrm{lock}}|n\rangle = \frac{\pi}{q}(l-n)c_q(n-l)$.

For pairs of phase-locked states an antisymmetric commutator C_q^{pairs} similar to (21) is obtained with $k_q(n,l)$ in place of $c_q(n-l)$.

3.2. Phase expectation value and variance

The finite quantum mechanical rules are encoded in the expectation values of the phase operator and phase variance.

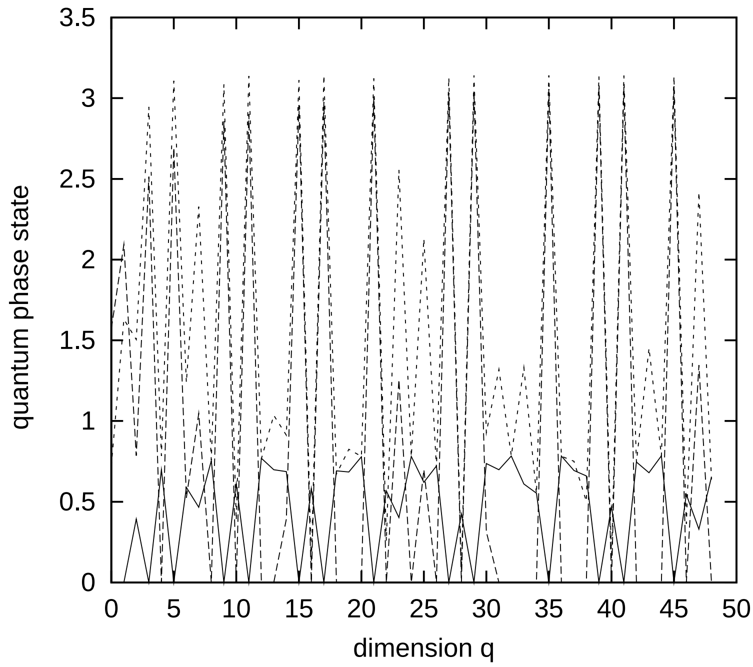

Figure 5. Oscillations in the expectation value (23) of the locked phase at $\beta = 1$ (dotted line) and their squeezing at $\beta = 0$ (plain line). The brokenhearted line which touches the horizontal axis is $\pi\Lambda(q)/\ln q$.

Rephrasing Pegg and Barnett, let us consider a pure phase state $|f\rangle = \sum_{n=0}^{q-1} u_n|n\rangle$ having u_n of the form

$$u_n = (1/\sqrt{q})\exp(in\beta), \tag{22}$$

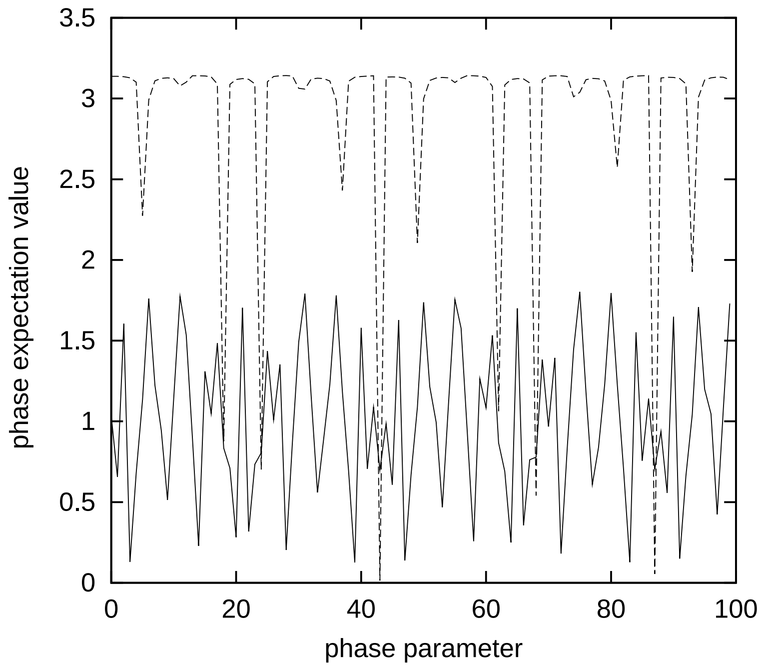

Figure 6. Phase expectation value versus the phase parameter β. Plain lines: $q = 15$. Dotted lines $q = 13$.

where β is a real phase parameter. One defines the phase probability distribution $\langle\theta'_p|f\rangle^2$, the phase expectation value $\langle\Theta_q^{\mathrm{lock}}\rangle = \sum_p \theta_p \langle\theta'_p|f\rangle^2$, and the phase variance $(\Delta\Theta_q^2)^{\mathrm{lock}} = \sum_p(\theta_p - \langle\Theta_q^{\mathrm{lock}}\rangle)^2\langle\theta'_p|f\rangle^2$. One gets

$$\langle\Theta_q^{\mathrm{lock}}\rangle = \frac{\pi}{q^2}\sum_{n,l} c_q(l-n)\exp[i\beta(n-l)], \tag{23}$$

$$(\Delta\Theta_q^2)^{\mathrm{lock}} = 4\langle\tilde{\Theta}_q^{\mathrm{lock}}\rangle + \frac{\langle\Theta_q\rangle^2}{\pi}(\langle\Theta_q\rangle - 2\pi), \tag{24}$$

with the modified expectation value $\langle\tilde{\Theta}_q^{\mathrm{lock}}\rangle = \frac{\pi}{q^2}\sum_{n,l}\tilde{c}_q(l-n)\exp[i\beta(n-l)]$, and the modified Ramanujan sums $\tilde{c}_q(n) = \sum_p (p/q)^2\exp(2i\pi m\frac{p}{q})$.

Fig. 5 illustrates the phase expectation value versus the dimension q for two different values of the phase parameter β. For $\beta = 1$ they are peaks at dimensions $q = p^r$ which are powers of a prime number p. The most significant peaks are fitted by the function $\pi\Lambda(q)/\ln q$, where $\Lambda(q)$ is the Mangoldt function introduced in (9) of Sect.2. This observation provides the link between the arithmetical hyperbolic viewpoint and the quantum one. A deepest explanation based on the relation with quantum statistical mechanics and the work of Bost and Connes can be found in [17]. For $\beta = 0$ the peaks are smoothed out due to the averaging over the Ramanujan sums matrix. Fig. 6 shows the phase expectation value versus the phase parameter β. For the case of the prime number $q = 13$, the mean value is high with absorption like lines at isolated values of β. For the case of the dimension $q = 15$ which is not a prime power the phase expectation is much lower in value and much more random.

Fig. 7 illustrates the phase variance versus the dimension q. Again the case

$\beta = 1$ leads to peaks at prime powers. Like the expectation value in Fig. 5, it is thus reminiscent of the Mangoldt function. Mangoldt function $\Lambda(n)$ is defined as $\ln p$ if n is the power of a prime number p and 0 otherwise. It arises in the frame of prime number theory [1] from the logarithmic derivative of the Riemann zeta function $\zeta(s)$ as $-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=0}^{\infty} \frac{\Lambda(n)}{n^s}$. Its average value oscillates about 1 with an error term which is explicitely related to the positions of zeros of $\zeta(s)$ on the critical line $s = \frac{1}{2}$. The error term shows a power spectral density close to that of $1/f$ noise [1]. It is stimulating to recover results reminding prime number theory in the new context of quantum phase-locking.

Finally, the phase variance is considerably smoothed out for $\beta = \pi$ and is much lower than the classical limit $\pi^2/3$. The parameter β can thus be interpreted as a squeezing parameter since it allows to define quantum phase-locked states having weak phase variance for a whole range of dimensions.

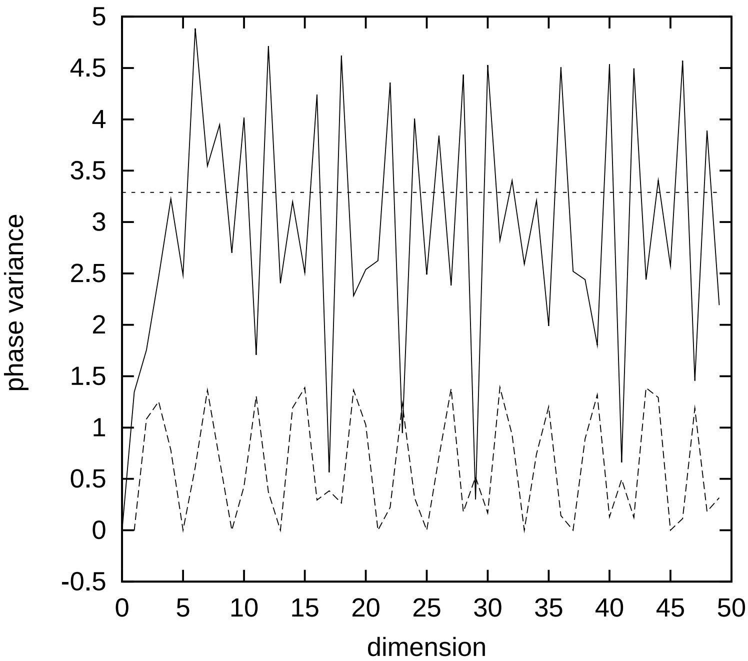

Figure 7. Phase variance versus the dimension q of the Hilbert space. Plain lines: $\beta = 1$. Dotted lines: $\beta = \pi$.

3.3. Towards discrete phase entanglement

The expectation value of quantum phase states can be rewritten using the projection operator of individual phase states $\pi_p = |\theta'_p\rangle\langle\theta'_p|$ as follows

$$\langle\Theta_q^{\mathrm{lock}}\rangle = \sum_p \theta_p \langle f|\theta'_p\rangle\langle\theta'_p|f\rangle = \sum_p \theta_p \langle f|\pi_p|f\rangle. \tag{25}$$

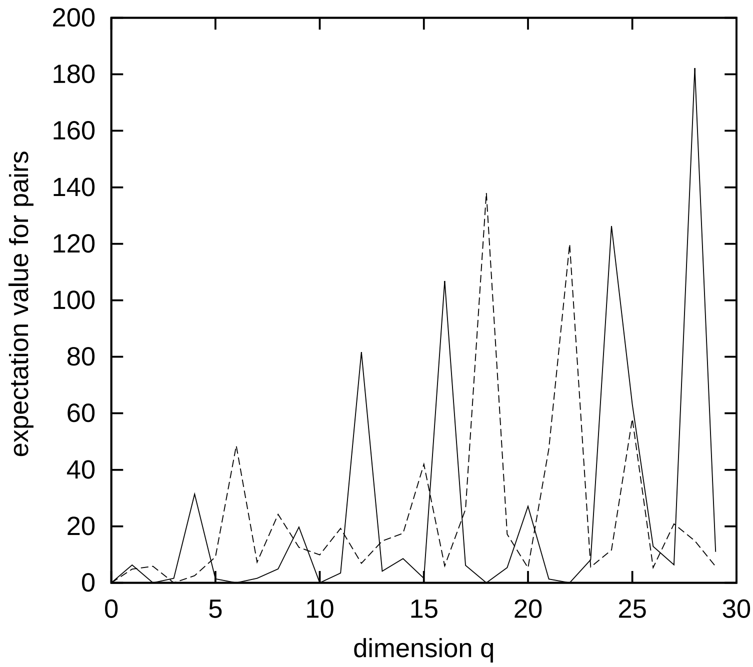

Figure 8. Phase expectation value versus the dimension q for pairs of phase-locked states. Plain lines: $\beta = 0$. Dotted lines: $\beta = 1$.

This suggests a definition of expectation values for pairs based on the product $\pi_p\pi_{\bar{p}}$ as follows

$$\langle \Theta_q^{\mathrm{pairs}} \rangle = \sum_{p,\bar{p}} \theta_p \langle f|\pi_p\pi_{\bar{p}}|f\rangle. \tag{26}$$

It is inspired by the quantum calculation of correlations in Bell's theorem [16]. Using pure phase states as in (22) we get

$$\langle \Theta_q^{\mathrm{pairs}} \rangle = \frac{2\pi}{q^2} \sum_{n,l} \tilde{k_q}(n,l) \exp[i\beta(n-l)], \tag{27}$$

where we introduced generalized Kloosterman sums $\tilde{k}_q(n,l) = \sum_{p,\bar{p}} p \exp[\frac{2i\pi}{q}(p-\bar{p})(l-n)]$. These sums are in general complex numbers (and are not Gaussian integers). The expectation value is real as expected. In Fig. 8 it is represented versus the dimension q for two different values, $\beta = 0$ and $\beta = 1$, respectively. Note that the pair correlation (26) is very strongly dependent on q and becomes quite huge at some values.

This result suggests that a detailed study of Bell's type inequalities based on quantum phase-locked states, and their relationship to the properties of numbers, should be undertaken. Calculations involving fully entangled states

$$|f\rangle = \frac{1}{q} \sum_{p,\bar{p}} |\theta_p, 1\rangle \otimes |\theta_{\bar{p}}, 2\rangle, \tag{28}$$

have to be carried out. This is left for future work.

Table 1. $(Z/7Z)^*$ is a cyclic group of order $\phi(7) = 6$.

α	1	2	3	4	5	6	7	8
3^α	3	2	6	4	5	1	3	2

Table 2. $(Z/3^2Z)^*$, is a cyclic group of order $\phi(9) = 6$.

α	1	2	3	4	5	6	7	8
2^α	2	4	8	7	5	1	2	4

Table 3. $(Z/8Z)^*$ has a largest cyclic group of order $\lambda(8) = 2$.

α	1	2	3	4	5	6	7	8
2^α	3	1	3	1	3	1	3	1

3.4. The discrete phase: cycles in Z/qZ

There is a scalar viewpoint for the above approach, which emphasizes well the intricate order of the group Z/qZ, the group of integers modulo q. One asks the question: what is the largest cycle in that group. For that purpose one looks at the primitive roots, which are the solutions g of the equation

$$g^\alpha \equiv 1 (\mathrm{mod}\ q), \tag{29}$$

such that the equation is wrong for any $1 \leq \alpha < q - 1$ and true only for $\alpha = q - 1$. If q=p, a prime number, and $p = 7$, the largest period is thus $\phi(p)$=p-1=6, and the cycle is as given in Table I. If $q = 2$, 4, $q = p^r$, a power a prime number > 2, or $q = 2p^r$, twice the power of a prime number > 2, then a primitive root exists, and the largest cycle in the group is $\phi(q)$. For example $g = 2$ and $q = 3^2$ leads to the period $\phi(9) = 6 < q - 1 = 8$, as shown in Table II. Otherwise there is no primitive root. The period of the largest cycle in Z/qZ can still be calculated and is called the Carmichael Lambda function $\lambda(q)$. It is shown in Table III for the case $g = 3$ and $q = 8$. It is $\lambda(8) = 2 < \phi(8) = 4 < 8 - 1 = 7$. Fig. 9 shows the properly normalized period for the cycles in Z/qZ. Its fractal character can be appreciated by looking at the corresponding power spectral density shown in Fig. 10. It has the form of a $1/f^\alpha$ noise, with $\alpha = 0.70$. For a more refined link between primitive roots g, cyclotomy and Ramanujan sums see also [18].

4. Conclusion

In conclusion, we explained how useful could be the concepts of prime number theory in explaining various features of phase-locking at the classical and quantum level. In the classical realm we reminded the hyperbolic geometry of phase, which occurs when

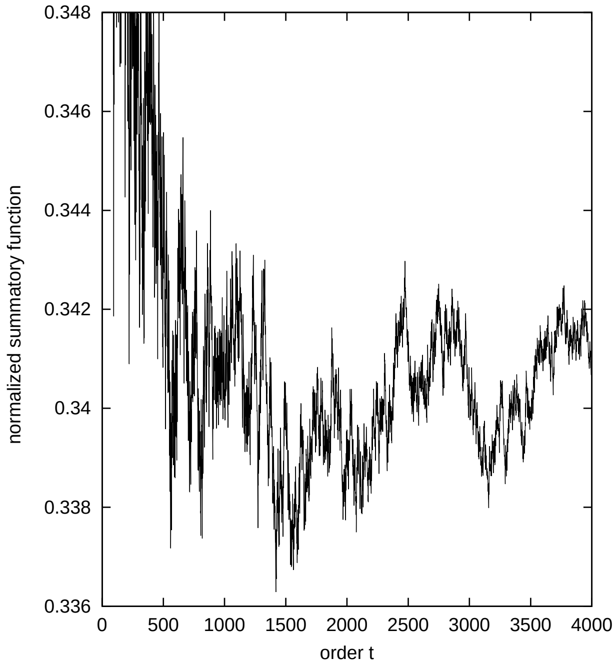

Figure 9. Normalized Carmichael lambda function: $(\sum_1^t \lambda(n))/t^{1.90}$.

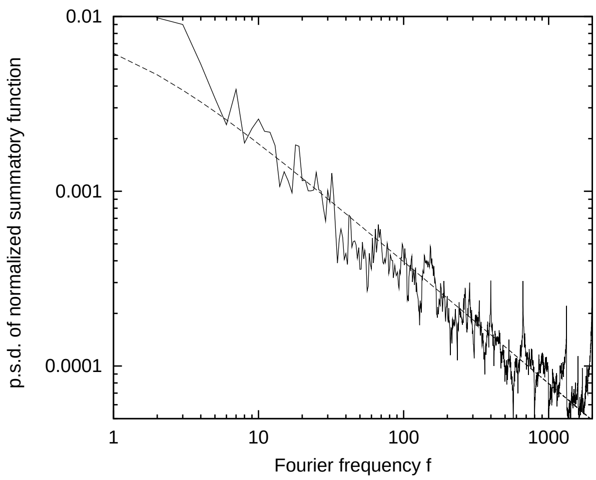

Figure 10. FFT of the normalized Carmichael lambda function. The staight line has slope -0.70.

one accounts for all harmonics in the mixing and low-pass filtering process, how $1/f$ frequency noise is produced and how it is related to Mangoldt function, and thus to the critical zeros of Riemann zeta function. Then we studied several properties resulting from introducing phase-locking in Pegg-Barnett quantum phase formalism. The idea of quantum teleportation was initially formulated by Bennett et al in finite-dimensional Hilbert space [19], but, yet independently of this, one can conjecture that cyclotomic aspects in phase-locking could play an important role in many fundamental tests of quantum mechanics related to quantum entanglement. Munro and Milburn [20] already conjectured that the best way to see the quantum nature of correlations in entangled states is through the measurement of the observable canonically conjugate to photon number, i.e. the quantum phase. In their paper dealing with the Greenberger-Horne-Zeilinger quantum correlations, they presented a homodyne scheme requiring discrete phase measurement. We expect that the interplay between quantum mechanics and number theory will appear repetitively in the coming attempts to manipulate quantum information [21].

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