Wigner measures in the discrete setting: high-frequency analysis of sampling & reconstruction operators

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Abstract

The goal of this article is that of understanding how the oscillation and concentration effects developed by a sequence of functions in $\mathbb{R}^d$ are modified by the action of Sampling and Reconstruction operators on regular grids. Our analysis is performed in terms of Wigner and defect measures, which provide a quantitative description of the high frequency behavior of bounded sequences in $L^2\left(\mathbb{R}^d\right)$. We actually present explicit formulas that make possible to compute such measures for sampled/reconstructed sequences. As a consequence, we are able to characterize sampling and reconstruction operators that preserve or filter the high-frequency behavior of specific classes of sequences. The proofs of our results rely on the construction and manipulation of Wigner measures associated to sequences of discrete functions.

Key words. Wigner Measures; Sampling and Reconstruction; High-frequency analysis; Concentration and Oscillation; Weak Convergence; Weak Compactness; Shift-invariant Spaces.

MSC. 42C15; 94A12; 65D05; 46E35; 46E39.

Contents

1 Introduction

1.1 Statement of the problem: oscillation and concentration under the effect of sampling and reconstruction

A central problem in Numerical Analysis and Signal Theory is that of reconstructing a function $u(x)$ defined in $\mathbb{R}^d$ from a discrete set of measurements taken on an uniform grid of step size h. This discrete values are typically obtained by applying to the function u a **sampling operator** S_φ^h of the following type:

$$S_\varphi^h u\left(n\right) := \frac{1}{h^d}\int_{\mathbb{R}^d} u\left(x\right)\overline{\varphi\left(\frac{x}{h}-n\right)}dx,$$

for some **sampling function** φ. One then tries to recover u by means of a **reconstruction** (or **interpolation**) **operator** T_ψ^h through the formula:

$$T_\psi^h S_\varphi^h u\left(x\right) := \sum_{n\in\mathbb{Z}^d} S_\varphi^h u\left(n\right)\psi\left(\frac{x}{h}-n\right), \tag{1}$$

where ψ is some fixed **reconstruction function**. This process usually only provides an approximation of the original function u, with an error that vanishes as h tends to zero. Such reconstruction schemes have been the object of intensive study both from the point of view of Approximation Theory and Numerical Analysis.

Here we shall be concerned with the **high-frequency** approximation properties of those operators, that is, we shall study how a reconstruction scheme such as (1) is able to capture (or filter) **oscillation** and **concentration**-like phenomena on the functions it is intended to approximate. More generally, me shall be interested in clarifying how the high frequency behavior of a sequence of reconstructed functions depends on the profiles φ, ψ and the sampling rate h chosen.

Before giving a more precise statement of our objectives, let us first illustrate the above discussion with two specific examples: consider $f_k\left(x\right) := k^{d/2}\rho\left(k\left(x-x_0\right)\right)$ and $g_k\left(x\right) := \rho\left(x\right)e^{ikx\cdot\xi^0}$ with $\rho\in L^2\left(\mathbb{R}^d\right)$; the sequence (f_k) concentrates around the point x_0 as $k\to\infty$, whereas (g_k) oscillates in the direction ξ^0. The results we shall present in this paper are aimed to understand to what extent the sequences $\left(T_\psi^{h_k}S_\varphi^{h_k}f_k\right)$ and $\left(T_\psi^{h_k}S_\varphi^{h_k}g_k\right)$ reproduce the same behavior as (f_k) and (g_k) (i.e., if concentration and oscillation persist), for a given sequence (h_k) of positive reals that tends to zero (the sampling steps) and some choice of φ and ψ.

Perhaps, the simplest convenient setting to formulate our results is provided by the notion of **defect measure**, an object that gives a quantitative description of what we shall understand by concentration and oscillation effects and whose definition we next recall. Let (u_k) be a weakly converging sequence in the space $L^2\left(\mathbb{R}^d\right)$; denote by u its weak limit and remark that the densities $\left|u_k-u\right|^2$ are uniformly bounded in $L^1\left(\mathbb{R}^d\right)$. Helly's compactness Theorem then ensures that some subsequence $\left(\left|u_{k_n}-u\right|^2\right)$ weakly converges in the set of positive Radon measures;[1]

[1]From now on, we shall use the term *measure* as an abbreviation of the longer *Radon measure*. Recall that the space of Radon measures $\mathcal{M}\left(\mathbb{R}^d\right)$ is identified, by Riesz's Theorem, with the space of continuous linear functionals on $C_c\left(\mathbb{R}^d\right)$.

or, in other words, that there exists a positive measure ν on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \phi(x) |u_{k_n}(x) - u(x)|^2 dx \to \int_{\mathbb{R}^d} \phi(x) d\nu(x) \qquad \text{as } n \to \infty,$$

for every $\phi \in C_c(\mathbb{R}^d)$. When the above convergence takes place without extracting a subsequence we say that ν is the **defect measure** of the sequence (u_k).

Immediately from this definition one deduces the following general principle: if ν is the defect measure of a sequence (u_k) and $\omega \subset \mathbb{R}^d$ is a bounded Borel set, then there is an equivalence between $\nu(\omega) = 0$ and the fact that $u_k|_\omega$ converges strongly to $u|_\omega$ in $L^2(\omega)$. Thus, the support of ν is precisely the set where strong convergence fails, that is, the set where oscillations and concentrations take place.

But defect measures are also able to detect concentration and oscillatory phenomena and give quantitative information about them. Consider the sequences (f_k), (g_k) previously defined; they both weakly converge to zero in $L^2(\mathbb{R}^d)$ and it is easy to check that their respective defect measures are $\|\rho\|^2_{L^2(\mathbb{R}^d)} \delta_{x_0}$ and $|\rho(x)|^2 dx$. Notice that, in the first case, the defect measure actually captures the concentration of the sequence around the point $x = x_0$. In the complementary of that point, where the sequence converges strongly to zero, the measure vanishes. In the second example, the defect measure is uniformly distributed on $\mathbb{R}^d$, this being consistent with the fact that strong convergence does not take place in any subset of $\mathbb{R}^d$.

Let us point out that the analysis of concentration and oscillation effects developed by a sequence of functions is a central issue in many problems of the Calculus of Variations and Partial Differential Equations. A number of applications of defect measures may be found in the analysis of variational problems with loss of compactness performed by P.-L. Lions in [11, 12].[2]

Consider a sequence (u_k), weakly converging to zero in $L^2(\mathbb{R}^d)$; sample it using a profile φ and form the reconstructed sequence

$$v_k := T_\psi^{h_k} S_\varphi^{h_k} u_k,$$

for some given ψ and some sequence (h_k) of positive reals tending to zero. The functions v_k are bounded in $L^2(\mathbb{R}^d)$ and tend weakly to zero provided φ and ψ satisfy suitable hypotheses (see Lemma 3.1 in Section 3 below). Suppose furthermore that the densities $|v_k|^2$ weakly converge to the defect measure $\nu_{\varphi,\psi}$.

One of the main issues addressed in this article is that of understanding the relations existing between the defect measure $\nu_{\varphi,\psi}$, the profiles φ, ψ and the sequences (u_k), (h_k). Among these, we point out:

[2]We also refer to L.C. Evans' notes [4] for an exposition of some additional applications as well as a discussion of other measure-theoretical objects (such as, for example, Young measures) designed to study the failure of strong convergence.

A. Is there a formula, valid for any sequence (u_k), relating $\nu_{\varphi,\psi}$ to the defect measure ν only in terms of the profiles φ and ψ?

B. Given (u_k), characterize the profiles φ and ψ such that $\nu_{\varphi,\psi} = 0$. This is the problem of **filtering** since, as we have discussed before, $\nu_{\varphi,\psi} = 0$ is equivalent to the strong convergence to zero of the sequence $\left(T_{\psi}^{h_k} S_{\varphi}^{h_k} u_k\right)$.

C. Similarly, characterize the profiles φ and ψ such that $\nu_{\varphi,\psi} = \nu$ for a given (u_k).

D. Finally, characterize the profiles that give $\nu_{\varphi,\psi} = \nu$ *for every* (u_k).

We shall prove that the answer to question **A** is negative. This is due to the fact that the measure $\nu_{\varphi,\psi}$ is sensitive to the characteristic directions of oscillation of the sequence (u_k), whereas ν is unable to distinguish them. As we have seen above, the defect measure of the oscillating sequence (g_k) equals $|\rho(x)|^2 dx$ independently of the vector ξ^0; that is not the case for $\nu_{\varphi,\psi}$. Indeed, under additional assumptions on φ and ψ we prove (see Theorem 1.3 and Corollary 1.4):

$$\nu_{\varphi,\psi}(x) = \sum_{k \in \mathbb{Z}^d} \left| \widehat{\psi}\left(\xi^0 + 2\pi k\right) \right|^2 \left| \widehat{\varphi}\left(\xi^0\right) \right|^2 |\rho(x)|^2 dx.$$

Thus the measure $\nu_{\varphi,\psi}$ is ξ^0-dependent and cannot be expressed solely in terms of ν, φ and ψ. Note that $\nu_{\varphi,\psi}$ is identically zero as soon as any of $\sum_{k \in \mathbb{Z}^d} \left| \widehat{\psi}\left(\xi^0 + 2\pi k\right) \right|^2$ or $\widehat{\varphi}\left(\xi^0\right)$ is null. Analogously, the profiles that give $\nu_{\varphi,\psi} = \nu$ are precisely those which satisfy

$$\sum_{k \in \mathbb{Z}^d} \left| \widehat{\psi}\left(\xi^0 + 2\pi k\right) \right|^2 \left| \widehat{\varphi}\left(\xi^0\right) \right|^2 = 1.$$

Therefore, in order to understand how $\nu_{\varphi,\psi}$ is built, we must have at our disposal an object that is able to distinguish between oscillatory phenomena at different directions.

1.2 Wigner measures

This refinement is provided by the theory of **Wigner measures**.[3] Given a bounded sequence in $L^2\left(\mathbb{R}^d\right)$ one associates to it a measure $\mu(x,\xi)$ on $\mathbb{R}^d \times \mathbb{R}^d$ which describes the concentration and oscillation effects (these are the respective

[3]This object is present in the work of E.P. Wigner on semiclassical quantum mechanics [20]. Recently, Wigner measures have gained interest since the works of P. Gérard [6], P.-L. Lions & Th. Paul [13], P. Markowich, N. Mauser & F. Poupaud [14] among others. Related objects are the **Microlocal defect measures** or **H-measures**, introduced independently by P. Gérard [5] and L. Tartar [18].

roles of the variables x and ξ) occurring at some characteristic length-scale. This measure takes into account the characteristic speeds as well as the directions of propagation of oscillations. One way of defining them consists in replacing the density $|u(x)|^2$ involved in the definition of the defect measure by the phase space (microlocal) density:

$$m^{\varepsilon}[u](x,\xi) := \frac{1}{(2\pi\varepsilon)^d}\overline{u(x)}\widehat{u}(\xi/\varepsilon)\, e^{ix\cdot\xi/\varepsilon}, \tag{2}$$

where $\widehat{u}$ is the Fourier transform of u and ε is a positive constant. The $(2\pi)^{-d}$ factor in the definition of $m^{\varepsilon}[u]$ is placed to have:

$$\int_{\mathbb{R}^d} m^{\varepsilon}[u](x,\xi)\, d\xi = |u(x)|^2, \qquad \int_{\mathbb{R}^d} m^{\varepsilon}[u](x,\xi)\, dx = \frac{|\widehat{u}(\xi/\varepsilon)|^2}{(2\pi\varepsilon)^d}. \tag{3}$$

Thus, the function $m^{\varepsilon}[u]$ may be looked at as joint physical space-Fourier space "density", in spite of the fact that $m^{\varepsilon}[u]$ is not positive in general. However, limits of these quantities are positive measures:

Theorem 1.1 *Let (u_k) be a bounded sequence in $L^2(\mathbb{R}^d)$ and let (ε_k) be a sequence of positive numbers tending to zero. Then it is possible to extract a subsequence (u_{k_n}) such that, for every test function $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d\times\mathbb{R}^d} a(x,\xi)\, m^{\varepsilon_{k_n}}[u_{k_n}](x,\xi)\, dx d\xi = \int_{\mathbb{R}^d\times\mathbb{R}^d} a(x,\xi)\, d\mu(x,\xi), \tag{4}$$

where μ is a finite positive measure on $\mathbb{R}^d \times \mathbb{R}^d$.

A measure $\mu \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d)$ is called the **Wigner measure** of the sequence (u_k) at scale (ε_k) whenever the limit (4) holds without extracting a subsequence. Different proofs of Theorem 1.1 may be found in [8, 13, 7]. Let us point out that other quadratic densities may used to define Wigner measures. For instance, in [13] μ is obtained by replacing $m^{\varepsilon}[u]$ in the limit (4), by the more familiar **Wigner transform**:

$$w^{\varepsilon}[u](x,\xi) := \int_{\mathbb{R}^d} u\left(x - \varepsilon\frac{p}{2}\right)\overline{u\left(x + \varepsilon\frac{p}{2}\right)} e^{ip\cdot\xi}\frac{dp}{(2\pi)^d}. \tag{5}$$

It is also possible to consider **Wave-packet (Husimi) transforms**. Of course, all this methods are equivalent (the same limit is obtained), cf. the discussion in [8].

The Wigner measure encodes all the information contained in the defect measure provided the sequence (u_k) oscillates at frequencies of the order of ε_k^{-1}. More precisely (see [8, 13]):

Proposition 1.2 *If μ is the Wigner measure at scale (ε_k) of a sequence (u_k) and ν is the measure obtained as the weak limit in $\mathcal{M}_+\left(\mathbb{R}^d\right)$ of the densities $|u_k|^2\,dx$, then the identity*

$$\nu(x) = \int_{\mathbb{R}^d} \mu(x, d\xi)$$

holds provided (u_k) is $\boldsymbol{\varepsilon_k}$***-oscillatory****:*

$$\limsup_{k\to\infty} \int_{|\xi|>R/\varepsilon_k} |\widehat{u_k}(\xi)|^2\,d\xi \to 0 \qquad \text{as } R\to\infty. \tag{6}$$

Notice that condition (6) actually expresses that the energy of the Fourier transform of u_k is concentrated in a ball of radius R/ε_k, which should be understood as the requirement that the sequence (u_k) does not oscillate at length scales finer than ε_k.

To illustrate this discussion it may be helpful to look at explicit computations. The Wigner measure at scale (ε_k) of the concentrating sequence (f_k) defined at the beginning of this section is given by:

$$\mu(x,\xi) = \begin{cases} \|\rho\|^2_{L^2\left(\mathbb{R}^d\right)} \delta_{x_0}(x)\otimes\delta_0(\xi) & \text{if } \varepsilon_k k\to 0, \\ \delta_{x_0}(x)\otimes|\widehat{\rho}(\xi)|^2 \dfrac{d\xi}{(2\pi)^d} & \text{if } \varepsilon_k = k^{-1}, \\ 0 & \text{if } \varepsilon_k k\to\infty, \end{cases} \tag{7}$$

while for the oscillating sequence (g_k) it can be checked to be:

$$\mu(x,\xi) = \begin{cases} |\rho(x)|^2\,dx\otimes\delta_0(\xi) & \text{if } \varepsilon_k k\to 0, \\ |\rho(x)|^2\,dx\otimes\delta_{\xi^0}(\xi) & \text{if } \varepsilon_k = k^{-1}, \\ 0 & \text{if } \varepsilon_k k\to\infty. \end{cases} \tag{8}$$

These examples show the importance of the choice of the scale (ε_k). When this scale is taken to be coarser than the characteristic length-scale k^{-1} of oscillation/concentration, it is no longer true that the projection on the first component of their Wigner measures coincides with the defect measure. On the other hand, in the case $\varepsilon_k k\to 0$ (the scale chosen is much smaller than the actual oscillation scale) the Wigner measure is not able to capture the direction of oscillation. Hence, to obtain a complete description, the scale (ε_k) must be taken of the same order than that of the oscillations.

Wigner measures turn out to be the correct tool for comparing the high frequency behavior of the sequences (u_k) and $\left(T^{h_k}_\psi S^{h_k}_\varphi u_k\right)$.

1.3 Computation of Wigner and defect measures

Given a sequence of sampling steps (h_k), it is clear that the functions $T_{\psi}^{h_k} S_{\varphi}^{h_k} u_k$ will not develop oscillation and concentration effects of characteristic sizes asymptotically smaller that h_k. Most commonly, these functions will form an h_k-oscillatory sequence;[4] consequently, only Wigner measure at scales coarser or of the same order than (h_k) will be considered.

In order to establish explicit formulas, we shall require additional hypotheses on φ, ψ and on the Wigner measures involved. Nevertheless, in order to simplify the statement of our results, in this introduction we shall impose the following (more restrictive) condition on the admissible profiles:

$$|\gamma(x)| \leq C(1+|x|)^{-d-\varepsilon}, \qquad \text{for every } x \in \mathbb{R}^d \text{ and some } C, \varepsilon > 0. \tag{9}$$

More general results may be found in Section 7.

We prove the following:

Theorem 1.3 *Let φ, ψ satisfy (9). Suppose (u_k) is a bounded sequence in $L^2(\mathbb{R}^d)$ and that μ is its Wigner measure at scale (h_k). Suppose moreover that the measures*

$$|\widehat{\varphi}(\xi + 2\pi n)|^2 \mu(x, \xi + 2\pi n) \tag{10}$$

are mutually singular for $n \in \mathbb{Z}^d$.

Then the Wigner measure at scale (h_k) of the sequence $\left(T_{\psi}^{h_k} S_{\varphi}^{h_k} u_k\right)$ is given by:

$$\mu_{\varphi,\psi}(x,\xi) = \left|\widehat{\psi}(\xi)\right|^2 \sum_{k \in \mathbb{Z}^d} |\widehat{\varphi}(\xi + 2\pi n)|^2 \mu(x, \xi + 2\pi n).$$

From this, one deduces:

Corollary 1.4 *If, moreover, $\left|T_{\psi}^{h_k} S_{\varphi}^{h_k} u_k\right|^2 dx$ weakly converges to a measure $\nu_{\varphi,\psi}$ then:*

$$\nu_{\varphi,\psi}(x) = \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} \left|\widehat{\psi}(\xi + 2\pi k)\right|^2 |\widehat{\varphi}(\xi)|^2 \mu(x, d\xi).$$

This shows, in particular, that a formula relating $\nu_{\varphi,\psi}$ and the weak limit ν of $|u_k|^2 dx$ does not exist unless (u_k) is h_k-oscillatory and μ is of the form $\nu(x) \otimes \sigma(\xi)$. It also shows that $\nu = \nu_{\varphi,\psi}$ if and only if $\sum_{k \in \mathbb{Z}^d} \left|\widehat{\psi}(\xi + 2\pi k)\right|^2 |\widehat{\varphi}(\xi)|^2 = 1$ for μ-almost every $\xi \in \mathbb{R}^d$. Consequently, there do not exist profiles φ, ψ satisfying (9) such that ν equals $\nu_{\varphi,\psi}$ for every h_k-oscillatory sequence (u_k).

[4] However, this may fail for some pathological examples (see paragraph 5.3).

On the other hand, Theorem 1.3 implies that question **A** above does have a positive answer in terms of Wigner measures, at least when restricted to the class of sequences which satisfy (10). That condition, roughly speaking, imposes a restriction on the size of the region in frequency space where an admissible sequence fails to converge strongly to zero. Below, we shall compare it with that appearing in Shannon's sampling Theorem.

The above results will be obtained as corollaries of the more general Theorems 7.1 and 7.3. Profiles that belong to negative-order Sobolev spaces or that fail to satisfy the localization hypothesis (9) are allowed. However, this will require to impose compatibility conditions on the Wigner measure μ.

As an illustration of the range of results that will be obtained in this more general setting, we present an **asymptotic version** of **Shannon's sampling Theorem**.[5] It corresponds to taking as sampling profile $\varphi = \delta_0$, the Dirac delta at the origin, and as reconstruction function $\widehat{\psi} := \mathbf{1}_Q$, where $Q := [-\pi, \pi)^d$. Notice that $S^h_{\delta_0} u\,(n) = u\,(hn)$ is the discretization operator, whereas the T^h_ψ corresponds to band-limited reconstruction.

Theorem 1.5 *Let (u_k) be a bounded sequence in $L^2\left(\mathbb{R}^d\right)$ and denote by μ its Wigner measure at scale (h_k). Suppose, in addition, that $u_k \in H^s\left(\mathbb{R}^d\right)$ for some $s > d/2$ and*

$$\begin{array}{ll} i) & \left(1 - h_k^2 \Delta_x\right)^{s/2} u_k \quad \text{are uniformly bounded in } L^2\left(\mathbb{R}^d\right). \\ ii) & \mu\left(\mathbb{R}^d \times (\partial Q + 2\pi n)\right) = 0 \quad \text{for } n \in \mathbb{Z}^d. \\ iii) & \mu\,(x, \xi + 2\pi n)\,, \quad n \in \mathbb{Z}^d, \quad \text{are mutually singular measures.} \end{array} \tag{11}$$

Then, the Wigner measure at scale (h_k) of $\left(T^{h_k}_\psi S^{h_k}_{\delta_0} u_k\right)$ is

$$\mu_{\delta_0,\psi}\,(x,\xi) = \mathbf{1}_Q\,(\xi) \sum_{n \in \mathbb{Z}^d} \mu\,(x, \xi + 2\pi n)\,. \tag{12}$$

Moreover, if $\left|T^{h_k}_\psi S^{h_k}_{\delta_0} u_k\right|^2 dx$ and $|u_k|^2\,dx$ weakly converge to ν_S and ν, respectively, then

$$\nu_S\,(x) = \int_{\mathbb{R}^d} \mu\,(x, d\xi) = \nu\,(x)\,.$$

Thus, unlike the operators considered in Theorem 1.3, the composition of discretization and band-limited reconstruction preserves the defect measure for a large class of sequences.

Notice that, by the Sobolev imbedding Theorem, $S^{h_k}_{\delta_0} u_k$ is well-defined. Actually, (11.i) ensures that the sequence of discretizations is square-summable and,

[5]See paragraph 3.1 for a statement of Shannon's original sampling Theorem.

consequently, that $\left(T_{\psi}^{h_k} S_{\delta_0}^{h_k} u_k\right)$ is bounded in $L^2\left(\mathbb{R}^d\right)$ and h_k-oscillatory (for a more complete result, we refer to Lemma 3.1). Condition (11.ii) appears because $\widehat{\psi}$ is not continuous; we shall discuss its necessity in paragraph 4.4. Finally, (11.iii) should be understood as the analog of Shannon's original band-limited condition in this context.

To conclude this short description, let us present how the above results may be refined when the sequence (u_k) is known to be ε_k-oscillatory and the sampling rate (h_k) is taken to satisfy $h_k/\varepsilon_k \to 0$. As it can be expected, much more precision is gained:

Theorem 1.6 *Suppose φ, ψ satisfy (9) and (u_k) is an ε_k-oscillatory, bounded sequence in $L^2\left(\mathbb{R}^d\right)$. If μ is its Wigner measure at scale (ε_k) then the corresponding measure of the sequence $\left(T_{\psi}^{h_k} S_{\varphi}^{h_k} u_k\right)$ is*

$$\mu_{\varphi,\psi} = \left|\widehat{\psi}\left(0\right)\right|^2 \left|\widehat{\varphi}\left(0\right)\right|^2 \mu.$$

Moreover, if the densities a $\left|T_{\psi}^{h_k} S_{\varphi}^{h_k} u_k\right|^2 dx$ and $\left|u_k\right|^2 dx$ weakly converge to $\nu_{\varphi,\psi}$ and ν respectively then

$$\nu_{\varphi,\psi}\left(x\right) = \sum_{n\in\mathbb{Z}^d} \left|\widehat{\psi}\left(2\pi n\right)\right|^2 \left|\widehat{\varphi}\left(0\right)\right|^2 \nu\left(x\right).$$

This Theorem holds under much more general conditions on φ and ψ (see Theorem 7.6) and gives a positive answer to question **A** provided we consider only ε_k-oscillatory sequences.

An immediate consequence of the above result is that zero-mean sampling profiles φ (i.e. with $\widehat{\varphi}\left(0\right) = 0$, as a wavelet, for instance) completely filter any oscillations that occur at scales much coarser than the sampling rate h_k. For such a profile, $\nu_{\varphi,\psi} = 0$ for every ε_k-oscillatory sequence. An analogous phenomenon occurs for reconstruction profiles satisfying $\widehat{\psi}\left(2\pi n\right) = 0$ for every $n \in \mathbb{Z}^d$.

On the other hand, a sufficient condition to have equality between $\nu_{\varphi,\psi}$ and ν is that $\left|\widehat{\varphi}\left(0\right)\right| = \left|\widehat{\psi}\left(0\right)\right| = 1$ and $\left|\widehat{\psi}\left(2\pi n\right)\right| = 0$ for $n \neq 0$.

1.4 Strategy of proof: Wigner measures in the discrete setting

The proof of the results we have presented above will be achieved by analyzing separately the sampling and reconstruction operators S_{φ}^{h} and T_{ψ}^{h}. In order to develop, it is necessary to deal with a concept of **Wigner measure associated to**

a sequence of discrete functions. We shall introduce it by means of a discrete analogous of the transform $m^{\varepsilon}[\cdot]$. We detail this in the following paragraph.

To a discrete square-summable function $U \in L^2\left(h\mathbb{Z}^d\right)$, where $L^2\left(h\mathbb{Z}^d\right)$ stands for the space of the functions U defined on $\mathbb{Z}^d$ with values in $\mathbb{C}$ such that the norm

$$\|U\|_h := \left(h^d \sum_{n\in\mathbb{Z}^d} |U_n|^2\right)^{1/2}$$

is finite, we associate:

$$M^{\varepsilon}[U](x,\xi) := \frac{h^{2d}}{(2\pi\varepsilon)^d} \sum_{m\in\mathbb{Z}^d} \overline{U_m}\widehat{U}\left(\frac{h}{\varepsilon}\xi\right) e^{im\cdot(h/\varepsilon)\xi}\delta_{hm}(x). \tag{13}$$

Here, δ_{hm} is the Dirac mass centered at the point hm and $\widehat{U}$ denotes the discrete Fourier transform:

$$\widehat{U}(\xi) := \sum_{n\in\mathbb{Z}^d} U_n e^{-in\cdot\xi},$$

which, as is well known, is a $2\pi\mathbb{Z}^d$-periodic function in $L^2_{\mathrm{loc}}\left(\mathbb{R}^d\right)$. The discrete transform $M^{\varepsilon}[U]$ may be related to the continuous $m^{\varepsilon}[u]$ by noticing that

$$M^{\varepsilon}[U] = m^{\varepsilon}\left[T^h_{\delta_0}U\right], \quad \text{where} \quad T^h_{\delta_0}U(x) = h^d \sum_{k\in\mathbb{Z}^d} U^h_n \delta_{hn}(x). \tag{14}$$

This is meaningful, since $m^{\varepsilon}[u]$ is well-defined for any tempered distribution $u \in \mathcal{S}'\left(\mathbb{R}^d\right)$.

In order to simplify our language we make the following definition:

Definition 1.7 *Let $h = (h_k)$ be a scale. We shall call a sequence $\left(U^{h_k}\right)$* $\mathbf{h}_k$-***bounded*** *if and only if $U^{h_k} \in L^2\left(h_k\mathbb{Z}^d\right)$ and $\left\|U^{h_k}\right\|_{h_k} \leq C$ for every $k \in \mathbb{N}$.*

One has the following convergence result (which is not a direct consequence of Theorem 1.1):

Proposition 1.8 *Let (h_k), (ε_k) be scales such that (h_k/ε_k) is bounded and let $\left(U^{h_k}\right)$ be an h_k-bounded sequence of discrete functions. Then $\left(M^{\varepsilon_k}\left[U^{h_k}\right]\right)$ is bounded in $\mathcal{S}'\left(\mathbb{R}^d\times\mathbb{R}^d\right)$ and given any of its convergent subsequences $\left(U^{h_{k_n}}\right)$ there exists a positive measure μ such that,*

$$\lim_{n\to\infty} \left\langle M^{\varepsilon_{k_n}}\left[U^{h_{k_n}}\right], a\right\rangle_{\mathcal{S}'\times\mathcal{S}} = \int_{\mathbb{R}^d\times\mathbb{R}^d} a(x,\xi)\, d\mu(x,\xi), \tag{15}$$

for every $a \in \mathcal{S}\left(\mathbb{R}^d\times\mathbb{R}^d\right)$.

This will be proved as a Corollary of the more general Proposition 3.4, which in turn follows from the analysis of Wigner measures in negative-order Sobolev spaces that is performed in Section 8. As in the continuous setting, we say that a measure μ is the **Wigner measure at scale** (ε_k) of a sequence of discrete functions $\left(U^{h_k}\right)$ if the limit (15) holds for the whole sequence.

Remark 1.9 *i) When (h_k/ε_k) is unbounded, it may happen that $M^{\varepsilon_{k_n}}\left[U^{h_{k_n}}\right]$ is not bounded in $\mathcal{S}'\left(\mathbb{R}^d \times \mathbb{R}^d\right)$.*

ii) If $h_k/\varepsilon_k \to c > 0$ then μ is not finite. Indeed, it is periodic (with respect to the lattice $(2\pi/c)\,\mathbb{Z}^d$) in the ξ variable.

iii) However, when $h_k/\varepsilon_k \to 0$, the Wigner measure μ is finite, as in the continuous case.

With this tool at our disposal, we are able to compare the Wigner measure of a sequence of discrete functions $\left(U^{h_k}\right)$ with that of a reconstructed sequence $\left(T_\psi^{h_k}U^{h_k}\right)$. Analogously, we may compute the Wigner measures of sequences of sampled discrete functions $\left(S_\varphi^{h_k}u_k\right)$ in terms of those corresponding to the original sequence (u_k). These are respectively the contents of Theorems 4.6 and 4.2.

1.5 Plan of the article

Results and assumptions concerning the operators S_φ^h and T_ψ^h are collected in Section 3.

In Section 4, the problem of computing Wigner measures for sequences of sampled or reconstructed functions is addressed. Formulas for Wigner measures at scales of the same order than the sampling/reconstruction step (h_k) are presented in Theorems 4.6 and 4.2. Theorems 1.3 and 1.5 then easily follow from those two results. We also point out the relationships existing between these Wigner measures and the concept of **Wigner series** introduced in [14, 9].

The problem of the computation of defect measures of sequences of the from $\left(T_\psi^{h_k}U^{h_k}\right)$ is considered in Section 5; the main results are presented in Proposition 5.8 and Corollary 5.9.

In Section 6 we investigate Wigner measures at scales (ε_k) satisfying $h_k/\varepsilon_k \to 0$. Explicit formulas are presented in Theorems 6.1 and 6.2, from which Theorem 1.6 immediately follows.

The composition of sampling and reconstruction is studied in Section 7, the main results of this article are proven there.

Finally, Section 8 contains the elements from the Theory of Wigner measures on which the proofs of most of the results of this article are based on. Propositions 8.1 and 8.3, which extend the Theory of Wigner measures to sequences in Sobolev spaces of negative order, are systematically used throughout this paper.

2 Notations and conventions

We briefly present some notation that will be used throughout this article.

$B(x;R)$ will denote the open ball with radius R of $\mathbb{R}^d$ centered at the point x. We shall set

$$Q := [-\pi, \pi)^d,$$

and $\mathbf{1}_A$ will denote the characteristic function of a set $A \subseteq \mathbb{R}^d$.

We write Γ to denote de lattice $2\pi\mathbb{Z}^d$. A function f defined on $\mathbb{R}^d$ is Γ-periodic if $f(x+\gamma) = f(x)$ for every $\gamma \in \Gamma$ and every $x \in \mathbb{R}^d$.

We adopt the following convention for the Fourier transform:

$$\widehat{u}(\xi) := \int_{\mathbb{R}^d} u(x)\, e^{-ix\cdot\xi} dx.$$

Given a measurable function $\varphi(\xi)$, the **Fourier multiplier** of symbol φ is the operator $\varphi(D_x)$ formally defined by

$$\varphi(D_x)\, u(x) := \int_{\mathbb{R}^d} \varphi(\xi)\, \widehat{u}(\xi)\, e^{ix\cdot\xi} \frac{d\xi}{(2\pi)^d} = \check{\varphi} * u(x),$$

$\check{\varphi}$ being the inverse Fourier transform of φ.

A particularly important Fourier multiplier is the **Bessel potential** $\langle D_x \rangle$, of symbol

$$\langle \xi \rangle := \left(1 + |\xi|^2\right)^{1/2}.$$

Next, we recall the definition of some function spaces.

As usual, $\mathcal{S}\left(\mathbb{R}^d\right)$ denotes the space of **rapidly decreasing functions** and $\mathcal{S}'\left(\mathbb{R}^d\right)$ stands for its dual, the space of **tempered distributions**.

Given $r \in \mathbb{R}$, $H^r\left(\mathbb{R}^d\right)$, the **Sobolev space** of order r, consists of the distributions $u \in \mathcal{S}'\left(\mathbb{R}^d\right)$ such that $\langle D_x \rangle^r u \in L^2\left(\mathbb{R}^d\right)$.

The weighted space $L^2\left(\mathbb{R}^d; \langle x \rangle^r\right)$ is that of the functions $u \in L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)$ such that

$$\|u\|_{L^2\left(\mathbb{R}^d;\langle x\rangle^r\right)} := \left(\int_{\mathbb{R}^d} |u(x)|^2 \langle x \rangle^r\, dx\right)^{1/2} < \infty.$$

The analogous definition is understood for $L^\infty\left(\mathbb{R}^d; \langle x \rangle^r\right)$.

By $C^\infty\left(\mathbb{R}^d; \langle x \rangle^r\right)$ we intend the space of functions $u \in C^\infty\left(\mathbb{R}^d\right)$ such that

$$\|\partial_x^\alpha u\|_{L^\infty\left(\mathbb{R}^d;\langle x\rangle^r\right)} < \infty \qquad \text{for every multiindex } \alpha \in \mathbb{N}^d.$$

$C_0\left(\mathbb{R}^d\right)$ denotes the spaces of continuous functions on $\mathbb{R}^d$ vanishing at infinity.

Given an open set $\Omega \subseteq \mathbb{R}^d$, $\mathcal{M}_+(\Omega)$ is the set of **positive Radon measures** on Ω, which can be identified through Riesz's Theorem to the set of positive functionals on $C_c(\Omega)$, the space of continuous functions on Ω with compact support.

Finally, in order to lighten our writing, *we shall write $\mathcal{S}$ and $\mathcal{S}'$ instead of $\mathcal{S}\left(\mathbb{R}^d_x \times \mathbb{R}^d_\xi\right)$ and $\mathcal{S}'\left(\mathbb{R}^d_x \times \mathbb{R}^d_\xi\right)$ respectively.*

For a measurable function $f : \mathbb{R}^d \to \mathbb{C}$, we use the notation

$$D_f := \left\{x \in \mathbb{R}^d \ : \ f \text{ is not continuous at } x\right\}.$$

An important, perhaps non-standard, definition is that of a scale:

Definition 2.1 *A **scale** (ε_k) is a sequence of positive numbers that tends to zero as $k \to \infty$.*

Given two scales (h_k) and (ε_k), the notations $h_k \ll \varepsilon_k$ and $h_k \sim \varepsilon_k$ will be used to indicate that $\lim_{k\to\infty} h_k/\varepsilon_k = 0$ and $\lim_{k\to\infty} h_k/\varepsilon_k = c > 0$ respectively.

3 Sampling and reconstruction

3.1 Definitions and examples

The sampling and reconstruction operators we are going to consider are next described. Given a distribution $\varphi \in \mathcal{S}'\left(\mathbb{R}^d\right)$ we set for every $n \in \mathbb{Z}^d$ and $h > 0$,

$$\varphi^h_n(x) := \varphi\left(\frac{x}{h} - n\right).$$

The **reconstruction** (or **synthesis**) **operator** T^h_φ, acting on discrete functions U of $\mathbb{Z}^d$ is defined to be

$$T^h_\varphi U(x) := \sum_{n\in\mathbb{Z}^d} U_n \varphi^h_n(x). \tag{16}$$

This expression is well-defined for finitely supported discrete functions. When φ is a continuous function such that $\varphi(0) = 1$ and $\varphi(k) = 0$ for $k \in \mathbb{Z}^d \setminus \{0\}$ then $T^h_\varphi U$ is actually a function that **interpolates** the discrete values U_n on the grid $h\mathbb{Z}^d$, i.e. $T^h_\varphi U(hn) = U_n$ for all $n \in \mathbb{Z}^d$.

Analogously, the **sampling** (or **analysis**) **operator** S^h_φ, a priori only acting on functions $u \in \mathcal{S}\left(\mathbb{R}^d\right)$, is defined as follows: $S^h_\varphi u$ is the discrete function given by

$$S^h_\varphi u(n) := h^{-d}\left\langle \overline{\varphi^h_n}, u\right\rangle_{\mathcal{S}'\left(\mathbb{R}^d\right)\times\mathcal{S}\left(\mathbb{R}^d\right)}.$$

When $\varphi = \delta_0$, we obtain the usual **discretization** operator: $S^h_{\delta_0} u(n) = h^{-d} u(hn)$ for every $n \in \mathbb{Z}^d$.

Indeed, these sampling/reconstruction schemes include several well-known such procedures on regular grids. Among many others we may cite:

• **Cardinal B-Splines.** The B-spline of order zero is the function $\varphi(x) := \mathbf{1}_{[-1/2,1/2]^d}(x)$; the function $T_\varphi^h U$ is just the piecewise constant interpolation of the discrete function U on the grid $h\mathbb{Z}^d$. The B-spline of order 1,

$$\varphi(x) = \mathbf{1}_{[-1/2,1/2]^d} * \mathbf{1}_{[-1/2,1/2]^d} = \prod_{j=1}^{d} (1 - |x_j|)_+ ,$$

gives rise to the piecewise linear interpolation operator. Analogously, B-splines of order $r \in \mathbb{N}$ are defined iterating this convolution r times. These are $C^{r-1}(\mathbb{R}^d)$ functions supported in $[-r/2, r/2]^d$, taking the value 1 at the origin. More details may be found, for instance, in [1].

• **Band-limited sampling/reconstruction.** This corresponds to the profile

$$\varphi(\xi) := \prod_{j=1}^{d} \operatorname{sinc}(\xi_j) ,$$

where the **cardinal sine function** is defined by

$$\operatorname{sinc}(t) := \frac{\sin \pi t}{\pi t}.$$

It is easy to check that $\widehat{\varphi}(\xi) = \mathbf{1}_Q(\xi)$. This profile is relevant because of **Shannon's sampling Theorem**: *a function u belongs to the space*

$$V^h := \left\{ u \in L^2(\mathbb{R}^d) : \operatorname{supp} \widehat{u} \subset [-\pi/h, \pi/h)^d \right\} = \operatorname{range}(T_\varphi^h)$$

if and only if

$$u = \sum_{n \in \mathbb{Z}^d} u(hn) \varphi_n^h.$$

In particular, such functions are determined by their values on the grid $h\mathbb{Z}^d$.

• **Wavelets.** Take again $h_k := 2^{-k}$ for every $k \in \mathbb{Z}$. A function $\psi \in L^2(\mathbb{R}^d)$ is a wavelet provided $\left\{ \psi_n^{h_k} : n \in \mathbb{Z}^d, k \in \mathbb{Z} \right\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$. For more details on wavelets and the closely related **MultiResolution Analyses**, the reader may see [10, 16].

Additional examples and references (from the viewpoint of Signal Theory), may be found in the survey [19].

3.2 Boundedness properties

In order to ensure that the sampling and reconstruction operators are bounded, we shall make the assumption (**BP**) below:

$$\begin{gathered}\text{There exist } s \in \mathbb{R} \text{ and } B > 0 \text{ such that } \varphi \in H^s(\mathbb{R}) \text{ and} \\ \tau_{\langle D_x\rangle^s\varphi}(\xi) := \sum_{k\in\mathbb{Z}^d} \left|\langle \xi + 2\pi k\rangle^s \widehat{\varphi}(\xi + 2\pi k)\right|^2 \le B \quad \text{for a.e. } \xi \in \mathbb{R}^d.\end{gathered} \tag{\textbf{BP}}$$

Lemma 3.1 *Suppose $\varphi \in \mathcal{S}'\left(\mathbb{R}^d\right)$. Then the following are equivalent:*

*i) φ satisfies (**BP**).*

ii) There exists $B > 0$ such that

$$\left\|\langle hD_x\rangle^s T_\varphi^h U\right\|_{L^2\left(\mathbb{R}^d\right)} \le \sqrt{B}\, \|U\|_{L^2\left(h\mathbb{Z}^d\right)} \tag{17}$$

holds uniformly for $h > 0$ and $U \in L^2\left(h\mathbb{Z}^d\right)$.

iii) There exists $B > 0$ such that

$$\left\|S_\varphi^h u\right\|_{L^2\left(h\mathbb{Z}^d\right)} \le \sqrt{B}\, \left\|\langle hD_x\rangle^{-s} u\right\|_{L^2\left(\mathbb{R}^d\right)} \tag{18}$$

holds uniformly for $h > 0$ and $u \in H^{-s}\left(\mathbb{R}^d\right)$.

Moreover, whenever i), ii) or iii) is fulfilled, the smallest constant B for which any of the above assertion holds is precisely $\left\|\tau_{\langle D_x\rangle^s\varphi}\right\|_{L^\infty(Q)}$.

Proof. To see why i) and ii) are equivalent, first observe that, given any $\varphi \in H^s\left(\mathbb{R}^d\right)$ the following identity holds

$$T_\varphi^h = \langle hD_x\rangle^{-s} T_{\langle D_x\rangle^s\varphi}^h. \tag{19}$$

To check this, simply notice that

$$\widehat{T_\varphi^h U}(\xi) = h^d \widehat{\varphi}(h\xi) \sum_{n\in\mathbb{Z}^d} U_n e^{-ihn\cdot\xi} = \widehat{\varphi}(h\xi)\, h^d \widehat{U}(h\xi),$$

hence

$$\widehat{T_\varphi^h U}(\xi) = \langle h\xi\rangle^{-s} \langle h\xi\rangle^s \widehat{\varphi}(h\xi)\, h^d \widehat{U}(h\xi) = \widehat{\langle hD_x\rangle^{-s} T_{\langle D_x\rangle^s\varphi}^h U}(\xi).$$

Since $\langle D_x\rangle^s \varphi \in L^2\left(\mathbb{R}^d\right)$ and

$$\left\|\langle hD_x\rangle^s T^h_\varphi U\right\|_{L^2\left(\mathbb{R}^d\right)} = \left\|T^h_{\langle D_x\rangle^{-s}\varphi} U\right\|_{L^2\left(\mathbb{R}^d\right)}$$

it suffices to deal with the case $s = 0$. But it is a well-known result (see for instance [2, 17]) that for $\varphi \in L^2\left(\mathbb{R}^d\right)$, i) and ii) are equivalent and that $\left\|T^h_\varphi\right\| = \|\tau_\varphi\|_{L^\infty\left(\mathbb{R}^d\right)}$ whenever T^h_φ is bounded.

Statements ii) and iii) are equivalent because of the following duality relation:

$$\left(\langle hD_x\rangle^s T^h_\varphi U, \langle hD_x\rangle^{-s} u\right)_{L^2\left(\mathbb{R}^d\right)} = \left(U, S^h_\varphi u\right)_{L^2\left(h\mathbb{Z}^d\right)},$$

which holds for every $u \in H^{-s}\left(\mathbb{R}^d\right)$ and $U \in L^2\left(h\mathbb{Z}^d\right)$. This is simple to check:

$$\begin{aligned}\left(\langle hD_x\rangle^s T^h_\varphi U, \langle hD_x\rangle^{-s} u\right)_{L^2\left(\mathbb{R}^d\right)} &= \sum_{n\in\mathbb{Z}^d} U_n \int_{\mathbb{R}^d} \langle hD_x\rangle^s \varphi^h_n(x)\, \overline{\langle hD_x\rangle^{-s} u(x)}dx \\ &= \sum_{n\in\mathbb{Z}^d} U_n \left\langle \varphi^h_n, \overline{u}\right\rangle_{H^s\left(\mathbb{R}^d\right)\times H^{-s}\left(\mathbb{R}^d\right)} \\ &= h^d \sum_{n\in\mathbb{Z}^d} U^h_n \overline{S^h_\varphi u(n)}.\end{aligned}$$

■

Remark 3.2 *For $s \leq 0$, estimate (17) implies that*

$$\left\|\langle \varepsilon D_x\rangle^s T^h_\varphi U^h\right\|_{L^2\left(\mathbb{R}^d\right)} \leq \sqrt{B}\left\|U^h\right\|_{L^2\left(h\mathbb{Z}^d\right)}, \tag{20}$$

a soon as $h/\varepsilon \leq 1$, as it can be easily checked taking Fourier transforms.

A sufficient condition for (**BP**) in terms of decay on φ is next given:

Lemma 3.3 *Suppose $\varphi \in H^s\left(\mathbb{R}^d\right)$ satisfies, for some $\varepsilon > 0$,*

$$\int_{\mathbb{R}^d} \left|\langle D_x\rangle^s \varphi(x)\right|^2 (1+|x|)^{d+\varepsilon}\, dx < \infty \tag{21}$$

*Then $\widehat{\varphi}$ and $\tau_{\langle D_x\rangle^s\varphi}$ are continuous functions. In particular, (**BP**) always holds for such a φ.*

Proof. It follows the lines of [16], Lemma II.7. Under condition (21), $\langle\xi\rangle^s \widehat{\varphi} \in H^{d/2+\varepsilon/2}\left(\mathbb{R}^d\right)$; Sobolev's imbedding Theorem then ensures that $\langle\xi\rangle^s \widehat{\varphi}$ is a continuous function and hence so is $\widehat{\varphi}$. The continuity of $\tau_{\langle D_x\rangle^s\varphi}$ is a consequence of the fact that, whenever $\chi \in C_c^\infty\left(\mathbb{R}^d\right)$ satisfies $\sum_{n\in\mathbb{Z}^d} |\chi(\xi + 2\pi n)| \geq 1$, the expression

$$\left[\sum_{n\in\mathbb{Z}^d} \|u\chi(\cdot + 2\pi n)\|^2_{H^s\left(\mathbb{R}^d\right)}\right]^{1/2}$$

defines an equivalent norm in $H^s\left(\mathbb{R}^d\right)$, $s \geq 0$. This actually proves that

$$\sum_{n\in\mathbb{Z}^d} \sup_{\xi\in\mathbb{R}^d} \left|\langle\xi\rangle^s \widehat{\varphi}(\xi)\, \chi(\xi + 2\pi n)\right|^2 < \infty.$$

In particular, the series defining $\tau_{\langle D_x\rangle^s\varphi}$ is uniformly convergent and the claim then follows. ■

Condition (21) automatically holds for profiles φ such that

$$\left|\langle D_x\rangle^s \varphi(x)\right| \leq C(1 + |x|)^{-d-\varepsilon}, \qquad \text{for every } x \in \mathbb{R}^d \text{ and some } C, \varepsilon > 0; \quad (22)$$

in particular, the hypothesis (9) we assumed in the introduction implies (**BP**) for $s = 0$.

Now we can prove a general result from which Proposition 1.8 immediately follows:

Proposition 3.4 *Suppose φ satisfies (***BP***) and we are given scales (h_k), (ε_k) such that (h_k/ε_k) is bounded. If $\left(U^{h_k}\right)$ is an h_k-bounded sequence of discrete functions then the distributions $m^{\varepsilon_k}\left[T_\varphi^{h_k} U^{h_k}\right]$ are uniformly bounded in $\mathcal{S}'$. Moreover, the limit of any weakly convergent subsequence is a positive measure.*

The proof this is a direct consequence of Remark 3.2 and the general result established in Proposition 8.1.

3.3 Bases and projections

Below, we recall some results from Approximation Theory that will be needed in the sequel. These results deal with the range in $H^s\left(\mathbb{R}^d\right)$ of the reconstruction operator T_φ^h, which we denote V_φ^h.

The space V_φ^h is a **Principal Shift Invariant (PSI)** space. When any of the conditions of Lemma 3.1 are satisfied, the family $\left\{h^{-d/2}\varphi_n^h : n \in \mathbb{Z}^d\right\}$ is said to form a **Bessel system** for V_φ^h.

The next Lemma clarifies how the function $\tau_{\langle D_x\rangle^s\varphi}$ characterizes further basis properties of the functions φ_n^h.

Lemma 3.5 *Let* $\varphi \in \mathcal{S}'\left(\mathbb{R}^d\right)$ *satisfy (***BP***). Then*

i) $\left\{h^{-d/2}\varphi_n^h : n \in \mathbb{Z}^d\right\}$ *is an orthonormal basis of* V_φ^h *if and only if*

$$\tau_{\langle D_x\rangle^s\varphi}(\xi) = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^d.$$

ii) $\left\{h^{-d/2}\varphi_n^h : n \in \mathbb{Z}^d\right\}$ *is a* ***Riesz basis*** [6] *of* V_φ^h *if and only if there exist constants* $A, B > 0$ *such that*

$$A \le \tau_{\langle D_x\rangle^s\varphi}(\xi) \le B \qquad \text{for a.e. } \xi \in \mathbb{R}^d.$$

Proof. The operator $\langle D_x\rangle^s : H^s\left(\mathbb{R}^d\right) \to L^2\left(\mathbb{R}^d\right)$ is unitary. Hence $\left\{h^{-d/2}\varphi_n^h : n \in \mathbb{Z}^d\right\}$ is an orthonormal (resp. Riesz) basis of V_φ^h if and only if $\left\{h^{-d/2}\left(\langle D_x\rangle^s\varphi\right)_n^h : n \in \mathbb{Z}^d\right\}$ is an orthonormal (resp. Riesz) basis of the range of $T_{\langle D_x\rangle^s\varphi}^h$. Thus, the Lemma needs only to be proved for profiles $\varphi \in L^2\left(\mathbb{R}^d\right)$ and this is a well-known result (see, for instance, [17]). ■

We shall also need the following expression for the orthogonal projection onto V_φ^h:

Lemma 3.6 *Let* $\varphi \in \mathcal{S}'\left(\mathbb{R}^d\right)$ *satisfy (***BP***). The orthogonal projection* $P_\varphi^h : H^s\left(\mathbb{R}^d\right) \to V_\varphi^h$ *equals* $P_\varphi^h = T_\varphi^h S_{\widetilde{\langle D_x\rangle^s\varphi}}^h \langle hD_x\rangle^s$, *where, for* $f \in L^2\left(\mathbb{R}^d\right)$, $\tilde{f} \in L^2\left(\mathbb{R}^d\right)$ *is defined by:*

$$\widehat{\tilde{f}}(\xi) := \begin{cases} \dfrac{\widehat{f}(\xi)}{\tau_f(\xi)}, & \text{if } \tau_f(\xi) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof of the result for $s = 0$ may be found in [2], Theorem 2.9. We can reduce ourselves to this case by noticing that

$$P_\varphi^h = \langle hD_x\rangle^{-s} P_{\langle hD_x\rangle^s\varphi}^h \langle hD_x\rangle^s,$$

since, as we have seen in (19), the range of T_φ^h equals that of $\langle hD_x\rangle^{-s} T_{\langle hD_x\rangle^s\varphi}^h$ and $\langle hD_x\rangle^s$ is an orthogonal mapping. Using the L^2-result we obtain:

$$P_\varphi^h = \langle hD_x\rangle^{-s} T_{\langle hD_x\rangle^s\varphi}^h S_{\widetilde{\langle D_x\rangle^s\varphi}}^h \langle hD_x\rangle^s = T_\varphi^h S_{\widetilde{\langle D_x\rangle^s\varphi}}^h \langle hD_x\rangle^s,$$

as claimed. ■

[6] This means that there exist constants $A, B > 0$ such that

$$A\|U\|_{L^2(h\mathbb{Z}^d)}^2 \le \left\|T_\varphi^h U\right\|_{H^s(\mathbb{R}^d)}^2 \le B\|U\|_{L^2(h\mathbb{Z}^d)}^2$$

for all $U \in L^2\left(h\mathbb{Z}^d\right)$. This is equivalent to the existence of a linear isomorphism $R : V^h \to V^h$ such that $\left\{h^{-d/2}R\varphi_n^h : n \in \mathbb{Z}^d\right\}$ forms an orthonormal basis of $H^s\left(\mathbb{R}^d\right)$. This property is sometimes also referred as that $\left(\varphi_n^h\right)_{n\in\mathbb{Z}^d}$ form a **stable frame** in $H^s\left(\mathbb{R}^d\right)$.

4 High frequency analysis: $h \sim \varepsilon$

4.1 Reduction to the case $h = \varepsilon$

In this section we analyze the effect of sampling and reconstruction on Wigner measures at scales (ε_k), of the same order of the sampling/reconstruction rate (h_k) (i.e., such that (h_k/ε_k) is bounded).

First notice that it suffices to treat the case $\varepsilon_k = h_k$; the more general one can be obtained by a proper rescaling. This is due to the following identity:

$$m^{\varepsilon}[u](x,\xi) = (h/\varepsilon)^d m^h [u](x,(h/\varepsilon)\xi),$$

which clearly implies:

Lemma 4.1 *Suppose $h_k/\varepsilon_k \to c > 0$. Then $m^{\varepsilon_k}[u_k]$ converges in $\mathcal{S}'$ if and only if $m^{h_k}[u_k]$ does. Their respective limits μ_c and μ are related through:*

$$\mu_c(x,\xi) = c^d \mu(x,c\xi). \tag{23}$$

When $h_k = \varepsilon_k$, the transforms $M^{h_k}\left[U^{h_k}\right]$ are Γ-periodic in the variable ξ; hence, so are their limiting Wigner measures

4.2 Sampling

We start by exploring the effect of sampling on the structure of Wigner measures. The computation of the Wigner measure at scale (h_k) of a sequence of samples $\left(S_{\varphi}^{h_k} u_k\right)$ is done in the following theorem; it is applicable whenever the hypothesis (**D**) below is fulfilled:

$$\operatorname*{essup}_{\xi\in Q} \sum_{|n|\geq R} \left|\langle \xi + 2\pi n\rangle^s \widehat{\varphi}(\xi+2\pi n)\right|^2 \to 0 \qquad \text{as } R\to\infty. \tag{\textbf{D}}$$

Notice that profiles with the property (21) immediately verify (**D**).

Before stating our result, it is important to notice that the Fourier transform of a profile φ satisfying condition (**BP**) is an element of $L^2_{\mathrm{loc}}\left(\mathbb{R}^d\right)$. In particular, it is only defined modulo a set of zero Lebesgue measure. Thus, when dealing with pointwise properties of $\widehat{\varphi}$, we shall systematically assume that a precise representative of the class of $\widehat{\varphi}$ has, once for all, been chosen.

For instance, the Wigner measures μ of the sequences (u_k) in Theorem 4.2 below, will be assumed to satisfy conditions (**MS**) and (**ND**):

$$|\widehat{\varphi}(\xi+2\pi n)|^2 \mu(x,\xi+2\pi n), \quad n\in\mathbb{Z}^d, \quad \text{are mutually singular measures.} \tag{\textbf{MS}}$$

$$\mu\left(\mathbb{R}^d \times \overline{D_{\widehat{\varphi}}}\right) = 0, \tag{ND}$$

where, recall, $D_{\widehat{\varphi}}$ stands for the set of discontinuity points of $\widehat{\varphi}$. These conditions must be understood to hold for the same representative of $\widehat{\varphi}$.

Theorem 4.2 *Let (h_k) be a scale and take φ satisfying (**BP**) and (**D**). Let (u_k) be a sequence in $H^{-s}\left(\mathbb{R}^d\right)$ such that $\left(\langle h_k D_x\rangle^{-s} u_k\right)$ is bounded and suppose that $m^{h_k}[u_k]$ converges to a Wigner measure μ that fulfills (**ND**), (**MS**).*

Then $M^{h_k}\left[S_\varphi^{h_k} u_k\right]$ converges to the Wigner measure μ^φ given by:

$$\mu^\varphi(x,\xi) = \sum_{n\in\mathbb{Z}^d} |\widehat{\varphi}(\xi + 2\pi n)|^2 \mu(x, \xi + 2\pi n). \tag{24}$$

Remark 4.3 *i) As pointed out above, formula (28) holds for the same precise representative of the Fourier transform $\widehat{\varphi}$ which was chosen in (**ND**) and (**MS**).*

*ii) The necessity of hypotheses (**ND**) and (**MS**) will be discussed in paragraph 4.4.*

*iii) Condition (**D**) may be replaced by the assumption that $\left(\langle h_k D_x\rangle^{-s} u_k\right)$ is h_k-oscillatory. This will be made clear in the proof of the Theorem.*

iv) The boundedness of $\langle h_k D_x\rangle^{-s} u_k$ implies that (u_k) is h_k-oscillatory.

The proof of this Theorem is postponed to the end of this section.

The expression (24) may be related to the concept of **Wigner series** introduced in [14, 9]. Recall that given $u \in \mathcal{S}'\left(\mathbb{R}^d\right)$, the **Wigner series** of u at scale ε is defined by:

$$w_S^\varepsilon[u](x,\xi) := \frac{1}{(2\pi)^d} \sum_{n\in\mathbb{Z}^d} u(x - \varepsilon\pi n)\,\overline{u}(x + \varepsilon\pi n)\, e^{in\cdot\xi}.$$

It is easy to check that $w_S^\varepsilon[u](x,\xi) = \sum_{n\in\mathbb{Z}^d} w^\varepsilon[u](x, \xi + 2\pi n)$.[7]

When (u_k) is bounded in $L^2\left(\mathbb{R}^d\right)$, ε_k-oscillatory and possesses a Wigner measure at scale (ε_k) then the following relation holds:

$$\lim_{k\to\infty} \int_{\mathbb{R}^d\times\mathbb{R}^d} a(x,\xi)\, w_S^{\varepsilon_k}[u_k](x,\xi)\, dx d\xi = \int_{\mathbb{R}^d\times\mathbb{R}^d} \sum_{n\in\mathbb{Z}^d} a(x, \xi + 2\pi n)\, d\mu(x,\xi), \tag{25}$$

for $a \in \mathcal{S}$, see [3].

Theorem 4.2 has a simple interpretation in terms of Wigner series: the measure μ^φ may be obtained as the limit of the Wigner series

$$w_S^{h_k}[\widehat{\varphi}(h_k D_x) u_k].$$

[7]See (5) for the definition of the Wigner transform $w^\varepsilon[u]$.

This is due to the fact that, under any of the hypotheses (**D**), $(\widehat{\varphi}(h_k D_x) u_k)$ is h_k-oscillatory. Besides, as a consequence of Proposition 8.3, the Wigner measure at scale (h_k) of $(\widehat{\varphi}(h_k D_x) u_k)$ is given by $|\widehat{\varphi}(\xi)|^2 \mu(x,\xi)$. The assertion then follows from (25).

As was already mentioned in the Introduction, condition (**MS**) is a restriction on the support of the measure $|\widehat{\varphi}(\xi)|^2 \mu(x,\xi)$. Two extremal cases in which it is trivially satisfied are the following:

i) $\widehat{\varphi}|_{\mathbb{R}^d \setminus Q} \equiv 0$, in this case (**MS**) holds independently of what μ is.

ii) The sequence (u_k) is **asymptotically band-limited** i.e. its Wigner measures at scale (h_k) is concentrated on the cube $\overline{Q}$. For those sequences, condition (**MS**) only involves the behavior of μ on the boundary ∂Q: it essentially expresses that the restrictions of μ to parallel sides of ∂Q do not overlap (i.e. are mutually singular). A sufficient condition for this is, for instance,

$$\limsup_{k\to\infty} \int_{\mathbb{R}^d \setminus Q_R} \left| \widehat{u_k}\left(\frac{\xi}{h_k}\right)\right|^2 \frac{d\xi}{(2\pi h_k)^d} \to 0 \qquad \text{as } R \to \infty, \tag{26}$$

where $Q_R := [-\pi, \pi - 1/R)^d$.

Remark 4.4 *In any of the above cases, we have:*

$$\mathbf{1}_Q(\xi)\, \mu^{\varphi}(x,\xi) = |\widehat{\varphi}(\xi)|^2 \mu(x,\xi).$$

Hence, the restriction of μ^{φ} to $\mathbb{R}^d \times Q$ coincides with μ if and only if $|\widehat{\varphi}(\xi)|^2 = 1$ for μ-almost every $\xi \in \overline{Q}$.

The specific choice $\varphi = \delta_0$ corresponds to the analysis of **discretization**, for then $S^h_{\delta_0} u(n) = u(hn)$. Theorem 4.2 takes the following simple form:

Corollary 4.5 *Let (h_k) be a scale and let (u_k) be a sequence in $H^s(\mathbb{R}^d)$, for some $s > d/2$, such that $(\langle h_k D_x \rangle^s u_k)$ is bounded. If μ is its Wigner measure at scale (h_k) and the measures $\mu(x, \xi + 2\pi n)$ are mutually singular then Wigner measure μ^{δ_0} corresponding to the sequence of discretizations is the periodization:*

$$\mu^{\delta_0}(x,\xi) = \sum_{n\in\mathbb{Z}^d} \mu(x, \xi + 2\pi n).$$

In other words, μ^{δ_0} is the limit of the Wigner series $w^{h_k}_S[u_k]$.

This Corollary is particularly useful in the explicit computation of Wigner measures for discrete functions. As an example, consider the concentrating and oscillating sequences we defined in the Introduction, $f_k(x) = k^{d/2}\rho(k(x - x_0))$

and $g_k(x) := \rho(x) e^{ikx\cdot\xi^0}$ with $\rho \in L^2(\mathbb{R}^d)$. Using identities (7) and (8) we obtain, for (f_k) and (g_k) respectively:

$$\mu^{\delta_0}(x,\xi) = \delta_{x_0}(x) \otimes \sum_{n\in\mathbb{Z}} |\widehat{\rho}(\xi + 2\pi n)|^2 \frac{d\xi}{(2\pi)^d},$$

if, for instance, $\operatorname{supp}\widehat{\rho} \subset Q$, and

$$|\rho(x)|^2 dx \otimes \sum_{n\in\mathbb{Z}^d} \delta_{\xi^0+2\pi n}(\xi), \tag{27}$$

with no assumption on ρ.

4.3 Reconstruction

Now we deal with the reconstruction operator T^h_φ; it modifies the high-frequency behavior of a sequence of discrete functions in the following way:

Theorem 4.6 *Let (h_k) be a scale and (U^{h_k}) be an h_k-bounded sequence; take φ satisfying (**BP**). If $M^{h_k}[U^{h_k}]$ converges to the Wigner measure μ which verifies (**ND**) then $m^{h_k}[T^{h_k}_\varphi U^{h_k}]$ converges to a Wigner measure μ_φ given by:*

$$\mu_\varphi(x,\xi) = |\widehat{\varphi}(\xi)|^2 \mu(x,\xi). \tag{28}$$

The proof of Theorem 4.6 is based on explicit formulas for the Fourier transforms of $T^h_\varphi U$. As we have already seen,

$$\widehat{T^h_\varphi U}(\xi) = \widehat{\varphi}(h\xi) h^d \widehat{U}(h\xi), \tag{29}$$

for any $U \in L^2(h\mathbb{Z}^d)$. The following Remark ensures that Proposition 8.3 can be applied in the proof below.

Remark 4.7 *If $\varphi \in H^s(\mathbb{R}^d)$ satisfies (**BP**) then $\widehat{\varphi} \in L^\infty(\mathbb{R}^d; \langle\xi\rangle^s)$.*

Proof of Theorem 4.6. Just notice that (29) can be rewritten as:

$$T^h_\varphi U^h = \widehat{\varphi}(hD_x) T^h_{\delta_0} U^h.$$

The hypotheses made on φ and μ allow us to apply Proposition 8.3 (see Remark 4.7) and conclude ■

Identity (28) expresses how the measure μ is modulated by the profile φ; the necessity of the hypothesis (**ND**) for this result is discussed in paragraph 4.4 as well.

Since μ is Γ-periodic in ξ, formula (28) suggests that μ may be compared to the periodization of μ_φ with respect to the variable ξ.

Corollary 4.8 *Let φ, $\left(U^{h_k}\right)$, μ and μ_φ be as in Theorem 4.6. Then the periodization*

$$\mu_{\varphi,s}(x,\xi) := \sum_{n\in\mathbb{Z}^d} \langle \xi + 2\pi n\rangle^{2s} \mu_\varphi (x, \xi + 2\pi n) \tag{30}$$

is a well-defined[8] *measure, Γ-periodic in ξ, that satisfies:*

$$\mu_{\varphi,s}(x,\xi) = \tau_{\langle D_x\rangle^s \varphi}(\xi)\, \mu(x,\xi) . \tag{31}$$

In particular:

i) If $\tau_{\langle D_x\rangle^s \varphi}(\xi) = 1$ except for ξ in a set of zero μ-measure then $\mu_{\varphi,s} = \mu$.

ii) $\tau_{\langle D_x\rangle^s \varphi} \equiv 1$ if and only if the identity $\mu_{\varphi,s} = \mu$ holds for every sequence $\left(U^{h_k}\right)$.

Proof. Since $|\langle\xi\rangle^s \widehat{\varphi}(\xi)|^2$ is a nonnegative continuous function, the series defining $\tau_{\langle D_x\rangle^s \varphi}(\xi)$ converges absolutely for every ξ on the support of μ (which consists of continuity points for $\widehat{\varphi}(\xi)$). Thus, by the dominated convergence Theorem:

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} a(x,\xi)\, \tau_{\varphi,s}(\xi)\, d\mu(x,\xi) = \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}^d\times\mathbb{R}^d} a(x,\xi) \left|\langle \xi+2\pi n\rangle^s \widehat{\varphi}(\xi+2\pi n)\right|^2 d\mu(x,\xi)$$

for every $a \in C_c\left(\mathbb{R}^d \times \mathbb{R}^d\right)$. Taking now into account (28) and the fact that μ is Γ-periodic in ξ, we find that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} a(x,\xi)\, \tau_{\varphi,s}(\xi)\, d\mu(x,\xi) = \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}^d\times\mathbb{R}^d} a(x,\xi) \langle \xi+2\pi n\rangle^{2s} d\mu_\varphi(x,\xi+2\pi n)$$

and the first part of the result follows.

Statement i) as well as the "only if" part of ii) are trivial. To obtain the necessity in ii), just consider sequences of discrete functions whose Wigner measures are of the form $\mu(x,\xi) = \nu(x) \otimes \sum_{n\in\mathbb{Z}^d} \delta_{\xi^0+2\pi n}$ (as (27), for instance). Clearly, for $\mu_{\varphi,s} = \mu$ to hold for such a measure, we must have $\tau_{\langle D_x\rangle^s \varphi}\left(\xi^0\right) = 1$. ■

Remark 4.9 *i) Because of Lemma 3.5, if relation $\mu_{\varphi,s} = \mu$ holds for every h_k-bounded sequence of discrete functions then the profile φ has the property: $\left\{h^{-d/2}\varphi_n^h : n \in \mathbb{Z}^d\right\}$ is an orthonormal family in $H^s\left(\mathbb{R}^d\right)$ for every $h > 0$.*

ii) However, the converse is not true, if φ gives rise to an orthonormal family then $\tau_{\langle D_x\rangle^s \varphi}(\xi) = 1$ holds outside a set of null Lebesgue measure. If μ is supported on that set, identity $\mu_{\varphi,s} = \mu$ may not hold.

[8] The limit defining the sum (30) is understood to exist for the weak convergence of measures in $\mathcal{M}_+\left(\mathbb{R}^d \times \mathbb{R}^d\right)$.

As in the preceding section, our result has an interpretation in terms of Wigner series. Under the conditions of Theorem 4.6, the measure $\mu_{\varphi,s}$ may be obtained as the limit as $k \to \infty$ of the functions

$$w_S^{h_k}\left[\langle h_k D_x\rangle^s T_\varphi^{h_k} U^{h_k}\right],$$

provided $\left(\langle h_k D_x\rangle^s T_\varphi^{h_k} U^{h_k}\right)$ is h_k-oscillatory. Note however, that this may not be the case for certain profiles φ (see paragraph 5.3).

In particular, Corollary 4.8 shows that the limits of $w_S^{h_k}\left[T_\varphi^{h_k} U^{h_k}\right]$ and $M^{h_k}\left[U^{h_k}\right]$ coincide if we chose, for instance, $\varphi := \mathbf{1}_{[-1/2,1/2)^d}$.

4.4 The necessity of the hypotheses of Theorems 4.2 and 4.6

Formulas (28) and (24) may not hold when $\widehat{\varphi}$ is not continuous and the Wigner measure μ does not vanish on the closure of the set of discontinuity points $D_{\widehat{\varphi}}$. We illustrate this with two one-dimensional examples where

$$\varphi(x) = \frac{\sin \pi x}{\pi x}.$$

We will chose $\mathbf{1}_Q$ as the representative of $\widehat{\varphi}$ for which the counterexamples will be built.

1. Necessity of condition (ND) in Theorem 4.6. Take U^h to be the sequence discrete function of $L^2\left(h\mathbb{Z}^d\right)$ given by their Fourier transforms:

$$\widehat{U^h}(\xi) := \frac{1}{h}\sum_{n\in\mathbb{Z}} \mathbf{1}_{(-1,1)}\left(\frac{\xi - (2n+1)\pi}{h}\right).$$

Then, denoting by μ the Wigner measure at scale h of $\left(U^h\right)$,

$$\left|\widehat{\varphi}(\xi)\right|^2 \mu(x,\xi) = \frac{\sin^2(x)}{\pi^2 x^2} dx \otimes \delta_{-\pi}(\xi).$$

This measure differs from μ_φ, which is given by:

$$\mu_\varphi(x,\xi) = \frac{\sin^2(x/2)}{\pi^2 x^2} dx \otimes \left[\delta_\pi(\xi) + \delta_{-\pi}(\xi)\right].$$

Remark 4.10 *i) The particular choice of the representative of $\widehat{\varphi}$ does not play a role. Theorem 4.6 still fails if we take as representative of $\widehat{\varphi}$ the characteristic functions of $(-\pi,\pi)^d$ or $[-\pi,\pi]^d$.*

ii) In particular, this example shows that even the two projections on x and ξ of the measures μ and μ_φ may differ.

iii) This also shows that the periodization in ξ of μ_φ does not necessarily coincide with μ, even when $\tau_\varphi = 1$ as is the case here. Thus the conclusion of Corollary 4.8 may fail when $\widehat{\varphi}$ is not continuous.

Our counterexample to Theorem 4.2 is essentially the same as the previous one:

2. Necessity of condition (ND) in Theorem 4.2. Define

$$\widehat{v^h}(\xi) := \mathbf{1}_{(-1,1)}(\xi + \pi/h).$$

Then, denoting by μ the Wigner measure at scale h of $\left(v^h\right)$,

$$\sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2\pi n)|^2 \mu(x, \xi + 2\pi n) = \frac{\sin^2(x)}{\pi^2 x^2} dx \otimes \sum_{n\in\mathbb{Z}} \delta_{(2n+1)\pi}(\xi)$$

and this is different from μ^φ, which is precisely:

$$\mu^\varphi(x, \xi) = \frac{\sin^2(x/2)}{\pi^2 x^2} dx \otimes \sum_{n\in\mathbb{Z}} \delta_{(2n+1)\pi}(\xi).$$

Finally, we investigate hypothesis (**MS**). Now we set $\varphi := \delta_0$.

3. Necessity of condition (MS) in Theorem 4.2. Define

$$\widehat{v^h}(\xi) := \mathbf{1}_Q(h\xi) \sum_{n\in\mathbb{Z}} \mathbf{1}_{(-1,1)}(\xi - (2n+1)\pi).$$

Clearly, as in our first example, the periodization of the Wigner measure of $\left(v^h\right)$ is:

$$\sum_{k\in\mathbb{Z}^d} \mu(x, \xi + 2\pi n) = \frac{\sin^2(x/2)}{\pi^2 x^2} dx \otimes \sum_{k\in\mathbb{Z}} \delta_{(2n+1)\pi}(\xi).$$

However, the sequence of discretizations $\left(S^h_{\delta_0} v^h\right)$ has the following one:

$$\mu^{\delta_0}(x, \xi) = \frac{\sin^2(x)}{\pi^2 x^2} dx \otimes \sum_{n\in\mathbb{Z}} \delta_{(2n+1)\pi}(\xi).$$

The proof of these counterexamples easily follows from (7), identity (55) and Lemma 8.13.

4.5 A Poisson summation formula and proof of Theorem 4.2

The computation of the Fourier transform of $S_\varphi^h u$ is given by the following identity:

Lemma 4.11 *Let φ satisfy (**BP**) and $u \in H^{-s}\left(\mathbb{R}^d\right)$. Then the Fourier transform of $S_\varphi^h u$ is:*

$$h^d \sum_{n\in\mathbb{Z}^d} S_\varphi^h u\left(n\right) e^{-ihn\cdot\xi} = \sum_{n\in\mathbb{Z}^d} \overline{\widehat{\varphi}\left(h\xi + 2\pi n\right)}\widehat{u}\left(\xi + \frac{2\pi}{h}n\right), \tag{32}$$

the convergence of the first series being in $L^2_{loc}\left(\mathbb{R}^d\right)$ while the second takes place in $L^1_{loc}\left(\mathbb{R}^d\right)$.

Proof. Begin by noticing that $\overline{\widehat{\varphi}}\widehat{u} \in L^1\left(\mathbb{R}^d\right)$ and thus

$$\Pi^h\left(\xi\right) := \sum_{n\in\mathbb{Z}^d} \overline{\widehat{\varphi}\left(h\xi + 2\pi n\right)}\widehat{u}\left(\xi + 2\pi/hn\right)$$

is a well-defined $\left(2\pi/h\right)\mathbb{Z}^d$-periodic $L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)$ function, the series defining it being absolutely convergent in $L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)$. We can compute its Fourier coefficients:

$$\begin{aligned}\int_{[-\pi/h,\pi/h)^d} \Pi^h\left(\xi\right) e^{ihn\cdot\xi} \frac{h^d d\xi}{\left(2\pi\right)^d} &= \sum_{k\in\mathbb{Z}^d} \int_Q \overline{\widehat{\varphi}\left(\xi + 2\pi k\right)}\widehat{u}\left(\frac{\xi + 2\pi k}{h}\right) e^{in\cdot\xi} \frac{d\xi}{\left(2\pi\right)^d} \\ &= \int_{\mathbb{R}^d} \overline{\widehat{\varphi}\left(\xi\right)}\widehat{u}\left(\frac{\xi}{h}\right) e^{in\cdot\xi} \frac{d\xi}{\left(2\pi\right)^d} \\ &= \int_{\mathbb{R}^d} \overline{h^d\widehat{\varphi}\left(h\xi\right) e^{-ihn\cdot\xi}}\widehat{u}\left(\xi\right) \frac{d\xi}{\left(2\pi\right)^d} \\ &= \left\langle \overline{\varphi_n^h}, u\right\rangle_{\mathcal{S}'\times\mathcal{S}} = h^d S_\varphi^h u\left(n\right).\end{aligned}$$

Lemma 3.1 proves that $S_\varphi^h u$ is square-summable and, consequently,

$$\Pi^h\left(\xi\right) = \sum_{n\in\mathbb{Z}^d} h^d S_\varphi^h u\left(n\right) e^{-ihn\cdot\xi},$$

the sum being understood in the L^2-sense. This is precisely formula (32). ■

Remark 4.12 *Identity (32) may be viewed as a generalization of **Poisson summation formula**. Taking as φ the Dirac delta δ_0, we obtain:*

$$h^d \sum_{n\in\mathbb{Z}^d} u\left(hn\right) e^{-ihn\cdot\xi} = \sum_{n\in\mathbb{Z}^d} \widehat{u}\left(\xi + \frac{2\pi}{h}n\right),$$

for every $u \in H^s\left(\mathbb{R}^d\right)$ with $s > d/2$.

Proof of Theorem 4.2. The proof will be done in two steps:

Step 1: We first establish the result for sequences such that $\widehat{\varphi}(\xi)\,\widehat{u_k}(\xi/h_k)$ has support in a ball $B(0;R)$ for every $k\in\mathbb{N}$. We claim that the following formula holds:

$$T^{h_k}_{\delta_0}S^{h_k}_{\varphi}u_k(x)=\sum_{|n|\leq R+\pi\sqrt{d}}e^{-2\pi in\cdot x/h_k}\overline{\widehat{\varphi}}(h_kD_x)\,u_k(x)\,.$$

This is obtained by applying the inverse Fourier transform at both sides of identity (32), and remarking that only summands satisfying $|n|\leq R+\pi\sqrt{d}$ must be considered because of the condition on the support of $\widehat{\varphi}\widehat{u_k}(\cdot/h_k)$. The Wigner measures of the functions

$$e^{-2\pi in\cdot x/h_k}\overline{\widehat{\varphi}}(h_kD_x)\,u_k(x)$$

are precisely (cf. Proposition 8.3 and Remark 4.7):

$$|\widehat{\varphi}(\xi+2\pi n)|^2\,\mu(x,\xi+2\pi n)\,.$$

By hypothesis, they are mutually singular so, by Lemma 8.13, we deduce that the measure μ^{φ} obtained as the limit of $m^{h_k}\left[T^{h_k}_{\delta_0}S^{h_k}_{\varphi}u_k\right]$ is given by (24).

Step 2: We prove the result in the general case by taking advantage of hypothesis (**D**). Let $\chi\in C_c^{\infty}\left(\mathbb{R}^d\right)$ be a cut-off function identically equal to one in the unit ball $B(0;1)$. Denote by $S^{h_k}_{\varphi,R}u_k$ the truncation given by:

$$\begin{aligned}\widehat{S^{h_k}_{\varphi,R}u_k}(\xi)&:=S^{h_k}_{\varphi}\chi\left(\widehat{\frac{h_kD_x}{R}}\right)u_k(\xi)\\&=\frac{1}{(h_k)^d}\sum_{n\in\mathbb{Z}^d}\overline{\widehat{\varphi}(\xi+2\pi n)}\chi\left(\frac{\xi+2\pi n}{R}\right)\widehat{u_k}\left(\frac{\xi+2\pi n}{h_k}\right);\end{aligned}$$

Then, by the first step we have just proved, $M^{h_k}\left[S^{h_k}_{\varphi,R}u\right]$ converges to

$$\mu^{\varphi}_R(x,\xi):=\sum_{n\in\mathbb{Z}^d}\left|\widehat{\varphi}(\xi+2\pi n)\,\chi\left(\frac{\xi+2\pi n}{R}\right)\right|^2\mu(x,\xi+2\pi n)\,.\qquad(33)$$

We claim that (**D**) implies the following:

$$\limsup_{k\to\infty}\left\|S^{h_k}_{\varphi}u_k-S^{h_k}_{\varphi,R}u_k\right\|^2_{L^2\left(h_k\mathbb{Z}^d\right)}\to 0\qquad\text{as }R\to\infty.\qquad(34)$$

It is sufficient to realize that

$$S^{h_k}_{\varphi}u_k-S^{h_k}_{\varphi,R}u_k=S^{h_k}_{\psi_R}u_k$$

for $\widehat{\psi_R} := \overline{\chi(\cdot/R)\widehat{\varphi}}$. The norm of $S^{h_k}_{\psi_R}$ is precisely (cf. Lemma 3.1)

$$\operatorname*{essup}_{\xi\in Q}\sum_{n\in\mathbb{Z}^d}\left|\langle\xi+2\pi n\rangle^s\,\widehat{\varphi}\,(\xi+2\pi n)\,\chi\left(\frac{\xi+2\pi n}{R}\right)\right|^2$$

which tends to zero as $R\to 0$.

Lemma 8.12 then ensures that μ^φ_R weakly converge to μ^φ. Identity (33) means that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} a\,(x,\xi)\,d\mu^\varphi_R\,(x,\xi) = \int_{\mathbb{R}^d\times\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d} a\,(x,\xi+2\pi n)\,|\widehat{\varphi}\,(\xi)|^2\,|\chi\,(\xi/R)|^2\,d\mu\,(x,\xi)$$

for every test function $a\in\mathcal{S}$. Passing to limits as $R\to\infty$ in the above identity we obtain the claimed result.

Notice that the same argument may be applied if, instead of condition (**D**), we have that $\left(\langle h_k D_x\rangle^{-s}u_k\right)$ is h_k-oscillatory. This is because (34) may be estimated from above by

$$\limsup_{k\to\infty}\left\|\langle h_k D_x\rangle^{-s}\left(1-\chi\left(\frac{h_k D_x}{R}\right)\right)u_k\right\|^2_{L^2\left(\mathbb{R}^d\right)}\to 0\qquad\text{as } R\to\infty$$

because of Lemma 3.1 and the h_k-oscillation hypothesis. ■

5 Computation of defect measures

5.1 Relations between defect and Wigner measures in the discrete setting

In this paragraph, we establish the analog of Proposition 1.2 in the discrete setting. In particular, we present conditions that ensure that the projection on the x-component of a Wigner measure may be obtained as the limit of quadratic densities of the type:

$$E^h\left[U^h\right](x) := h^d\sum_{n\in\mathbb{Z}^d}\left|U^h_n\right|^2\delta_{hn}\,(x)\,.$$

Proposition 5.1 *Let (h_k) be a scale and $\left(U^{h_k}\right)$ be an h_k-bounded sequence. Suppose that $\left(M^{h_k}\left[U^{h_k}\right]\right)$ converges to μ as $k\to\infty$. Then, for every $\phi\in C_c\left(\mathbb{R}^d\right)$,*

$$\int_{\mathbb{R}^d\times Q}\phi\,(x)\,d\mu\,(x,\xi) = \lim_{k\to\infty}(h_k)^d\sum_{n\in\mathbb{Z}^d}\phi\,(h_k n)\left|U^{h_k}_n\right|^2. \qquad (35)$$

If (ε_k) is a scale such that $h_k \ll \varepsilon_k$ and the transforms $M^{\varepsilon_k}\left[U^{h_k}\right]$ converge to μ, then (35) holds provided $\left(U^{h_k}\right)$ is ***ε_k-oscillatory,*** *i.e.:*

$$\limsup_{k\to\infty} (h_k)^d \int_{Q\setminus B(0;h_k/\varepsilon_k R)} \left|\widehat{U^{h_k}}(\xi)\right|^2 d\xi \to 0 \qquad \text{as } R\to\infty. \tag{36}$$

In view of Proposition 5.1, one could think that Wigner measures at scales coarser than h_k are unnecessary. However, as the next result shows, if $\left(U^{h_k}\right)$ is ε_k-oscillatory for such a scale then the Wigner measure at scale (h_k) does not give any information about the oscillation effects.

Proposition 5.2 *Let (h_k) and (ε_k) be scales such that $h_k \ll \varepsilon_k$. For every ε_k-oscillatory, h_k-bounded sequence $\left(U^{h_k}\right)$ such that $M^{h_k}\left[U^{h_k}\right] \rightharpoonup \mu$ as $k\to\infty$ we have*

$$\mu(x,\xi) = \nu(x)\otimes\sum_{k\in\mathbb{Z}^d}\delta_{2\pi k}(\xi)$$

where ν is the weak limit in $\mathcal{M}_+\left(\mathbb{R}^d\right)$ of the measures $E^{h_k}\left[U^{h_k}\right]$.

The Wigner measure also gathers the information on the densities $\left|\mathcal{F}^{\varepsilon_k}U^{h_k}(\xi)\right|^2$; indeed, these converge to the projection on ξ of the Wigner measure provided that no energy is lost at infinity.

Proposition 5.3 *Let (h_k) and (ε_k) be scales such that h_k/ε_k is bounded. Suppose that $\left(U^{h_k}\right)$ is* ***compact at infinity****:*

$$\limsup_{k\to\infty}(h_k)^d \sum_{|h_k n|>R}\left|U_n^{h_k}\right|^2 \to 0, \qquad \text{as } R\to\infty, \tag{37}$$

and that $M^{\varepsilon_k}\left[U^{h_k}\right] \rightharpoonup \mu$ as $k\to\infty$. Then

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\psi(\xi)\,d\mu(x,\xi) = \lim_{k\to\infty}\int_{\mathbb{R}^d}\psi(\xi)\left|\mathcal{F}^{\varepsilon_k}U^{h_k}(\xi)\right|^2 d\xi$$

for every $\psi\in C_c\left(\mathbb{R}^d\right)$.

The proof of Propositions 5.1 and 5.3 requires the following preliminary result, which explains how the transform $M^{\varepsilon}\left[U^h\right]$ of a discrete function U^h can be localized:

Lemma 5.4 *Let $U^h \in L^2\left(h\mathbb{Z}^d\right)$ and $\varphi,\ \phi\in C_c^\infty\left(\mathbb{R}^d\right)$. Then for every $a\in\mathcal{S}\left(\mathbb{R}^d\times\mathbb{R}^d\right)$ the following holds:*

$$\lim_{k\to\infty}\left|\left\langle M^{\varepsilon_k}\left[U^{h_k}\right], |\phi(x)|^2\varphi(\xi)\right\rangle_{\mathcal{S}'\times\mathcal{S}} - (h_k)^d\int_{\mathbb{R}^d}\left|\widehat{\phi U^{h_k}}(\xi)\right|^2\varphi\left(\frac{\varepsilon_k}{h_k}\xi\right)\frac{d\xi}{(2\pi)^d}\right| = 0.$$

Proof. First remark that, as a consequence of relation (14) and Lemma 8.5 we have

$$\lim_{k\to\infty}\left|\left\langle M^{\varepsilon_k}\left[U^{h_k}\right],\left|\phi\left(x\right)\right|^2\varphi\left(\xi\right)\right\rangle_{\mathcal{S}'\times\mathcal{S}}-\left\langle M^{\varepsilon_k}\left[\phi U^{h_k}\right],\psi\left(x\right)\varphi\left(\xi\right)\right\rangle_{\mathcal{S}'\times\mathcal{S}}\right|=0 \quad (38)$$

for every test function $\psi \in C_c^\infty\left(\mathbb{R}^d\right)$ such that $\psi\left(x\right)=1$ for $x\in\operatorname{supp}\phi$. Now, (52.i) and (14) together with Plancherel's formula for the discrete Fourier transform yield:

$$\begin{aligned}\left\langle M^{\varepsilon_k}\left[\phi U^{h_k}\right],\psi\left(x\right)\varphi\left(\xi\right)\right\rangle_{\mathcal{S}'\times\mathcal{S}} &= \left\langle \phi\left(x\right)T_{\delta_0}^{h_k}U^{h_k},\varphi\left(\varepsilon_k D_x\right)\phi\left(x\right)T_{\delta_0}^{h_k}U^{h_k}\right\rangle_{\mathcal{S}'\times\mathcal{S}}\\ &= \left(h_k\right)^{2d}\int_{\mathbb{R}^d}\left|\widehat{\phi U^{h_k}}\left(h_k\xi\right)\right|^2\varphi\left(\varepsilon_k\xi\right)\frac{d\xi}{\left(2\pi\right)^d}\end{aligned}$$

and the result follows. ■

Proof of Proposition 5.1. Identity (35) in the case $h_k=\varepsilon_k$ is a direct consequence of the identity

$$\int_Q M^h\left[U\right]\left(x,\xi\right)d\xi = E^h\left[U^h\right]\left(x\right)$$

and that, due to the Γ-periodicity in ξ of $M^{h_k}\left[U^{h_k}\right]$ and μ, one has

$$\lim_{k\to\infty}\int_{\mathbb{R}^d\times Q}\phi\left(x\right)M^{h_k}\left[U^{h_k}\right]\left(x,\xi\right)dxd\xi=\int_{\mathbb{R}^d\times Q}\phi\left(x\right)d\mu\left(x,\xi\right)$$

for every $\phi\in C_c^\infty\left(\mathbb{R}^d\right)$.

Next we analyze the case $h_k/\varepsilon_k\to 0$. Given functions $\phi,\chi\in C_c^\infty\left(\mathbb{R}^d\right)$, using Lemma 5.4 and periodization in the variable ξ we find:

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\left|\phi\left(x\right)\right|^2\chi\left(\xi\right)d\mu\left(x,\xi\right)=\lim_{k\to\infty}\left(h_k\right)^d\int_Q\left|\widehat{\phi U^{h_k}}\left(\xi\right)\right|^2\sum_{n\in\mathbb{Z}^d}\chi\left(\frac{\varepsilon_k}{h_k}\left(\xi+2\pi n\right)\right)\frac{d\xi}{\left(2\pi\right)^d}. \quad (39)$$

Choose a function $\chi\in C_c^\infty\left(\mathbb{R}^d\right)$ such that

$$\begin{aligned}&\chi\left(\xi\right)=1 \text{ for } \left|\xi\right|\le 1,\\ &\chi\left(\xi\right)=0 \text{ for } \left|\xi\right|\ge 2,\\ &0\le\chi\left(\xi\right)\le 1 \text{ for } \xi\in\mathbb{R}^d,\end{aligned}$$

and set $\chi_R\left(\xi\right):=\chi\left(\xi/R\right)$ for every $R>0$. With such a test function and $h_k/\varepsilon_k<\pi/R$ we have $\chi_R\left(\frac{\varepsilon_k}{h_k}\left(\xi+2\pi n\right)\right)=\chi_R\left(\frac{\varepsilon_k}{h_k}\xi\right)\le 1$ for every $\xi\in Q$. Then, taking this into account in (39) and using Plancherel's formula, the following is obtained:

$$\lim_{k\to\infty}\left|(h_k)^d\int_Q\left|\widehat{\phi U^{h_k}}(\xi)\right|^2\frac{d\xi}{(2\pi)^d}-\int_{\mathbb{R}^d\times\mathbb{R}^d}|\phi(x)|^2\chi_R(\xi)\,d\mu(x,\xi)\right|\le M(R),$$

where

$$M(R):=\limsup_{k\to\infty}(h_k)^d\int_{\mathbb{R}^d}\left(1-\chi_R\left(\frac{\varepsilon_k}{h_k}(\xi+2\pi n)\right)\right)\left|\widehat{\phi U^{h_k}}(\xi)\right|^2\frac{d\xi}{(2\pi)^d}.$$

Identity (35) is obtained by letting R tend to ∞, noticing that (36) implies that $M(R)\to 0$ as $R\to\infty$. ■

Proof of Proposition 5.2. Since $\left(U^{h_k}\right)_{k\in\mathbb{N}}$ is ε_k-oscillatory, we have

$$\limsup_{k\to\infty}\int_{Q\setminus B(0;\delta)}\left|\widehat{\phi U^{h_k}}(\xi)\right|^2d\xi=0$$

for every $\delta>0$ and $\phi\in C_c^\infty\left(\mathbb{R}^d\right)$. Using Lemma 5.4 below, we obtain, for every $\varphi\in C_c^\infty(Q\setminus\{0\})$,

$$0=\lim_{k\to\infty}\int_Q\varphi(\xi)\left|\widehat{\phi U^{h_k}}(\xi)\right|^2\frac{d\xi}{(2\pi)^d}=\int_{\mathbb{R}^d\times Q}|\phi(x)|^2\varphi(\xi)\,d\mu(x,\xi).$$

In particular, μ is concentrated on the set $\mathbb{R}^d\times\{0\}$. Since $\mu(\cdot\times Q)=\nu(x)$ by Proposition 5.1 we find that, because of the periodicity, $\mu\left(\mathbb{R}^d\times\cdot\right)=\nu\left(\mathbb{R}^d\right)\sum_{k\in\mathbb{Z}^d}\delta_{2\pi k}(\xi)$; and this restricts μ to be equal to $\nu\otimes\sum_{k\in\mathbb{Z}^d}\delta_{2\pi k}$. ■

Proof of Proposition 5.3. Since the densities $\left|\mathcal{F}^{\varepsilon_k}U^{h_k}\right|^2$ are uniformly bounded in $L^1\left(\mathbb{R}^d\right)$ (and consequently in $\mathcal{M}\left(\mathbb{R}^d\right)$) it suffices to prove the result for test functions $\psi\in C_c^\infty\left(\mathbb{R}^d\right)$. Let χ and χ_R be defined as in the proof of Proposition 5.1. Because of Lemma 5.4 the following holds for every $\psi\in\mathcal{S}\left(\mathbb{R}^d\right)$:

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}|\chi_R(x)|^2\psi(\xi)\,d\mu(x,\xi)=\lim_{k\to\infty}\int_{\mathbb{R}^d}\psi(\xi)\left|\mathcal{F}^{\varepsilon_k}\chi_R U^{h_k}(\xi)\right|^2d\xi.$$

Since $|\chi_R(x)|^2\to 1$ as $R\to\infty$ for every $x\in\mathbb{R}^d$ we only have to show that

$$\lim_{R\to\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d}|\chi_R(x)|^2\psi(\xi)\,d\mu(x,\xi)=\lim_{k\to\infty}\int_{\mathbb{R}^d}\psi(\xi)\left|\mathcal{F}^{\varepsilon_k}U^{h_k}(\xi)\right|^2d\xi.$$

This appears as a consequence of the identity

$$\begin{aligned}\int_{\mathbb{R}^d}\psi(\xi)\left(\left|\mathcal{F}^{\varepsilon_k}U^{h_k}(\xi)\right|^2-\left|\mathcal{F}^{\varepsilon_k}\chi_R U^{h_k}(\xi)\right|^2\right)d\xi&=\int_{\mathbb{R}^d}\psi(\xi)\left[\mathcal{F}^{\varepsilon_k}\left(U^{h_k}-\chi_R U^{h_k}\right)(\xi)\right]\overline{\mathcal{F}^{\varepsilon_k}U^{h_k}(\xi)}d\xi\\&+\int_{\mathbb{R}^d}\psi(\xi)\,\mathcal{F}^{\varepsilon_k}\chi_R U^{h_k}(\xi)\,\overline{\left[\mathcal{F}^{\varepsilon_k}\left(U^{h_k}-\chi_R U^{h_k}\right)(\xi)\right]}d\xi\end{aligned}$$

which implies

$$\limsup_{k\to\infty}\left|\int_{\mathbb{R}^d}\psi(\xi)\left(\left|\mathcal{F}^{\varepsilon_k}U^{h_k}(\xi)\right|^2-\left|\mathcal{F}^{\varepsilon_k}\chi_R U^{h_k}(\xi)\right|^2\right)d\xi\right|\leq C_\psi\limsup_{k\to\infty}\left\|U^{h_k}-\chi_R U^{h_k}\right\|^2_{L^2\left(h\mathbb{Z}^d\right)}.$$

Since the U^{h_k} are compact at infinity, the second term in the above estimate tends to zero as R tends to infinity and thus:

$$\left|\limsup_{k\to\infty}\int_{\mathbb{R}^d}\psi(\xi)\left|\mathcal{F}^{\varepsilon_k}U^{h_k}(\xi)\right|^2-\int_{\mathbb{R}^d\times\mathbb{R}^d}\left|\chi_R(x)\right|^2\psi(\xi)\,d\mu(x,\xi)\right|\to 0\qquad\text{as } R\to\infty.$$

One easily deduces from this that the measures $\left|\mathcal{F}^{\varepsilon_k}U^{h_k}(\xi)\right|^2 d\xi$ converge in $\mathcal{M}_+\left(\mathbb{R}^d\right)$ to the measure $\int_{\mathbb{R}^d}\mu(dx,\cdot)$ as claimed. ■

5.2 Defect measures of reconstructed sequences

Let $\left(U^{h_k}\right)$ be h_k-bounded and $\varphi\in H^s\left(\mathbb{R}^d\right)$ some profile satisfying (**BP**). As a consequence of Lemma 3.1, the sequence of densities

$$\left|\langle h_k D_x\rangle^s T_\varphi^{h_k}U^{h_k}\right|^2$$

is uniformly bounded in $L^1\left(\mathbb{R}^d\right)$. Hence, Helly's compactness Theorem ensures that, extracting a subsequence if necessary, there exists a measure $\nu_\varphi\in\mathcal{M}_+\left(\mathbb{R}^d\right)$ such that

$$\lim_{k\to\infty}\int_{\mathbb{R}^d}\phi(x)\left|\langle h_k D_x\rangle^s T_\varphi^{h_k}U^{h_k}(x)\right|^2dx=\int_{\mathbb{R}^d\times Q}\phi(x)\,d\nu_\varphi(x).$$

The main issue addressed in this section is that of clarifying how ν_φ depends of the sequence $\left(U^{h_k}\right)$ and the profile φ. We shall see that a formula relating ν_φ and the limit of $E^{h_k}\left[U^{h_k}\right]$ does not exist in general. However such a formula may be established in terms of the Wigner measure of $\left(U^{h_k}\right)$.

Suppose that $M^{h_k}\left[U^{h_k}\right]$ converges to μ. Then, Theorem 4.6 may be applied to obtain that, provided $\mu\left(\mathbb{R}^d\times\overline{D_{\widehat{\varphi}}}\right)=0$, one has

$$m^{h_k}\left[T_\varphi^{h_k}U^{h_k}\right]\rightharpoonup\left|\widehat{\varphi}(\xi)\right|^2\mu(x,\xi).$$

In general, we are only able to ensure (see Proposition 1.7 in [9]):

$$\nu_\varphi(x)\geq\int_{\mathbb{R}^d}\left|\langle\xi\rangle^s\widehat{\varphi}(\xi)\right|^2\mu(x,d\xi),$$

but equality holds whenever $\left(\langle h_k D_x\rangle^s T_\varphi^{h_k}U^{h_k}\right)$ is h_k-oscillatory. However, this is not immediate since there exist profiles φ for which $\left(\langle h_k D_x\rangle^s T_\varphi^{h_k}U^{h_k}\right)$ may fail to be h_k-oscillatory for some $\left(U^{h_k}\right)$ (an example is provided at the end of this section). Nevertheless,

Proposition 5.5 *$\left(\langle h_k D_x\rangle^s T_\varphi^{h_k} U^{h_k}\right)$ is h_k-oscillatory whenever (**D**) holds.*

This immediately follows from:

Lemma 5.6 *$\left(\langle h_k D_x\rangle^s T_\varphi^{h_k} U^{h_k}\right)$ is h_k-oscillatory if and only if*

$$\limsup_{k\to\infty} (h_k)^d \int_Q \sigma_\varphi^R(\xi) \left|\widehat{U^{h_k}}(\xi)\right|^2 d\xi \to 0 \qquad \text{as } R\to\infty,$$

where

$$\sigma_\varphi^R(\xi) := \sum_{|n|\geq R} \left|\langle \xi + 2\pi n\rangle^s \widehat{\varphi}(\xi+2\pi n)\right|^2.$$

Proof. Start by noticing that:

$$\int_{|\xi|\geq R/h_k} \left|\widehat{\langle h_k D_x\rangle^s T_\varphi^{h_k} U^{h_k}}(\xi)\right|^2 d\xi = \int_{|\xi|\geq R} (h_k)^d \left|\langle\xi\rangle^s \widehat{\varphi}(\xi)\widehat{U^{h_k}}(\xi)\right|^2 d\xi.$$

Periodizing in ξ we get:

$$\int_Q \sigma_\varphi^{R+\sqrt{d}\pi}(\xi)\left|\widehat{U^{h_k}}(\xi)\right|^2 d\xi \leq \int_{|\xi|\geq R}\left|\langle\xi\rangle^s \widehat{\varphi}(\xi)\widehat{U^{h_k}}(\xi)\right|^2 d\xi \leq \int_Q \sigma_\varphi^{R-\sqrt{d}\pi}(\xi)\left|\widehat{U^{h_k}}(\xi)\right|^2 d\xi,$$

and the claim follows. ■

Hence, h_k-oscillation is obtained if φ decays at infinity at an uniform rate. For more general φ, it is still possible to obtain sufficient conditions; however, these depend on the particular sequence of discrete functions to be reconstructed.

Proposition 5.7 *Suppose*

$$\begin{aligned} &i) \quad \mu\left(\mathbb{R}^d \times \overline{D_{\tau_{\langle D_x\rangle^s\varphi}}}\right) = 0, \\ &ii) \quad \left(U^{h_k}\right) \text{ is compact at infinity.} \end{aligned} \tag{40}$$

Then $\left(\langle h_k D_x\rangle^s T_\varphi^{h_k} U^{h_k}\right)$ is h_k-oscillatory.

Proof. For the sake of simplicity, we prove the result for $s=0$; the proof in the general case being identical. Taking into account the periodicity of the densities involved, Proposition 5.3 ensures that

$$\lim_{k\to\infty} (h_k)^d \int_Q \psi(\xi)\left|\widehat{U^{h_k}}(\xi)\right|^2 d\xi = \int_{\mathbb{R}^d\times Q} \psi(\xi)\, d\mu(x,\xi) \tag{41}$$

for every $\psi \in C_c\left(\mathbb{R}^d\right)$ and the claim follows. Since $\mu\left(\mathbb{R}^d \times \overline{D_{\tau_\varphi}}\right) = 0$, necessarily $\mu\left(\mathbb{R}^d \times \overline{D_{\sigma_\varphi^R}}\right)$ is null for every $R > 0$. From classical results on weak convergence of measures, one deduces that relation (41) also holds for $\psi = \sigma_\varphi^R$. Hence, by the dominated convergence Theorem,

$$\lim_{R\to\infty}\lim_{k\to\infty}(h_k)^d\int_Q \sigma_\varphi^R(\xi)\left|\widehat{U^{h_k}}(\xi)\right|^2 d\xi = \lim_{R\to\infty}\int_{\mathbb{R}^d\times Q}\sigma_\varphi^R(\xi)\,d\mu(x,\xi) = 0,$$

and the result follows. ■

What we have proved so, combined with Proposition 1.2 far is gathered in the next proposition.

Proposition 5.8 *Suppose at least one of (***D***) or (40) is satisfied and that* $\mu\left(\mathbb{R}^d \times \overline{D_{\widehat{\varphi}}}\right) = 0$. *Then*

$$\lim_{k\to\infty}\int_{\mathbb{R}^d}\phi(x)\left|\langle h_k D_x\rangle^s T_\varphi^{h_k}U^{h_k}\right|^2 dx = \int_{\mathbb{R}^d\times\mathbb{R}^d}\phi(x)\left|\langle\xi\rangle^s\widehat{\varphi}(\xi)\right|^2 d\mu(x,\xi) \qquad (42)$$

for every $\phi \in C_c\left(\mathbb{R}^d\right)$.

As anticipated above, (42) shows that the knowledge of the weak limit of the measures $E^{h_k}\left[U^{h_k}\right]$ and the profile φ are not enough, in general, to reconstruct the weak limit of the densities $\left|\langle h_k D_x\rangle^s T_\varphi^{h_k}U^{h_k}\right|^2 dx$. However, when the sequence of discrete functions under consideration is ε_k-oscillatory for some scale coarser than the reconstruction step h_k, there does exist a formula that relates both limits:

Corollary 5.9 *Let* $\left(U^{h_k}\right)$ *be an* h_k*-bounded,* ε_k*-oscillatory sequence such that* $\left(E^{h_k}\left[U^{h_k}\right]\right)$ *weakly converges to a measure* ν. *Suppose moreover that* $\widehat{\varphi}$ *is continuous at* Γ *and that any of (***D***) or (40) is satisfied. Then the densities* $\left|\langle h_k D_x\rangle^s T_\varphi^{h_k}U^{h_k}\right|^2$ *weakly converge to the measure*

$$\nu_\varphi(x) = \left(\sum_{n\in\mathbb{Z}^d}\left|\langle 2\pi n\rangle^s\widehat{\varphi}(2\pi n)\right|^2\right)\nu(x).$$

Proof. Using Proposition 5.2, we find that any Wigner measure at scale h_k of $\left(U^{h_k}\right)$ equals

$$\mu(x,\xi) = \nu(x)\otimes\sum_{n\in\mathbb{Z}^d}\delta_{2\pi n}(\xi).$$

Since $\mu\left(\mathbb{R}^d\times\overline{D_{\widehat{\varphi}}}\right) = 0$, Proposition 5.8 is applicable and gives:

$$\left|\langle h_k D_x\rangle^s T_\varphi^{h_k}U^{h_k}\right|^2 dx \rightharpoonup \tau_{\langle D_x\rangle^s\varphi}(0)\,\nu(x) \qquad \text{as } k\to\infty,$$

as claimed. ■

Remark that condition (40.i) reduces in this setting to the requirement that $\tau_{\langle D_x\rangle^s\varphi}$ is continuous at $\xi = 0$.

5.3 A counterexample to h-oscillation

Here we exhibit a function $\varphi \in L^2(\mathbb{R})$ satisfying (**BP**) with $\widehat{\varphi}$ is continuous but

$$\left\|\sigma_{\varphi}^{R}\right\|_{L^\infty(Q)} = 1 \qquad \text{for every } R > 0.$$

With such a profile, we show that there exist a sequence of discrete functions $\left(U^h\right)$ such that $\left(T_\varphi^h U^h\right)$ is not h-oscillatory

To construct φ, define $t_n := e^{-n}$ for $n = 0, 1, 2...$ and let ψ_n be the piecewise linear function given for $n \geq 1$ by

$$\psi_n(t) = \begin{cases} \dfrac{t - t_{n+1}}{t_n - t_{n+1}} & \text{if } t \in (t_{n+1}, t_n), \\ \dfrac{t - t_{n-1}}{t_n - t_{n-1}} & \text{if } t \in (t_n, t_{n-1}), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\sum_{n=1}^{\infty} \psi_n(t) = 1$ for $t \in (0, t_1)$, and the sum vanishes for $t \leq 0$. Defining

$$\widehat{\varphi}(\xi) := \sqrt{\sum_{n=1}^{\infty} \psi_n(\xi - 2\pi n)}$$

we obtain $\varphi \in L^2\left(\mathbb{R}^d\right)$, $\widehat{\varphi} \in C\left(\mathbb{R}^d\right)$ and $\tau_\varphi(\xi) = \sum_{n=1}^{\infty} \psi_n(\xi)$.

Moreover

$$\sigma_\varphi^n(\xi) = \begin{cases} 1 & \text{if } \xi \in (0, x_{n+1}), \\ 0 & \text{if } \xi \leq 0. \end{cases}$$

Thus $\left\|\sigma_\varphi^R\right\|_{L^\infty(Q)} = 1$ for every $R > 0$.

If we chose discrete functions $U^h \in L^2(h\mathbb{Z})$ such that

$$\widehat{U^h}(\xi) = h^{-1} \sum_{n \in \mathbb{Z}} \mathbf{1}_{(0,h)}(\xi + 2\pi n)$$

then for the φ above constructed we obtain

$$\lim_{h \to 0} \int_Q \sigma_\varphi^R(\xi)\, h \left|\widehat{U^h}(\xi)\right|^2 d\xi = 1 \qquad \text{for every } R > 0.$$

This proves that $\left(T_\varphi^h U^h\right)$ is not h-oscillatory.

6 High frequency analysis: $h \ll \varepsilon$

Here we shall investigate the structure of Wigner measures at scales (ε_k) asymptotically coarser than the sampling/reconstruction rate (h_k).

In the next two Theorems, we suppose that φ satisfies (**BP**) and $\widehat{\varphi}$ *is continuous in a neighborhood of* $\xi = 0$. Moreover, (h_k) and (ε_k) will be scales such that $h_k \ll \varepsilon_k$.

Theorem 6.1 *Suppose that* $\left(U^{h_k}\right)$ *is* h_k*-bounded and* $M^{\varepsilon_k}\left[U^{h_k}\right]$ *converges to the Wigner measure* μ. *Then* $m^{\varepsilon_k}\left[T_\varphi^{h_k} U^{h_k}\right]$ *converges to a measure* μ_φ *given by:*

$$\mu_\varphi (x, \xi) = \left|\widehat{\varphi}(0)\right|^2 \mu (x, \xi) . \tag{43}$$

The proof of this result is completely analogous to that of Theorem 4.6.

Concerning the sampling operators, the situation is much similar:

Theorem 6.2 *Let* (u_k) *be a sequence in* $H^{-s}\left(\mathbb{R}^d\right)$ *such that* $\left(\langle h_k D_x\rangle^{-s} u_k\right)$ *is bounded in* $L^2\left(\mathbb{R}^d\right)$ *and* ε_k*-oscillatory.*

i) Then $\left(S_\varphi^{h_k} u_k\right)$ *is* ε_k*-oscillatory.*

ii) Suppose moreover that $m^{\varepsilon_k}[u_k]$ *converges to a Wigner measure* μ. *Then* $M^{\varepsilon_k}\left[S_\varphi^{h_k} u_k\right]$ *converges to the Wigner measure* μ^φ *given by:*

$$\mu^\varphi = \left|\widehat{\varphi}(0)\right|^2 \mu. \tag{44}$$

Proof. To prove the first part of the Theorem, begin by noticing that, by the Cauchy-Schwarz inequality and Lemma 4.11, for almost every $\xi \in \mathbb{R}^d$:

$$\begin{aligned}
\left|\widehat{S_\varphi^{h_k} u_k}(\xi)\right|^2 &= \left|\frac{1}{(h_k)^d} \sum_{n\in\mathbb{Z}^d} \overline{\widehat{\varphi}(\xi + 2\pi n)} \widehat{u_k}\left(\frac{\xi + 2\pi n}{h_k}\right)\right|^2 \\
&\leq \frac{\left\|\tau_{\langle D_x\rangle^s \varphi}\right\|_{L^\infty(Q)}}{(h_k)^{2d}} \sum_{n\in\mathbb{Z}^d} \left|\langle \xi + 2\pi n\rangle^{-s} \widehat{u_k}\left(\frac{\xi + 2\pi n}{h_k}\right)\right|^2 .
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{Q\setminus B(0;h_k/\varepsilon_k R)} (h_k)^d \left|\widehat{S_\varphi^{h_k} u_k}(\xi)\right|^2 d\xi &\leq \left\|\tau_{\langle D_x\rangle^s \varphi}\right\|_{L^\infty(Q)} \int_{Q\setminus B(0;R/\varepsilon_k)} \sum_{n\in\mathbb{Z}^d} \left|\langle h_k\xi + 2\pi n\rangle^{-s} \widehat{u_k}(\xi + 2\pi n)\right|^2 d\xi \\
&\leq \left\|\tau_{\langle D_x\rangle^s \varphi}\right\|_{L^\infty(Q)} \int_{\mathbb{R}^d\setminus B(0;R/\varepsilon_k)} \left|\langle h_k\xi\rangle^{-s} \widehat{u_k}(\xi)\right|^2 d\xi.
\end{aligned}$$

and this clearly proves that $\left(S_\varphi^{h_k} u_k\right)$ is ε_k-oscillating as soon as $\left(\langle h_k D_x\rangle^{-s} u_k\right)$ is.

The proof of identity (44) is essentially identical to that of Theorem 4.2. A completely analogous argument to that used in the step 2 of that proof allows us to consider only sequences such that $\widehat{\varphi}\left(h_k/\varepsilon_k\cdot\right)\widehat{u_k}\left(\cdot/\varepsilon_k\right)$ is supported in a ball $B\left(0;R\right)$. This hypothesis together with Lemma 4.11 implies that, for h_k/ε_k small enough,

$$(h_k)^d\,\widehat{S_\varphi^{h_k}u_k}\left(\frac{h_k}{\varepsilon_k}\xi\right)=\overline{\widehat{\varphi}\left(\frac{h_k}{\varepsilon_k}\xi\right)}\widehat{u_k}\left(\xi/\varepsilon_k\right),$$

that is, only one summand is involved. Then the result follows from Proposition 8.3 exactly as in the proof of Theorem 4.2. ■

We conclude with a simple remark:

Corollary 6.3 *Under the assumptions and notations of Theorems 6.1 and 6.2:*

i) If φ has zero mean (i.e. $\widehat{\varphi}\left(0\right)=0$) then the Wigner measure at scale $\left(\varepsilon_k\right)$ of any sequence $\left(T_\varphi^{h_k}U^{h_k}\right)$ or $\left(S_\varphi^{h_k}u_k\right)$ vanishes identically. In particular, this is the case if φ is a wavelet.[9]

ii) $\mu_\varphi=\mu^\varphi=\mu$ always holds for profiles such that $\left|\widehat{\varphi}\left(0\right)\right|=1$.

7 Wigner measures of Sampled/Reconstructed sequences

Now we are able to describe Wigner measures of sequences of the form $T_\psi^hS_\varphi^hu$. In its full generality, our result requires several compatibility hypothesis, that we describe below. First of all,

$$\begin{array}{ll}\text{i)} & \psi \text{ and } \varphi \text{ satisfy } (\mathbf{BP}) \text{ with exponents } s' \text{ and } s \text{ respectively.}\\ \text{ii)} & \varphi \text{ satisifes } (\mathbf{D}).\end{array}\tag{45}$$

The admissible sequences will be assumed to be such that:

$$u_k\in H^{-s}\left(\mathbb{R}^d\right)\ \text{ and }\ \left(\left\langle h_kD_x\right\rangle^{-s}u_k\right)\ \text{ is bounded in } L^2\left(\mathbb{R}^d\right),\tag{46}$$

and their Wigner measures must satisfy the following compatibility conditions for some precise representatives of $\widehat{\psi}$ and $\widehat{\varphi}$:

$$\begin{array}{ll}\text{i)} & \mu \text{ fulfills } (\mathbf{ND}).\\ \text{ii)} & \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\overline{D_{\widehat{\psi}}}}\left(\xi+2\pi n\right)\left|\widehat{\varphi}\left(\xi\right)\right|^2d\mu\left(x,\xi\right)=0,\quad n\in\mathbb{Z}^d.\\ \text{iii)} & \mu \text{ satisfies } (\mathbf{MS}).\end{array}\tag{47}$$

Combining Theorems 4.6 and 4.2 we obtain:

[9] See, for instance, [10], Proposition 2.1.

Theorem 7.1 *Let ψ and φ be functions satisfying (45); let (h_k) be a scale and (u_k) be a sequence satisfying (46). Suppose moreover that $m^{h_k}[u_k]$ converges to a Wigner measure μ that satisfies (47).*

Then $m^{h_k}\left[T_\psi^{h_k} S_\varphi^{h_k} u_k\right]$ converges to the measure $\mu_{\varphi,\psi}$ given by:

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} a(x,\xi)\, d\mu_{\varphi,\psi}(x,\xi) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \sum_{n\in\mathbb{Z}^d} a(x,\xi+2\pi n) \left|\widehat{\psi}(\xi+2\pi n)\right|^2 |\widehat{\varphi}(\xi)|^2\, d\mu(x,\xi) \tag{48}$$

for every $a \in C_c\left(\mathbb{R}^d\times\mathbb{R}^d\right)$.

Proof. Hypothesis (47.ii) expresses that the closure of the set of discontinuity points of $\widehat{\psi}$, is a null set for the Wigner measure of $S_\varphi^{h_k}u_k$, $\sum_{k\in\mathbb{Z}^d} |\widehat{\varphi}(\xi+2\pi n)|^2 \mu(x,\xi+2\pi n)$. Hence, Theorem 4.6 is applicable and we conclude that the distributions $m^{h_k}\left[T_\psi^{h_k} S_\varphi^{h_k} u_k\right]$ converge to the measure

$$\left|\widehat{\psi}(\xi)\right|^2 \sum_{n\in\mathbb{Z}^d} |\widehat{\varphi}(\xi+2\pi n)|^2 \mu(x,\xi+2\pi n).$$

Since $\left|\widehat{\psi}(\xi)\right|^2$ is integrable with respect to the finite measure $|\widehat{\varphi}|^2\mu$ (this is again due to (47.ii)), its periodization is integrable as well and formula (48) follows. ■

Remark 7.2 *i) When $\widehat{\psi}$ and $\widehat{\varphi}$ verify (21) hypotheses (45), (47.i) and (47.ii) are immediately satisfied.*

ii) (45.ii) may be replaced by the requirement that $\left(\langle h_k D_x\rangle^{-s} u_k\right)$ is h_k-oscillatory.

From formula (48) one sees at once that, taking $\psi = \varphi = \delta_0$ one has that $\mu_{\varphi,\psi}$ is the periodization in ξ of the Wigner measure μ. Hence, $\mu_{\varphi,\psi}$ coincides with the limit of the Wigner series corresponding to (u_k).

When $\widehat{\psi}$ and $|\widehat{\varphi}|^2\mu$ vanish off Q it is easy to check that formula (48) takes the simple form:

$$\mu_{\varphi,\psi}(x,\xi) = \left|\widehat{\psi}(\xi)\right|^2 |\widehat{\varphi}(\xi)|^2 \mu(x,\xi).$$

It is also clear that, as soon as $|\widehat{\varphi}(\xi)|^2\mu(x,\xi)$ is not null outside Q, the measures $\mu_{\varphi,\psi}$ and μ will in general differ.

Concerning defect measures, combining Proposition 5.8 and the previous theorem, we obtain:

Theorem 7.3 *Under the notations of Theorem 7.1 the following holds: if*

$$\left|\langle h_k D_x\rangle^{s'} T_\psi^{h_k} S_\varphi^{h_k} u_k\right|^2 dx \text{ weakly converges to a measure } \nu_{\varphi,\psi}$$

*and ψ verifies (**D**) then:*

$$\nu_{\varphi,\psi}(x) = \int_{\mathbb{R}^d_\xi} \sum_{n\in\mathbb{Z}^d} \left| \langle \xi + 2\pi n\rangle^{s'} \widehat{\psi}(\xi + 2\pi n)\right|^2 |\widehat{\varphi}(\xi)|^2 \mu(x, d\xi). \qquad (49)$$

Remark 7.4 *i) The conclusion of the Theorem still holds if condition "ψ satisfies (**D**)" is replaced by (40).*

ii) Theorems 1.3 and 1.5 follow immediately from Theorems 7.1 and 7.3.

With formula (49) at our disposal, we are now able to answer, in a quite general way, the questions **A**-**D** addressed in the introduction. Of course, the answer to **A** is negative, since, in general, μ is not trivial in its ξ component; concerning the problem of filtering, we immediately get the necessary and sufficient condition:

$$c_{\varphi,\psi}(\xi) := |\widehat{\varphi}(\xi)|^2 \sum_{n\in\mathbb{Z}^d} \left| \langle \xi + 2\pi n\rangle^{s'} \widehat{\psi}(\xi + 2\pi n)\right|^2 = 0 \qquad \text{for } \mu\text{-a.e. } \xi \in \mathbb{R}^d.$$

Analogously, $c_{\varphi,\psi}(\xi) = 1$ for μ-a.e. $\xi \in \mathbb{R}^d$ characterizes the profiles that give $\nu_{\varphi,\psi} = \nu$. To answer **D**, we must, of course, assume that $\widehat{\varphi}$ and $\tau_{\langle D_x\rangle^{s'}\widehat{\psi}}$ are continuous (which, as we know, is the case if (21) holds). In that case, we have and equality $\nu_{\varphi,\psi} = \nu$ for every admissible sequence if and only if

$$|\widehat{\varphi}(\xi)|^2 = \frac{1}{\tau_{\langle D_x\rangle^{s'}\psi}(\xi)} \qquad \text{for every } \xi \in \mathbb{R}^d \text{ with } \tau_{\langle D_x\rangle^{s'}\psi}(\xi) \neq 0.$$

The sampling profile φ, cannot be an $L^2(\mathbb{R}^d)$ function, since $|\widehat{\varphi}|^2$ is necessarily periodic. When $\varphi = \delta_0$ and ψ generates an orthonormal basis in the sense of Lemma 3.5 we always have $\nu_{\varphi,\psi} = \nu$. If ψ merely generates a Riesz basis, $A\nu \leq \nu_{\varphi,\psi} \leq B\nu$ holds instead.

The above results may be used to compute Wigner measures of the **orthogonal projections** $P_\psi^{h_k} u_k$ of a given sequence (u_k) on the shift-invariant space defined by the range of $T_\psi^{h_k}$. As we have seen in Lemma 3.6, P_ψ^h may be written as the composition of T_ψ^h with $S_\varphi^h \langle hD_x\rangle^s$ for a sampling profile $\varphi := \widetilde{\langle D_x\rangle^s \psi}$. Hence, Theorem 7.1 gives:

Corollary 7.5 *For ψ satisfying (21) and (u_k) such that (46) and (**MS**) holds, the defect measures of the sequence $\left(P_\psi^{h_k} u_k\right)$ is given by:*

$$\nu_{P_\psi}(x) = \int_{\mathbb{R}^d} \frac{\mathbf{1}_\psi(\xi)}{\tau_{\langle D_x\rangle^s\psi}(\xi)} \left|\langle \xi\rangle^s \widehat{\psi}(\xi)\right|^2 \mu(x, d\xi),$$

where $\mathbf{1}_\psi(\xi)$ denotes the characteristic function of the set of $\xi \in \mathbb{R}^d$ such that $\tau_{\langle D_x\rangle^s\psi}(\xi) \neq 0$.

In particular, when ψ gives rise to an orthonormal family, we obtain the simple formula (cf. Lemma 3.5):

$$\nu_{P_\psi}(x) = \int_{\mathbb{R}^d} \left| \langle \xi \rangle^s \widehat{\psi}(\xi) \right|^2 \mu(x, d\xi).$$

To conclude, we shall see how the above results may be refined when the sequence $\left(\langle h_k D_x \rangle^{-s} u_k\right)$ is assumed to be ε_k-oscillatory at some scale $h_k \ll \varepsilon_k$. The assumptions of φ and ψ are weaker:

$$\begin{array}{ll} \text{i)} & \psi \text{ and } \varphi \text{ satisfy } (\mathbf{BP}) \text{ with exponents } s' \text{ and } s \text{ respectively.} \\ \text{ii)} & \widehat{\psi},\ \widehat{\varphi} \text{ are continuous in a neighborhood of } \xi = 0. \\ \text{iii)} & \tau_{\langle D_x \rangle^{s'} \psi} \text{ is continuous at } \xi = 0. \end{array} \tag{50}$$

Theorems 6.1, 6.2 and Corollary 5.9 give then:

Theorem 7.6 *Let ψ and φ be functions satisfying (50); let (h_k), (ε_k) be scales with $h_k \ll \varepsilon_k$ and let (u_k) be a sequence such that (46) holds and $\left(\langle h_k D_x \rangle^{-s} u_k\right)$ is ε_k-oscillatory. Suppose moreover that $m^{\varepsilon_k}[u_k]$ converges to a Wigner measure μ.*

Then $m^{\varepsilon_k}\left[T_\psi^{h_k} S_\varphi^{h_k} u_k\right]$ converges to the measure $\mu_{\varphi,\psi}$ given by:

$$\mu_{\varphi,\psi}(x,\xi) = \left| \widehat{\psi}(0) \right|^2 \left| \widehat{\varphi}(0) \right|^2 \mu(x,\xi).$$

Moreover, if $\left| \langle h_k D_x \rangle^{s'} T_\psi^{h_k} S_\varphi^{h_k} u_k \right|^2 dx$ weakly converges to a measure $\nu_{\varphi,\psi}$ then:

$$\nu_{\varphi,\psi}(x) = \sum_{n \in \mathbb{Z}^d} \left| \langle 2\pi n \rangle^{s'} \widehat{\psi}(2\pi n) \right|^2 \left| \widehat{\varphi}(0) \right|^2 \nu(x),$$

where ν is the weak limit of the densities $\left| \langle h_k D_x \rangle^{-s} u_k \right|^2 dx$.

Hence, when a sequence possesses a characteristic oscillation scale (ε_k) (that is the meaning of the ε_k-oscillation condition), choosing a sampling/reconstruction rate (h_k) asymptotically finer than (ε_k) allows to completely capture its oscillation/concentration behavior (modulo a constant that only depends on ψ and φ).

Filtering in that case can only be achieved by means of a sampling profile φ with zero mean ($\widehat{\varphi}(0) = 0$) or a reconstruction profile such that $\widehat{\psi}$ vanishes at Γ.

8 Tools from the theory of Wigner measures

The main tools from the theory of Wigner measures used in this article are Propositions 8.1 and 8.3 below. The first of these is an extension of Theorem 1.1 to bounded sequences in Sobolev spaces:

Proposition 8.1 *Let (ε_k) be a scale and (u_k) be a sequence of functions in $H^{-s}\left(\mathbb{R}^d\right)$ for some $s \geq 0$ satisfying:*

$$\left\| \langle \varepsilon_k D_x \rangle^{-s} u_k \right\|_{L^2\left(\mathbb{R}^d\right)} \text{ are uniformly bounded in } k. \tag{51}$$

Then the sequence of distributions $(m^{\varepsilon_k}[u_k])$ is uniformly bounded in $\mathcal{S}'$. Moreover, any of its weakly converging subsequences tends to a positive measure.

As we have done so far, a measure $\mu \in \mathcal{M}_+\left(\mathbb{R}^d \times \mathbb{R}^d\right)$ will be called the **Wigner measure at scale** (ε_k) of a sequence (u_k) (satisfying the hypotheses of Proposition 8.1) provided $m^{\varepsilon_k}[u_k] \rightharpoonup \mu$ in $\mathcal{S}'$ as $k \to \infty$.

Remark 8.2 *i) When $s > 0$, condition (51) is stronger than just requiring that (u_k) is bounded in $H^{-s}\left(\mathbb{R}^d\right)$.*

ii) Let (h_k) be a scale such that $h_k \ll \varepsilon_k$. If $\left\| \langle h_k D_x \rangle^{-s} u_k \right\|_{L^2\left(\mathbb{R}^d\right)} \leq C$ for every $k \in \mathbb{N}$ then $\left\| \langle \varepsilon_k D_x \rangle^{-s} u_k \right\|_{L^2\left(\mathbb{R}^d\right)}$ is uniformly bounded as well.

iii) The same result holds if $m^{\varepsilon}[\cdot]$ is replaced by the Wigner transform (5).

The second main result of this section a localization formula for Wigner measures which is used several times in this article:

Proposition 8.3 *Let (ε_k), (h_k) be scales and let (u_k) be a sequence in $H^{-s}\left(\mathbb{R}^d\right)$, $s \geq 0$, satisfying (51). Suppose that ϕ is a Borel function such that $\phi \in L^\infty\left(\mathbb{R}^d; \langle \xi \rangle^r\right)$ for some $r \in \mathbb{R}$. If $m^{\varepsilon_k}[u_k]$ converges to μ then $m^{\varepsilon_k}[\phi(h_k D_x) u_k]$ converges to a Wigner measure μ_ϕ which has the following properties:*

i) If $h_k = \varepsilon_k$ and $\mu\left(\mathbb{R}^d \times \overline{D_\phi}\right) = 0$, D_ϕ being the set of points where ϕ is not continuous, then

$$\mu_\phi(x,\xi) = |\phi(\xi)|^2 \mu(x,\xi).$$

ii) If $h_k \ll \varepsilon_k$ and ϕ is continuous in a neighborhood of $\xi = 0$ then

$$\mu_\phi = |\phi(0)|^2 \mu.$$

When applied to $\phi(\xi) := \langle \xi \rangle^s$, this result gives:

Remark 8.4 *Let* (ε_k)*,* (h_k) *and* (u_k) *be as in Proposition 8.3. Suppose* $m^{\varepsilon_k}[u_k]$ *converges to* μ*. Then* $m^{\varepsilon_k}\left[\langle h_k D_x\rangle^{-s} u_k\right]$ *converges to the measure* μ_s *given by:*

$$\begin{aligned} &\mu_s(x,\xi) = \langle\xi\rangle^{-2s}\mu(x,\xi)\,, && \text{if } h_k = \varepsilon_k, \\ &\mu_s = \mu, && \text{if } h_k \ll \varepsilon_k. \end{aligned}$$

In particular (cf. Theorem 1.1), $\langle\xi\rangle^{-2s}\mu$ *(resp.* μ*) is a finite measure when* $h_k = \varepsilon_k$ *(resp.* $h_k \ll \varepsilon_k$*).*

For the convenience of the reader, we give detailed proofs of both results; they follow the ideas present in the existing literature on the subject ([6, 13, 8, 9]). Proposition 8.1 will be proved in paragraph 8.2. We shall essentially show that truncation of the high frequencies of a sequence satisfying (51) implies ξ-variable localization of the corresponding $m^\varepsilon[\cdot]$. Then we conclude by applying Theorem 1.1 to the localized sequence.

Proposition 8.3 is proved in paragraph 8.3; to conclude this section, we describe two results useful for the computation of Wigner measures (Lemmas 8.12 and 8.13).

8.1 First properties of $m^\varepsilon[u]$

We begin by discussing three alternative ways of computing $m^\varepsilon[u]$ that may be used when u is merely a tempered distribution. First remark that, given a $u \in \mathcal{S}'(\mathbb{R}^d)$, it makes sense to consider the distribution $m^\varepsilon[u]$ given by (2), since the Fourier transform of u is well-defined. Actually $m^\varepsilon[u] \in \mathcal{S}'$.

1. The action of $m^\varepsilon[u]$ on a test function $a \in \mathcal{S}$ is given by any of the formulas (see [7]):

$$\langle m^\varepsilon[u], a\rangle_{\mathcal{S}'\times\mathcal{S}} = \begin{cases} \langle \overline{u}, a(x,\varepsilon D_x)u\rangle_{\mathcal{S}'(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)}\,, & \text{(i)} \\ \displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{\varepsilon^d}k_a\left(x,\frac{x-p}{\varepsilon}\right)u(p)\,\overline{u(x)}dpdx. & \text{(ii)} \end{cases} \tag{52}$$

where $a(x,\varepsilon D_x)$ is the **semiclassical pseudodifferential operator** of symbol a:

$$a(x,\varepsilon D_x)u(x) = \int_{\mathbb{R}^d} a(x,\varepsilon\xi)\,\widehat{u}(\xi)\,e^{ix\cdot\xi}\frac{d\xi}{(2\pi)^d}, \tag{53}$$

and the kernel $k_a(x,p)$ is the inverse Fourier transform of a with respect to ξ:

$$k_a(x,p) := \int_{\mathbb{R}^d} a(x,\xi)\,e^{ip\cdot\xi}\frac{d\xi}{(2\pi)^d}.$$

Formula (52.i) makes sense because the operator $a(x, \varepsilon D_x)$ maps continuously $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$ whenever $a \in \mathcal{S}$ (see, for instance, [15]). The integral in (52.ii) must, of course, be understood in distributional sense.

2. The distribution $m^\varepsilon[u]$ may be computed through the rescaled Fourier transform

$$\mathcal{F}^\varepsilon u(\xi) := \frac{1}{(2\pi\varepsilon)^{d/2}} \widehat{u}\left(\frac{\xi}{\varepsilon}\right), \tag{54}$$

using the identity:

$$m^\varepsilon[u](x, \xi) = \overline{m^\varepsilon[\mathcal{F}^\varepsilon u](\xi, -x)}. \tag{55}$$

This follows from a direct computation from the definition (2).

3. Now we present two localization formulas:

Lemma 8.5 *Let $u \in \mathcal{S}'(\mathbb{R}^d)$, $\phi \in C^\infty(\mathbb{R}^d; \langle x\rangle^r)$ for some $r \in \mathbb{R}$ and $a \in \mathcal{S}$. Then there exists $r_1^\sigma, r_2^\sigma \in \mathcal{S}$ such that:*

$$\langle m^\varepsilon[\phi u], a\rangle_{\mathcal{S}'\times\mathcal{S}} = \left\langle |\phi(x)|^2 m^\varepsilon[u], a\right\rangle_{\mathcal{S}'\times\mathcal{S}} + \varepsilon \langle m^\varepsilon[u], r_1^\varepsilon\rangle_{\mathcal{S}'\times\mathcal{S}},$$

$$\langle m^\varepsilon[\phi(hD_x)u], a\rangle_{\mathcal{S}'\times\mathcal{S}} = \left\langle \left|\phi\left(\frac{h}{\varepsilon}\xi\right)\right|^2 m^\varepsilon[u], a\right\rangle_{\mathcal{S}'\times\mathcal{S}} + \frac{h}{\varepsilon}\left\langle m^\varepsilon[u], r_2^{h/\varepsilon}\right\rangle_{\mathcal{S}'\times\mathcal{S}}.$$

Moreover, the test functions r_1^σ, r_2^σ are uniformly bounded in $\mathcal{S}$ for $0 < \sigma \leq 1$.

This holds as a consequence of standard results on symbolic calculus for semiclassical pseudodifferential operators; see for instance [15]. Remark that Proposition 8.3 is not a consequence of this result, since the multiplier $\phi(hD_x)$ there may have a non-smooth symbol.

8.2 Boundedness of the transforms $m^\varepsilon[u]$

The next lemmas are used to establish the boundedness in $\mathcal{S}'$ of the sequence $(m^{\varepsilon_k}[u_k])$ provided (u_k) satisfies the hypotheses of Proposition 8.1.

Lemma 8.6 *For every $u \in L^2(\mathbb{R}^d; \langle x\rangle^r)$ and $a \in \mathcal{S}$ the following estimate holds:*

$$\left|\langle m^\varepsilon[u], a\rangle_{\mathcal{S}'\times\mathcal{S}}\right| \leq \|u\|^2_{L^2(\mathbb{R}^d;\langle x\rangle^r)} \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} \left|k_a(x,p)\langle x - \varepsilon p\rangle^{-r/2} \langle x\rangle^{-r/2}\right| dp,$$

Proof. Use formula (52.ii) to write

$$\langle m^\varepsilon[u], a\rangle_{\mathcal{S}'\times\mathcal{S}} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k_a(x,p)\, u(x - \varepsilon p)\, \overline{u(x)} dp dx,$$

noticing that this integral makes sense as $k_a \in \mathcal{S}$. Multiply and divide the integrand above by $\langle x - \varepsilon p \rangle^{r/2} \langle x \rangle^{r/2}$ to obtain, by Hölder's inequality,

$$\left| \langle m^\varepsilon [u], a \rangle_{\mathcal{S}' \times \mathcal{S}} \right| \leq \int_{\mathbb{R}^d} \sup_{x \in \mathbb{R}^d} \left| k_a(x,p) \langle x - \varepsilon p \rangle^{-r/2} \langle x \rangle^{-r/2} \right| \int_{\mathbb{R}^d} \left| u_r(x - \varepsilon p) \overline{u_r(x)} \right| dx dp,$$

where we have set $u_r(x) := \langle x \rangle^{r/2} u(x)$. The conclusion follows from another application of Hölder's inequality. ■

If $u \in H^{-s}(\mathbb{R}^d)$ then $\mathcal{F}^\varepsilon u \in L^2\left(\mathbb{R}^d; \langle \xi \rangle^{-2s}\right)$. Clearly,

$$\left\| \langle \varepsilon D_x \rangle^{-s} u \right\|^2_{L^2(\mathbb{R}^d)} = \left\| \mathcal{F}^\varepsilon u \right\|^2_{L^2\left(\mathbb{R}^d; \langle \xi \rangle^{-2s}\right)}. \tag{56}$$

Thus, taking identity (55) into account, we obtain using the preceding lemma:

$$\left| \langle m^\varepsilon [u], a \rangle_{\mathcal{S}' \times \mathcal{S}} \right| \leq \left\| \langle \varepsilon D_x \rangle^{-s} u \right\|^2_{L^2(\mathbb{R}^d)} \int_{\mathbb{R}^d} \sup_{\xi \in \mathbb{R}^d} \left| \widehat{a}(q,\xi) \langle \xi + \varepsilon q \rangle^s \langle \xi \rangle^s \right| \frac{dq}{(2\pi)^d}, \tag{57}$$

where $\widehat{a}(q,\xi)$ denotes the Fourier transform in x of the function $a(x,\xi)$.

Lemma 8.7 *For every $s \geq 0$ there exists a constant $C_{s,d} > 0$ such that*

$$\left| \langle m^\varepsilon [u], a \rangle_{\mathcal{S}' \times \mathcal{S}} \right| \leq C_{s,d} \left\| \langle \varepsilon D_x \rangle^{-s} u \right\|^2_{L^2(\mathbb{R}^d)} \int_{\mathbb{R}^d} \sup_{\xi \in \mathbb{R}^d} \left| \widehat{a}(q,\xi) \langle \xi \rangle^{2s} \right| \langle \varepsilon q \rangle^s dq, \tag{58}$$

holds for every $u \in H^{-s}(\mathbb{R}^d)$ and every $a \in \mathcal{S}$.

Proof. This is obtained through the simple inequality $\langle \xi + q \rangle^s \leq C_{s,d} \langle \xi \rangle^s \langle q \rangle^s$, which holds when $s \geq 0$. ■

Notice that whenever $a \in \mathcal{S}$, the integrals $\int_{\mathbb{R}^d} \sup_{\xi \in \mathbb{R}^d} \left| \widehat{a}(q,\xi) \langle \xi \rangle^{2s} \right| \langle \varepsilon q \rangle^s dq$ are uniformly bounded for $0 < \varepsilon \leq 1$. Consequently,

Corollary 8.8 *Let (ε_k) and (u_k) satisfy the hypotheses of Proposition 8.1. Then the sequence $(m^{\varepsilon_k}[u_k])$ is bounded in $\mathcal{S}'$.*

Estimate (58) gives immediately the following:

Remark 8.9 *Lemma 8.7 shows that $m^\varepsilon [u]$ acts continuously on test functions a in the closure of $\mathcal{S}$ for the norm:*

$$[a]_s := \int_{\mathbb{R}^d} \sup_{\xi \in \mathbb{R}^d} \left| \widehat{a}(q,\xi) \langle \xi \rangle^{2s} \right| \langle q \rangle^s dq < \infty. \tag{59}$$

This closure contains the space

$$\Sigma^s := \left\{ \langle D_x \rangle^s \langle \xi \rangle^{2s} a \in C_0\left(\mathbb{R}^d \times \mathbb{R}^d\right) : [a]_s < \infty \right\}. \tag{60}$$

Remark 8.10 *Consequently, if (u_k) is as in Proposition 8.1 and $(m^{\varepsilon_k}[u_k])$ converges weakly in $\mathcal{S}'$ then $\langle m^{\varepsilon_k}[u], a\rangle$ converges as well for every $a \in \Sigma^s$.*

Proof of Proposition 8.1. The boundedness of the sequence $(m^{\varepsilon_k}[u_k])$ was proved in Corollary 8.8. Suppose now that the distributions $m^{\varepsilon_k}[u_k]$ weakly converge to some $\mu \in \mathcal{S}'$. We next show by means of a localization argument that μ is a positive distribution and thus, due to Schwartz's Theorem, a positive Radon measure.

Take $\phi \in \mathcal{S}\left(\mathbb{R}^d_\xi\right)$; Lemma 8.5 gives

$$\lim_{k\to\infty} \langle m^{\varepsilon_k}[\phi(\varepsilon_k D_x) u_k], a\rangle_{\mathcal{S}'\times\mathcal{S}} = \int_{\mathbb{R}^d\times\mathbb{R}^d} a(x,\xi) |\phi(\xi)|^2 d\mu(x,\xi)$$

for every $a \in \mathcal{S}$. Since $(\phi(\varepsilon_k D_x) u_k)$ is a bounded sequence in $L^2\left(\mathbb{R}^d\right)$, Theorem 1.1 ensures that $|\phi(\xi)|^2 \mu$ is a positive Radon measure (and hence a positive distribution). But $\phi \in \mathcal{S}\left(\mathbb{R}^d_\xi\right)$ is arbitrary, so μ itself is positive and we obtain the desired result. ■

Notice that a very similar proof would give a version of Proposition 8.1 in the context of weighted spaces $L^2\left(\mathbb{R}^d; \langle x\rangle^r\right)$.

8.3 Proof of Proposition 8.3

The key ingredient in the proof of the Proposition is the following auxiliary result:

Lemma 8.11 *Under the assumptions of Proposition 8.3 and for every $a \in \mathcal{S}$, if any of the following conditions hold:*

i) $h_k = \varepsilon_k$ and a vanishes on the set of discontinuity points of ϕ.

ii) $h_k \ll \varepsilon_k$ and ϕ is continuous at $\xi = 0$.

Then

$$\lim_{k\to\infty} \left| \left\langle m^{\varepsilon_k}[\phi(h_k D_x) u_k] - \left|\phi\left(\frac{h_k}{\varepsilon_k}\xi\right)\right|^2 m^{\varepsilon_k}[u_k], a \right\rangle \right| = 0. \tag{61}$$

Proof. Take $a \in \mathcal{S}$ and set $\Phi_k(\xi) := \phi(h_k/\varepsilon_k \xi)$. From relations (55), (52.i) and (57) we obtain:

$$\left| \left\langle m^{\varepsilon_k}[\phi(h_k D_x) u_k] - \left|\phi\left(\frac{h_k}{\varepsilon_k}\xi\right)\right|^2 m^{\varepsilon_k}[u_k], a \right\rangle \right| \le M_k(a) \left\| \langle \varepsilon_k D_x\rangle^{-s} u_k \right\|^2_{L^2\left(\mathbb{R}^d\right)}$$

where

$$M_k(a) := \int_{\mathbb{R}^d} \sup_{\xi\in\mathbb{R}^d} \left|\widehat{a}(q,\xi)\,\Phi_k(\xi)\left[\Phi_k(\xi+\varepsilon_k q)-\Phi_k(\xi)\right]\langle\xi+\varepsilon_k q\rangle^s\langle\xi\rangle^s\right| \frac{dq}{(2\pi)^d}; \tag{62}$$

recall that $\widehat{a}(q,\xi)$ stands for the Fourier transform of $a(x,\xi)$ in x.

We now must prove that $M_k(a)\to 0$ as $k\to\infty$. This will be done by first checking that for test functions a belonging to the smaller class:

$$\widehat{\mathcal{D}} := \left\{a\in\mathcal{S} : \widehat{a}\in C_c^\infty\left(\mathbb{R}^d\times\mathbb{R}^d\right)\right\}.$$

Take $R>0$ such that $\operatorname{supp} a$ is contained in $B(0;R)\times B(0;R)$.

When $k\in\mathbb{N}$ is sufficiently large, $\varepsilon_k\le 1$ and

$$\frac{h_k}{\varepsilon_k}(\xi+\varepsilon_k q)\in B(0;2R\sup h_k/\varepsilon_k) \qquad \text{for every } q,\xi\in B(0;R). \tag{63}$$

Suppose now that i) holds. If C_ϕ denotes the set of points where ϕ is continuous, then $\Phi_k=\phi$ is uniformly continuous over $C_\phi\cap B(0;R)$ and, consequently,

$$\sup_{q,\xi\in B(0;R)} \mathbf{1}_{C_\phi}(\xi)\left|\phi(\xi+\varepsilon_k q)-\phi(\xi)\right|\to 0 \qquad \text{as } k\to\infty$$

because of (63).

On the other hand, when $h_k/\varepsilon_k\to 0$ and ϕ is continuous at $\xi=0$, again as a consequence of (63),

$$\sup_{\xi,q\in B(0;R)}\left|\phi\left(\frac{h_k}{\varepsilon_k}(\xi+\varepsilon_k q)\right)-\phi\left(\frac{h_k}{\varepsilon_k}\xi x\right)\right|\le 2\sup_{\xi\in B(0;2h_k/\varepsilon_k R)}\left|\phi(\xi)-\phi(0)\right|\to 0 \qquad \text{as } k\to\infty.$$

Thus, in either case,

$$\sup_{\xi\in\mathbb{R}^d}\left|\widehat{a}(q,\xi)\,\Phi_k(\xi)\left[\Phi_k(\xi+\varepsilon_k q)-\Phi_k(\xi)\right]\langle\xi+\varepsilon_k q\rangle^s\langle\xi\rangle^s\right|\to 0 \qquad \text{as } k\to\infty,$$

for every $q\in\mathbb{R}^d$. Lebesgue's dominated convergence Theorem gives the convergence to zero of the integrals (62). The density of $\widehat{\mathcal{D}}$ in $\mathcal{S}$ concludes the proof of the Lemma. ■

Proof of Proposition 8.3. To prove i) and ii) it only needs to be checked that, for any $a\in C_c^\infty\left(\mathbb{R}^d\times\mathbb{R}^d\right)$ (if $\varepsilon_k=h_k$, we further require that $a|_{\mathbb{R}^d\times\overline{D_\phi}}\equiv 0$), the functions $\left|\phi(h_k/\varepsilon_k\xi)\right|^2 a(x,\xi)$ belong to the class Σ^s. If so, then

$$\lim_{k\to\infty}\left\langle\left|\phi\left(\frac{h_k}{\varepsilon_k}\xi\right)\right|^2 m^{\varepsilon_k}[u_k],a\right\rangle_{\mathcal{S}'\times\mathcal{S}} = \int_{\mathbb{R}^d\times\mathbb{R}^d}\left|\phi(c\xi)\right|^2 a(x,\xi)\,d\mu,$$

holds with $c := \lim h_k/\varepsilon_k$, because of Remark 8.10. The conclusion would then follow from identity (61).

First, notice that $|\phi(h_k/\varepsilon_k\cdot)|^2 a$ are compactly supported and infinitely differentiable in x. When $\varepsilon_k = h_k$ we must verify that $|\phi|^2 a \in \Sigma^s$ which is clearly the case if $a|_{\mathbb{R}^d \times \overline{D_\phi}} \equiv 0$, for then $|\phi|^2 a$ is continuous in ξ.

On the other hand, if $h_k \ll \varepsilon_k$ and ϕ is merely continuous in a ball $B(0;\delta)$ then, for k large enough, $\operatorname{supp} a \subset B(0; h_k/\varepsilon_k \delta)$ and consequently, $\phi(h_k/\varepsilon_k\cdot)$ is continuous on $\operatorname{supp} a$. ■

8.4 Additional properties.

The next approximation result is sometimes useful in the computation of Wigner measures:

Lemma 8.12 *Let (u_k) and (u_k^N) be sequences in $H^{-s}(\mathbb{R}^d)$, $s \geq 0$, satisfying (51) with the same bound and*

$$\limsup_{k\to\infty} \left\| \langle \varepsilon_k D_x \rangle^{-s} \left(u_k - u_k^N\right) \right\|_{L^2(\mathbb{R}^d)} \to 0 \qquad \text{as } N \to \infty.$$

Suppose that $m^{\varepsilon_k}[u_k]$ and $m^{\varepsilon_k}[u_k^N]$ converge respectively to μ and μ_N. Then

$$\mu_N \rightharpoonup \mu \qquad \text{in } \mathcal{M}_+\left(\mathbb{R}^d \times \mathbb{R}^d\right) \text{ as } N \to \infty.$$

Proof. This is a simple consequence of the identity:

$$\left\langle m^{\varepsilon_k}[u_k] - m^{\varepsilon_k}\left[u_k^N\right], a\right\rangle_{\mathcal{S}'\times\mathcal{S}} = \left\langle \overline{u_k^N}, a(x, \varepsilon_k D_x)\left(u_k - u_k^N\right)\right\rangle_{\mathcal{S}'\times\mathcal{S}} + \\ + \left\langle \overline{\left(u_k - u_k^N\right)}, a(x, \varepsilon_k D_x) u_k \right\rangle_{\mathcal{S}'\times\mathcal{S}}.$$

This gives an estimate:

$$\left|\left\langle m^{\varepsilon_k}[u_k] - m^{\varepsilon_k}\left[u_k^N\right], a\right\rangle_{\mathcal{S}'\times\mathcal{S}}\right| \leq C \left\| \langle \varepsilon_k D_x \rangle^{-s} \left(u_k - u_k^N\right) \right\|_{L^2(\mathbb{R}^d)};$$

taking limits as $k \to \infty$ we obtain:

$$\left| \int_{\mathbb{R}^d\times\mathbb{R}^d} a(x,\xi)\,(d\mu - d\mu_N) \right| \leq C \limsup_{k\to\infty} \left\| \langle \varepsilon_k D_x \rangle^{s} \left(u_k - u_k^N\right) \right\|_{L^2(\mathbb{R}^d)}$$

and the result follows, since the measures μ_N and μ are equibounded. ■

We conclude this section with an almost orthogonality result:

Lemma 8.13 *Let* (u_k) *and* (v_k) *be sequences in* $H^{-s}\left(\mathbb{R}^d\right)$, $s \geq 0$, *satisfying (51) for some scale* (ε_k). *Suppose their Wigner measures at scale* (ε_k), μ *and* ν *are mutually singular. Then* $m^{\varepsilon_k}[u_k + v_k]$ *converges to* $\mu + \nu$.

Proof. A proof of this result for $s = 0$ may be found in [6] or [13]. For the general case, it suffices to take into account Remark 8.4 to conclude that the Wigner measures of $\langle \varepsilon_k D_x \rangle^{-s} u_k$ and $\langle \varepsilon_k D_x \rangle^{-s} v_k$ are $\langle \xi \rangle^{-2s} \mu$ and $\langle \xi \rangle^{-2s} \nu$. These are clearly mutually singular and thus the aforementioned L^2-version of the present result gives

$$m^{\varepsilon_k}\left[\langle \varepsilon_k D_x \rangle^{-s}(u_k + v_k)\right] \rightharpoonup \langle \xi \rangle^{-2s} \mu + \langle \xi \rangle^{-2s} \nu$$

and finally

$$m^{\varepsilon_k}[u_k + v_k] \rightharpoonup \langle \xi \rangle^{2s}\left(\langle \xi \rangle^{-2s} \mu + \langle \xi \rangle^{-2s} \nu\right) = \mu + \nu$$

as claimed. ■

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