Lift of C_∞ and L_∞ morphisms to G_∞ morphisms

Grégory Ginot, Gilles Halbout

LIFTS OF C_∞ AND L_∞-MORPHISMS TO G_∞-MORPHISMS

Grégory Ginot[(a)] , Gilles Halbout[(b)]

[(a)] Laboratoire Analyse Géométrie et Applications, Université Paris 13 et Ecole Normale Supèrieure de Cachan, France
[(b)] Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, Strasbourg, France

Abstract. Let $\mathfrak{g}_2$ be the Hochschild complex of cochains on $C^\infty(\mathbb{R}^n)$ and $\mathfrak{g}_1$ be the space of multivector fields on $\mathbb{R}^n$. In this paper we prove that given any G_∞-structure (i.e. Gerstenhaber algebra up to homotopy structure) on $\mathfrak{g}_2$, and any C_∞-morphism φ (i.e. morphism of commutative, associative algebra up to homotopy) between $\mathfrak{g}_1$ and $\mathfrak{g}_2$, there exists a G_∞-morphism Φ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ that restricts to φ. We also show that any L_∞-morphism (i.e. morphism of Lie algebra up to homotopy), in particular the one constructed by Kontsevich, can be deformed into a G_∞-morphism, using Tamarkin's method for any G_∞-structure on $\mathfrak{g}_2$. We also show that any two of such G_∞-morphisms are homotopic.

0-Introduction

Let M be a differential manifold and $\mathfrak{g}_2 = (C^\bullet(A,A), b)$ be the Hochschild cochain complex on $A = C^\infty(M)$. The classical Hochschild-Kostant-Rosenberg theorem states that the cohomology of $\mathfrak{g}_2$ is the graded Lie algebra $\mathfrak{g}_1 = \Gamma(M, \wedge^\bullet TM)$ of multivector fields on M. There is also a graded Lie algebra structure on $\mathfrak{g}_2$ given by the Gerstenhaber bracket. In particular $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are also Lie algebras up to homotopy (L_∞-algebra for short). In the case $M = \mathbb{R}^n$, using different methods, Kontsevich ([Ko1] and [Ko2]) and Tamarkin ([Ta]) have proved the existence of Lie homomorphisms "up to homotopy" (L_∞-morphisms) from $\mathfrak{g}_1$ to $\mathfrak{g}_2$. Kontsevich's proof uses graph complex and is related to multizeta functions whereas Tamarkin's construction uses the existence of Drinfeld's associators. In fact Tamarkin's L_∞-morphism comes from the restriction of a Gerstenhaber algebra up to homotopy homomorphism (G_∞-morphism) from $\mathfrak{g}_1$ to $\mathfrak{g}_2$. The G_∞-algebra structure on $\mathfrak{g}_1$ is induced by its classical Gerstenhaber algebra structure and a far less trivial G_∞-structure on $\mathfrak{g}_2$ was proved to exist by Tamarkin [Ta] and relies on a Drinfeld's associator. Tamarkin's G_∞-morphism also restricts into a commutative, associative up to homotopy morphism (C_∞-morphism for short). The C_∞-structure on $\mathfrak{g}_2$ (given by

Keywords : Deformation quantization, star-product, homotopy formulas, homological methods.
AMS Classification : Primary 16E40, 53D55, Secondary 18D50, 16S80.

restriction of the G_∞-one) highly depends on Drinfeld's associator, and any two choices of a Drinfeld associator yields *a priori* different C_∞-structures. When M is a Poisson manifold, Kontsevich and Tamarkin homomorphisms imply the existence of a star-product (see [BFFLS1] and [BFFLS2] for a definition). A connection between the two approaches has been given in [KS] but the morphisms given by Kontsevich and Tamarkin are not the same. The aim of this paper is to show that, given any G_∞-structure on $\mathfrak{g}_2$ and any C_∞-morphism φ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$, there exists a G_∞-morphism Φ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ that restricts to φ. We also show that any L_∞-morphism can be deformed into a G_∞-one.

In the first section, we fix notation and recall the definitions of L_∞ and G_∞-structures. In the second section we state and prove the main theorem. In the last section we show that any two G_∞-morphisms given by Tamarkin's method are homotopic.

Remark : In the sequel, unless otherwise is stated, the manifold M is supposed to be $\mathbb{R}^n$ for some $n \geq 1$. Most results could be generalized to other manifolds using techniques of Kontsevich [Ko1] (also see [TS], [CFT]).

1-C_∞, L_∞ and G_∞-structures

For any graded vector space $\mathfrak{g}$, we choose the following degree on $\wedge^\bullet \mathfrak{g}$: if $X_1, \dots, X_k$ are homogeneous elements of respective degree $|X_1|, \dots |X_k|$, then

$$|X_1 \wedge \cdots \wedge X_k| = |X_1| + \cdots + |X_k| - k.$$

In particular the component $\mathfrak{g} = \wedge^1 \mathfrak{g} \subset \wedge^\bullet \mathfrak{g}$ is the same as the space $\mathfrak{g}$ with degree shifted by one. The space $\wedge^\bullet \mathfrak{g}$ with the deconcatenation cobracket is the cofree cocommutative coalgebra on $\mathfrak{g}$ with degree shifted by one (see [LS], Section 2). Any degree one map $d^k : \wedge^k \mathfrak{g} \to \mathfrak{g}$ ($k \geq 1$) extends into a derivation $d^k : \wedge^\bullet \mathfrak{g} \to \wedge^\bullet \mathfrak{g}$ of the coalgebra $\wedge^\bullet \mathfrak{g}$ by cofreeness property.

Definition 1.1. *A vector space $\mathfrak{g}$ is endowed with a L_∞-algebra (Lie algebras "up to homotopy") structure if there are degree one linear maps $m^{1,\dots,1}$, with k ones : $\wedge^k \mathfrak{g} \to \mathfrak{g}$ such that if we extend them to maps $\wedge^\bullet \mathfrak{g} \to \wedge^\bullet \mathfrak{g}$, then $d \circ d = 0$ where d is the derivation*

$$d = m^1 + m^{1,1} + \cdots + m^{1,\dots,1} + \cdots .$$

For more details on L_∞-structures, see [LS]. It follows from the definition that a L_∞-algebra structure induces a differential coalgebra structure on $\wedge^\bullet \mathfrak{g}$ and that the map $m^1 : \mathfrak{g} \to \mathfrak{g}$ is a differential. If $m^{1,\dots,1} : \wedge^k \mathfrak{g} \to \mathfrak{g}$ are 0 for $k \geq 3$, we get the usual definition of (differential if $m^1 \neq 0$) graded Lie algebras.

For any graded vector space $\mathfrak{g}$, we denote $\underline{\mathfrak{g}^{\otimes n}}$ the quotient of $\mathfrak{g}^{\otimes n}$ by the image of all shuffles of length n (see [GK] or [GH] for details). The graded vector space $\oplus_{n \geq 0} \underline{\mathfrak{g}^{\otimes n}}$ is a quotient coalgebra of the tensor coalgebra $\oplus_{n \geq 0} \mathfrak{g}^{\otimes n}$. It is well known that this coalgebra $\oplus_{n \geq 0} \underline{\mathfrak{g}^{\otimes n}}$ is the cofree Lie coalgebra on the vector space $\mathfrak{g}$ (with degree shifted by minus one).

Definition 1.2. *A C_∞-algebra (commutative and asssociative "up to homotopy" algebra) structure on a vector space $\mathfrak{g}$ is given by a collection of degree one linear maps m^k : $\underline{\mathfrak{g}^{\otimes k}} \to \mathfrak{g}$ such that if we extend them to maps $\oplus \underline{\mathfrak{g}^{\otimes \bullet}} \to \oplus \underline{\mathfrak{g}^{\otimes \bullet}}$, then $d \circ d = 0$ where d is the derivation*

$$d = m^1 + m^2 + m^3 + \cdots .$$

In particular a C_∞-algebra is an A_∞-algebra.

For any space $\mathfrak{g}$, we denote $\wedge^\bullet \underline{\mathfrak{g}^{\otimes\bullet}}$ the graded space

$$\wedge^\bullet \underline{\mathfrak{g}^{\otimes\bullet}} = \bigoplus_{m\geq 1,\ p_1+\cdots+p_n=m} \underline{\mathfrak{g}^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}^{\otimes p_n}}.$$

We use the following grading on $\wedge^\bullet \underline{\mathfrak{g}^{\otimes\bullet}}$: for $x_1^1, \cdots, x_n^{p_n} \in \mathfrak{g}$, we define

$$|\underline{x_1^1 \otimes \cdots \otimes x_1^{p_1}} \wedge \cdots \wedge \underline{x_n^1 \otimes \cdots \otimes x_n^{p_n}}| = \sum_{i_1}^{p_1} |x_1^{i_1}| + \cdots + \sum_{i_n}^{p_1} |x_n^{i_n}| - n.$$

Notice that the induced grading on $\wedge^\bullet \mathfrak{g} \subset \wedge^\bullet \underline{\mathfrak{g}^{\otimes\bullet}}$ is the same than the one introduced above. The cobracket on $\oplus\underline{\mathfrak{g}^{\otimes\bullet}}$ and the coproduct on $\wedge^\bullet\mathfrak{g}$ extend to a cobracket and a coproduct on $\wedge^\bullet\underline{\mathfrak{g}^{\otimes\bullet}}$ which yield a Gerstenhaber coalgebra structure on $\wedge^\bullet\underline{\mathfrak{g}^{\otimes\bullet}}$. It is well known that this coalgebra structure is cofree (see [Gi], Section 3 for example).

Definition 1.3. *A G_∞-algebra (Gerstenhaber algebra "up to homotopy") structure on a graded vector space $\mathfrak{g}$ is given by a collection of degree one maps*

$$m^{p_1,\ldots,p_n} \;:\; \underline{\mathfrak{g}^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}^{\otimes p_n}} \to \mathfrak{g}$$

indexed by $p_1, \ldots p_n \geq 1$ such that their canonical extension: $\wedge^\bullet\underline{\mathfrak{g}^{\otimes\bullet}} \to \wedge^\bullet\underline{\mathfrak{g}^{\otimes\bullet}}$ satisfies $d \circ d = 0$ where

$$d = \sum_{m\geq 1,\ p_1+\cdots p_n=m} m^{p_1,\ldots,p_n}.$$

Again, as the coalgebra structure of $\wedge^\bullet\underline{\mathfrak{g}^{\otimes\bullet}}$ is cofree, the map d makes $\wedge^\bullet\underline{\mathfrak{g}^{\otimes\bullet}}$ into a differential coalgebra. If the maps $m^{p_1,\ldots,p_n}$ are 0 for $(p_1, p_2, \ldots) \neq (1, 0, \ldots)$, $(1, 1, 0, \ldots)$ or $(2, 0, \ldots)$, we get the usual definition of (differential if $m^1 \neq 0$) Gerstenhaber algebra.

The space of multivector fields $\mathfrak{g}_1$ is endowed with a graded Lie bracket $[-,-]_S$ called the Schouten bracket (see [Kos]). This Lie algebra can be extended into a Gerstenhaber algebra, with commutative structure given by the exterior product: $(\alpha, \beta) \mapsto \alpha \wedge \beta$

Setting $d_1 = m_1^{1,1} + m_1^2$, where $m_1^{1,1} : \wedge^2\mathfrak{g}_1 \to \mathfrak{g}_1$, and $m_1^2 : \underline{\mathfrak{g}_1^{\otimes 2}} \to \mathfrak{g}_1$ are the extension of the Schouten bracket and the exterior product, we find that $(\mathfrak{g}_1, d_1)$ is a G_∞-algebra.

In the same way, one can define a differential Lie algebra structure on the vector space $\mathfrak{g}_2 = C(A, A) = \bigoplus_{k\geq 0} C^k(A, A)$, the space of Hochschild cochains (generated by differential k-linear maps from A^k to A), where $A = C^\infty(M)$ is the algebra of smooth differential functions over M. Its bracket $[-,-]_G$, called the Gerstenhaber bracket, is defined, for $D, E \in \mathfrak{g}_2$, by

$$[D, E]_G = \{D|E\} - (-1)^{|E||D|}\{E|D\},$$

where

$$\{D|E\}(x_1, \ldots, x_{d+e-1}) = \sum_{i\geq 0} (-1)^{|E|\cdot i} D(x_1, \ldots, x_i, E(x_{i+1}, \ldots, x_{i+e}), \ldots).$$

The space $\mathfrak{g}_2$ has a grading defined by $| D |= k \Leftrightarrow D \in C^{k+1}(A, A)$ and its differential is $b = [m, -]_G$, where $m \in C^2(A, A)$ is the commutative multiplication on A.

Tamarkin (see [Ta] or also [GH]) stated the existence of a G_∞-structure on $\mathfrak{g}_2$ (depending on a choice of a Drinfeld associator) given by a differential $d_2 = m_2^1 + m_2^{1,1} + m_2^2 + \cdots + m_2^{p_1,\dots,p_n} + \cdots$, on $\wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}}$ satisfying $d_2 \circ d_2 = 0$. Although this structure is non-explicit, it satisfies the following three properties :

$$\begin{aligned} (a) \quad & m_2^1 \text{ is the extension of the differential } b \\ (b) \quad & m_2^{1,1} \text{ is the extension of the Gerstenhaber bracket } [-,-]_G \\ & \text{and } m_2^{1,1,\dots,1} = 0 \\ (c) \quad & m_2^2 \text{ induces the exterior product in cohomology and the} \\ & \text{collection of the } (m^k)_{k\geq 1} \text{ defines a } C_\infty\text{-structure on } \mathfrak{g}_2. \end{aligned} \tag{1.1}$$

Definition 1.4. *A L_∞-morphism between two L_∞-algebras $(\mathfrak{g}_1, d_1 = m_1^1 + \dots)$ and $(\mathfrak{g}_2, d_2 = m_2^1 + \dots)$ is a morphism of differential coalgebras*

$$\varphi \ : \ (\wedge^\bullet \mathfrak{g}_1, d_1) \to (\wedge^\bullet \mathfrak{g}_2, d_2). \tag{1.2}$$

Such a map φ is uniquely determined by a collection of maps $\varphi^n : \wedge^n \mathfrak{g}_1 \to \mathfrak{g}_2$ (again by cofreeness properties). In the case $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are respectively the graded Lie algebra $(\Gamma(M, \wedge TM),\ [-,-]_S)$ and the differential graded Lie algebra $(C\,(A,A)\,,\ [-,-]_G)$, the formality theorems of Kontsevich and Tamarkin state the existence of a L_∞-morphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that φ^1 is the Hochschild-Kostant-Rosenberg quasi-isomorphism.

Definition 1.5. *A morphism of C_∞-algebras between two C_∞-algebras $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$ is a map $\phi : (\oplus \underline{\mathfrak{g}_1^{\otimes \bullet}}, d_1) \to (\oplus \underline{\mathfrak{g}_2^{\otimes \bullet}}, d_2)$ of codifferential coalgebras.*

A C_∞-morphism is in particular a morphism of A_∞-algebras and is uniquely determined by maps $\partial^k : \underline{\mathfrak{g}^{\otimes k}} \to \mathfrak{g}$.

Definition 1.6. *A morphism of G_∞-algebras between two G_∞-algebras $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$ is a map $\phi : (\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}, d_1) \to (\wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}}, d_2)$ of codifferential coalgebras.*

There are coalgebras inclusions $\wedge^\bullet \mathfrak{g} \to \wedge^\bullet \underline{\mathfrak{g}^{\otimes \bullet}}$, $\oplus \underline{\mathfrak{g}^{\otimes \bullet}} \to \wedge^\bullet \underline{\mathfrak{g}^{\otimes \bullet}}$ and it is easy to check that any G_∞-morphism between two G_∞-algebras $(\mathfrak{g}, \sum m^{p_1,\dots,p_n})$, $\left(\mathfrak{g}', \sum m'^{p_1,\dots,p_n}\right)$ restricts to a L_∞-morphism $\left(\wedge^\bullet \mathfrak{g}, \sum m^{1,\dots,1}\right) \to \left(\wedge^\bullet \mathfrak{g}', \sum m'^{1,\dots,1}\right)$ and a C_∞-morphism $\left(\oplus \underline{\mathfrak{g}^{\otimes \bullet}}, \sum m^k\right) \to \left(\oplus \underline{\mathfrak{g}'^{\otimes \bullet}}, \sum m'^k\right)$. In the case $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are as above, Tamarkin's theorem states that there exists a G_∞-morphism between the two G_∞ algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ (with the G_∞ structure he built) that restricts to a C_∞ and a L_∞-morphism.

2-Main theorem

We keep the notations of the previous section, in particular $\mathfrak{g}_2$ is the Hochschild complex of cochains on $C^\infty(M)$ and $\mathfrak{g}_1$ its cohomology. Here is our main theorem.

Theorem 2.1. *Given any G_∞-structure d_2 on $\mathfrak{g}_2$ satisfying the three properties of (1.1), and any C_∞-morphism φ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that φ^1 is the Hochschild-Kostant-Rosenberg map, there exists a G_∞-morphism $\Phi : (\mathfrak{g}_1, d_1) \to (\mathfrak{g}_2, d_2)$ that restricts to φ.*

Also, given any L_∞-morphism γ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that γ^1 is the Hochschild-Kostant-Rosenberg map, there exists a G_∞-structure $(\mathfrak{g}_1, d_1')$ on $\mathfrak{g}_1$ and G_∞-morphism $\Gamma : (\mathfrak{g}_1, d_1') \to (\mathfrak{g}_2, d_2)$ that restricts to γ. Moreover there exists a G_∞-morphism $\Gamma' : (\mathfrak{g}_1, d_1) \to (g_1, d_1)$.

In particular, Theorem 2.1 applies to the formality map of Kontsevich and also to any C_∞-map derived (see [Ta], [GH]) from any B_∞-structure on $\mathfrak{g}_2$ lifting the Gerstenhaber structure of $\mathfrak{g}_1$.

Let us first recall the proof of Tamarkin's formality theorem (see [GH] for more details):

1. First one proves there exists a G_∞-structure on $\mathfrak{g}_2$, with differential d_2, as in (1.1).
2. Then, one constructs a G_∞-structure on $\mathfrak{g}_1$ given by a differential d_1' together with a G_∞-morphism Φ between $(\mathfrak{g}_1, d_1')$ and $(\mathfrak{g}_2, d_2)$.
3. Finally, one constructs a G_∞-morphism Φ' between $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_1, d_1')$.

The composition $\Phi \circ \Phi'$ is then a G_∞-morphism between $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$,thus restricts to a L_∞-morphism between the differential graded Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$.
We suppose now that, in the first step, we take any G_∞-structure on $\mathfrak{g}_2$ given by a differential d_2 and we suppose we are given a C_∞-morphism φ and a L_∞-morphism γ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ satisfying $\gamma^1 = \varphi^1 = \varphi_{\mathrm{HKR}}$ the Hochschild-Kostant-Rosenberg quasi-isomorphism.
Proof of Theorem 2.1:

The Theorem will follow if we prove that steps 2 and 3 of Tamarkin's construction are still true with the extra conditions that the restriction of the G_∞-morphism Φ (resp. Φ') on the C_∞-structures is the C_∞-morphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ (resp. id).

Let us recall (see [GH]) that the constructions of Φ and d_1' can be made by induction. For $i = 1, 2$ and $n \geq 0$, let us set

$$V_i^{[n]} = \bigoplus_{p_1+\cdots+p_k=n} \underline{\mathfrak{g}_i^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}_i^{\otimes p_k}}$$

and $V_i^{[\leq n]} = \sum_{k\leq n} V_i^{[k]}$. Let $d_2^{[n]}$ and $d_2^{[\leq n]}$ be the sums

$$d_2^{[n]} = \sum_{p_1+\cdots+p_k=n} d_2^{p_1,\ldots,p_k} \qquad \text{and} \qquad d_2^{[\leq n]} = \sum_{p\leq n} d_2^{[p]}.$$

Clearly, $d_2 = \sum_{n\geq 1} d_2^{[n]}$. In the same way, we denote $d_1' = \sum_{n\geq 1} {d'}_1^{[n]}$ with

$${d'}_1^{[n]} = \sum_{p_1+\cdots+p_k=n} d_1^{'p_1,\ldots,p_k} \qquad \text{and} \qquad {d'}_1^{[\leq n]} = \sum_{1\leq k\leq n} {d'}_1^{[k]}.$$

We know from Section 1 that a morphism $\Phi : (\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes\bullet}}, d_1') \to (\wedge^\bullet \underline{\mathfrak{g}_2^{\otimes\bullet}}, d_2)$ is uniquely determined by its components $\Phi^{p_1,\ldots,p_k} : \underline{\mathfrak{g}_1^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}_1^{\otimes p_k}} \to \mathfrak{g}_2$. Again, we have $\Phi = \sum_{n\geq 1} \Phi^{[n]}$ with

$$\Phi^{[n]} = \sum_{p_1+\cdots+p_k=n} \Phi^{p_1,\ldots,p_k} \qquad \text{and} \qquad \Phi^{[\leq n]} = \sum_{1\leq k\leq n} \Phi^{[k]}.$$

We want to construct the maps ${d'}_1^{[n]}$ and $\Phi^{[n]}$ by induction with the initial condition

$$ {d'}_1^{[1]} = 0 \qquad \text{and} \qquad \Phi^{[1]} = \varphi_{\mathrm{HKR}}, $$

where $\varphi_{\mathrm{HKR}} : (\mathfrak{g}_1, 0) \to (\mathfrak{g}_2, b)$ is the Hochschild-Kostant-Rosenberg quasi-isomorphism (see [HKR]) defined, for $\alpha \in \mathfrak{g}_1,\ f_1, \cdots, f_n \in A$, by

$$ \varphi_{\mathrm{HKR}} \ : \ \alpha \mapsto \big((f_1, \dots, f_n) \mapsto \ \langle \alpha, df_1 \wedge \cdots \wedge df_n \rangle \big). $$

Moreover, we want the following extra conditions to be true:

$$ \Phi^{k\geq 2} = \varphi^k, \qquad {d'}_1^2 = d_1^2, \qquad {d'}_1^{k\geq 3} = 0. \tag{2.3} $$

Now suppose the construction is done for $n-1$ ($n \geq 2$), i.e., we have built maps $({d'}_1^{[i]})_{i\leq n-1}$ and $(\Phi^{[i]})_{i\leq n-1}$ satisfying conditions (2.3) and

$$ \Phi^{[\leq n-1]} \circ {d'}_1^{[\leq n-1]} = d_2^{[\leq n-1]} \circ \ \Phi^{[\leq n-1]} \text{ on } V_1^{[\leq n-1]} \text{ and } {d'}_1^{[\leq n-1]} \circ {d'}_1^{[\leq n-1]} = 0 \text{ on } V_1^{[\leq n]}. \tag{2.4} $$

In [GH], we prove that for any such $({d'}_1^{[i]})_{i\leq n-1}$ and $(\Phi^{[i]})_{i\leq n-1}$, one can construct ${d'}_1^{[n]}$ and $\Phi^{[n]}$ such that condition (2.4) is true for n instead of $n-1$. To complete the proof of Theorem 2.1 (step 2), we have to show that ${d'}_1^{[n]}$ and $\Phi^{[n]}$ can be chosen to satisfy conditions (2.3). In the equation 2.4, the terms ${d'}_1^k$ and Φ^k only act on V_1^k. So one can replace Φ^n with φ^n, ${d'}_1^2$ with d_1^2 (or ${d'}_1^i, i \geq 3$ with 0) provided conditions (2.4) are still satisfied on V_1^n. The other terms acting on V_1^n in the equation (2.4) only involve terms $\Phi^m = \varphi^m$ and ${d'}_1^m$. Then conditions (2.4) on $V_1^{1,\dots,1}$ are the equations that should be satisfied by a C_∞-morphism between the C_∞-algebras $(\mathfrak{g}_1, {d'}_1^{1,1} = d_1^{1,1})$ and $(\mathfrak{g}_2, \sum_{k\geq 1} d_2^k)$ restricted to V_1^n. Hence by hypothesis on φ the conditions hold.

Similarly the construction of Φ' can be made by induction. Let us recall the proof given in [GH]. Again a morphism $\Phi' : (\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes\bullet}}, d_1) \to (\wedge^\bullet \underline{\mathfrak{g}_2^{\otimes\bullet}}, d'_1)$ is uniquely determined by its components ${\Phi'}^{p_1,\dots,p_k} : \underline{\mathfrak{g}_1^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}_1^{\otimes p_k}} \to \mathfrak{g}_1$. We write $\Phi' = \sum_{n\geq 1} {\Phi'}^{[n]}$ with

$$ {\Phi'}^{[n]} = \sum_{p_1+\cdots+p_k=n} {\Phi'}^{p_1,\dots,p_k} \qquad \text{and} \qquad {\Phi'}^{[\leq n]} = \sum_{1\leq k\leq n} {\Phi'}^{[k]}. $$

We construct the maps ${\Phi'}^{[n]}$ by induction with the initial condition ${\Phi'}^{[1]} = \mathrm{id}$. Moreover, we want the following extra conditions to be true:

$$ {\Phi'}^n = 0 \text{ for } n \geq 2. \tag{2.5} $$

Now suppose the construction is done for $n-1$ ($n \geq 2$), i.e., we have built maps $({\Phi'}^{[i]})_{i\leq n-1}$ satisfying conditions (2.5) and

$$ {\Phi'}^{[\leq n-1]} d_1^{[\leq n]} = {d'_1}^{[\leq n]} {\Phi'}^{[\leq n-1]} \text{ on } V_1^{[\leq n]}. \tag{2.6} $$

In [GH], we prove that for any such $({\Phi'}^{[i]})_{i\leq n-1}$, one can construct ${\Phi'}^{[n]}$ such that condition (2.6) is true for n instead of $n-1$ in the following way : making the equation $\Phi' d_1 = d'_1\Phi'$ on $V_1^{[n+1]}$ explicit, we get

$$ {\Phi'}^{[\leq n]}\, d_1^{[\leq n+1]} = {d'_1}^{[\leq n+1]}\, {\Phi'}^{[\leq n]}. \tag{2.7}$$

If we now take into account that $d_1^{[i]} = 0$ for $i \neq 2$, ${d'_1}^{[1]} = 0$ and that on $V_1^{[n+1]}$ we have ${\Phi'}^{[k]} d_1^{[l]} = {d'_1}^{[\leq k]}{\Phi'}^{[l]} = 0$ for $k + l > n + 2$, the identity (2.7) becomes

$$ {\Phi'}^{[\leq n]}\, d_1^{[2]} = \sum_{k=2}^{n+1} {d'_1}^{[k]}\, {\Phi'}^{[\leq n-k+2]}.$$

As ${d'_1}^{[2]} = {d_1}^{[2]}$, (2.7) is equivalent to

$$ {d_1}^{[2]}{\Psi'}^{[\leq n]} - {\Phi'}^{[\leq n]}{d_1}^{[2]} = \left[{d_1}^{[2]}, {\Phi'}^{[\leq n]}\right] = -\sum_{k=3}^{n+1} {d'_1}^{[k]}{\Phi'}^{[\leq n-k+2]}.$$

Notice that $d_1^{[2]} = m_1^{1,1} + m_1^2$. Then the construction will be possible when the term $\sum_{k=3}^{n+1} {d'_1}^{[k]}{\psi'}^{[\leq n-k+2]}$ is a couboundary in the subcomplex of $(\mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes\bullet}}), [d_1^{[2]}, -])$ consisting of maps which restrict to zero on $\oplus_{n\geq 2}\underline{\mathfrak{g}_1}^{\otimes n}$. It is always a cocycle by straightforward computation (see [GH]) and the subcomplex is acyclic because both $(\mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes\bullet}}), [d_1^{[2]}, -])$ and the Harrison cohomology of $\mathfrak{g}_1$ are trivial according to Tamarkin [Ta] (see also [GH] Proposition 5.1 and [Hi] 5.4).

In the case of the L_∞-morphism γ, the first step is similar: the fact that γ is a L_∞-map enables us to build a G_∞-structure $(\mathfrak{g}_1, d'_1)$ on $\mathfrak{g}_1$ and a G_∞-morphism $\Gamma : (\mathfrak{g}_1, d'_1) \to (\mathfrak{g}_2, d_2)$ such that:

$$ \Gamma^{1,\dots,1} = \gamma^{1,\dots,1}, \qquad {d'}_1^{1,1} = d_1^{1,1}, \qquad {d'}_1^{1,1,\dots,1} = 0. \tag{2.8}$$

For the second step, we have to build a map Γ' satisfying the equation

$$ {d_1}^{[2]}{\Gamma'}^{[\leq n]} - {\Gamma'}^{[\leq n]}{d_1}^{[2]} = \left[{d_1}^{[2]}, {\Gamma'}^{[\leq n]}\right] = -\sum_{k=3}^{n+1} {d'_1}^{[k]}{\Gamma'}^{[\leq n-k+2]}$$

on $V_1^{[n+1]}$ for any $n \geq 1$. Again, because Tamarkin has prooved that the complex $(\mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes\bullet}}), [d_1^{[2]}, -])$ is acyclic (we are in the case $M = \mathbb{R}^n$), the result follows from the fact that $\sum_{k=3}^{n+1} {d'_1}^{[k]}{\Gamma'}^{[\leq n-k+2]}$ is a cocycle. The difference with the C_∞-case is that the ${\Gamma'}^{1,\dots,1}$ could be non zero. ■

3-The difference between two G_∞-maps

In this section we investigate the difference between two differents G_∞-formality maps. We fix once for all a G_∞-structure on $\mathfrak{g}_2$ (given by a differential d_2) satisfying the conditions (1.1) and a morphism of G_∞-algebras $T : (\mathfrak{g}_1, d_1) \to (\mathfrak{g}_2, d_2)$ such that $T^1 : \mathfrak{g}_1 \to \mathfrak{g}_2$ is φ_{HKR}. Let $K : (\mathfrak{g}_1, d_1) \to (\mathfrak{g}_2, d_2)$ be any other G_∞-morphism with $K^1 = \varphi_{\mathrm{HKR}}$ (for example any lift of a Kontsevich formality map or any G_∞-maps lifting another C_∞-morphism).

Theorem 3.1. *There exists a map* $h : \wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}} \to \wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}}$ *such that*

$$T - K = h \circ d_1 + d_2 \circ h.$$

In other words the formality G_∞-morphisms K and T are homotopic.
The maps T and K are elements of the cochain complex $\Big(\mathrm{Hom}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}, \wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}}), \delta\Big)$ with differential given, for all $f \in \mathrm{Hom}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}, \wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}}), |f| = k$, by

$$\delta(f) = d_2 \circ f - (-1)^k f \circ d_1.$$

We first compare this cochain complex with the complexes $\Big(\mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}), [d_1; -]\Big)$ and $\Big(\mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}}), [d_2; -]\Big)$ (where $[-;-]$ is the graded commutator of morphisms). There are morphisms

$$T_* : \mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}) \to \mathrm{Hom}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}, \wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}}),\ T^* : \mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}}) \to \mathrm{Hom}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}, \wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}})$$

defined, for $f \in \mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}})$ and $g \in \mathrm{Hom}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}, \wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}})$, by

$$T_*(f) = T \circ f,\ T^*(g) = g \circ T.$$

Lemma 3.2. *The morphisms*

$$T_* : \Big(\mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}), [d_1; -]\Big) \to \Big(\mathrm{Hom}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}, \wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}}), \delta\Big) \leftarrow \Big(\mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}}), [d_2; -]\Big) : T^*$$

of cochain complexes are quasi-isomorphisms.

Remark: This lemma holds for every manifold M and any G_∞-morphism $T : (\mathfrak{g}_1, d_1) \to (g_2, d_2)$.

Proof :. First we show that T_* is a morphism of complexes. Let $f \in \mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}})$ with $|f| = k$, then

$$\begin{aligned} T_*([d_1; f]) =& T \circ d_1 \circ f - (-1)^k T \circ f \circ d_1 \\ =& d_2 \circ (T \circ f) - (-1)^k (T \circ f) \circ d_1 \\ =& \delta(T_*(f)). \end{aligned}$$

Let us prove now that T_* is a quasi-isomorphism. For any graded vector space $\mathfrak{g}$, the space $\wedge^\bullet \underline{\mathfrak{g}^{\otimes \bullet}}$ has the structure of a filtered space where the m-level of the filtration is $F^m(\wedge^\bullet \underline{\mathfrak{g}^{\otimes \bullet}}) = \oplus_{p_1 + \cdots + p_n - 1 \le m} \underline{\mathfrak{g}^{\otimes p_1}} \wedge \ldots \underline{\mathfrak{g}^{\otimes p_n}}$. Clearly the differential d_1 and d_2 are compatible with the filtrations on $\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}$ and $\wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}}$, hence $\mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}})$ and $\mathrm{Hom}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}, \wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}})$ are filtered cochain complex. This yields two spectral sequences (lying in the first quadrant) $E_\bullet^{\bullet;\bullet}$ and $\widetilde{E}_\bullet^{\bullet;\bullet}$ which converge respectively toward the cohomology $H^\bullet(\mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}))$ and $H^\bullet(\mathrm{Hom}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}, \wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}}))$. By standard spectral sequence techniques it is enough to prove that the map $T_*^0 : E_0^{\bullet,\bullet} \to \widetilde{E}_0^{\bullet,\bullet}$ induced by T_* on the associated graded is a quasi-isomorphism.

The induced differentials on $E_0^{\bullet,\bullet}$ and $\widetilde{E}_0^{\bullet,\bullet}$ are respectively $[d_1^1, -] = 0$ and $d_2^1 \circ (-) - (-) \circ d_1^1 = b \circ (-)$ where b is the Hochschild coboundary. By cofreeness property we have the following two isomorphisms

$$E_0^{\bullet,\bullet} \cong \mathrm{End}(\mathfrak{g}_1), \qquad \widetilde{E}_0^{\bullet,\bullet} \cong \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_2).$$

The map $T_*^0 : E_0^{\bullet,\bullet} \to \widetilde{E}_{\bullet\bullet}^0$ induced by T_* is $\varphi_{\mathrm{HKR}} \circ (-)$. Let $p : \mathfrak{g}_2 \to \mathfrak{g}_1$ be the projection onto the cohomology, *i.e.* $p \circ \varphi_{\mathrm{HKR}} = \mathrm{id}$. Let $u : \mathfrak{g}_1 \to \mathfrak{g}_2$ be any map satisfying $b(u) = 0$ and set $v = p \circ u \in \mathrm{End}(\mathfrak{g}_1)$. One can choose a map $w : \mathfrak{g}_1 \to \mathfrak{g}_2$ which satisfies for any $x \in \mathfrak{g}_1$ the following identity

$$\varphi_{\mathrm{HKR}} \circ p \circ u(x) - u(x) = b \circ w(x).$$

It follows that $\varphi_{\mathrm{HKR}}(v)$ has the same class of homology as u which proves the surjectivity of T_*^0 in cohomology. The identity $p \circ \varphi_{\mathrm{HKR}} = \mathrm{id}$ implies easily that T_*^0 is also injective in cohomology which finish the proof of the lemma for T_*.

The proof that T^* is also a quasi-isomorphism is analogous.

Proof of Theorem 3.1:.
It is easy to check that $T - K$ is a cocycle in $\Big(\mathrm{Hom}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes\bullet}}, \wedge^\bullet \underline{\mathfrak{g}_2^{\otimes\bullet}}), \delta\Big)$. The complex of cochain $\Big(\mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes\bullet}}), [d_1, -]\Big) \cong \Big(\mathrm{Hom}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes\bullet}}, \mathfrak{g}_1), [d_1, -]\Big)$ is trigraded with $|\,|_1$ being the degree coming from the graduation of $\mathfrak{g}_1$ and any element x lying in $\underline{\mathfrak{g}_1^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}_1^{\otimes p_q}}$ satisfies $|x|_2 = q - 1$, $|x|_3 = p_1 + \dots p_q - q$. In the case $M = \mathbb{R}^n$, the cohomology $H^\bullet \Big(\mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes\bullet}}), [d_1, -]\Big)$ is concentrated in bidegree $(|\,|_2, |\,|_3) = (0, 0)$ (see [Ta], [Hi]). By Lemma 3.2, this is also the case for the cochain complex $\Big(\mathrm{Hom}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes\bullet}}, \wedge^\bullet \underline{\mathfrak{g}_2^{\otimes\bullet}}), \delta\Big)$. Thus, its cohomology classes are determined by complex morphisms $(\mathfrak{g}_1, 0) \to (\mathfrak{g}_2, d_2^1)$ and it is enough to prove that T and K determine the same complex morphism $(\mathfrak{g}_1, 0) \to (\mathfrak{g}_2, d_2^1 = b)$ which is clear because T^1 and K^1 are both equal to the Hochschild-Kostant-Rosenberg map. ■

Remark. *It is possible to have an explicit formula for the map h in Theorem .3.1. In fact the quasi-isomorphism coming from Lemma 3.2 can be made explicit using explicit homotopy formulae for the Hochschild-Kostant-Rosenberg map (see [Ha] for example) and deformation retract techniques (instead of spectral sequences) as in [Ka]. The same techniques also apply to give explicit formulae for the quasi-isomorphism giving the acyclicity of* $\Big(\mathrm{End}(\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes\bullet}}), [d_1; -]\Big)$ *in the proof of theorem 3.1 (see [GH] for example)*

References

[BFFLS1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Quantum mechanics as a deformation of classical mechanics*, Lett. Math. Phys. **1** (1975), 521–530.

[BFFLS2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Deformation theory and quantization, I and II*, Ann. Phys. **111** (1977), 61–151.

[CFT] A. S. Cattaneo, G. Felder, L. Tomassini, *From local to global deformation quantization of Poisson manifolds*, Duke Math. J. **115** (2002), 329–352.

[Gi] G. Ginot, *Homologie et modèle minimal des algèbres de Gerstenhaber*, Ann. Math. Blaise Pascal **11** (2004), 91–127.

[GH] G. Ginot, G. Halbout, *A formality theorem for Poisson manifold*, Lett. Math. Phys. **66** (2003), 37–64.

[GK] V. Ginzburg, M. Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1994), 203–272.

[Ha] G. Halbout, *Formule d'homotopie entre les complexes de Hochschild et de de Rham*, Compositio Math. **126** (2001), 123–145.

[Hi] V. Hinich, *Tamarkin's proof of Kontsevich's formality theorem*, Forum Math. **15** (2003), 591–614.

[HKR] G. Hochschild, B. Kostant and A. Rosenberg, *Differential forms on regular affine algebras*, Transactions AMS **102** (1962), 383-408.

[Ka] C. Kassel, *Homologie cyclique, caractère de Chern et lemme de perturbation*, J. Reine Angew. Math. **408** (1990), 159–180.

[Ko1] M. Kontsevich, *Formality conjecture. Deformation theory and symplectic geometry*, Math. Phys. Stud. **20** (1996), 139–156.

[Ko2] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), 157–216.

[KS] M. Kontsevich, Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Math. Phys. Stud. **21** (2000), 255–307.

[Kos] J. L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie in "Elie Cartan et les mathématiques d'aujourd'hui"*, Astérisque (1985), 257-271.

[LS] T. Lada, J. D. Stasheff,, *Introduction to SH Lie algebras for physicists*, nternat. J. Theoret. Phys. **32** (1993), 1087-1103.

[Ta] D. Tamarkin, *Another proof of M. Kontsevich's formality theorem*, math. QA/9803025.

[TS] D. Tamarkin, B Tsygan, *Noncommutative differential calculus, homotopy BV algebras and formality conjectures*, Methods Funct. Anal. Topology **6** (2000), 85–97.

(a) Laboratoire Analyse Géométrie et Applications, Université Paris 13,
Centre des Mathématiques et de Leurs Applications, ENS Cachan,
61, av. du Président Wilson, 94230 Cachan, France
e-mail:ginot@cmla.ens-cachan.fr

(b) Institut de Recherche Mathématique Avancée,
Université Louis Pasteur et CNRS
7, rue René Descartes, 67084 Strasbourg, France
e-mail:halbout@math.u-strasbg.fr