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The largest singletons in weighted set partitions and its applications

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Recently, Deutsch and Elizalde studied the largest fixed points of permutations. Motivated by their work, we consider the analogous problems in weighted set partitions. Let $A_{n,k}(t)$ denote the total weight of partitions on $[n+1] = \{1, 2, \ldots, n+1\}$ with the largest singleton $\{k+1\}$. In this paper, explicit formulas for $A_{n,k}(t)$ and many combinatorial identities involving $A_{n,k}(t)$ are obtained by umbral operators and combinatorial methods. In particular, the permutation case leads to an identity related to tree enumerations, namely,

$$\sum_{k=0}^{n} \binom{n}{k} D_{k+1}(n+1)^{n-k} = n^{n+1},$$

where $D_k$ is the number of permutations of $[k]$ with no fixed points.

Keywords: Set partition, Bell polynomial, Permutation, Derangement.

1 Introduction

A partition of a set $[n] = \{1, 2, \ldots, n\}$ is a collection $\pi = \{B_1, B_2, \ldots, B_r\}$ of nonempty and mutually disjoint subsets of $[n]$, called blocks, whose union is $[n]$. For a block $B$, we denote by $|B|$ the size of the block $B$, that is the number of the elements in the block $B$. A block $B$ will be called singleton if $|B| = 1$. If $\{k\}$ is a singleton of a partition, we denote it by $k$ for short. If $|B| = j$, we assign a weight $t_j$ for $B$.

The weight $w(\pi)$ of a partition $\pi$ is defined to be the product of the weight of each block of $\pi$.

It is well known that the weight of partitions of $[n]$ with $r$ blocks is the partial Bell polynomial $B_{n,r}(t_1, t_2, \ldots)$ on the variables $\{t_j\}_{j \geq 1}$, that is

$$B_{n,r}(t_1, t_2, \ldots) = \sum_{\alpha \in \mathcal{R}(r)} \frac{n!}{r_1!r_2! \cdots r_n!} \left(\frac{t_1}{1!}\right)^{r_1} \left(\frac{t_2}{2!}\right)^{r_2} \cdots \left(\frac{t_n}{n!}\right)^{r_n},$$

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where the summation $\kappa_n(r)$ is over all the nonnegative integer solutions of $r_1 + r_2 + \cdots + r_n = r$ and $r_1 + 2r_2 + \cdots + nr_n = n$. The total weight for partitions of $[n]$ is the complete Bell polynomial

$$Y_n(t) = \sum_{k=0}^{n} B_{n,k} t^k,$$

which has the exponential generating function

$$Y(t; x) = \sum_{n \geq 0} Y_n(t_1, t_2, \ldots) \frac{x^n}{n!} = \exp \left( \sum_{j \geq 1} \frac{t_j x^j}{j!} \right).$$

Let $A_{n,k}$ denote the set of partitions of $[n+1]$ with the largest singleton $k+1$. Let $A_{n,k}(t)$ denote the total weight of partitions in $A_{n,k}$. Clearly,

$$A_{0,0}(t) = t_1 Y_n(0,0,\ldots) \quad\text{and}\quad A_{n,0}(t) = t_1 Y_n(t_2, t_3, \ldots),$$

where $Y_n(0,0,\ldots)$ is the weight of partitions of $[n]$ without singletons.

Recently, Deutsch and Elizalde [4] studied the largest fixed points of permutations, which is the special case when $t_j = (j - 1)!$ for $j \geq 1$. Later, Sun and Wu [17] considered the largest singletons in set partitions, which is the special case when $t_j = 1$ for $j \geq 1$.

In this paper we will investigate the largest singletons in weighted set partitions generally. The next section is devoted to studying the properties of $A_{n,k}(t)$, involving its explicit formulas and many combinatorial identities for $A_{n,k}(t)$. In the third section, we consider the permutation case, i.e., the special case when $t_j = (j - 1)!$ for $j \geq 1$, and derive a surprising identity analogous to the Riordan identity related to tree enumerations.

## 2 The properties of $A_{n,k}(t)$

According to the definition of $A_{n,k}(t)$, for any weighted partition $\pi$ of $[n+1]$ with the largest singleton $k+1$, if $k$ is also a singleton, delete the singleton $k+1$ and subtracting one from all the entries larger than $k+1$, we obtain a partition of $[n]$ with the largest singleton $k$. This contributes the weight $t_1 A_{n-1,k-1}(t)$; if $k$ is not a singleton, exchange $k$ and $k+1$, we obtain a partition of $[n+1]$ with the largest singleton $k$. This contributes the weight $A_{n,k-1}(t)$. Consequently, we obtain a recurrence for $n, k \geq 1$,

$$A_{n,k}(t) = A_{n,k-1}(t) + t_1 A_{n-1,k-1}(t) \quad (1)$$

with the initial conditions $A_{n,0}(t) = t_1 Y_n(0, t_2, \ldots)$ for $n \geq 0$.

**Lemma 2.1** The bivariate exponential generating function for $A_{n+k,k}(t)$ is given by

$$A(t; x, y) = \sum_{n,k \geq 0} A_{n+k,k}(t) \frac{x^n y^k}{n!} = t_1 e^{-x_1} Y(t; x + y).$$

**Proof:** Define

$$A_k(t; x) = \sum_{n \geq 0} A_{n+k,k}(t) \frac{x^n}{n!}. $$
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Clearly, \( A_0(t; x) = t_1 e^{-xt_1} Y(t; x) \). From (1), one can derive that

\[
A_k(t; x) = t_1 A_{k-1}(t; x) + \frac{\partial}{\partial x} A_{k-1}(t; x),
\]

which produces

\[
A_k(t; x) = (t_1 + \frac{\partial}{\partial x}) A_{k-1}(t; x) = (t_1 + \frac{\partial}{\partial x})^k A_0(t; x).
\]

Then

\[
A(t; x, y) = \sum_{k \geq 0} A_k(t; x) \frac{y^k}{k!} = \sum_{k \geq 0} \frac{y^k(t_1 + \frac{\partial}{\partial x})^k}{k!} A_0(t; x)
\]

\[
e^{yt_1+y \frac{\partial}{\partial x}} t_1 e^{-xt_1} Y(t; x) = t_1 e^{yt_1} e^{y \frac{\partial}{\partial x}} e^{-xt_1} Y(t; x)
\]

\[
= t_1 e^{yt_1} e^{-(x+y)t_1} Y(t; x+y) = t_1 e^{-xt_1} Y(t; x+y).
\]

This completes the proof. \( \square \)

**Theorem 2.2**  For any integers \( n, m \geq 0 \) and any indeterminate \( \lambda \), there hold

\[
\begin{align*}
\sum_{k=0}^{n} \binom{k+\lambda-1}{k} A_{n+m,k+n}(t) &= \sum_{k=0}^{n} \binom{n+\lambda}{k} \binom{n+\lambda-k-1}{n-k} A_{m+k,n}(t) t_1^{n-k}, \\
\sum_{k=0}^{n} \binom{k+\lambda-1}{k} A_{n+m,k+n}(t) &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n+\lambda}{k} \sum_{k=0}^{n} \binom{-\lambda}{k} \binom{\lambda}{Y_{t_k}-t_1}^{n-k}.
\end{align*}
\]

**Proof:** With the umbra \( Y_t \), given by \( Y^n_t = Y_n(t) \), \( Y(t; x) \) may be written as \( Y(t; x) = e^{xt_1} \). (See, for example, \([7,12,13]\)). Then, by Lemma 2.1, we have

\[
A(t; x, y) = t_1 e^{Y_t(x+y)-t_1 x} = t_1 e^{(Y_t-x_1) x} e^{Y_t y}.
\]

When comparing the coefficient of \( \frac{x^n y^k}{n! k!} \), \( A_{n+k,k}(t) \) can be represented umbrally as

\[
A_{n+k,k}(t) = t_1 Y_t^n (Y_t - t_1)^n.
\]

Let \([x^n]f(x)\) denote the coefficient of \( x^n \) in the formal power series \( f(x) \). Then we get

\[
\begin{align*}
\sum_{k=0}^{n} \binom{k+\lambda-1}{k} A_{n+m,k+n}(t) &= \sum_{k=0}^{n} \sum_{k=0}^{n} (-1)^{n-k} \binom{-\lambda}{k} \binom{\lambda}{Y_t-n-k} t_1 Y_t^n (Y_t - t_1)^n-k \\
&= t_1 Y_t^n (Y_t - t_1)^n \sum_{k=0}^{n} \binom{-\lambda}{k} \left( -\frac{Y_t}{Y_t - t_1} \right)^k
\end{align*}
\]
\[ \begin{align*}
&= t_1 Y_t^n (Y_t - t_1)^n \sum_{k=0}^{n} x^k \left( 1 - \frac{x Y_t}{Y_t - t_1} \right)^{-\lambda} \\
&= t_1 Y_t^n (Y_t - t_1)^n \left( 1 - \frac{x Y_t}{Y_t - t_1} \right)^{-\lambda}
\end{align*} \]

which proves (2).

By the identity
\[
\binom{n}{k} \binom{k}{i} = \binom{n}{i} \binom{n-i}{k-i},
\]
and Vandermonde’s convolution identity
\[
\sum_{k=0}^{n} \binom{n}{k} \binom{b}{n-k} = \binom{a+b}{n},
\]
we have
\[
\sum_{k=0}^{n} \binom{k+\lambda-1}{k} A_{n+m,m+k}(t)
\]
\[
= \sum_{k=0}^{n} \binom{n+\lambda}{k} \binom{\lambda}{n-k} t_1 Y_t^n (Y_t - t_1)^{k} (-t_1)^{n-k}
\]
\[
= \sum_{k=0}^{n} \binom{n+\lambda}{k} \binom{\lambda}{n-k} \sum_{i=0}^{k} \binom{k}{i} t_1 Y_t^{m+i} (-t_1)^{n-i}
\]
\[
= \sum_{i=0}^{n} t_1 Y_t^{m+i} (-t_1)^{n-i} \sum_{k=i}^{n} \binom{n+\lambda}{k} \binom{\lambda}{n-k} \binom{k}{i}
\]
\[
= \sum_{i=0}^{n} \binom{n+\lambda}{i} t_1 Y_t^{m+i} (-t_1)^{n-i} \sum_{k=i}^{n} \binom{n+\lambda-i}{k} \binom{\lambda}{n-k} \binom{k}{i}
\]
\[
= \sum_{i=0}^{n} \binom{n+\lambda}{i} t_1 Y_t^{m+i} (-t_1)^{n-i}
\]
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\[ \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{k} \gamma_{m+k}(t^{n-k+1}), \]

which proves (3). \( \square \)

The case \( \lambda = 0 \) in (3) yields an explicit formula for \( A_{n+m,m}(t) \).

**Corollary 2.3** For any integers \( n, m \geq 0 \), there holds

\[ A_{n+m,m}(t) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{k} t_1^{n-k+1} \gamma_{m+k}(t). \] (5)

**Corollary 2.4** For any integers \( n, m \geq 0 \), there hold

\[ \sum_{k=0}^{n} A_{n+m,m+k}(t) = \frac{t_1 \gamma_{n+m+1}(t) - A_{n+m+1,m}(t)}{t_1}, \] (6)

\[ \sum_{k=0}^{n} (k+1) A_{n+m,m+k}(t) = \frac{A_{n+m+2,m}(t) - t_1 \gamma_{n+m+2}(t) + (n+2)t_1^2 \gamma_{n+m+1}(t)}{t_1^2}, \] (7)

\[ \sum_{k=0}^{n} (n-k+1) A_{n+m,m+k}(t) = \frac{t_1 \gamma_{n+m+2}(t) - A_{n+m+2,m}(t) - (n+2)t_1 A_{n+m+1,m}(t)}{t_1^2}. \] (8)

**Proof:** By combining (5) with \( n \) replaced by \( n+1 \) (resp. \( n+2 \)) and with the case \( \lambda = 1 \) (resp. \( \lambda = 2 \)) in (3), we obtain (6) (resp. (7)). Moreover, (8) can be easily obtained from (6) and (7). \( \square \)

**Theorem 2.5** For any integers \( n, m, k \geq 0 \), there holds

\[ A_{n+m+k,m+k}(t) = \sum_{j=0}^{m} \binom{m-j}{j} t_1^{m-j} A_{n+k+j,k}(t). \] (9)

**Proof:** Here we provide a combinatorial proof. For any \( \pi \in A_{n+m+k,m+k} \), suppose that \( \pi \) has exactly \( m-j \) singletons in \( \{k+1, \ldots, k+m\} \), which contributes the weight \( t_1^{m-j} \), and there are \( \binom{m-j}{j} \) ways to do this. The remaining \( j \) elements in \( \{k+1, \ldots, k+m\} \) can not be singletons in \( \pi \). These \( j \) elements can be regarded as the roles that greater than \( m+k+1 \), so the remaining \( n+k+j+1 \) elements can be partitioned with the largest singleton \( m+k+1 \); these cases contribute the weight \( A_{n+k+j,k}(t) \). Thus the total weight of such partitions is \( \binom{m-j}{j} t_1^{m-j} A_{n+k+j,k}(t) \). Summing up all the possible cases yields (9). \( \square \)

**Theorem 2.6** For any integers \( n, m \geq 0 \) and any indeterminate \( y \), there hold

\[ \sum_{k=0}^{n} \binom{n}{k} A_{n+m+m+k}(t) y^k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \gamma_{m+k}(t)(y+1)^k t_1^{n-k+1}, \] (10)

\[ \sum_{k=0}^{n} \binom{n}{k} A_{m+k,m}(t) y^{n-k} = t_1 \sum_{k=0}^{n} \binom{n}{k} \gamma_{m+k}(t)(y-t_1)^{n-k}. \] (11)
Proof: By (11), we have
\[
\sum_{k=0}^{n} \binom{n}{k} A_{n+m,m+k}(t)y^k = \sum_{k=0}^{n} \binom{n}{k} t_1 Y_{t_1}^{m+k}(Y_t - t_1)^{n-k} y^k \\
= t_1 Y_{t_1}^{m}((y+1)Y_t - t_1)^{n} \\
= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (y+1)^k Y_{t_1}^{m+k} t_1^{n-k+1} \\
= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (y+1)^k Y_{m+k}(t) t_1^{n-k+1},
\]
which proves (10). Similarly, (11) can be obtained, but here we provide a combinatorial proof.

Let \(X_{n,m} = \bigcup_{k=0}^{n} X_{n,m,k}\) and \(X_{n,m,k}\) denote the set of pairs \((\pi, S)\) such that

- \(S\) is an \((n-k)\)-subset of \([m+2, n+m+1] = \{m+2, \ldots, n+m+1\}\), and each element of \(S\) is colored by \(t_1\) or \(y - t_1\);
- \(\pi\) is a partition of the set \([n+m+1] - S\) with the largest singleton \(m+1\), and each element of \([n+m+1] - S\) is only colored by 1.

Let \(Y_{n,m} = \bigcup_{k=0}^{n} Y_{n,m,k}\) and \(Y_{n,m,k}\) denote the set of pairs \((\pi, S)\) such that

- \(S\) is an \((n-k)\)-subset of \([m+2, n+m+1]\) and each element of \(S\) is only colored by \(y - t_1\);
- \(\pi\) is a partition of the set \([n+m+1] - S\) such that \(m+1\) must be a singleton, and each element of \([n+m+1] - S\) is only colored by 1.

The weight of \((\pi, S)\) is defined to be the product of the weight of \(\pi\) and the color of each element of \([n+m+1]\). Clearly, the weights of \(X_{n,m}\) and \(Y_{n,m}\) are counted respectively by the left and right sides of (11).

Given any pair \((\pi, S) \in X_{n,m}\), \(S\) can be partitioned into two parts \(S_1\) and \(S_2\) such that each element of \(S_1\) is colored by \(y - t_1\) and each element of \(S_2\) is colored by \(t_1\). Regard each element of \(S_2\) as a singleton which is weighted by \(t_1\) and colored by 1, together with \(\pi\), we obtain a partition \(\pi_1\) of \([n+m+1] - S_1\) such that \(m+1\) is always a singleton. Then the pair \((\pi_1, S_1)\) lies in \(Y_{n,m}\).

Conversely, for any pair \((\pi_1, S_1) \in Y_{n,m}\), let \(S\) denote the union of \(S_1\) and the singletons of \(\pi_1\) greater than \(m+1\), then \(\pi_1\) can be partitioned into two parts \(\pi\) and \(\pi'\) such that \(\pi\) is a partition of \([n+m+1] - S\) with the largest singleton \(m+1\) and \(\pi'\) is the singletons of \(\pi_1\) greater than \(m+1\). By regarding \(\pi'\) as a subset of \([m+2, n+m+1]\) in which each element is colored by \(t_1\), together with \(S_1\). Then we obtain an \((n-k)\)-subset of \([m+2, n+m+1]\) for some \(k\) such that each element of \(S\) is colored by \(t_1\) or \(y - t_1\). Then the pair \((\pi, S)\) lies in \(X_{n,m}\).

Clearly we find a bijection between \(X_{n,m}\) and \(Y_{n,m}\), which proves (11). \(\Box\)

The cases \(y = -1\) in (10) and \(y = t_1\) in (11) lead to
Corollary 2.7 For any integers \( n, m \geq 0 \), there hold

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_{n+m,m+k}(t) = \mathcal{Y}_m(t)t_1^{n+1},
\]

\[
\sum_{k=0}^{n} \binom{n}{k} A_{m+k,m}(t)t_1^{n-k-1} = \mathcal{Y}_{m+n}(t).
\]

The case \( y = \frac{t_1}{y+1} \) in (11), together with (10) generates the following result which has a combinatorial interpretation.

Corollary 2.8 For any integers \( n, m \geq 0 \), there holds

\[
\sum_{k=0}^{n} \binom{n}{k} A_{m+k,m}(t)(y+1)^{k}(yt_1)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} A_{n+m,m+k}(t)y^k. \tag{12}
\]

Proof: Let \( X^*_{n,m} = \bigcup_{j=0}^{n} X^*_{n,m,k} \) and \( X^*_{n,m,k} \) denote the set of pairs \((\pi, S)\) such that

- \( \pi \) is a partition of the set \([n + m + 1]\) containing at least the singleton \( m + 1 \);
- \( S \) is an \((n-k)\)-subset of \([m + 2, n + m + 1]\) which is also the set of singletons of \( \pi \) greater than \( m + 1 \), each element of \( S \) is only colored by \( y \) and each element of \([m+2,n+m+1]-S\) is colored by \( 1 \) or \( y \);
- each element of \([m+1]\) is only colored by \( 1 \).

Let \( X^*_{n,m} = \bigcup_{k=0}^{n} X^*_{n,m,k} \) and \( Y^*_{n,m,k} \) denote the set of pairs \((\pi, S)\) such that

- \( S \) is a \( k\)-subset \( \{i_1, i_2, \ldots, i_k\}\) of \([m+2,n+m+1]\) in increasing order, each element of \( S \) is only colored by \( y \) and each element of \([n+m+1]-S\) is only colored by \( 1 \);
- \( \pi \) is a partition of the set \([n + m + 1]\) such that \( i_k \) must be the largest singleton if \( S \) is not empty and \( m + 1 \) must be the largest singleton if \( S \) is empty;
- each element of \([m+2,n+m+1]-S\) must not be a singleton.

The weight of \((\pi, S)\) is defined to be the product of the weight of \( \pi \) and the colors of all elements in \([n+m+1]\). Clearly, any \((\pi, S) \in X^*_{n,m}\) can be obtained as follows. First choose an \((n-k)\)-subset \( S \) of \([m+2,n+m+1]\), there are \( \binom{n}{k} \) ways to do this. Regard each element of \( S \) as a singleton with color \( y \). Then color each element of \([m+2,n+m+1]-S\) by \( 1 \) or \( y \), namely, each element of \([m+2,n+m+1]-S\) is colored by \( y + 1 \). Now partitioning \([n+m+1]-S\) such that the largest singleton is \( m + 1 \), together with the \( n-k \) singletons formed from \( S \), we get the partition \( \pi \) of \([n+m+1]\) such that \( m + 1 \) must be a singleton; Hence the total weight of pairs \((\pi, S) \in X^*_{n,m}\) is just the left hand side of (12).

Similarly, the total weight of pairs \((\pi, S) \in Y^*_{n,m}\) is just the right hand side of (12) if regarding each element of \([m+2,n+m+1]-S\) as the role greater than \( i_k \) when \( S \neq \emptyset \).

Now we can construct a bijection \( \varphi \) between \( X^*_{n,m} \) and \( Y^*_{n,m} \) which preserves the weights. For any \((\pi, S) \in X^*_{n,m}\), let \( S_1 \) denote the set of elements of \([n+m+1]\) with colors \( y \). Clearly, \( S \) is a subset of \( S_1 \).
Assume that $S_1 = \{i_1, i_2, \ldots, i_k\}$ for some $0 \leq k \leq n$ in increasing order. If $S_1$ is the empty set $\emptyset$, which implies that $S = \emptyset$ and all elements of $[n+m+1]$ are colored by 1, it is obvious that $(\pi, \emptyset) \in \mathcal{Y}_{n,m}^*$. Then define $\varphi(\pi, \emptyset) = (\pi, \emptyset)$. If $S_1$ is not the empty set, exchanging $m+1$ and $i_k$ in $\pi$, we obtain a partition $\pi_1$, it is easily to verify that $(\pi_1, S_1) \in \mathcal{Y}_{n,m}^*$ and has the same weight as $(\pi, S)$. Then define $\varphi(\pi, S) = (\pi_1, S_1)$.

Conversely, for any $(\pi_1, S_1) \in \mathcal{Y}_{n,m}^*$, if $S_1 = \emptyset$, so $\pi_1$ has the largest singleton $m+1$, then $(\pi_1, \emptyset) \in \mathcal{X}_{n,m}^*$, and define $\varphi^{-1}(\pi_1, \emptyset) = (\pi_1, \emptyset)$. If $S_1 \neq \emptyset$, assume that $S_1 = \{i_1, i_2, \ldots, i_k\}$ for some $1 \leq k \leq n$ in increasing order, let $S$ denote the set of all the elements in $S_1$ such that each forms a singleton of $\pi_1$. Now exchanging $m+1$ and $i_k$ in $\pi_1$, we obtain a partition $\pi$, it is easy verifiable that $(\pi, S) \in \mathcal{X}_{n,m}^*$, which has the same weight as $(\pi_1, S_1)$. Then define $\varphi^{-1}(\pi_1, S_1) = (\pi, S)$.

Clearly, $\varphi$ is indeed a bijection between $\mathcal{X}_{n,m}^*$ and $\mathcal{Y}_{n,m}^*$, which proves (12).

3 The special case for permutations

When the parameter $t$ in $A_{n,k}(t)$ takes some special value, that is to assign a special structure to each block of partitions of $[n+1]$. For example, the case $t = (1^0, 2^1, 3^2, \ldots)$ indicates that each block of partitions is assigned by a (rooted and labeled) tree structure, such partitions are equivalent to labeled forests; The case $t = (1, 1^0, 0^1, \ldots)$ leads to involutions on $[n+1]$.

In this section, we just present an interesting specialization, but leave others to inclined readers. Consider the special case when $t = (0!, 1!, 2!, \ldots)$, that is to assign a cycle structure to each block of partitions, such partitions is equivalent to permutations. Let $P_{n,k} = A_{n,k}(t)$ with $t = (0!, 1!, 2!, \ldots)$, i.e., $P_{n,k}$ is the number of permutations of $[n+1]$ with the largest fixed point $k+1$. From (5) and (9), one has the explicit formulas for $P_{n,k}$:

$$P_{n+k,k} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (k+j)! = \sum_{j=0}^{k} \binom{k}{j} D_{n+j}.$$

Clearly, $P_{n,n} = n! = \mathcal{Y}_n(0!, 1!, 2!, \ldots)$ and $P_{n,0} = D_n = \mathcal{Y}_n(0, 1!, 2!, \ldots)$, where $D_n$ is the derangement number of $[n]$, i.e., the number of permutations of $[n]$ without fixed points. See Table 1 for some small values of $P_{n,k}$.

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Table 1. The values of $P_{n,k}$ for $n$ and $k$ up to 6.
In fact, \( \{P_{n,k}\}_{n \geq k \geq 0} \) forms the difference table introduced by Euler, which has been investigated in depth in derangement theory [2, 5, 6, 8, 9]. Chen [1] gave two other interpretations for \( P_{n,k} \) using \( k \)-relative derangements on \([n]\) and skew derangements from \([n]\) to \(\{-k+1, \ldots, -1, 0, 1, \ldots, n-k\}\) for \(0 \leq k \leq n\). Moreover, Chen established a bijection between these two settings. Recently, Deutsch and Elizalde [4] gave a new interpretation of derangement number \( D_{n+2} \) as the sum of the values of the largest fixed points of all non-derangements of length \( n+1 \), namely,

\[
\sum_{k=0}^{n} (k+1)P_{n,k} = D_{n+2},
\]

which is the special case of (7) when \( t = (0!, 1!, 2!, \ldots) \) and \( m = 0 \).

Next, we can explore some new relations between \( P_{n,k} \) and other classical sequences such as Bell numbers or Fibonacci numbers.

**Example 3.1** By Lemma 2.1, one can derive the bivariate exponential generating function for \( P_{n+k,k} \), i.e.,

\[
P(x, y) = \sum_{n,k \geq 0} P_{n+k,k} \frac{x^n y^k}{n! k!} = \frac{e^{-x}}{1 - x - y}.
\]

Extracting the coefficient of \( \frac{x^n}{n!} \) in \( P(x, x^2) \), we have

\[
\sum_{k=0}^{[n/2]} \binom{n}{2k} \binom{2k}{k} ! P_{n-k,k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k! F_k,
\]

where \( F_k \) is the \( k \)-th Fibonacci number defined by

\[
\frac{1}{1-x-x^2} = \sum_{k \geq 0} F_k x^k.
\]

**Example 3.2** When \( t = (0!, 1!, 2!, \ldots) \), (10) and (11) reduce to

\[
\sum_{k=0}^{n} \binom{n}{k} P_{n+m,k} y^k = \sum_{k=0}^{n} (-1)^{-k} \binom{n}{k} (m+k)! (y+1)^k, \quad (13)
\]

\[
\sum_{k=0}^{n} \binom{n}{k} P_{m+k,m} y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (m+k)! (y-1)^{n-k}. \quad (14)
\]

It should be noted that (13) and (14) have close relations to the (re-normalized) Charlier polynomials \( C_n(u, v) \) [7] defined by

\[
C_n(u, v) = \sum_{k=0}^{n} \binom{n}{k} (u)_k v^{n-k},
\]

where \( (u)_k = u(u+1) \cdots (u+k-1) \) for \( k \geq 1 \) and \( (u)_0 = 1 \). In fact, (13) is \( \frac{(y+1)^n}{m!} C_n(m+1, -\frac{1}{y+1}) \) and (14) is equal to \( \frac{1}{m!} C_n(m+1, y-1) \).
Recall that, by (4), $P_{n,k}$ can be represented umbrally as

$$P_{n,k} = P^k(1-P)^{n-k},$$

where $P = Y_t$ with $t = (0!, 1!, 2!, \ldots)$. In particular, $D_n = (P - 1)^n$ and $n! = P^n$. Hence, the case $y = P - 1$ in (13) and the case $y = P$ in (14) generate

$$\sum_{k=0}^{n} \binom{n}{k} P_{n+m,m+k} D_k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (m+k)! k!,$$

$$\sum_{k=0}^{n} \binom{n}{k} P_{m+k,m}(n-k)! = \sum_{k=0}^{n} \binom{n}{k} (m+k)! D_{n-k}.$$

With the Bell umbra $B$ given by $B = Y_t$ with $t = (1, 1, 1, \ldots)$, the Bell number can be written as $B_n = B^n$ and $B^{n+1} = (B + 1)^n$. Then the case $y = B$ in (13) and the case $y = B + 1$ in (14) generate

$$\sum_{k=0}^{n} \binom{n}{k} P_{n+m,m+k} B_k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (m+k)! B_{k+1},$$

$$\sum_{k=0}^{n} \binom{n}{k} P_{m+k,m} B_{n-k+1} = \sum_{k=0}^{n} \binom{n}{k} (m+k)! B_{n-k}.$$

Using the Riordan identity [3][11] P173,

$$\sum_{k=0}^{n} \binom{n}{k} (k+1)! (n+1)^{n-k} = (n+1)^{n+1},$$

the case in (13) with $m = 1$ and $y = -\frac{n+2}{n+1}$ and the case in (14) with $m = 1$ and $y = n+2$ generate respectively

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} P_{n+1,k+1} (n+2)^k (n+1)^{n-k} = (n+1)^{n+1},$$

$$\sum_{k=0}^{n} \binom{n}{k} (D_k + D_{k+1}) (n+2)^{n-k} = (n+1)^{n+1},$$

(15)

where we use the relation $P_{k+1,1} = D_k + D_{k+1}$. By the well-known recurrence $D_{k+2} = (k+1)(D_k + D_{k+1})$ for derangement numbers $D_k$, together with $D_1 = 0$, after routine computation, (15) is equivalent to

$$\sum_{k=0}^{n} \binom{n}{k} D_{k+1} (n+1)^{n-k} = n^{n+1},$$

(16)

which was also obtained by Riordan [10]. In a forthcoming paper [18], using functional digraph theory, we will give a combinatorial interpretation for a more general identity involving the Riordan identity and (16) as special cases.
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References


