Spin^c geometry of Kähler manifolds and the Hodge Laplacian on minimal Lagrangian submanifolds

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SPIN^c GEOMETRY OF KÄHLER MANIFOLDS AND THE HODGE LAPLACIAN ON MINIMAL LAGRANGIAN SUBMANIFOLDS

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Abstract. From the existence of parallel spinor fields on Calabi-Yau, hyper-Kähler or complex flat manifolds, we deduce the existence of harmonic differential forms of different degrees on their minimal Lagrangian submanifolds. In particular, when the submanifolds are compact, we obtain sharp estimates on their Betti numbers. When the ambient manifold is Kähler-Einstein with positive scalar curvature, and especially if it is a complex contact manifold or the complex projective space, we prove the existence of Kählerian Killing spinor fields for some particular spin^c structures. Using these fields, we construct eigenforms for the Hodge Laplacian on certain minimal Lagrangian submanifolds and give some estimates for their spectra. Applications on the Morse index of minimal Lagrangian submanifolds are obtained.

1. Introduction

Recently, connections between the spectrum of the classical Dirac operator on submanifolds of a spin Riemannian manifold and its geometry were investigated. Even when the submanifold is spin, many problems appear. In fact, it is known that the restriction of the spin bundle of a spin manifold M to a spin submanifold is a Hermitian bundle given by the tensorial product of the intrinsic spin bundle of the submanifold and certain bundle associated with the normal bundle of the immersion ([2, 3, 6]). In general, it is not easy to have a control on such a Hermitian bundle. Some results have been obtained ([2, 24, 25]) when the normal bundle of the submanifold is trivial, for instance for hypersurfaces. In this case, the spin bundle of the ambient space M

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restricted to the hypersurface gives basically the intrinsic spin bundle of the hypersurface and the induced Dirac operator is the intrinsic classical Dirac operator on the hypersurface.

Another interesting manageable case is the case where the ambient space M is a spin Kähler manifold and the spin submanifold is Lagrangian. In this case, the normal bundle of the immersion is isomorphic, through the complex structure of the ambient space, to the tangent bundle of the submanifold. Hence, it is natural to expect that the restriction of the spin bundle of the Kähler manifold should be a well-known intrinsic bundle associated with the Riemannian structure of the submanifold.

In fact, Ginoux has proved in [17] that, in some restrictive cases, this bundle is the complexified exterior bundle on the submanifold. On the other hand, Smoczyk [50] showed that non-trivial harmonic 2-forms on minimal Lagrangian submanifolds of hyper-Kähler manifolds are obtained as restrictions to the submanifold of some parallel exterior forms on the ambient space. These two works can be considered as the starting point of this paper which can be framed into a series of results where well-behaved geometric objects on submanifolds are obtained from parallel objects on the ambient manifold.

In this sense, such results should be seen as sophisticated versions of the old Takahashi theorem [52], where eigenfunctions for the Laplace operator on a minimal submanifold of a sphere are obtained from the parallel vector fields of the Euclidean space. Also, for example, Bär [2] got eigenspinors on a constant mean curvature hypersurface from parallel or Killing spinor field on the ambient space, and Savo tried recently to prove in [48] a Takahashi theorem for exterior forms, using the same idea. The paper is organized as follows:

Our goal is to find conditions on an ambient Kähler manifold and on a Lagrangian submanifold, in order to get information on the spectrum of the Hodge Laplacian acting on the bundle of differential forms

of the submanifold. For this, the main tools are to make use of the spin^c geometry of the Kähler ambient manifold and the symplectic geometry of the submanifold. In fact, to consider spin Kähler manifolds is quite restrictive, because important examples, as complex projective spaces of even complex dimension, are not spin. On the other hand, any Kähler manifold admits a spin^c structure. This allows us to work with such spin^c structures on Kähler manifolds, to look for suitable (parallel, Killing, Kählerian Killing,...) spinor fields for those structures and to investigate what do these fields induce on their Lagrangian submanifolds. In fact, they induce differential forms which are eigenforms for the Hodge Laplacian, provided that a certain condition on the symplectic geometry of the submanifold is satisfied. In this way we find an unexpected connection between spin^c geometry of the ambient Kähler manifold and some invariants of the symplectic geometry of its Lagrangian submanifolds. In order to get and clarify this relation we chose to lengthen the expository part of the paper.

There are two different successful situations: when the ambient space is a Calabi-Yau manifold (in a wide sense, that is, including hyper-Kähler and flat complex manifolds) and when it is a Kähler-Einstein manifold of positive scalar curvature.

In the first case, we consider the trivial spin structure, which carries parallel spinor fields, and the Lagrangian submanifold is assumed to be minimal. In this situation, the restriction to the submanifold of the spin bundle on the Calabi-Yau manifold is nothing but the complex exterior bundle, the restriction of the spin connection induces the Levi-Civitá connection and the induced Dirac operator is just the Euler operator on the submanifold (Proposition 20). Using these identifications, we see that the restriction to the submanifold of parallel spinor fields of the Calabi-Yau manifold are harmonic forms, which yield sharp topological restrictions on such submanifolds (Theorem 22), improving those obtained in [50].

When the ambient space is a Kähler-Einstein manifold of positive scalar curvature we have two different kind of problems: first, to choose certain spin^c structures supporting suitable spinor fields (there are no parallel spinor fields on these manifolds), and second to get conditions on the Lagrangian submanifold so that the restriction of the ambient spin^c bundle on the submanifold to be the exterior bundle and that the induced Dirac operator to be the Euler operator on the submanifold.

Using the cones on the maximal Maslov covering of a Kähler-Einstein manifold of positive scalar curvature (which are simply-connected Calabi-Yau manifolds), we find Kählerian Killing spinors for different spin^c structures on some of such manifolds: the complex projective space

and the Fano complex contact manifolds (Theorem 17). In these cases, to assume the minimality of the Lagrangian submanifold is not enough to guarantee that the restriction of some spin^c bundle is the exterior complex bundle. The reason is that there exists a very nice relation between the spin^c bundles on the Kähler-Einstein manifold which restricted to the Lagrangian submanifold are the exterior bundle and certain integer number associated with a minimal Lagrangian submanifold, studied by Oh [46] and Seidel [49], which has its origin in symplectic topology. This number will be called the *level* of the minimal Lagrangian submanifold. Since in the above references that number is only studied for *embedded* Lagrangian submanifolds, in Sections 3 and 4 we generalize their construction to immersed totally real submanifolds (Propositions 4 and 8), computing in Section 5 the level of certain minimal Lagrangian submanifolds of the complex projective space. In Theorem 24, we summarize the above results obtaining eigenforms of the Hodge Laplacian on some minimal Lagrangian submanifolds of the complex projective space and of Fano complex contact manifolds (twistor manifolds on quaternionic-Kähler manifolds).

Finally, as the Jacobi operator of a compact minimal Lagrangian submanifold of a Kähler manifold is an intrinsic operator acting on 1-forms of the submanifolds [44], in the last section we obtain some applications to study the index of such submanifolds (Corollaries 29 and 30).

2. Some bundles over almost-complex manifolds

Let M be an almost-complex manifold of dimension $2n$ and let J be its almost-complex structure. The complexified cotangent bundle decomposes into $\pm i$-eigenbundles for the complex linear extension of J

$$T^*M \otimes \mathbb{C} = T^*M^{1,0} \oplus T^*M^{0,1}$$

This decomposition induces a splitting of the space of complex valued p-forms, $p = 0, \dots, 2n$,

$$\Lambda^p(M) \otimes \mathbb{C} = \bigoplus_{r+s=p} \Lambda^{r,s}(M) \tag{2.1}$$

on M, where each subspace is determined by

$$\Lambda^{r,s}(M) = \Lambda^{r,0}(M) \otimes \Lambda^{0,s}(M)$$

and the spaces of *holomorphic* and *antiholomorphic* type forms are defined as follows

$$\Lambda^{r,0}(M) = \Lambda^r(T^*M^{1,0}) \qquad \Lambda^{0,s}(M) = \Lambda^s(T^*M^{0,1}).$$

They are vector bundles on M with complex ranks $\binom{n}{r}$ and $\binom{n}{s}$ respectively. In particular the vector bundles $\Lambda^{n,0}(M)$ and $\Lambda^{0,n}(M)$ are line bundles over M. The first one is usually called the *canonical bundle* of the manifold M and we will be denoted by

$$K_M = \Lambda^{n,0}(M). \tag{2.2}$$

We will deliberately not distinguish complex line bundles and $\mathbb{S}^1$-principal bundles over the manifold M. Their equivalence classes of isomorphisms are parametrized by the cohomology groups

$$c_1 : H^1(M, \mathbb{S}^1) \cong H^2(M, \mathbb{Z}), \tag{2.3}$$

where the isomorphism is given by mapping each line bundle or $\mathbb{S}^1$-principal bundle L onto its first Chern class $c_1(L)$. (Note that, this is not true if one considers real Chern classes, cfr. [58, page 108].) This is nothing but the boundary operator in the exact long cohomology sequence corresponding to the exact short sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{S}^1 \to 1.$$

Suppose that the first Chern class $c_1(K_M)$ of the canonical bundle of the almost-complex manifold M is a non-zero cohomology class. Let $p \in \mathbb{N}^*$ be the greatest number such that

$$\alpha = \frac{1}{p} c_1(K_M) \in H^2(M, \mathbb{Z}).$$

When M is a Kähler manifold this natural number p is called the *index* of the manifold. Instead, in symplectic geometry, this number is usually called the *minimal Chern number* of M (see [49]). We will use the first terminology in the general case. Take now an $\mathbb{S}^1$-principal $\mathcal{L} \to M$ bundle such that

$$c_1(\mathcal{L}) = \alpha = \frac{1}{p} c_1(K_M).$$

Then, the isomorphism (2.3) shows that

$$\mathcal{L}^p = \mathcal{L} \otimes \overset{p}{\cdots} \otimes \mathcal{L} = K_M, \tag{2.4}$$

that is, $\mathcal{L}$ is a p-th root of the canonical bundle K_M. Note that the space of such roots is an affine space over the group $H^1(M, \mathbb{Z}_p)$. This can be shown by considering the long exact sequence in the cohomology of M corresponding to the following short sequence

$$1 \to \mathbb{Z}_p \to \mathbb{S}^1 \xrightarrow{z \mapsto z^p} \mathbb{S}^1 \to 1.$$

Using this type of cohomology exact sequences one can see that, in general, if $\mathcal{N} \in H^1(M, \mathbb{S}^1)$ is an $\mathbb{S}^1$-bundle, its q-th tensorial power $\mathcal{N}^q$

can be identified with its quotient $\mathcal{N}^q = \mathcal{N}/\mathbb{Z}_q$, where $\mathbb{Z}_q$ is the group of the q-th roots of the unity acting on the principal bundle $\mathcal{N}$ through the $\mathbb{S}^1$-action. Then we have that $\mathcal{L}$ is a p-fold covering space of the canonical bundle K_M and so we have

$$\mathcal{L} \xrightarrow{\mathbb{Z}_p} \mathcal{L}^p = \mathcal{L}/\mathbb{Z}_p = K_M \xrightarrow{\mathbb{S}^1} M.$$

In fact, in the context of symplectic geometry, a *q-fold Maslov covering* of M (see [49, Section 2.b.]) can be defined as a q-th root of the power $(K_M)^2$. So, the bundle $\mathcal{L}$ is a $2p$-fold Maslov covering of M and, indeed it is a Maslov covering with maximal number of sheets.

Remark 1. Consider the case where M is the complex projective space $\mathbb{C}P^n$ endowed with its usual complex structure. It is well-known that

$$c_1(K_{\mathbb{C}P^n}) = (n+1)[\frac{1}{\pi}\Omega] \in H^2(\mathbb{C}P^n, \mathbb{Z}) \subset H^2(\mathbb{C}P^n, \mathbb{R}),$$

where Ω is the Kähler form

$$\Omega(u, v) = \langle Ju, v\rangle$$

with respect to the Fubini-Study metric of constant holomorphic sectional curvature 4. Since $[\frac{1}{\pi}\Omega]$ is a generator of $H^2(\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z}$, it is clear that $n+1$ is the index of $\mathbb{C}P^n$. The line bundle with first Chern class $[\frac{1}{\pi}\Omega]$ is the so called *universal bundle* over $\mathbb{C}P^n$ (see [18]), whose fiber over the point $[z]$, $z \in \mathbb{C}^{n+1} - \{0\}$, is just the line $\langle z\rangle \subset \mathbb{C}^{n+1}$. The corresponding principal $\mathbb{S}^1$-bundle $\mathcal{L}$ is nothing but the Hopf bundle $\mathbb{S}^{2n+1} \to \mathbb{C}P^n$. Then, in this case, $K_{\mathbb{C}P^n} = \mathcal{L}^{n+1} = \mathbb{S}^{2n+1}/\mathbb{Z}_{n+1}$, that is, the total space of this principal bundle is a lens space with parameters $n+1$ and 1 (see [59]). Consequently we have that $(K_{\mathbb{C}P^n})^2 = \mathcal{L}^{2n+2} = \mathbb{S}^{2n+1}/\mathbb{Z}_{2n+2}$.

Remark 2. A Kähler manifold M of dimension $2n = 4k+2$ is said to be a *complex contact manifold* when it carries a codimension 1 holomorphic subbundle $\mathcal{H}$ of $T^{1,0}M$ which is maximally non-integrable. Then the quotient line bundle $\mathcal{E} = T^{1,0}M/\mathcal{H}$ satisfies $\mathcal{E}^{-(k+1)} = K_M$ (see, for example, [34]). Hence the Chern class $c_1(K_M)$ is divisible by $k+1$ and the index p of M must be bigger than or equal to $k+1 = \frac{n+1}{2}$. Conversely, if a Kähler manifold of odd complex dimension $n = 2k+1$ has its first Chern class divisible by $k+1$, then it is a complex contact manifold and each $(k+1)$-th root of the anticanonical bundle $(K_M)^{-1}$ determines a corresponding complex contact structure. For instance, the complex projective space $\mathbb{C}P^{2k+1}$ is a complex contact manifold because the Hopf line bundle $\mathcal{L}$ satisfies $\mathcal{L}^{2k+2} = K_{\mathbb{C}P^n}$ and so $\mathcal{E} = \mathcal{L}^{-2}$ is a $(k+1)$-th root of the anticanonical bundle. But we already know that the index of $\mathbb{C}P^{2k+1}$ is $2k+2 = n+1$. This fact characterizes the

odd-dimensional complex projective space among all the complex contact manifolds. In fact, any other such manifold has index $k+1=\frac{n+1}{2}$ (see [7, Proposition 2.12]). Of course, there are a lot of non-trivial such Kähler manifolds. For example, the projectivized holomorphic cotangent bundle $\mathbb{P}(T^*N)$ of any $(k+1)$-dimensional complex manifold N, the product $\mathbb{C}P^k \times Q^{k+1}(\mathbb{C})$ of a complex projective space and a complex quadric, or the complex projective space $\mathbb{C}P^{2k+1}$ blown up along a subspace $\mathbb{C}P^{k-1}$. On the other hand, Bérard-Bergery [4] and Salamon [47] showed that the twistor space of any $4k$-dimensional quaternionic-Kähler manifold is a $(4k+2)$-dimensional complex contact manifold which admits a Kähler-Einstein metric of positive scalar curvature. Later LeBrun proved in [34, Theorem A] that any complex contact manifold admitting such a metric must be the twistor space of a quaternionic-Kähler manifold.

3. Totally real submanifolds

Let L be an n-dimensional manifold immersed into our $2n$-dimensional almost-complex manifold M. This immersed submanifold is called *totally real* when its tangent spaces have no complex lines, that is, when

$$T_xL \cap J_x(T_xL) = \{0\}, \qquad \forall x \in L. \tag{3.1}$$

The following result quotes one of the more relevant basic facts of this class of submanifolds:

Lemma 3. *Let $\iota : L \to M$ be a totally real immersion of an n-dimensional manifold L into an almost-complex $2n$-dimensional manifold M. Then the pull-back maps*

$$\iota^* : \Lambda^{r,0}(M)|_L \to \Lambda^r(L)\otimes\mathbb{C}, \qquad \iota^* : \Lambda^{0,s}(M)|_L \to \Lambda^s(L)\otimes\mathbb{C}$$

are vector bundle isomorphisms.

Proof : It suffices to check it at each point $x \in L$ and only for the map on the left side. Since the complex vector spaces $\Lambda^{r,0}(M)_x$ and $\Lambda^r(L)_x$ have the same dimension $\binom{n}{r}$, it is sufficient to prove that ι^*_x has trivial kernel. Suppose that $\omega \in \Lambda^{r,0}(M)_x$ and that

$$\omega(u_{i_1},\ldots,u_{i_r}) = 0$$

for a given basis $u_1,\ldots,u_n$ of T_xL. As ω is of $(r,0)$ type, we have

$$\omega(u_{i_1} - iJu_{i_1}, u_{i_2},\ldots,u_{i_r}) = 2\omega(u_{i_1},\ldots,u_{i_r}).$$

Then, by repeating the same argument at each entry of ω, we have

$$\omega(u_{i_1} - iJu_{i_1},\ldots,u_{i_r} - iJu_{i_r}) = 2^r\omega(u_{i_1},\ldots,u_{i_r}) = 0.$$

But condition (3.1) implies that the vectors $u_1 - iJu_1, \dots, u_n - iJu_n$ form a basis for $T^{1,0}_x(M)$ and so $\omega = 0$. Q.E.D.

As a consequence of Lemma 3, if the submanifold L is orientable, then the restricted line bundle $K_M|_L = \Lambda^{n,0}(M)|_L = \Lambda^n(L) \otimes \mathbb{C}$ is a trivial bundle and, in all cases, the same occurs for the tensorial 2-power $(K_M)^2|_L$ since the line bundle $\Lambda^n(L) \otimes \Lambda^n(L)$ is trivial on each n-dimensional manifold L. Hence, the restriction to the submanifold L of the maximal Maslov covering $\mathcal{L}$ over M verifies that $\mathcal{L}^{2p}|_L$ is trivial. So, it is clear that the immersion ι can be factorized through $\mathcal{L}^{2p}$, that is, we have a lift

$$\begin{array}{ccc} & & (K_M)^2 = \mathcal{L}^{2p} = \mathcal{L}/\mathbb{Z}_{2p} \\ & \overset{\jmath}{\nearrow} & \downarrow \\ L & \overset{\iota}{\to} & M. \end{array}$$

As ι is an immersion, then such any lift $\jmath$ is another immersion which is transverse to the fundamental vector field (see [31]) produced by the $\mathbb{S}^1$-action on $\mathcal{L}^{2p}$. Note that each lift $\jmath$ is a section of $(K_M)^2|_L$ and so, from Lemma 3, it provides a non-zero n-form, determined up to a sign, on the submanifold. Reciprocally, each non-trivial section of $(\Lambda^n(M) \otimes \mathbb{C})^2$ yields a lift $\jmath$.

Fix a lift $\jmath : L \to \mathcal{L}^{2p}$ of the totally real immersion $\iota : L \to M$. For each positive integer q such that $q|2p$, denote by $P_{q,2p} : \mathcal{L}^q \to \mathcal{L}^{2p}$ the corresponding $\frac{2p}{q}$-fold covering map. We may consider the composition

$$\pi_1(L) \xrightarrow{\jmath_*} \pi_1(\mathcal{L}^{2p}) \longrightarrow \frac{\pi_1(\mathcal{L}^{2p})}{(P_{q,2p})_*(\pi_1(\mathcal{L}^q))} \cong \mathbb{Z}_{\frac{2p}{q}}. \tag{3.2}$$

Moreover, denote by $n(L, \jmath)$ the smallest integer q such that this composition of group homomorphisms is zero. Then we have $n(L, \jmath)|2p$ and that $\jmath$ can be lifted to $\mathcal{L}^{n(L,\jmath)}$ with the property $\jmath$ cannot be lifted to $\mathcal{L}^q$ for any $q < n(L, \jmath)$. Note that, when L is orientable, then any lift $\jmath$ factorizes through $\mathcal{L}^p$ and then $n(L, \jmath)|p$.

The above facts can be summarized as:

Proposition 4. *Let $\iota : L \to M$ be a totally real immersion of an n-dimensional manifold into an almost-complex $2n$-dimensional manifold M with index $p \in \mathbb{N}^*$. Then ι may be lifted to $\mathcal{L}^{2p}$, where $\mathcal{L}$ is any maximal Maslov covering of M. For each such lift $\jmath : L \to \mathcal{L}^{2p}$, there exists a positive integer $n(L, \jmath)|2p$ such that $\jmath$ can be lifted to $\mathcal{L}^{n(L,\jmath)}$ and this is the smallest index power for which this lift is possible. That is, there is an immersion $\widetilde{\jmath} : L \to \mathcal{L}^{n(L,\jmath)}$ transverse to the fundamental*

field of the bundle such that

$$\begin{tikzcd} & \mathcal{L}^{n(L,\jmath)} \arrow[d] \\ L \arrow[ur, "\tilde{\jmath}"] \arrow[d, equal] & \mathcal{L}^{2p} \arrow[d] \\ L \arrow[ur, "\jmath"] \arrow[r, "\iota"] & M \end{tikzcd}$$

is commutative. Moreover $n(L,\jmath)|p$ if and only if L is orientable.

Remark 5. (Cf. [49, Lemmae 2.3 and 2.6]) If q is an integer with $1 < q|n(L,\jmath)$, we consider the short exact sequence

$$\begin{array}{ccccccccc} & & & & z & \longmapsto & z^q & & \\ 1 & \longrightarrow & \mathbb{Z}_q & \longrightarrow & \mathbb{S}^1 & \longrightarrow & \mathbb{S}^1 & \longrightarrow & 1 \end{array}$$

and the corresponding long exact sequence in the cohomology

$$\begin{array}{ccccccccc} & & & & \iota^*\mathcal{L}^{\frac{n(L,\jmath)}{q}} & \longmapsto & \iota^*\mathcal{L}^{n(L,\jmath)} = 1 & & \\ \cdots & \longrightarrow & H^1(L,\mathbb{Z}_q) & \longrightarrow & H^1(L,\mathbb{S}^1) & \longrightarrow & H^1(L,\mathbb{S}^1) & \longrightarrow & \cdots \end{array}$$

From it, we can deduce that there exists a cohomology class in $H^1(L,\mathbb{Z}_q)$ which is mapped onto the bundle $\mathcal{L}^{\frac{n(L,\jmath)}{q}}|_L$, that is, this bundle trivializes over a q-fold covering of L. This covering cannot be trivial, i.e., the number of its connected components is less than q. Otherwise the bundle $\mathcal{L}^{\frac{n(L,\jmath)}{q}}|_L$ itself would be trivial and this is not possible because $\frac{n(L,\jmath)}{q} < n(L,\jmath)$. Hence we have

$$1 < q, \quad q|n(L,\jmath) \Longrightarrow H^1(L,\mathbb{Z}_q) \neq 0.$$

Moreover, as $H^1(L,\mathbb{Z}_r)$ is a subgroup of $H^1(L,\mathbb{Z}_s)$, provided that $r|s$, we have that

$$1 < n(L,\jmath)|q \Longrightarrow H^1(L,\mathbb{Z}_q) \neq 0.$$

As a consequence, as $n(L,\jmath)|2p$, we have that

$$q|2p,\ H^1(L,\mathbb{Z}_q) = 0 \Longrightarrow n(L,\jmath)|\frac{2p}{q}.$$

In particular, if $H^1(L,\mathbb{Z}_{2p}) = 0$, then we must have that $n(L,\jmath) = 1$ for any lift $\jmath$ of the immersion ι. In other words, each $\jmath$ may be lifted to the maximal Maslov covering $\mathcal{L}$.

Remark 6. Since the principal $\mathbb{S}^1$-bundle $\mathcal{L}^{n(L,\jmath)}|_L$ is trivial, we have from the isomorphism (2.3) that $c_1(\mathcal{L}^{n(L,\jmath)}|_L) = 0$, that is, $n(L,\jmath)\iota^*\alpha = 0$ in the group $H^2(L,\mathbb{Z})$. As a consequence, if $\iota^*\alpha \neq 0$, the order of $\iota^*\alpha \in H^2(L,\mathbb{Z})$ divides $n(L,\jmath)$. Note that in the case $\iota^*\alpha = 0$ we have

that the immersion $\iota : L \to M$ may be lifted to the maximal Maslov covering $\mathcal{L}$, but we have not $n(L, \jmath) = 1$ necessarily for every $\jmath$ (if $H^1(L, \mathbb{Z}_{2p}) \neq 0$) as we can see in Example 3 of Section 5. This same example shows that $n(L, \jmath)$ depends on the lift $\jmath$ fixed and not only on the totally real immersion ι.

4. Lagrangian submanifolds

Suppose now that the $2n$-dimensional almost-complex manifold M is endowed with a Riemannian metric $\langle\ ,\ \rangle$. Then the totally real submanifold L is another Riemannian manifold with the metric induced from M through the immersion $\iota : L \to M$. If $x \in L$, we have an n-form ω_x defined, up to a sign, on T_xL by means of the condition $\omega_x(e_1, \ldots, e_n) = 1$ for each orthonormal basis $\{e_1, \ldots, e_n\}$. Then $\omega \otimes \omega$ is a global nowhere vanishing section of the line bundle $\Lambda^n(L) \otimes \Lambda^n(L)$ on L. When L is orientable, ω is nothing but the volume form of its metric. Hence, using Lemma 3, we obtain a nowhere vanishing section of the restricted line bundle $(\Lambda^{n,0}(M))^2|_L$. This means that, in this case, we have a privileged lift $\jmath_0 : L \to \mathcal{L}^{2p} = (K_M)^2$ for the totally real immersion ι.

> *We will write n_L for the integer $n(L, \jmath_0)$ given in Proposition 4 where $\jmath_0$ is the squared volume form of the metric induced on L and we will call it the* level *of the totally real submanifold L.*

It is clear from (3.2) that this level is invariant for deformations of the immersion ι through totally real immersions, that is, it is an invariant of the (totally real) isotopy class of ι.

Remark 7. One can see that, according to the above definition, the level n_L coincides with the quotient $\frac{2p}{N_L}$, where N_L is the integer defined by Seidel in his work on *grading* Lagrangian submanifolds of symplectic manifolds (see [49]). We point out that in [49], this integer is called the *minimal Chern number* of the pair (M, L) while it is called *minimal Maslov number* in [9]. In fact, the integer N_L is the positive generator of the subgroup $\mu(H_1(L, \mathbb{Z})) \subset \mathbb{Z}$, where $\mu \in H^1(L, \mathbb{Z})$ is the Maslov class of L (see for example [11]), if the subgroup is non-zero, and $N_L = 2p$ if it vanishes. Note that these authors deal with embedded submanifolds and, consequently, they employ cohomology of pairs, while in this paper we consider immersed submanifolds.

From now on, we assume that the almost-complex ambient manifold is Kähler. This means that the almost-complex structure J is an

isometry for the metric $\langle\ ,\ \rangle$ and that its Levi-Cività connection ∇ parallelizes J. This ∇ uniquely extends to each exterior bundle $\Lambda^p(M)\otimes\mathbb{C}$ and, in the Kähler case, since J is parallel, the decomposition (2.1) is invariant under this extension, that is, each $\Lambda^{r,s}(M)\otimes\mathbb{C}$ is endowed with an induced connection, also denoted by ∇. In particular, we can dispose of a Levi-Cività connection on the canonical bundle K_M defined in (2.2). This connection may be induced in a natural way on the maximal Maslov covering $\mathcal{L}$ which is, according to (2.4) a p-fold covering of K_M and so on each tensorial power $\mathcal{L}^q$. Then it is natural to inquire under which conditions the lift $\jmath_0$ to $\mathcal{L}^{2p}$ of the immersion ι is horizontal with respect to this Levi-Cività connection. Note that, if such a $\jmath_0$ is horizontal, then any higher lift until the corresponding $\mathcal{L}^{n_L}$ will be horizontal too. We will give an answer when the metric of M is Einstein. This answer will point out the importance of a fashionable family of totally real submanifolds.

Proposition 8. *Let $\iota : L \to M$ be a totally real immersion of an n-dimensional manifold L into a $2n$-dimensional non scalar-flat Kähler-Einstein manifold M of index $p \in \mathbb{N}^*$ and $\jmath_0 : L \to \mathcal{L}^{2p} = (K_M)^2$ be the lift determined by the squared volume form of the induced metric. Then $\jmath_0$ (or any other higher lift of $\jmath_0$) is horizontal with respect to the Levi-Cività connection of M if and only if ι is a minimal Lagrangian immersion.*

Proof : Our assertion is an improvement of Proposition 2.2 in [46] (see also [8]) and the proof follows from the corresponding one there. In fact, if $\jmath_0$ is horizontal, one has a global nowhere vanishing section for the restricted line bundle $(K_M)^2|_L$ on the submanifold L. So the Levi-Cività connection on the restriction of the canonical bundle K_M must be flat. But it is well known that the curvature form of this connection is projected onto the Ricci form of the Kähler metric of M. Since this metric is Einstein and non scalar-flat, we have that $\iota^*\Omega^M = 0$, where Ω^M stands for the Kähler two-form on M. Hence, the immersion is Lagrangian. But now, from Proposition 2.2 of [46] (take into account that Oh uses a different convention that ours for the mean curvature and that we work on the *squared* K_M), we have in this case

$$\nabla_u \jmath_0 = 2n\sqrt{-1}\,\langle JH, u\rangle\, \jmath_0 \qquad \forall u \in TL, \tag{4.1}$$

where H is the mean curvature vector field of the immersion ι. Then $\jmath_0$ is parallel if and only if H is identically zero, that is, ι is minimal. Q.E.D.

Remark 9. From Proposition 8 above, one deduces that if the totally real submanifold L is Lagrangian and minimal, then the restricted line

bundle $(K_M)^2|_L$ is geometrically trivial, that is, it is flat and has trivial holonomy. The minimality of the submanifold L is not necessary in order to geometrically trivialize $(K_M)^2|_L$, because perhaps one could find another parallel section different from $\jmath_0$. Using (4.1), one sees that a suitable modification of $\jmath_0$ through a change of gauge $f : L \to \mathbb{S}^1$ is parallel if and only if $id \log f$ coincides with the 1-form given by

$$u \in TL \longmapsto 2n\langle JH, u\rangle.$$

This means that the squared line bundle $(K_M)^2$ is geometrically trivial on the submanifold L if and only if the so called *mean curvature form*

$$\eta(u) = \frac{n}{\pi}\langle JH, u\rangle \qquad \forall u \in TL,$$

which is a closed form (see [12]), has integer periods, that is, the cohomology class $[\eta] \in H^1(L, \mathbb{R})$ belongs to the image of the subgroup $H^1(L, \mathbb{Z})$. With the same effort, one can check in fact that $\mathcal{L}^q|_L$ is geometrically trivial if and only $n_L|q|2p$ and $\frac{q}{2p}[\eta] \in H^1(L, \mathbb{Z})$. In particular, this happens when L is minimal or when the mean curvature form is exact (see [44]). This result is also true when the ambient manifold is Ricci-flat, provided that the submanifold is supposed to be Lagrangian and not only totally real. In fact, in Ricci-flat ambient manifolds, this mean curvature class coincides with the Maslov class (see [35, 11] and notice that there the mean curvature is not normalized) which is defined as an integer class.

Remark 10. As another consequence of this Proposition 8,

> *The level n_L of a minimal Lagrangian immersion $\iota : L \to M$ into a non scalar-flat Kähler-Einstein manifold of index p, defined at the beginning of this Section 4, is nothing but the smallest number $q|2p$ such that ι admits a horizontal (also called* Legendrian*) lift to the Maslov covering $\mathcal{L}^q$.*

In fact, such a horizontal lift followed by the covering map $P_{q,2p} : \mathcal{L}^q \to \mathcal{L}^{2p} = (K_M)^2$ is also a horizontal map and so differs from $\jmath_0$ only by a constant change of gauge. Taking into account that, according to [11], the minimal Maslov number is computable in terms of symplectic energy for minimal Lagrangian submanifolds, our level n_L coincides also with the integer defined by Oh in [46], although his definition, made in terms of cohomology of pairs, was valid only for embeddings. In particular, the line bundles $\mathcal{L}^{qn_L}|_L$ with $q = 1, \dots, \frac{2p}{n_L}$ are not only trivial bundles but trivial as flat bundles too. In other words, they are flat bundles with trivial holonomy.

5. Some examples in $\mathbb{C}P^n$

Now we are going to examine in some detail the most relevant examples of minimal Lagrangian submanifolds of the complex projective space, emphasizing on the computation of their respective levels. In fact, all the examples considered below are compact and have parallel second fundamental form. Examples 3 and 4 are Riemannian products. The other ones are the only compact irreducible examples of minimal Lagrangian submanifolds in $\mathbb{C}P^n$ with parallel second fundamental form (see [42, 43]).

1. Consider the totally geodesic embedding $\mathbb{S}^n \hookrightarrow \mathbb{S}^{2n+1}$ of the n-dimensional unit sphere into the $(2n+1)$-dimensional unit sphere induced from the standard inclusion

$$(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mapsto (x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \equiv \mathbb{R}^{2n+2}.$$

This map induces a well-known 2:1 immersion ι from the sphere $\mathbb{S}^n$ into the complex projective space $\mathbb{C}P^n$ given by

$$\iota : (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \mapsto [x_1, \ldots, x_{n+1}] \in \mathbb{C}P^n$$

which is a Lagrangian with respect to the usual symplectic structure constructed from the complex structure and the Fubini-Study metric (of holomorphic sectional curvature 4). Moreover the metric induced by ι on the sphere $\mathbb{S}^n$ is the standard round metric of curvature 1. It is clear from the definition that ι factorizes in the following way

$$(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \overset{\jmath}{\mapsto} (x_1, \ldots, x_{n+1}) \in \mathbb{S}^{2n+1} \overset{\pi}{\mapsto} [x_1, \ldots, x_{n+1}] \in \mathbb{C}P^n.$$

The Hopf projection π is a submersion with respect to the metrics involved on $\mathbb{S}^{2n+1}$ and $\mathbb{C}P^n$ (see [31]) and the Levi-Cività connection on the sphere $\mathbb{S}^{2n+1}$, viewed as a Maslov $(2n+2)$-covering of the squared canonical bundle $K_{\mathbb{C}P^n}$, coincides with the Levi-Cività connection of the standard unit metric. Then, as the immersion $\jmath$ is also isometric, we have that $\jmath$ is a horizontal lift of ι. As a conclusion

$$n_{\mathbb{S}^n} = 1.$$

2. The immersion $\iota : \mathbb{S}^n \to \mathbb{C}P^n$ above factorizes through an immersion

$$\iota_2 : \mathbb{S}^n/\mathbb{Z}_2 = \mathbb{R}P^n \longrightarrow \mathbb{S}^{2n+1}/\mathbb{S}^1 = \mathbb{C}P^n$$

given by

$$\iota_2 : \{x_1, \ldots, x_{n+1}\} \in \mathbb{R}P^n \longmapsto [x_1, \ldots, x_{n+1}] \in \mathbb{C}P^n.$$

It is easy to see that ι_2 is a totally geodesic Lagrangian embedding and that $\jmath_2 : \mathbb{R}P^n \to \mathbb{R}P^{2n+1}$ defined by

$$\jmath_2 : \{x_1, \ldots, x_{n+1}\} \in \mathbb{R}P^n \longmapsto \{x_1, \ldots, x_{n+1}\} \in \mathbb{R}P^{2n+1}$$

is an isometric and hence horizontal lift of ι_2. As a consequence, we have that $n_{\mathbb{R}P^n}|2$. But, on the other hand, we have that $c_1(\mathcal{L}|_{\mathbb{S}^n/\mathbb{Z}_2}) = (\iota_2)^*\alpha$ and that

$$(\iota_2)^* : H^2(\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z} \longrightarrow H^2(\mathbb{R}P^n, \mathbb{Z}) \cong \mathbb{Z}_2$$

is surjective. Then the line bundle $\mathcal{L}|_{\mathbb{R}P^n}$ is topologically non-trivial and hence the level of ι_2 cannot be 1. As a conclusion

$$n_{\mathbb{R}P^n} = 2.$$

3. Consider the $(n+1)$-dimensional flat torus

$$\mathbb{T}^{n+1} = \mathbb{S}^1\left(\frac{1}{\sqrt{n+1}}\right) \times \overset{n+1}{\cdots} \times \mathbb{S}^1\left(\frac{1}{\sqrt{n+1}}\right) \subset \mathbb{S}^{2n+1}.$$

The action of the circle $\mathbb{S}^1$ on the sphere $\mathbb{S}^{2n+1}$ fixes $\mathbb{T}^{n+1}$ and induces an embedding ι from the corresponding quotient $\mathbb{T}^{n+1}/\mathbb{S}^1$, which is isometric to a certain flat torus $\mathbb{T}^n$, into the complex projective space $\mathbb{C}P^n$ such that

$$\begin{array}{ccc} \mathbb{T}^{n+1} & \hookrightarrow & \mathbb{S}^{2n+1} \\ \downarrow & & \downarrow \\ \mathbb{T}^n & \xrightarrow{\iota} & \mathbb{C}P^n \end{array}$$

is commutative. This is the well-known embedding of the Lagrangian Clifford torus, which is a minimal embedding.

We define a map $\jmath : \mathbb{T}^n \to \mathbb{S}^{2n+1}/\mathbb{Z}_{n+1}$ by

$$\{z_1, \ldots, z_{n+1}\} \in \frac{\mathbb{T}^{n+1}}{\mathbb{S}^1} \overset{\jmath}{\mapsto} \left\{\frac{1}{\sqrt[n+1]{z_1 \cdots z_{n+1}}}(z_1, \ldots, z_{n+1})\right\} \in \frac{\mathbb{S}^{2n+1}}{\mathbb{Z}_{n+1}}.$$

One can check that this $\jmath$ is a horizontal lift of the embedding ι and so it coincides with the horizontal lift $\jmath_0$ determined by the volume form of $\mathbb{T}^n$ up to product by an $(n+1)$-th root of the unity. On the other hand, it is not difficult to see that the induced group homomorphism

$$\jmath_* : \pi_1(\mathbb{T}^n) \cong \mathbb{Z} \oplus \overset{n}{\cdots} \oplus \mathbb{Z} \longrightarrow \pi_1\left(\mathbb{S}^{2n+1}/\mathbb{Z}_{n+1}\right) \cong \mathbb{Z}_{n+1}$$

is surjective. Then, from definition (3.2), $n(\mathbb{T}^n, \jmath_0) = n(\mathbb{T}^n, \jmath) = n+1$. Hence

$$n_{\mathbb{T}^n} = n+1.$$

It is important to notice that, in this case, the cohomology homomorphism

$$\iota^* : H^2(\mathbb{C}P^n, \mathbb{Z}) \longrightarrow H^2(\mathbb{T}^n, \mathbb{Z})$$

is trivial, and so the restriction of any line bundle $\mathcal{L}^q = \mathbb{S}^{2n+1}/\mathbb{Z}_q$ to the torus $\mathbb{T}^n$ is trivial too. That is, the embedding ι may be lifted to the sphere $\mathbb{S}^{2n+1}$ but not horizontally (see Remark 6).

4. For $n \geq 3$, consider the product $\mathbb{S}^1 \times \mathbb{S}^{n-1} \subset \mathbb{C} \times \mathbb{R}^n \subset \mathbb{C} \times \mathbb{C}^n$. The map determined by

$$(z, x) \in \mathbb{S}^1 \times \mathbb{S}^{n-1} \longmapsto \left[\frac{1}{\sqrt{n+1}}\left(\sqrt{n}\frac{x}{\sqrt[n+1]{z}}, \sqrt[n+1]{z^n}\right)\right] \in \mathbb{S}^{2n+1}/\mathbb{S}^1 = \mathbb{C}P^n$$

gives a minimal Lagrangian immersion $\iota : \mathbb{S}^1 \times \mathbb{S}^{n-1} \to \mathbb{C}P^n$ belonging to a family of highly symmetric minimal Lagrangian immersions studied in [10]. But, from this same definition, we see that ι is the projection of a horizontal immersion $\jmath : \mathbb{S}^1 \times \mathbb{S}^{n-1} \to \mathbb{S}^{2n+1}/\mathbb{Z}_{n+1}$ defined changing the squared brackets denoting homogeneous coordinates by the curly brackets representing the equivalence classes in the quotient of the sphere $\mathbb{S}^{2n+1}$. In a way similar to Example 3 above, one easily sees that the induced homomorphism

$$\jmath_* : \pi_1(\mathbb{S}^1 \times \mathbb{S}^{n-1}) \cong \mathbb{Z} \longrightarrow \pi_1\left(\mathbb{S}^{2n+1}/\mathbb{Z}_{n+1}\right) \cong \mathbb{Z}_{n+1}$$

is surjective. Consequently we have

$$n_{\mathbb{S}^1 \times \mathbb{S}^{n-1}} = n + 1.$$

The immersion ι above is not an embedding. In fact, we observe that $\iota(z, x) = \iota(z', x')$ if and only if $z' = -z$ and $x' = -x$. So, the immersion ι yields an embedding

$$\tilde{\iota} : (\mathbb{S}^1 \times \mathbb{S}^{n-1})/\mathbb{Z}_2 \longrightarrow \mathbb{C}P^n$$

where $\mathbb{Z}_2$ denotes the group spanned by the diffeomorphism $(z, x) \in \mathbb{S}^1 \times \mathbb{S}^{n-1} \mapsto (-z, -x) \in \mathbb{S}^1 \times \mathbb{S}^{n-1}$. It is clear that $\tilde{\iota}$ is a minimal Lagrangian embedding which is a very well-known example already studied by Naitoh in [42, Lemma 6.2]. When n is even, the horizontal lift $\jmath$ of ι may be induced on the quotient $(\mathbb{S}^1 \times \mathbb{S}^{n-1})/\mathbb{Z}_2$, but when n is odd this is impossible. In this case, the same expression defines an alternative lift

$$\tilde{\jmath} : \{(z, x)\} \in \frac{\mathbb{S}^1 \times \mathbb{S}^{n-1}}{\mathbb{Z}_2} \mapsto \left\{\frac{1}{\sqrt{n+1}}\left(\sqrt{n}\frac{x}{\sqrt[n+1]{z}}, \sqrt[n+1]{z^n}\right)\right\} \in \frac{\mathbb{S}^{2n+1}}{\mathbb{Z}_{2n+2}}.$$

This is also a horizontal lift inducing a surjective homomorphism between the corresponding fundamental groups. Hence

$$n_{(\mathbb{S}^1 \times \mathbb{S}^{n-1})/\mathbb{Z}_2} = \begin{cases} n+1 & \text{if } n \text{ is even} \\ 2n+2 & \text{if } n \text{ is odd.} \end{cases}$$

Of course, in this last case, the quotient $(\mathbb{S}^1 \times \mathbb{S}^{n-1})/\mathbb{Z}_2$ is not orientable.

5. The special unitary group $\mathrm{SU}(m)$ can be immersed into the unit sphere of the vector space $\mathfrak{gl}(m, \mathbb{C})$ of complex m-matrices as follows

$$\jmath : A \in \mathrm{SU}(m) \longmapsto \frac{1}{\sqrt{m}} A \in \mathbb{S}^{2m^2-1} \subset \mathfrak{gl}(m, \mathbb{C}).$$

Naitoh saw in [43] that this $\jmath$ is Legendrian and so it induces a minimal Lagrangian immersion $\iota = \pi \circ \jmath : \mathrm{SU}(m) \longrightarrow \mathbb{C}P^{m^2-1}$ whose level is *a fortiori*

$$n_{\mathrm{SU}(m)} = 1.$$

This immersion ι is not an embedding because you can see that, if $A, B \in \mathrm{SU}(m)$, $\iota(A) = \iota(B)$ if and only if $B = zA$ and $z \in \mathbb{Z}_m \subset \mathbb{S}^1$. Then one can consider an embedding

$$\tilde{\iota} : \{A\} \in \mathrm{SU}(m)/\mathbb{Z}_m \longmapsto \left[\frac{1}{\sqrt{m}} A\right] \in \mathbb{C}P^{m^2-1}$$

which is also minimal and Lagrangian. But it is obvious that

$$\tilde{\jmath} : \{A\} \in \mathrm{SU}(m)/\mathbb{Z}_m \longmapsto \left\{\frac{1}{\sqrt{m}} A\right\} \in \mathbb{S}^{2m^2-1}/\mathbb{Z}_m$$

is a horizontal lift of that embedding inducing an isomorphism

$$\tilde{\jmath}_* : \pi_1\left(\mathrm{SU}(m)/\mathbb{Z}_m\right) \cong \mathbb{Z}_m \longrightarrow \pi_1\left(\mathbb{S}^{2m^2-1}/\mathbb{Z}_m\right) \cong \mathbb{Z}_m$$

between their fundamental groups. This shows that

$$n_{\mathrm{SU}(m)/\mathbb{Z}_m} = m.$$

6. The symmetric space $\mathrm{SU}(m)/\mathrm{SO}(m)$ can be embedded into the unit sphere of the vector space $\mathrm{Sym}(m, \mathbb{C})$ of symmetric complex matrices in the following minimal Legendrian way

$$\jmath : A \cdot \mathrm{SO}(m) \in \frac{\mathrm{SU}(m)}{\mathrm{SO}(m)} \longmapsto \frac{1}{\sqrt{m}} AA^t \in \mathbb{S}^{m^2+m-1} \subset \mathrm{Sym}(m, \mathbb{C})$$

described in [43]. So this $\jmath$ yields a minimal Lagrangian immersion in the corresponding complex projective space $\iota = \pi \circ \jmath : \mathrm{SU}(m)/\mathrm{SO}(m) \to \mathbb{C}P^{\frac{1}{2}(m^2+m-2)}$. Henceforth

$$n_{\mathrm{SU}(m)/\mathrm{SO}(m)} = 1.$$

Like in the case above ι itself is not an injective map, but it induces an embedding

$$\tilde{\iota} : \{A \cdot \mathrm{SO}(m)\} \in \frac{\mathrm{SU}(m)}{\mathrm{SO}(m) \cdot \mathbb{Z}_m} \longmapsto \left[\frac{1}{\sqrt{m}} AA^t\right] \in \mathbb{C}P^{\frac{1}{2}(m^2+m-2)}$$

which can be horizontally lifted as follows

$$\tilde{\jmath} : \{A \cdot \mathrm{SO}(m)\} \in \frac{\mathrm{SU}(m)}{\mathrm{SO}(m) \cdot \mathbb{Z}_m} \longmapsto \left\{\frac{1}{\sqrt{m}} AA^t\right\} \in \frac{\mathbb{S}^{m^2+m-1}}{\mathbb{Z}_m}.$$

The map between the corresponding fundamental groups induced by this lift is nothing but $\mathbb{Z}_m \xrightarrow{2\cdot} \mathbb{Z}_m$. So its image depends on the parity of m and we have

$$n_{\mathrm{SU}(m)/\mathrm{SO}(m)\cdot\mathbb{Z}_m} = \begin{cases} m \text{ if } m \text{ is odd} \\ \frac{m}{2} \text{ if } m \text{ is even.} \end{cases}$$

7. Let $\mathfrak{so}(2m,\mathbb{C})$ be the Lie algebra of skew-symmetric complex $2m$-matrices and consider the minimal Legendrian embedding of the symmetric space $\mathrm{SU}(2m)/\mathrm{Sp}(m)$ into its unit sphere given by

$$\jmath : A\cdot\mathrm{Sp}(m) \in \mathrm{SU}(2m)/\mathrm{Sp}(m) \longmapsto \frac{1}{\sqrt{2m}} AJA^t \in \mathbb{S}^{4m^2-1} \subset \mathfrak{so}(2m,\mathbb{C}),$$

where we have put

$$J = \begin{pmatrix} 0_m & -I_m \\ I_m & 0_m \end{pmatrix}.$$

Projecting to the complex projective space we obtain a minimal Lagrangian immersion

$$\iota = \pi\circ\jmath : \mathrm{SU}(2m)/\mathrm{Sp}(m) \longrightarrow \mathbb{C}P^{2m^2-1}$$

whose horizontal lift is the $\jmath$ above. Hence

$$n_{\mathrm{SU}(2m)/\mathrm{Sp}(m)} = 1.$$

Studying again the equivalence relation associated with the map ι we can construct a minimal Lagrangian embedding by dividing by a suitable discrete subgroup, namely

$$\tilde\iota : \{A\cdot\mathrm{Sp}(m)\} \in \frac{\mathrm{SU}(2m)}{\mathrm{Sp}(m)\cdot\mathbb{Z}_{2m}} \longmapsto \left[\frac{1}{\sqrt{2m}}AJA^t\right] \in \mathbb{C}P^{2m^2-1}.$$

We can see immediately that this embedding is the projection of the following horizontal map

$$\tilde\jmath : \{A\cdot\mathrm{Sp}(m)\} \in \frac{\mathrm{SU}(2m)}{\mathrm{Sp}(m)\cdot\mathbb{Z}_{2m}} \longmapsto \left\{\frac{1}{\sqrt{2m}}AJA^t\right\} \in \frac{\mathbb{S}^{4m^2-1}}{\mathbb{Z}_{2m}}.$$

As a consequence, by studying the induced map $\tilde\jmath_*$ in the homotopy, we conclude (see [57]) that

$$n_{\mathrm{SU}(2m)/\mathrm{Sp}(m)\cdot\mathbb{Z}_{2m}} = \begin{cases} 2m \text{ if } m \text{ is odd} \\ m \text{ if } m \text{ is even.} \end{cases}$$

8. We are going to compute the level for the last example of minimal Lagrangian immersions in the complex projective space with parallel second fundamental form, studied by Naitoh in [43]. Let $\mathbf{C}$ be the Cayley algebra over $\mathbb{R}$ endowed with the canonical conjugation and let

$$H(3,\mathbf{C}) = \{A \in \mathfrak{gl}(3,\mathbf{C}) \mid \overline{A}^t = A\}$$

which is a 27-dimensional real vector space. One can define on this matrix space a $\mathbf{C}$-valued determinant det and a $\mathbf{C}$-inner product $((\ ,\ ))$ (see [43] for details). Consider now the complexification $V = H(3, \mathbf{C}) \otimes \mathbb{C}$ and extend det and $((\ ,\ ))$ $\mathbb{C}$-linearly on V. Then, if τ denotes the complex conjugation, $(\ ,\ ) = ((\tau\ ,\ ))$ is a positive definite Hermitian product on V. In this setting, the Lie group E_6 can be viewed as the group of $\mathbf{C}$-linear $(\ ,\ )$-isometries of V preserving det and the Lie subgroup $F_4 \subset E_6$ as the isotropy subgroup of the identity matrix $I \in V$. Naitoh considered the following minimal Legendrian embedding

$$\jmath : g \cdot F_4 \in E_6/F_4 \longmapsto \frac{1}{\sqrt{3}} g(I) \in \mathbb{S}^{53} \subset V$$

which induces a minimal Lagrangian immersion $\iota = \pi \circ \jmath : E_6/F_4 \to \mathbb{C}P^{26}$ with

$$n_{E_6/F_4} = 1.$$

It is clear that the group of cubic roots of the unity $\mathbb{Z}_3 \subset \mathbb{S}^1$ acts on E_6 as follows

$$(zg)(A) = zg(A) \qquad \forall z \in \mathbb{Z}_3,\ g \in E_6,\ A \in V$$

and that $\iota(g \cdot F_4) = \iota(h \cdot F_4)$ if and only if $h = zg$ for some $z \in \mathbb{Z}_3$. Then we get a minimal Lagrangian embedding $\tilde{\iota} : E_6/F_4 \cdot \mathbb{Z}_3 \longrightarrow \mathbb{C}P^{26}$ that we can horizontally lift to a map

$$\tilde{\jmath} : \{g \cdot F_4\} \in E_6/F_4 \cdot \mathbb{Z}_3 \longmapsto \left\{ \frac{1}{\sqrt{3}} g(I) \right\} \in \mathbb{S}^{53}/\mathbb{Z}_3.$$

As a conclusion

$$n_{E_6/F_4 \cdot \mathbb{Z}_3} = 3.$$

6. Structures on the Maslov coverings and their cones

Suppose now that the Kähler-Einstein $2n$-dimensional manifold M of index $p \in \mathbb{N}^*$ is compact and has *positive* scalar curvature. Up to a change of scale, we will consider that its value is exactly $4n(n+1)$, just the value taken on the complex projective space $\mathbb{C}P^n$ with constant holomorphic sectional curvature 4. A result by Kobayashi (see [30]) implies that, in this case, M is simply-connected. Then, a suitable use of the Thom-Gysin and the homotopy sequences for the principal circle bundle $\mathcal{L} \to M$ established in [3] that the maximal Maslov covering $\mathcal{L}$ of the manifold M is also simply-connected. Moreover, we already pointed out that the Levi-Cività connection on M yields a corresponding $\mathbb{S}^1$-connection on the principal $\mathbb{S}^1$-bundle $\mathcal{L} \to M$, given by a 1-form

$i\theta \in \Gamma(\Lambda^1(\mathcal{L}) \otimes i\mathbb{R})$. With the help of the 1-form θ, one defines a unique Riemannian metric on $\mathcal{L}$ by

$$\langle\ ,\ \rangle_{\mathcal{L}} = \pi^*\langle\ ,\ \rangle + \frac{p^2}{(n+1)^2}\theta \otimes \theta \tag{6.1}$$

such that the projection $\pi : \mathcal{L} \to M$ is an oriented submersion and that the fundamental vector field V defined by the free $\mathbb{S}^1$-action on $\mathcal{L}$ has constant length $p/(n+1)$, that is, the fibers are totally geodesic circles of length $2p\pi/(n+1)$. Thus the tangent spaces of the Kähler manifold M can be seen as subspaces of the tangent spaces of $\mathcal{L}$, as follows

$$T_z\mathcal{L} = T_{\pi(z)}M \oplus \langle V_z\rangle, \qquad \forall z \in \mathcal{L}. \tag{6.2}$$

This metric was originally considered by Hatekeyama [20]. Indeed, the bundle $\mathcal{L}$ is nothing but the Boothby-Wang fibration on M, which is under our hypotheses, a Hodge manifold. The main properties of the Riemannian manifold $\mathcal{L}$ and the submersion π will be listed in the following theorem of [3] (see also [6, Proposition 2.39 and Lemma 9.10]).

Proposition 11. [3, Example 1, page 84] *Let $\mathcal{L}$ be the maximal Maslov covering of a Kähler-Einstein $2n$-dimensional compact manifold with scalar curvature $4n(n+1)$ and index $p \in \mathbb{N}^*$. Then $\mathcal{L}$ is an orientable simply-connected compact $(2n+1)$-manifold and there exist a unique orientation and a unique Riemannian metric on $\mathcal{L}$ such that the projection $\mathcal{L} \to M$ is an oriented Riemannian submersion and the fibers are totally geodesic circles of length $2p\pi/(n+1)$. This metric is Einstein with scalar curvature $2n(2n+1)$ and supports a Sasakian structure on $\mathcal{L}$ whose characteristic field is the normalized fundamental vector field of the $\mathbb{S}^1$-fibration.*

Let U be the normalized fundamental field of the fibration, that is, the unit oriented field in the direction of the fundamental field V on $\mathcal{L}$. The Sasakian structure is given by the fact that the vector field U is a Killing field for the metric (6.1) and the sectional curvature of any section containing U is one. For other better known definitions and more details on the notion of a Sasakian structure on a Riemannian manifold one can consult [5] and [7]. Of course, the simplest example of a Sasakian manifold is provided by the standard odd-dimensional sphere $\mathbb{S}^{2n+1}$, which is obtained, via the Proposition 11 above, from $\mathbb{C}P^n$ endowed with the Fubini-Study metric of holomorphic sectional curvature 4. Probably, the most geometric definition of a Sasakian structure (and also the most useful for our purposes) is the following:

Let $\mathcal{C}$ be the cone $\mathbb{R}^+ \times \mathcal{L}$ over the bundle $\mathcal{L}$ endowed with the metric

$$\langle\ ,\ \rangle_{\mathcal{C}} = dr^2 + r^2 \langle\ ,\ \rangle_{\mathcal{L}}. \tag{6.3}$$

Then $\mathcal{L}$ is isometrically embedded in $\mathcal{C}$ as the hypersurface $r = 1$ and we have

$$T_{rz}\mathcal{C} = T_z\mathcal{L} \oplus \langle \partial_r \rangle, \qquad \forall r \in \mathbb{R}^+,\ z \in \mathcal{L}. \tag{6.4}$$

Using (6.2) and (6.4), we have that

$$T_{rz}\mathcal{C} = T_{\pi(z)}M \oplus \langle \{U_z, \partial_r\} \rangle.$$

This orthogonal decomposition allows us to define an almost complex structure J_{rz} on $T_{rz}\mathcal{C}$ by

$$J_{rz}|_{T_{\pi(z)}M} = J_{\pi(z)}, \qquad J_{rz}\,(\partial_r) = U_z.$$

The fact that $\mathcal{L}$ with the metric (6.1) is Sasakian, is equivalent to the fact that the cone $\mathcal{C}$, with the metric (6.3) and the almost structure above, is an $(2n+2)$-dimensional Kähler manifold (see [1, Lemma 4] or [7, Definition-Proposition 2]). Moreover, the fact that $\mathcal{L}$ is Einstein is equivalent to $\mathcal{C}$ be Ricci-flat. Since $\mathcal{L}$, and so $\mathcal{C}$, is simply-connected, we conclude that the holonomy group of the cone $\mathcal{C}$ is contained in $\mathrm{SU}(n+1)$, that is, $\mathcal{C}$ is a Calabi-Yau manifold. We may summarize all these assertions in the following:

Proposition 12. [1, 6, 7] *Let $\mathcal{L}$ be the maximal Maslov covering of a Kähler-Einstein $2n$-dimensional compact manifold with scalar curvature $4n(n+1)$ and $\mathcal{C}$ the corresponding cone. Then $\mathcal{C}$ is a simply-connected Calabi-Yau $(2n+2)$-dimensional manifold.*

7. Spin^c structures

We have seen that the cone $\mathcal{C}$ over the maximal Maslov covering $\mathcal{L}$ of M is a Calabi-Yau $(2n+2)$-dimensional simply-connected manifold. Hence $\mathcal{C}$ is a spin manifold and it has, up to an isomorphism, a unique spin structure (see any of [3, 6, 14, 32] for generalities about spin structures). We will denote by $\Sigma\mathcal{C}$ the corresponding complex spinor bundle. This is a Hermitian vector bundle with rank 2^{n+1} endowed with a Levi-Cività spin connection $\nabla^{\mathcal{C}}$ and a Clifford multiplication $\gamma^{\mathcal{C}} : T\mathcal{C} \to \mathrm{End}\,(\Sigma\mathcal{C})$. It is well-known that, if $u_1, \dots, u_{2n+2}$ is an arbitrary positively oriented orthonormal basis in $T\mathcal{C}$, then

$$\omega^{\mathcal{C}} = i^{n+1}\gamma^{\mathcal{C}}(u_1)\cdots\gamma^{\mathcal{C}}(u_{2n+2})$$

does not depend on the chosen basis and it is called the spin volume form on $\mathcal{C}$. Hence, we can put

$$\omega^{\mathcal{C}}_{rz} = i^{n+1}\gamma^{\mathcal{C}}(e_1)\gamma^{\mathcal{C}}(Je_1)\cdots\gamma^{\mathcal{C}}(e_n)\gamma^{\mathcal{C}}(Je_n)\gamma^{\mathcal{C}}(\partial_r)\gamma^{\mathcal{C}}(U_{rz}), \tag{7.1}$$

where $e_1, Je_1 \ldots, e_n, Je_n$ is any oriented orthonormal basis of $T_{\pi(z)}M$ and $r \in \mathbb{R}^+$, $z \in \mathcal{L}$. It is also known that $\nabla^{\mathcal{C}} \omega^{\mathcal{C}} = 0$ and $(\omega^{\mathcal{C}})^2 = 1$ and that it provides an orthogonal and parallel decomposition into ± 1-eigensubbundles, that is, into the so-called subbundles of *positive or negative chirality*,

$$\Sigma\mathcal{C} = \Sigma^+\mathcal{C} \oplus \Sigma^-\mathcal{C}.$$

The Kähler form $\Omega^{\mathcal{C}}$, acting by means of the Clifford multiplication on the spinors of $\Sigma\mathcal{C}$ in the following way

$$\Omega^{\mathcal{C}} = \sum_{i=1}^{n} \gamma^{\mathcal{C}}(e_i)\gamma^{\mathcal{C}}(Je_i) + \gamma^{\mathcal{C}}(\partial_r)\gamma^{\mathcal{C}}(U), \tag{7.2}$$

induces another orthogonal decomposition into parallel subbundles

$$\Sigma\mathcal{C} = \Sigma_0\mathcal{C} \oplus \cdots \oplus \Sigma_r\mathcal{C} \oplus \cdots \oplus \Sigma_{n+1}\mathcal{C}, \tag{7.3}$$

where each $\Sigma_r\mathcal{C}$ is the eigenbundle corresponding to the eigenvalue $(2r - n - 1)i$, $0 \leq r \leq n + 1$, and has rank $\binom{n+1}{r}$. One easily checks that the spin volume form $\omega^{\mathcal{C}}$ and the Kähler form $\Omega^{\mathcal{C}}$ commute and so they can be simultaneously diagonalized. In fact, we have

$$\Sigma^+\mathcal{C} = \bigoplus_{r \text{ even}} \Sigma_r\mathcal{C} \qquad \Sigma^-\mathcal{C} = \bigoplus_{r \text{ odd}} \Sigma_r\mathcal{C}.$$

The $\mathbb{S}^1$-bundle $\mathcal{L}$ is the hypersurface $r = 1$ in the cone $\mathcal{C}$. We consider the orientation determined by the orthonormal bases of the form $e_1, Je_1, \ldots, e_n, Je_n, U$ tangent to $\mathcal{L}$ where $e_1, Je_1, \ldots, e_n, Je_n$ are tangent to M. This orientation determines a unique spin structure on the hypersurface $\mathcal{L}$ from that of $\mathcal{C}$ (see [3, 2, 24, 54]) and each one of the restrictions $\Sigma^+\mathcal{C}|_{\mathcal{L}}$ and $\Sigma^-\mathcal{C}|_{\mathcal{L}}$ may be identified with a copy of the intrinsic spinor bundle $\Sigma\mathcal{L}$ of $\mathcal{L}$. From now on we will choose this one

$$\Sigma\mathcal{L} = \Sigma^+\mathcal{C}|_{\mathcal{L}}. \tag{7.4}$$

With this identification, the Clifford multiplication and the spin Levi-Cività connection of the spin structure on $\mathcal{L}$ are

$$\gamma^{\mathcal{L}} = -\gamma^{\mathcal{C}}\gamma^{\mathcal{C}}(\partial_r), \qquad \nabla^{\mathcal{L}} = \nabla^{\mathcal{C}} + \frac{1}{2}\gamma^{\mathcal{C}}\gamma^{\mathcal{C}}(\partial_r) = \nabla^{\mathcal{C}} - \frac{1}{2}\gamma^{\mathcal{L}}. \tag{7.5}$$

If we denote by $\omega^{\mathcal{L}}$ the complex spin volume form on $\mathcal{L}$, we have

$$\omega^{\mathcal{L}} = i^{n+1}\gamma^{\mathcal{L}}(e_1)\gamma^{\mathcal{L}}(Je_1)\cdots\gamma^{\mathcal{L}}(e_n)\gamma^{\mathcal{L}}(Je_n)\gamma^{\mathcal{L}}(U). \tag{7.6}$$

Using (7.5), the commutativity relations defining the Clifford algebras and (7.1), we obtain the equality $\omega^{\mathcal{L}} = \omega^{\mathcal{C}}|_{\mathcal{L}}$. Hence

$$\omega^{\mathcal{L}} = +1 \quad \text{on } \Sigma\mathcal{L}. \tag{7.7}$$

It is not difficult to see that the endomorphism of $\Sigma\mathcal{L}$, given by

$$\omega = i^n\gamma^{\mathcal{L}}(e_1)\gamma^{\mathcal{L}}(Je_1)\cdots\gamma^{\mathcal{L}}(e_n)\gamma^{\mathcal{L}}(Je_n)$$

is independent on the choice of the J-bases tangent to M and that $\omega^{\mathcal{L}} = i\omega\gamma^{\mathcal{L}}(U)$. Hence we have

$$\omega = i\gamma^{\mathcal{L}}(U), \qquad \omega^2 = 1. \tag{7.8}$$

We will think of the action of the endomorphism ω on the spinors of $\mathcal{L}$ as a *conjugation* and will write

$$\omega\psi = \overline{\psi}, \qquad \forall\psi \in \Gamma(\Sigma\mathcal{L}).$$

Also, the Kähler form on the holomorphic distribution determined by (6.2) on $\mathcal{L}$, acts on the spinor fields of $\mathcal{L}$ by means of the Clifford multiplication as

$$\Omega = \sum_{i=1}^{n} \gamma^{\mathcal{L}}(e_i)\gamma^{\mathcal{L}}(Je_i).$$

Indeed, taking into account (7.2) and the definition (7.5), we get the following relation between the Kähler forms Ω and $\Omega^{\mathcal{C}}$

$$\Omega = \Omega^{\mathcal{C}}|_{\mathcal{L}} - \gamma^{\mathcal{L}}(U) = \Omega^{\mathcal{C}}|_{\mathcal{L}} + i\omega. \tag{7.9}$$

Since the endomorphisms ω and Ω commute on $\Sigma\mathcal{L}$ and the eigenvalues of ω are clearly ± 1, we see that Ω has eigenvalues $(2s-n)i$ for $s = 0,\ldots,n$ and obtain an orthogonal decomposition into eigenbundles

$$\Sigma\mathcal{L} = \Sigma_0\mathcal{L} \oplus \cdots \oplus \Sigma_s\mathcal{L} \oplus \cdots \oplus \Sigma_n\mathcal{L},$$

where each $\Sigma_s\mathcal{L}$ has rank $\binom{n}{s}$. More precisely, the following relations between the eigenspaces of Ω and that of $\Omega^{\mathcal{C}}$ hold:

$$\begin{gathered} \Sigma_0\mathcal{C}|_{\mathcal{L}} = \Sigma_0\mathcal{L}, \qquad \Sigma_{n+1}\mathcal{C}|_{\mathcal{L}} = \Sigma_n\mathcal{L} \ \ \text{(if } n+1 \text{ is even)} \\ \Sigma_r\mathcal{C}|_{\mathcal{L}} = \Sigma_r\mathcal{L} \oplus \Sigma_{r-1}\mathcal{L}, \quad 0 < r < n+1, \ r \text{ even}. \end{gathered} \tag{7.10}$$

The first two relations can be checked easily. When $r = 0$ or $r = n+1$, $\Sigma_0\mathcal{C}$ and $\Sigma_{n+1}\mathcal{C}$ are line bundles which can be characterized as follows (see [14, Section 5.2])

$$\gamma^{\mathcal{C}} \circ J|_{\Sigma_0\mathcal{C}} = i\gamma^{\mathcal{C}}|_{\Sigma_0\mathcal{C}}, \qquad \gamma^{\mathcal{C}} \circ J|_{\Sigma_{n+1}\mathcal{C}} = -i\gamma^{\mathcal{C}}|_{\Sigma_{n+1}\mathcal{C}}.$$

Then, from definition (7.5), one obtains

$$\gamma^{\mathcal{L}}(Ju)\psi = i\gamma^{\mathcal{L}}(u)\psi, \qquad \gamma^{\mathcal{L}}(Ju)\varphi = -i\gamma^{\mathcal{L}}(u)\varphi$$

where $u \in T\mathcal{L}$, $u \perp U$, $\psi \in \Gamma(\Sigma_0\mathcal{C}|_{\mathcal{L}})$ and $\varphi \in \Gamma(\Sigma_{n+1}\mathcal{C}|_{\mathcal{L}})$.

It is important to point out that the above decomposition into eigenbundles of Ω is not parallel with respect to the connection $\nabla^{\mathcal{L}}$ defined

in (7.5). The reason is that the almost-complex structure J on the holomorphic distribution on $\mathcal{L}$ is not parallel with respect to the Levi-Cività connection, although it is parallel when it is induced on the Kähler manifold M. This is why we will consider an induced Clifford multiplication γ and a modified connection ∇ on the spinor bundle $\Sigma\mathcal{L}$, working only for tangent directions in the holomorphic distribution, which in some sense, reproduces the Levi-Cività connection of M. For any u a horizontal vector field according to the decomposition (6.2), i.e., $u \in T\mathcal{L},\ u \perp U$, set

$$\gamma(u) := \gamma^{\mathcal{L}}(u), \tag{7.11}$$

and

$$\nabla_u := \nabla^{\mathcal{L}}_u + \frac{1}{2}\gamma^{\mathcal{L}}(Ju)\gamma^{\mathcal{L}}(U) = \nabla^{\mathcal{L}}_u - \frac{i}{2}\gamma(Ju)\omega. \tag{7.12}$$

Now one can check that, this connection ∇ leaves invariant each eigen-subbundle $\Sigma_s\mathcal{L}$, with $0 \leq s \leq n$.

From [26], it is known that any Kähler manifold, and so M, admits a spin^c structure because its second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ is the reduction modulo 2 of the corresponding first Chern class $c_1(M) \in H^2(M, \mathbb{Z})$. Moreover, the set of spin^c structures on M is parametrized by the cohomology classes $\beta \in H^2(M, \mathbb{Z})$ such that $\beta \equiv c_1(M) \mod 2$ (see also [14, Proposition, p. 53]). Hence, having in mind the isomorphism (2.3), this set is classified by the equivalence classes of principal $\mathbb{S}^1$-bundles on M whose first Chern classes verify the above relation. The line bundle β is usually called the *auxiliary* or *determinant* line bundle of the given spin^c structure. Under this correspondence, the spin^c structures with auxiliary trivial line bundle, that is with $\beta = 0$, are just the spin structures. Moreover, given a spin^c structure on M with auxiliary line bundle $\beta \in H^2(M, \mathbb{Z})$, the oriented Riemannian submersion $\pi : \mathcal{L} \to M$ induces a spin^c structure on $\mathcal{L}$ whose auxiliary line bundle is $\pi^*(\beta) \in H^2(\mathcal{L}, \mathbb{Z})$ (one can see [6, Lemma 2.40]). On the other hand, the Thom-Gysin sequence associated with that fibration

$$0 \longrightarrow H^0(M, \mathbb{Z}) \cong \mathbb{Z} \overset{\cup(c_1(\mathcal{L})=\alpha)}{\longrightarrow} H^2(M, \mathbb{Z}) \overset{\pi^*}{\longrightarrow} H^2(\mathcal{L}, \mathbb{Z}) \longrightarrow 0$$

implies that $\ker \pi^* = \mathbb{Z}(\alpha)$. This means that the spin^c structures on the Kähler manifold M whose auxiliary line bundles are the tensorial powers $\mathcal{L}^q$, for $q \in \mathbb{Z}$, are exactly those inducing on $\mathcal{L}$, through the projection π, its unique spin structure. But, among these powers $\mathcal{L}^q$, only those satisfying the condition

$$c_1(\mathcal{L}^q) \equiv c_1(M) \mod 2$$

provide us a spin^c structure. Since $c_1(\mathcal{L}) = \alpha$ and $c_1(M) = -c_1(K_M) = -p\alpha$, this is equivalent to the equation

$$(q + p)\alpha \equiv 0 \mod 2 \qquad \text{in } H^2(M, \mathbb{Z}),$$

where p is the index of M. Now, as α is an indivisible class in $H^2(M, \mathbb{Z})$, this latter occurs if and only if

$$p + q \in 2\mathbb{Z}.$$

In other terms, we will be interested in the powers $\mathcal{L}^q$ such that $\mathcal{L}^q \otimes K_M$ has a square root.

We will denote by $\Sigma^q M$ the spinor bundle of the corresponding spin^c structure with auxiliary line bundle $\mathcal{L}^q$, $q \in -p + 2\mathbb{Z}$. All these spinor bundles are identifiable, via the projection π, with the spinor bundle $\Sigma\mathcal{L}$ of the spin manifold $\mathcal{L}$ (see [6, Lemma 2.40]), that is,

$$\pi^*\Sigma^q M = \Sigma\mathcal{L} \qquad \forall q \in \mathbb{Z}. \tag{7.13}$$

As M is Kähler, the spinor bundle $\Sigma^q M$ associated with any of its considered spin^c structures also decomposes into eigensubbundles corresponding to the spinorial action of its Kähler form Ω^M

$$\Sigma^q M = \Sigma^q_0 M \oplus \cdots \oplus \Sigma^q_s M \oplus \cdots \oplus \Sigma^q_n M.$$

Each $\Sigma^q_s M$ is associated with the eigenvalue $(2s - n)i$, $0 \leq s \leq n$ and has rank $\binom{n}{s}$. It is easy to see that $\pi^*\Omega^M = \Omega$ and so

$$\pi^*\Sigma^q_s M = \Sigma_s\mathcal{L} \qquad 0 \leq s \leq n.$$

Now, as in Section 4, for each integer $q \in \mathbb{Z}$, we will consider the Levi-Cività connection $i\theta^q \in \Gamma(\Lambda^1(\mathcal{L}^q) \otimes i\mathbb{R})$ on the line bundle $\mathcal{L}^q$, induced from the Kähler metric on M. It is not difficult to see that all these 1-forms are mapped, via the pull-backs of the q-fold coverings $P_{1,q} : \mathcal{L} \to \mathcal{L}^q$, on integer multiples of the same 1-form on $\mathcal{L}$. In fact, we have

$$P^*_{1,q}\theta^q = q\theta_1$$

where the connection $\theta_1 = \theta$ was already used in Section 6 to construct the metric on $\mathcal{L}$. We will denote by ∇^q the Hermitian connection on the spinor bundle $\Sigma^q M$, of any of the spin^c structures that we are considering on M, constructed from the Levi-Cività connection of M and from the connection θ^q on the auxiliary line bundle $\mathcal{L}^q$. Then (see for example [6, Theorem 2.41]), if we use the definitions (7.11) and (7.12), we conclude that all the ∇^q are identified with ∇ via the projection π, that is,

$$\pi^*\nabla^q_u\psi = \nabla_{u^*}\pi^*\psi, \qquad \forall u \in TM,\ \psi \in \Gamma(\Sigma^q M).$$

where $u^* \in T\mathcal{L}$ means the horizontal lift of vectors with respect to the decomposition (6.2).

As a consequence from (7.13), we see that the spinor fields $\psi \in \Gamma(\Sigma^q M)$ of the spin^c structure on M with auxiliary bundle $\mathcal{L}^q$ can be identified with certain spinor fields $\varphi = \pi^*\psi \in \Gamma(\Sigma\mathcal{L})$ on the Maslov covering $\mathcal{L}$. These fields are usually called *q-projectable* spinor fields and they have of course a particular behaviour with respect to the vertical direction of the submersion. This can be seen, for instance, in the second equation of [6, Theorem 2.41]. We have in fact

$$\nabla^{\mathcal{L}}_U\varphi + \frac{q}{2}i\theta(U)\varphi = \nabla^{\mathcal{L}}_U\varphi + \frac{q(n+1)}{2p}i\varphi = \frac{1}{2}\Omega\varphi$$

for every $\varphi \in \Gamma(\Sigma\mathcal{L})$, q-projectable. This is a necessary condition for the spinor fields φ on the manifold $\mathcal{L}$ to be obtained, via the projection $\pi : \mathcal{L} \to M$, from spinor fields $\psi \in \Gamma(\Sigma^q M)$ on M. It is not difficult to see that, in fact, this is also a sufficient condition.

Proposition 13. *A spinor field $\varphi \in \Gamma(\Sigma\mathcal{L})$ on the total space of the oriented Riemannian submersion $\pi : \mathcal{L} \to M$ is projectable onto a spinor field $\psi \in \Gamma(\Sigma^q M)$ corresponding to the spin^c structure of the Kähler manifold M with auxiliary line bundle $\mathcal{L}^q$, $q \in -p + 2\mathbb{Z}$, if and only if*

$$\nabla^{\mathcal{L}}_U\varphi + \frac{q(n+1)}{2p}i\varphi = \frac{1}{2}\Omega\varphi,$$

where p is the index of the Kähler manifold M, the endomorphism Ω is the Kähler form of the holomorphic distribution on $\mathcal{L}$ and U is the normalized fundamental vector field of the principal $\mathbb{S}^1$-fibration.

8. Parallel and Killing spinor fields

M. Wang classified in [56] the simply-connected spin Riemannian manifolds supporting non-trivial parallel spinor fields. Hitchin had already pointed out in [26] that they must be Ricci-flat and Wang proved that the only possible holonomies are 0, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, Spin_7 and G_2. Among them we find the Calabi-Yau manifolds, whose holonomy is contained in $\mathrm{SU}(n)$. Henceforth, that holonomy must be 0, and the manifold is flat, or $\mathrm{Sp}(n/2)$ with n even, and the manifold is hyper-Kähler, or just $\mathrm{SU}(n)$.

We already know that the cone $\mathcal{C}$ constructed over the maximal Maslov covering $\mathcal{L}$ of the Kähler-Einstein manifold M is a Calabi-Yau $(2n+2)$-dimensional manifold. Hence it carries a non-trivial parallel spinor field $\psi \in \Gamma(\Sigma\mathcal{C})$. Since the eigensubbundles (7.3) of $\Sigma\mathcal{C}$ for the

Kähler form $\Omega^{\mathcal{C}}$ are parallel, each r-component of ψ is also parallel and so, we will suppose that $\psi \in \Gamma(\Sigma_r\mathcal{C})$, for some $0 \leq r \leq n+1$. That is

$$\nabla^{\mathcal{C}}\psi = 0, \qquad \Omega^{\mathcal{C}}\psi = (2r-n-1)i\psi.$$

Suppose now that r is even. Then the identification (7.4) allows to see the restriction $\xi = \psi|_{\mathcal{L}}$ as a section of $\Gamma(\Sigma\mathcal{L})$, that is, as a spinor field on $\mathcal{L}$ which is also non-trivial since ψ has constant length. Taking into account (7.5) and (7.10), we have that

$$\xi \in \Gamma(\Sigma_r\mathcal{L} \oplus \Sigma_{r-1}\mathcal{L}), \qquad \nabla^{\mathcal{L}}\xi = -\frac{1}{2}\gamma^{\mathcal{L}}\xi,$$

with the convention $\Sigma_{-1}\mathcal{L} = \Sigma_{n+1}\mathcal{L} = \mathcal{L} \times \{0\}$. In particular, ξ is a real Killing spinor field on the spin manifold $\mathcal{L}$ (see [3] for a definition and references). Using this Killing equation satisfied by ξ for horizontal vectors $v \in T\mathcal{L}$, we obtain from (7.8), (7.11) and (7.12)

$$\nabla_v\xi = -\frac{1}{2}\gamma(v)\xi - \frac{i}{2}\gamma(Jv)\overline{\xi} \qquad \forall v \in T\mathcal{L},\, v \perp U.$$

Since the connection ∇ stabilizes each $\Sigma_s\mathcal{L}$ and the Clifford multiplication γ satisfies

$$\gamma(\Sigma_s\mathcal{L}) \subset \Sigma_{s-1}\mathcal{L} \oplus \Sigma_{s+1}\mathcal{L},$$

if $0 < r < n+1$ and one of the two components ξ_{r-1} or ξ_r of ξ vanished, then we would have $\nabla\xi = 0$ and $\gamma(Jv)\xi_r = i\gamma(v)\xi_r$ or $\gamma(Jv)\xi_{r-1} = -i\gamma(v)\xi_{r-1}$for any horizontal v. But this latter would imply that $\Omega\xi_r = -ni\xi_r$ or $\Omega\xi_{r-1} = ni\xi_{r-1}$ and so $r = 0$ in the first case or $r-1 = n$ in the second one, which is a contradiction. Then, when ξ has two components, its two components are non-trivial. On the other hand, taking now the unit vertical direction $U \in T\mathcal{L}$ and using (7.9), one has

$$\nabla^{\mathcal{L}}_U\xi = \frac{i}{2}\overline{\xi} = \frac{1}{2}\Omega\xi - \frac{2r-n-1}{2}i\xi.$$

This last equality, according Proposition 13, means that ξ is a q-projectable spinor field on $\mathcal{L}$ with

$$q = \frac{p}{n+1}(2r-n-1) = \frac{2rp}{n+1} - p.$$

From it, we will be able to get spinor fields of a spin^c structure of M provided that $q+p$ is even. In fact, only in this case, there will exist a spin^c structure on M whose auxiliary line bundle is $\mathcal{L}^q$. We now show that $q+p = \frac{2rp}{n+1}$ is divisible by 2. If the holonomy of $\mathcal{C}$ is 0, then $\mathcal{C} = \mathbb{C}^{n+1}$, $\mathcal{L} = \mathbb{S}^{2n+1}$ and $M = \mathbb{C}P^n$. Then $p = n+1$ and we have a parallel spinor for each $0 \leq r \leq n+1$. But then $\frac{2rp}{n+1} = 2r$ and we conclude. If the holonomy group of the Calabi-Yau cone $\mathcal{C}$ is $\mathrm{Sp}(k+1)$ with $2k+1 = n$, then $\mathcal{C}$ is a hyper-Kähler manifold. This implies that

$\mathcal{L}$ is 3-Sasakian and M is a complex contact manifold. In this case the index of M is exactly $p = k + 1$ (see, for example, Remark 2 and [7, Proposition 2.12]) and there are parallel spinor fields on $\mathcal{C}$ only for $0 \leq r \leq n + 1$ even (see [56]). But then $\frac{2rp}{n+1} = r$ is even and we also conclude. The last case is when the holonomy of $\mathcal{C}$ is just $\mathrm{SU}(n + 1)$. In this case we only know that $p < n + 1$ but the only parallel spinor fields carried by $\mathcal{C}$ are for $r = 0, n + 1$ (see [56]) and then $\frac{2rp}{n+1} = 0, 2p$ is also even.

If the parallel spinor field $\psi \in \Gamma(\Sigma_r\mathcal{C})$ has degree r odd, we would not work with the identification (7.4), but with

$$\Sigma\mathcal{L} = \Sigma^{-}\mathcal{C}|_{\mathcal{L}}.$$

In this case, one can check that all equalities involved are still true except (7.8) and (7.9) where the sign of ω has to be changed. Hence, only some minor changes are obtained in the conclusion. We will summarize this information in the following statement.

Theorem 14. *Let $\mathcal{C}$ be the cone over the maximal Maslov covering $\mathcal{L}$ of an $2n$-dimensional Kähler-Einstein compact manifold M with scalar curvature $4n(n+1)$ and index $p \in \mathbb{N}^*$. Each non-trivial parallel spinor field $\psi \in \Gamma(\Sigma_r\mathcal{C})$ with degree r, $0 \leq r \leq n + 1$ induces two non-trivial spinor fields $\xi_r \in \Gamma(\Sigma^q_r M)$ and $\xi_{r-1} \in \Gamma(\Sigma^q_{r-1}M)$ of the spin^c structure on M with auxiliary bundle $\mathcal{L}^q$, where $q = \frac{p}{n+1}(2r-n-1) \in \mathbb{Z}$, satisfying the first order system*

$$\begin{cases} \nabla^q_v\xi_r = -\frac{1}{2}\gamma(v)\xi_{r-1} + \frac{i}{2}\gamma(Jv)\xi_{r-1} \\ \nabla^q_v\xi_{r-1} = -\frac{1}{2}\gamma(v)\xi_r - \frac{i}{2}\gamma(Jv)\xi_r, \end{cases}$$

for all $v \in TM$. Such coupled spinor fields on the Kähler manifold M are usually called Kählerian Killing spinor fields.

Remark 15. Kählerian Killing spinor fields on Kähler manifolds play a rôle analogous to what of Killing spinors on Riemannian spin manifolds (see [28, 23]). Both two classes appear when one studies the so-called *limiting manifolds*, that is, those attaining the equality for the standard (Friedrich ([13]), Hijazi ([22]), Kirchberg ([27]), Herzlich-Moroianu ([21, 39, 40])) lower estimates for the eigenvalues of the Dirac operator. In the case of Kähler manifolds (see [28, 36, 37, 38], a usual strategy to classify manifolds admitting these particular fields is to construct the cone over a suitable principal $\mathbb{S}^1$-bundle in order to get manifolds carrying parallel spinor fields and to apply the corresponding results. In the proof of Theorem 14 we adopted the opposite approach. Also, details of the proof are given since we need an exact control on the

behaviour of the Kähler form on the Kählerian Killing spinor fields and on the spin^c structure.

Remark 16. Consider now the Dirac operator $D^q : \Gamma(\Sigma^q M) \to \Gamma(\Sigma^q M)$ acting on smooth spinor fields of the spin^c structure on M with auxiliary bundle $\mathcal{L}^q$. This operator is locally defined by

$$D^q = \sum_{i=1}^{2n} \gamma(e_i)\nabla^q_{e_i}$$

where $e_1, \ldots, e_{2n}$ is any orthonormal basis tangent to M. A direct consequence from Theorem 14 is that we can compute the images by D^q of the fields $\xi_r \in \Gamma(\Sigma^q_r M)$ and $\xi_{r-1} \in \Gamma(\Sigma^q_{r-1} M)$ obtained from the parallel spinor $\psi \in \Gamma(\Sigma_r \mathcal{C})$. We have

$$D^q \xi_r = 2(n+1-r)\xi_{r-1}, \qquad D^q \xi_{r-1} = 2r\xi_r.$$

Hence ξ_r and ξ_{r-1} are eigenspinors for the same eigenvalue of the second order operator $(D^q)^2$. In fact

$$(D^q)^2 \xi_{r,r-1} = 4r(n+1-r)\xi_{r,r-1}$$

and so the eigenvalue $\lambda_1(D^q)$ of D^q with least absolute value satisfies $\lambda_1(D^q)^2 \leq 4r(n+1-r)$. But we can observe that

$$4r(n+1-r) = \frac{n+1}{4n} \inf S - (n+1-2r)^2 = \frac{n+1}{4n} \inf S - q^2,$$

where S denotes the scalar curvature of M. If we take into account the generalization given in [21] to spin^c manifolds of Friedrich's estimate and the fact that the manifold M is Kähler, we could expect that a similar generalization would exist for Kirchberg's inequalities in the context of Kähler manifolds. It is natural to think that, in fact,

$$\lambda_1(D^q)^2 = 4r(n+1-r)$$

and the Kähler manifolds involved must be limiting manifolds for the Dirac operator of the spin^c structures with suitable auxiliary bundle $\mathcal{L}^q$. In fact, $q = p$ or $q = -p$ when the holonomy of $\mathcal{C}$ is $\mathrm{SU}(n+1)$; $q = 2s - k - 1$, $0 \leq s \leq k+1$, when the holonomy of $\mathcal{C}$ is $\mathrm{Sp}(k+1)$ with $n = 2k+1$; and $q = 2r - n - 1$, $0 \leq r \leq n+1$ when the M is the complex projective space.

If we bear in mind, like in Remark 16, the possible holonomies for the cone $\mathcal{C}$, the possible degrees of its parallel spinor fields, the corresponding multiplicities and the possible spin^c structures on the Kähler manifold M, we can rewrite Theorem 14 to show that it improves Theorem 9.11 in [6] (see also [40, 39, 29]).

Theorem 17. *Let M be an $2n$-dimensional Kähler-Einstein compact manifold with scalar curvature $4n(n+1)$.*

(1) *The vector space of Kählerian Killing spinors carried by the canonical spin^c structure on M is one (and all of them have degree n).*
(2) *The vector space of Kählerian Killing spinors carried by the anti-canonical spin^c structure on M is one (and all of them have degree 0).*
(3) *If M has a complex contact structure and so $n = 2k+1$, then, for each $0 \leq s \leq k+1$, the space of Kählerian Killing spinors $\psi \in \Gamma(\Sigma^{2s-k-1}_{2s-1} M \oplus \Sigma^{2s-k-1}_{2s} M)$ has dimension one.*
(4) *If M is the complex projective space $\mathbb{C}P^n$, then, for each $0 \leq r \leq n+1$, the space of Kählerian Killing spinors $\psi \in \Gamma(\Sigma^{2r-n-1}_{r-1} M \oplus \Sigma^{2r-n-1}_{r} M)$ has dimension $\begin{pmatrix} n+1 \\ r \end{pmatrix}$.*

9. Lagrangian submanifolds and induced Dirac operator

Let us come back to the situation that we studied in Section 4, i.e., consider a Lagrangian immersion $\iota : L \to M$ from an n-dimensional manifold L into the Kähler-Einstein manifold M. We will denote by $\Sigma^q L$ the restriction to the submanifold L of the spinor bundle $\Sigma^q M$ on M corresponding to the spin^c structure with auxiliary line bundle $\mathcal{L}^q$, with $q \in \mathbb{Z}$. That is,

$$\Sigma^q L = \Sigma^q M|_L. \tag{9.1}$$

It is clear that this $\Sigma^q L$ is a Hermitian vector bundle over L whose rank is 2^n and that it admits an orthogonal decomposition

$$\Sigma^q L = \Sigma^q_0 L \oplus \cdots \oplus \Sigma^q_s L \oplus \cdots \oplus \Sigma^q_n L, \tag{9.2}$$

where each summand is the restriction $\Sigma^q_s L = \Sigma^q_s M|_L$, with $0 \leq s \leq n$, hence a Hermitian bundle with rank $\begin{pmatrix} n \\ s \end{pmatrix}$. Since the immersion ι allows to view the tangent bundle TL as a subbundle of TM, we can consider the restrictions $\nabla^q : TL \otimes \Sigma^q L \to \Sigma^q L$ and $\gamma : TL \subset TM \to \mathrm{End}(\Sigma^q L)$ to the submanifold L of the connection and of the Clifford multiplication corresponding to the spin^c structure on M. The problem of these two restrictions is that they are compatible with respect to the Levi-Cività connection of M, but not with respect to the Levi-Cività ∇^L of the induced metric on L. To avoid this difficulty, we consider a

modified connection, also denoted by ∇^L on the bundle $\Sigma^q L$. We put

$$\nabla^{q,L}_v\psi = \nabla^q_v\psi - \frac{1}{2}\sum_{i=1}^n \gamma(e_i)\gamma(\sigma(v,e_i))\psi, \qquad v\in TL, \tag{9.3}$$

where σ is the second fundamental form of the immersion ι and $e_1,\dots,e_n$ is any basis tangent to L. Now one can verify that

$$\nabla^{q,L}_v\gamma(X)\psi = \gamma(\nabla^L_X Y)\psi + \gamma(Y)\nabla^{q,L}_v\psi, \qquad v\in TL,\ X\in\Gamma(TL).$$

It is immediate to check that this new connection $\nabla^{q,L}$ also parallelizes the Hermitian product because the endomorphism $\sum_{i=1}^n \gamma(e_i)\gamma(\sigma(v,e_i))$ is skew-Hermitian for every $v\in TL$. Even more interesting is the fact that ι being Lagrangian and minimal implies that $\nabla^{q,L}$ fixes each subbundle $\Sigma^q_s L$ in (9.2). To see this, first one must use the well-known property that the 3-form

$$\langle\sigma(X,Y),JZ\rangle, \qquad X,Y,Z\in\Gamma(TL)$$

is symmetric (see [42], for example). In this way, one obtains

$$\sum_{j=1}^n \gamma(e_j)\gamma(\sigma(v,e_j)) = \frac{1}{2i}\sum_{j,k=1}^n \langle\sigma(e_j,e_k),Jv\rangle\gamma(e_j-iJe_j)\gamma(e_k+iJe_k)$$

because one has

$$\sum_{j,k=1}^n \langle\sigma(e_j,e_k),Jv\rangle\gamma(e_j)\gamma(e_k) = n\langle H,Jv\rangle = 0.$$

Now recall that

$$\gamma(T^{1,0}M)(\Sigma^q_r M)\subset\Sigma^q_{r-1}M, \qquad \gamma(T^{0,1}M)(\Sigma^q_r M)\subset\Sigma^q_{r+1}M.$$

As a conclusion the Hermitian vector bundle $\Sigma^q L$ endowed with the metric connection $\nabla^{q,L}$ and the Clifford multiplication γ is a Dirac bundle (see any of [14, 32]) over the Riemannian manifold L. Moreover the orthogonal decomposition (9.2) is preserved by $\nabla^{q,L}$ and

$$\gamma(TL)(\Sigma^q_s L)\subset\Sigma^q_{s-1}L\oplus\Sigma^q_{s+1}L.$$

Let $D:\Gamma(\Sigma^q L)\to\Gamma(\Sigma^q L)$ be the Dirac operator of this Dirac bundle, called *induced Dirac operator*, defined locally, for any orthonormal basis $e_1,\dots,e_n$ tangent to L, by

$$D=\sum_{i=1}^n \gamma(e_i)\nabla^{q,L}_{e_i}.$$

Of course, when the submanifold L is endowed with a given spin structure, the corresponding intrinsic Dirac operator could be different than

D. ¿From the Gauß equation (9.3) one can deduce the following expression for D,

$$D\xi = \sum_{i=1}^{n} \gamma(e_i)\nabla^q_{e_i}\xi + \frac{n}{2}\gamma(H)\xi \qquad \forall \xi \in \Gamma(\Sigma^q L). \tag{9.4}$$

A first obvious consequence from formula (9.4) is that, if the immersion ι is *minimal*, then parallel spinor fields on the Kähler-Einstein manifold M yield harmonic spinor fields on the submanifold L. Note that this restriction is non-trivial, since the spinor field is parallel. Hence, looking at Wang's and Moroianu-Semmelmann's classifications [56, 41] of complete spin manifolds carrying non-trivial parallel spinors, we have the following result.

Proposition 18. *Let L be an n-dimensional minimal Lagrangian submanifold immersed into a complete Calabi-Yau manifold M of dimension $2n$ and let ΣL be the Dirac bundle induced over L from the spinor bundle of M.*

1. *There exists a non-trivial harmonic spinor in $\Gamma(\Sigma_0 L)$ and another one in $\Gamma(\Sigma_n L)$.*
2. *If M has a hyper-Kähler structure, and so $n = 2k$, for each $0 \leq s \leq k$, there exists a non-trivial harmonic spinor in $\Gamma(\Sigma_{2s} L)$.*
3. *If $M = \mathbb{C}^n/\Gamma$, where Γ is a discontinuously acting group of complex translations, then, for every $0 \leq r \leq n$, the space of harmonic spinors in $\Gamma(\Sigma_r L)$ has dimension at least $\binom{n}{r}$.*

The harmonicity of the spinor fields above is always referred to the induced Dirac operator D of the submanifold.

A second consequence from the same formula (9.4) is obtained when the spinor field $\xi = \xi_{r-1,r}$ is no longer parallel, but a Kählerian Killing spinor as those constructed in Theorem 14, when the Kähler-Einstein manifold M is compact and has positive scalar curvature. In this case, we have

$$\textstyle\sum_{j=1}^{n} \gamma(e_j)\nabla^q_{e_j}\xi_r = \frac{n}{2}\xi_{r-1} + \frac{i}{2}\Omega\xi_{r-1} = (n+1-r)\xi_{r-1}$$
$$\textstyle\sum_{j=1}^{n} \gamma(e_j)\nabla^q_{e_j}\xi_{r-1} = \frac{n}{2}\xi_r - \frac{i}{2}\Omega\xi_r = r\xi_r$$

and so, if the Lagrangian submanifold L is minimal, we obtain

$$D\xi_r = (n+1-r)\xi_{r-1} \qquad D\xi_{r-1} = r\xi_r. \tag{9.5}$$

Hence, the spinor fields ξ_r and ξ_{r-1} are eigenspinors for the second order operator D^2, namely

$$D^2\xi_{r-1,r} = r(n+1-r)\xi_{r-1,r}.$$

Theorem 19. *Let L be an n-dimensional minimal Lagrangian submanifold immersed into a Kähler-Einstein compact manifold M of dimension $2n$ and scalar curvature $4n(n+1)$. Let $\Sigma^q L$ be the Dirac bundle induced over L from the spinor bundle of M corresponding to the spin^c structure with auxiliary line bundle $\mathcal{L}^q$, $q \in \mathbb{Z}$.*

(1) *There exists a non-trivial harmonic spinor in $\Gamma(\Sigma_0^{-p} L)$ and another one in $\Gamma(\Sigma_n^p L)$, where p is the index of M.*
(2) *If M has a complex contact structure, and so $n = 2k+1$, for each $0 \leq s \leq k+1$, there exists a non-trivial spinor in $\Gamma(\Sigma_{2s}^{2s-k-1} L)$ and another one in $\Gamma(\Sigma_{2s-1}^{2s-k-1} L)$ which are eigenspinors of D^2 associated with the eigenvalue $4s(k+1-s)$.*
(3) *If $M = \mathbb{C}P^n$, then, for every $0 \leq r \leq n+1$, the eigenspace of D^2 in $\Gamma(\Sigma_{r-1}^{2r-n-1} L)$ or in $\Gamma(\Sigma_r^{2r-n-1} L)$, associated with the eigenvalue $r(n+1-r)$, have dimension at least $\binom{n+1}{r}$.*

The harmonicity of the spinor fields mentioned in (1) above is referred to the induced Dirac operator D of the submanifold L.

Proof : It only remains to point out that the restrictions of ξ_r and ξ_{r-1} in Theorem 14 to the submanifold L cannot be trivial. In fact, from (9.5), if one of them vanishes, then both of them have to vanish. But this is impossible because $\xi = \xi_{r-1} + \xi_r$ has constant length along the manifold M. Q.E.D.

10. The Hodge Laplacian on minimal Lagrangian submanifolds

In this section, we show that the Dirac bundles $\Sigma^q L$ constructed over a Lagrangian immersed submanifold from the spinor bundles $\Sigma^q M$ of different spin^c structures on a Kähler-Einstein manifold M can be identified with standard bundles. In fact, it is well-known (see [14, Section 3.4] for an explicit expression) that, for each possible r, we have an isometry of Hermitian bundles

$$\Sigma_r^q M = \Lambda^{0,r}(M) \otimes \Sigma_0^q M \tag{10.1}$$

and $\Sigma_0^q M$ is a square root of the tensorial product $\mathcal{L}^q \otimes K_M$, where $\mathcal{L}$ is the auxiliary line bundle of the spin^c structure and K_M is the canonical bundle of M. This fact was originally discovered by Hitchin [26] for spin structures. That is, the spinor fields of the spin^c structure can be thought of as anti-holomorphic r-forms taking values on the line bundle $\Sigma_0^q M$. Under this isometric identification, one can easily see that the connection ∇^q constructed on $\Sigma^q M$ from the Levi-Cività connection of

the Kähler metric on M and from the connection $i\theta_q$ on the line bundle $\mathcal{L}^q$, satisfies

$$\nabla^q|_{\Sigma^q_r M} = \nabla^M \otimes \nabla^q|_{\Sigma^q_0 M}, \tag{10.2}$$

where ∇^M is the usual Levi-Cività connection on the exterior bundle. This is a consequence of the compatibility of ∇^q and the Levi-Cività connection on M. One can also see (for example in [14, page 23] or in [32]) that the Clifford multiplication $\gamma : TM \to \mathrm{End}(\Lambda^{0,r}(M) \otimes \Sigma^q_0 M)$ can be viewed as follows

$$\gamma(v)(\beta \otimes \psi) = ((v^\flat \wedge \beta)^{0,r+1} - v \lrcorner \beta) \otimes \psi, \tag{10.3}$$

for all $v \in TM$, $\beta \in \Gamma(\Lambda^{0,r}(M))$ and $\psi \in \Gamma(\Sigma^q_0 M)$.

Now we consider the corresponding restrictions to the Lagrangian submanifold L. The definitions (9.1) and (9.2), the isometry (10.1) above and Lemma 3 imply another isometric identification of Hermitian bundles over L, namely

$$\Sigma^q_r L = \Lambda^r(L)_{\mathbb{C}} \otimes \Sigma^q_0 L.$$

The expressions above for the connection ∇^q and the Clifford product, together with the definition (9.3) of the modified connection $\nabla^{q,L}$ on $\Sigma^q L$, give

$$\nabla^{q,L}|_{\Sigma^q_r L} = \nabla^L \otimes \nabla^{q,L}|_{\Sigma^q_0 L},$$

where ∇^L is the Levi-Cività connection on the exterior bundle over L, and

$$\gamma(v)(\beta \otimes \psi) = (v^\flat \wedge \beta - v \lrcorner \beta) \otimes \psi,$$

for all $v \in TL$, $\beta \in \Gamma(\Lambda^r(L)_{\mathbb{C}})$ and $\psi \in \Gamma(\Sigma^q_0 L)$.

Hence, as a consequence of these last three identifications, if we could find some situations in which the line bundle $\Sigma^q_0 L$ is topologically trivial and admits a non-trivial parallel section, we would be able to identify the Dirac bundle $(\Sigma^q L, \nabla^{q,L}, \gamma)$ in a very pleasant manner.

Proposition 20. *Suppose that the restriction $\Sigma^q_0 L$ to a Lagrangian submanifold L of the 0-component $\Sigma^q_0 M$ of the spinor bundle of a spin^c structure of a Kähler manifold M is topologically trivial and admits a non-zero parallel section. Then the Dirac bundle $(\Sigma^q L, \nabla^{q,L}, \gamma)$ is isometrically isomorphic to the complex valued exterior bundle*

$$(\Lambda^*(L) \otimes \mathbb{C}, \nabla^L, \cdot^\flat \wedge - \cdot \lrcorner)$$

with its standard (see [32]*) Dirac bundle structure. Henceforth the Dirac operator D is nothing but the Euler operator $d + \delta$ and its square D^2 is the Hodge Laplacian Δ of L.*

We will present two situations where Proposition 20 above can be applied. The first one is when M is an n-dimensional Calabi-Yau manifold and $q = 0$, that is, we consider a spin structure on M. More precisely, we choose the trivial spin structure on M, in other words, the spin structure determined by taking as a square root of K_M, the bundle K_M itself. In this case, we have that $\Sigma_0 M = K_M$, which is topologically trivial and admits a non-trivial parallel section Θ (with respect to ∇^q). Take the restriction $\Upsilon = \Theta|_L \in \Gamma(\Sigma_0 L)$. From (9.3), we have for each $v \in TL$ that

$$\nabla_v^{0,L}\Upsilon = -\frac{1}{2}\sum_{j,k=1}^{n}\langle\sigma(e_j,e_k),Jv\rangle\gamma(e_j)\gamma(Je_k)\Upsilon.$$

But Θ is a 0-spinor and then $\gamma(Jv)\Theta = i\gamma(v)\Theta$ for every $v \in TM$. Then

$$\nabla_v^{0,L}\Upsilon = -\frac{i}{2}\sum_{i,j=1}^{n}\langle\sigma(e_i,e_j),Jv\rangle\gamma(e_i)\gamma(e_j)\Upsilon = -\frac{ni}{2}\langle JH,v\rangle\Upsilon.$$

So Υ is a non-trivial parallel section of $\Sigma_0 L$ if and only if L is a minimal submanifold.

Remark 21. As a consequence, we get to trivialize the restricted bundle $\Sigma_0 L$, but this is not a necessary condition in order to have this trivialization. In fact, from the equality above, we can conclude, as in Remark 9, that, when the ambient manifold M is a Calabi-Yau manifold, $\Sigma_0 L$ is geometrically trivial if and only if $\frac{1}{4}[\eta] \in H^1(L,\mathbb{Z})$, where η is the mean curvature form of the Lagrangian submanifold L, that in this case of Ricci-flat ambient space, coincides with the Maslov class. This explains why, in case $n = 1$ invoked in [17], one obtains the desired identification for the Lagragian immersion $z \in \mathbb{S}^1 \mapsto z^2 \in \mathbb{C}$ and not for the embedding $z \in \mathbb{S}^1 \mapsto z \in \mathbb{C}$.

Using Proposition 20, the assertions in Proposition 18 on the spectrum of the Dirac become now conditions on the Hodge Laplacian acting on differential forms of the Lagrangian submanifold.

Theorem 22. *Let L be an n-dimensional minimal Lagrangian submanifold immersed into a complete Calabi-Yau manifold M of complex dimension n.*

(1) *There exists a non-trivial harmonic complex n-form on L. So, if L is compact, then L is orientable. Hence L is a* special Lagrangian *submanifold (see* [19]*) of M.*
(2) *If M has a hyper-Kähler structure, and so $n = 2k$, for each $0 \le s \le k$, there exists a non-trivial harmonic complex $2s$-form*

on L. As a conclusion, if L is compact then the even Betti numbers of L satisfy $b_{2s}(L) \geq 1$.

(3) *If $M = \mathbb{C}^n/\Gamma$, where Γ is a discontinuously acting group of complex translations, then, for every $0 \leq r \leq n$, the space of harmonic complex r–forms on L has dimension at least $\binom{n}{r}$. In particular, all the Betti numbers of a compact minimal Lagrangian submanifold L immersed in a flat complex torus $\mathbb{T}^{2n}$ satisfy $b_r(L) \geq b_r(\mathbb{T}^n)$.*

Remark 23. We point out that part (2) in Theorem 22 is an improvement of the result by Smoczyk obtained in [50]. He proved Theorem 22 for $s = 1$ without using spin geometry. He showed that the equality $b_2(L) = 1$ implies that L is a complex submanifold for one of the complex structures of M. For the moment, we are not able to treat the case $b_{2s}(L) = 1$ with $s > 1$.

There is a second situation where Proposition 20 could be applied. It is the case, extensively studied in Section 7, of a Kähler-Einstein compact n-dimensional manifold with positive scalar curvature, normalized to be $4n(n+1)$. For integers r with $0 \leq r \leq n+1$ and such that $\frac{rp}{n+1} \in \mathbb{Z}$, where $p \in \mathbb{N}^*$ is the index of M, we have already considered the spin^c structures on M whose auxiliary bundles are $\mathcal{L}^q$, $q = \frac{2rp}{n+1} - p$, where $\mathcal{L}$ is a fix p-th root of the canonical bundle K_M of M. In this situation, we have that the line bundle of 0-spinors of this spin^c structure satisfies

$$(\Sigma_0^q M)^2 = \mathcal{L}^q \otimes K_M = \mathcal{L}^{\frac{2rp}{n+1}} = (\mathcal{L}^{\frac{rp}{n+1}})^2.$$

Recall that (see comments after (2.4)) the space of square roots of this last bundle is an affine space over $H^1(M, \mathbb{Z}_2)$. But this cohomology group is trivial since M is simply-connected under our hypotheses. This implies that the line bundle $\Sigma_0^q M$ is just $\mathcal{L}^{\frac{rp}{n+1}}$. Hence, the desired condition in Proposition 20 is that we can find a non-trivial parallel section on the line bundle $\mathcal{L}^{\frac{rp}{n+1}}|_L$ over the submanifold L. But this is equivalent to find a horizontal (or Legendrian) lift of the Lagrangian immersion $\iota : L \to M$ to the Maslov covering $\mathcal{L}^{\frac{rp}{n+1}}$ (see Section 3). In other terms, this means that the level n_L (see Section 4) of the submanifold divides the integer $\frac{rp}{n+1}$. On the other hand, just before Theorem 17, we considered the admissible values, under our hypotheses, for the index p and the integers r. We will use this information, together with Theorem 19, to translate the assertions about spinors and Dirac operator in the induced Dirac bundle $\Sigma^q L$ into results about exterior forms and Euler operator on L, when L is minimal.

Theorem 24. *Let L be an n-dimensional minimal Lagrangian submanifold immersed into a Kähler-Einstein compact manifold M of dimension $2n$ and scalar curvature $4n(n+1)$. Let $p \in \mathbb{N}^*$ be the index of M and $1 \leq n_L \leq 2p$ the level of this submanifold.*

1. *If $n_L | p$, then there exists a non-trivial complex harmonic n-form on L. In particular, when L is compact, if $n_L | p$, then L is orientable, as in Proposition 8.*
2. *If M has a complex contact structure, and so $n = 2k+1$, for each $0 \leq s \leq k+1$ such that $n_L | s$, there exists a non-trivial exact $2s$-form and another non-trivial co-exact $(2s-1)$-form on L which are eigenforms of the Hodge Laplacian Δ associated with the eigenvalue $4s(k+1-s)$.*
3. *If $M = \mathbb{C}P^n$, then, for every $0 \leq r \leq n+1$ such that $n_L | r$, the eigenspace of Δ on the subspace of exact forms of $\Gamma(\Lambda^r(L) \otimes \mathbb{C})$ or on the subspace of co-exact forms in $\Gamma(\Lambda^{r-1}(L) \otimes \mathbb{C})$ associated with the eigenvalue $r(n+1-r)$ have dimension at least $\binom{n+1}{r}$.*

Using the computation of levels for the minimal Lagrangian submanifolds of the complex projective space made in Section 5, we obtain some information about the spectra of the Hodge Laplacian on some symmetric spaces.

Corollary 25. *1. For each $r = 0, \dots, m^2$, the numbers $r(m^2 - r)$ are in the spectrum of the Hodge Laplacian for r-forms and $(r-1)$-forms on the Lie group $\mathrm{SU}(m)$ with multiplicity at least $\binom{m^2}{r}$.*

2. For each $r = 0, \dots, \frac{1}{2}m(m+1)$, the numbers $r(\frac{1}{2}m(m+1) - r)$ are in the spectrum of the Hodge Laplacian for r-forms and $(r-1)$-forms on the symmetric space $\frac{\mathrm{SU}(m)}{\mathrm{SO}(m)}$ with multiplicity at least $\binom{\frac{1}{2}m(m+1)}{r}$.

3. For each $r = 0, \dots, 2m^2$, the numbers $r(2m^2 - r)$ are in the spectrum of the Hodge Laplacian for r-forms and $(r-1)$-forms on the symmetric space $\frac{\mathrm{SU}(2m)}{\mathrm{Sp}(m)}$ with multiplicity at least $\binom{2m^2}{r}$.

4. For each $r = 0, \dots, 27$, the numbers $r(27 - r)$ are in the spectrum of the Hodge Laplacian for r-forms and $(r-1)$-forms on the symmetric space $\frac{\mathrm{E}_6}{\mathrm{F}_4}$ with multiplicity at least $\binom{27}{r}$.

Remark 26. Part (3) in Theorem 24 above was partially proved in [51] when the submanifold was Legendrian in the sphere $\mathbb{S}^{2n+1}$, that is, when

the level of the submanifold verifies $n_L = 1$. In fact, Smoczyk made a naïf mistake by calculating some differentials and he got the wrong eigenvalue $n+1-r$ instead of $r(n+1-r)$. Moreover, he obtained only $\binom{n}{r}$ as an estimate for the multiplicity of this eigenvalue. Our estimate $\binom{n+1}{r}$ for this multiplicity is sharp. For example, we studied in Example 1 of Section 5 the totally geodesic Lagrangian immersion of the sphere $\mathbb{S}^n$ into the projective space $\mathbb{C}P^n$. The induced metric is the standard round unit metric. For this metric, the spectrum of the Hodge Laplacian on the exterior bundle is well-known (see [15]). In fact, the first eigenvalue for exact r-forms is just $r(n+1-r)$ and its multiplicity is $\binom{n+1}{r}$. Analogously, the corresponding eigenvalue for co-exact forms is $(r+1)(n-r)$ and its multiplicity is $\binom{n+1}{r+1}$

Remark 27. Nicolas Ginoux started in his Ph. D. Thesis [16] the study of the spin geometry of Lagrangian *spin* submanifolds L in *spin* Kähler manifolds M and got upper estimates for the eigenvalues of the Dirac operator of the restriction to the submanifold of the spinor bundle on the ambient manifold. Later, in [17], Ginoux found some conditions in order to be able to identify this restriction with the exterior bundle of L and the Dirac operator with the Hodge operator. Our Proposition 20 also gives some answers to that problem of identification. When M is a Calabi-Yau manifold endowed with the trivial spin structure, we have observed before that $\Sigma_0 M = K_M$ and so $\Sigma_0 L = K_M|_L$. Hence, the identification works if and only if $\frac{1}{4}[\eta]$ is an integer cohomology class (see Remarks 9 and 21), where η is the mean curvature form. Then, when L is minimal, we have always the identification of the bundles and the identification of the operators. When M is Kähler-Einstein with positive scalar curvature and so index $p \in \mathbb{N}^*$, we have that M is spin if and only if p is even and, in this case, $\Sigma_0 M = \mathcal{L}^{\frac{p}{2}}$. Hence we have the identification of bundles when and only when $n_L|\frac{p}{2}$ and $\frac{1}{4}[\eta] \in H^1(L, \mathbb{Z})$. The identification of operators occurs if L is also minimal. In fact, we already knew (Proposition 8) that $n_L|p$ if and only if L is orientable.

11. Morse index of minimal Lagrangian submanifolds

Let L be a compact n-dimensional minimal Lagrangian submanifold immersed into a Kähler-Einstein manifold M of dimension $2n$ and scalar curvature c. Then L is a critical point for the volume functional

and it is natural to consider the second variation operator associated with such an immersion. This operator, known as the *Jacobi operator* of L, is a second order strongly elliptic operator acting on sections of the normal bundle $T^\perp L$, and it is defined as

$$J = \Delta^\perp + \mathcal{B} + \mathcal{R},$$

where $\Delta^\perp$ is the second order operator

$$\Delta^\perp = \sum_{i=1}^n \{\nabla^\perp_{e_i}\nabla^\perp_{e_i} - \nabla^\perp_{\nabla_{e_i}e_i}\},$$

the operators $\mathcal{B}$ and $\mathcal{R}$ are the endomorphisms

$$\mathcal{B}(\xi) = \sum_{i,j=1}^n \langle\sigma(e_i,e_j),\xi\rangle\sigma(e_i,e_j), \quad \mathcal{R}(\xi) = \sum_{i=1}^n \left(\bar{R}(\xi,e_i)e_i\right)^\perp,$$

where $\{e_1,\dots,e_n\}$ is an orthonormal basis tangent to L, σ is the second fundamental form of the submanifold L, $\bar{R}$ is the curvature tensor of M and $\perp$ denotes normal component. Let $Q(\xi) = -\int_L\langle J\xi,\xi\rangle dV$ be the quadratic form associated with the Jacobi operator J. We will denote by $\mathrm{ind}(L)$ and $\mathrm{nul}(L)$ the index and the nullity of the quadratic form Q, which are respectively the number of negative eigenvalues of J and the multiplicity of zero as a eingenvalue of J. The minimal submanifold L is called *stable* if $\mathrm{ind}(L) = 0$.

In [45], Oh studied the Jacobi operator for minimal Lagrangian submanifolds, proving that it is an intrinsic operator on the submanifold and can be written in terms of the Hodge Laplacian acting on 1-forms of L. In fact, if we consider the identification

$$T^\perp L = \Lambda^1(L), \qquad \xi \mapsto -(J\xi)^\flat,$$

then the Jacobi operator J is given by

$$J : \Gamma(\Lambda^1(L)) \longrightarrow \Gamma(\Lambda^1(L)), \qquad J = \Delta + c,$$

where Δ is the Hodge Laplacian operator. In particular,

> *Any compact minimal Lagrangian submanifold of a Kähler-Einstein manifold with negative scalar curvature ($c < 0$) is stable and its nullity is zero.*

Also, when the scalar curvature of the ambient manifold is zero, we have

> *Any compact minimal Lagrangian submanifold of a Ricci flat Kähler manifold ($c = 0$) is stable and its nullity coincides with its first Betti number,*

and when the scalar curvature is positive,

> *The index is the number of eigenvalues of* Δ *less than c, and the nullity is the multiplicity of c as an eigenvalue of* Δ.

As a consequence, if we denote by $b_1(L)$ the first Betti number of L, it follows that, when $c = 0$, $\mathrm{nul}(L) \geq b_1(L)$ and that, when $c > 0$, $\mathrm{ind}(L) \geq b_1(L)$.

When the ambient space is the complex projective space $\mathbb{C}P^n$, Lawson and Simons (see [33]) proved that *any compact minimal Lagrangian submanifold of* $\mathbb{C}P^n$ *is unstable* and Urbano proved in [55] that *the index of a compact orientable Lagrangian surface of* $\mathbb{C}P^2$ *is at least* 2 *and only the Clifford torus* $\mathbb{T}^2$ *(see Example 3 in Section 5) has index* 2. As a consequence of a result proved by Oh ([45]) we can get a lower bound for the nullity of this kind of submanifolds.

Corollary 28. *Let L be a compact minimal Lagrangian submanifold immersed in* $\mathbb{C}P^n$. *Then*

$$\mathrm{nul}(L) \geq \frac{n(n+3)}{2},$$

and the equality holds if and only if L is either the totally geodesic Lagrangian immersion of the unit sphere $\mathbb{S}^n$ *or the totally geodesic Lagrangian embedding of the real projective space* $\mathbb{R}P^n$ *(Examples 1 and 2 in Section 5).*

Proof : It is clear that if X is a Killing vector field in $\mathbb{C}P^n$, then the normal component of its restriction to the submanifold L is an eigenvector of the Jacobi operator J with eigenvalue 0. These normal vectors fields are called Jacobi vector fields on L. If Ω denotes the vector space of Killing vector fields on $\mathbb{C}P^n$, we will denote by $\Omega^\perp$ the vector space of the normal components of their restrictions to L. Then, Oh proved in [45, Proposition 4.1] that $\dim \Omega^\perp \geq n(n+3)/2$ and the equality implies that the submanifold is totally geodesic. Hence $\mathrm{ind}(L) \geq n(n+3)/2$ and the equality implies that L is totally geodesic. The proof is complete if we take into account that $2(n+1)$ is an eigenvalue of Δ (acting on 1-forms) with multiplicity $n(n+3)/2$ when L is the totally geodesic Lagrangian sphere $\mathbb{S}^n$ or the totally geodesic Lagrangian real projective space $\mathbb{R}P^n$ (see [15]). Q.E.D.

For other Kähler-Einstein manifolds with positive scalar curvature only very few results are known. We mention one due to Takeuchi [53], which says

If M is a Hermitian symmetric space of compact type and L is a compact totally geodesic Lagrangian submanifold embedded *in M, then L is stable if and only if L is simply-connected.*

As a consequence of Theorem 22, it follows:

Corollary 29. *Let L be a compact minimal Lagrangian submanifold immersed into a complex torus $\mathbb{C}^n/\Gamma$, where Γ is a lattice of complex translations. Then* $\mathrm{nul}(L) \geq n$.

Also, from Theorem 24, we have the following consequence.

Corollary 30. *Let L be a compact minimal Lagrangian submanifold immersed in a Kähler-Einstein compact manifold M of dimension $2n$ and scalar curvature $4n(n+1)$. If M has a complex contact structure and the level of L is one, i.e. $n_L = 1$, then L is unstable.*

Proof : In this case, the Jacobi operator is $J = \Delta + 2(n+1)$. But from Theorem 24, it follows that there exists a nontrivial 1-form on L which is an eigenform of Δ associated with the eigenvalue $2(n-1)$. Hence we obtain a negative eigenvalue of J. So the corollary follows. Q.E.D.

Corollary 31. *Let L be a compact minimal Lagrangian submanifold immersed in the complex projective space $\mathbb{C}P^n$. Then,*

(1) *If the level of L is one, i.e. $n_L = 1$, then*
$$\mathrm{ind}(L) \geq \frac{(n+1)(n+2)}{2},$$
and equality holds if and only if L is the totally geodesic Lagrangian immersion of the n-dimensional unit sphere $\mathbb{S}^n$ (Example 1 in Section 5).

(2) *If the level of L is two, i.e. $n_L = 2$, then*
$$\mathrm{ind}(L) \geq \frac{n(n+1)}{2},$$
and equality holds if and only if L is the totally geodesic Lagrangian embedding of the n-dimensional real projective space $\mathbb{R}P^n$ (Example 2 in Section 5).

Proof : If $n_L = 1$, then, taking in Theorem 24 $r = 1, 2$, we obtain two orthogonal eigenspaces of Δ in $\Gamma(\Lambda^1(L))$ of dimensions $n+1$ and $n(n+1)/2$ corresponding to the eigenvalues n and $2(n-1)$ respectively. These spaces are also eigenspaces of the Jacobi operator J with eigenvalues $-(n+2)$ and -4 respectively. Hence the inequality in (1) follows.

If $n_L = 2$, then, taking in Theorem 24 $r = 2$, we obtain an eigenspace of Δ in $\Gamma(\Lambda^1(L))$ of dimension $n(n+1)/2$ corresponding to the eigenvalue $2(n-1)$. This space is also an eigenspace of the Jacobi operator J with eigenvalue -4. Hence the inequality in (2) follows.

To study equality, we need to use [55, Theorem 1]. Such a result states that the first eigenvalue ρ_1 of Δ acting on 1-forms of any compact minimal Lagrangian submanifold of $\mathbb{C}P^n$ satisfies $\rho_1 \le 2(n-1)$ and that, if the equality holds, then L is totally geodesic. As the index of the totally geodesic real projective space $\mathbb{R}P^n$ is $n(n+1)/2$, this completes the proof of part (2).

To study equality in part (1) we need a modified version of [55, Theorem 1]. As in this case $n_L = 1$, L is orientable and hence from the Hodge decomposition Theorem, we have that $\Gamma(\Lambda^1(L)) = \mathcal{H}(L) \oplus d\,C^\infty(L) \oplus \delta\,\Gamma(\Lambda^2(L))$, where $\mathcal{H}(L)$ stands for harmonic 1-forms. To prove [55, Theorem 1], the author used certain *test* 1-forms that actually belong to $\mathcal{H}(L) \oplus \delta\,\Gamma(\Lambda^2(L))$. So really this Theorem says that the first eigenvalue ρ_1 of Δ acting on $\mathcal{H}(L) \oplus \delta\,\Gamma(\Lambda^2(L))$ satisfies $\rho_1 \le 2(n-1)$, with equality implying that L is totally geodesic. Now, if equality in (1) holds we have that the two first eigenvalues of Δ acting on 1-forms are n and $2(n-1)$ with multiplicities $n+1$ and $n(n+1)/2$ respectively. But the eigenspace corresponding to n is included in $dC^\infty(L)$. So we can use the modified version of the above result to get that L is totally geodesic. Q.E.D.

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