A note on acyclic vertex-colorings

Jean-Sébastien Sereni, Jan Volec

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Abstract

We prove that the acyclic chromatic number of a graph with maximum degree Δ is less than $2.835\Delta^{4/3}+\Delta$. This improves the previous upper bound, which was $50\Delta^{4/3}$. To do so, we draw inspiration from works by Alon, McDiarmid and Reed and by Esperet and Parreau.

1 Introduction

In 1973, Grübaum [8] considered proper colorings of graphs with an additional constraint: the subgraph induced by every pair of color classes is required to be acyclic. Such colourings are coined *acyclic colorings* and the least integer k such that a graph G admits an acyclic coloring with k colors is the *acyclic chromatic number* $\chi_a(G)$ of G.

Three years later, Erdős (see [1]) raised the question of determining the maximum possible value of $\chi_a(G)$ over all graphs G with maximum degree Δ. Let $\chi_a(\Delta)$ be this value. A first indication is given by the following observation: for every graph G, any proper coloring of G^2 is an acyclic coloring of G. Therefore, $\chi_a(\Delta) \leqslant \Delta^2+1$. However, Erdős conjectured a stronger statement, namely that $\chi_a(\Delta) = o(\Delta^2)$ as Δ tends to infinity.

This conjecture was confirmed about a quarter century later, by Alon, McDiarmid and Reed [2]. Relying on the Lovász Local Lemma [5], they established the following upper bound.

Theorem 1 (Alon, McDiarmid & Reed [2]). *For every positive integer Δ,*

$$\chi_a(\Delta) \leqslant 50\Delta^{4/3}.$$

* *Centre National de la Recherche Scientifique* (LORIA), Vandœuvre-lès-Nancy, France. E-mail: sereni@kam.mff.cuni.cz. This author's work was partially supported by the French *Agence Nationale de la Recherche* under reference ANR 10 JCJC 0204 01.

† Mathematics Institute and DIMAP, University of Warwick, Coventry CV4 7AL, UK. E-mail: honza@ucw.cz. This author's work was supported by a grant of the French Government.

This upper bound, more than confirming Erdős's conjecture, turns out to be of order very close to that of $\chi_a(\Delta)$. Indeed, Alon, McDiarmid and Reed [2] further proved that

$$\chi_a(\Delta) = \Omega\left(\frac{\Delta^{4/3}}{(\log \Delta)^{1/3}}\right).$$

Our goal is to exploit the recent advances regarding algorithmic versions of the Local Lemma, inspired by the incompressibility arguments. In 2009 Moser [11] and, in 2010, Moser and Tardos [12] designed strong algorithmic versions of the Local Lemma. More importantly for our purposes, while preparing his talk for the Symposium on Theory of Computing, Moser found a simpler proof of his result from 2009. The technique used in this proof became known as the "entropy compression" argument; the reader is referred to Fortnow's website [7] and Tao's blog [17] for more details.

Independently, Schweitzer [16] pursued a similar line of research, explaining how to obtain constructive bounds on van der Warden numbers. His work was subsequently improved by Kulich and Kemeňová [10] to precisely match the known non-constructive results.

All these ideas inspired new adaptations and more efficient uses of the essence of the Local Lemma to tackle various combinatorial questions, in particular graph colouring problems [4, 6, 9, 14, 15] and problems related to pattern avoidance [13]. We draw inspiration from the original work of Alon, McDiarmid & Reed [2] and a recent result of Esperet & Parreau [6] to establish the following upper bound.

$$\chi_a(\Delta) \leqslant \frac{9}{2^{5/3}} \cdot \Delta^{4/3} + \Delta < 2.83483 \cdot \Delta^{4/3} + \Delta.$$

2 Proof of the Upper Bound

We shall use certain standard estimates on the number of Dyck words with all descent of even lengths. A *partial Dyck word* is a bit string w such that no prefix of w contains more ones than zeros. A *Dyck word* is a partial Dyck word of length $2t$ with exactly t zeros. A *descent* in a partial Dyck word is a maximal sequence of consecutive ones.

The following lemma is a special case of [6, Lemmas 7 and 8] for Dyck words with all descents of even length. It follows from a folklore bijection between Dyck words and plane trees, and the asymptotic results for counting such trees; see, e.g., [3, Theorem 5]. More details are found in the work of Esperet and Parreau [6].

Lemma 2. *There exists an absolute constant C_{DYCK} such that the number of Dyck words of length $2t$ with all descents of even length is at most*

$$C_{\mathrm{DYCK}} \cdot \frac{\left(3\sqrt{3}/2\right)^t}{t^{3/2}}.$$

We also recall a special case of [6, Lemma 6].

Lemma 3. *Let r be a non-negative integer. The number of partial Dyck words with exactly t zeros, exactly $(t-r)$ ones, and all descents of even length is at most*

$$C_{\mathrm{DYCK}} \cdot \frac{\left(3\sqrt{3}/2\right)^{t+r}}{(t+r)^{3/2}}.$$

We are now ready to present our main result.

Theorem 4. *Fix a positive integer Δ and a real κ such that $\kappa \geqslant 2/\Delta^{2/3}$. If G is a graph with maximum degree Δ, then the acyclic chromatic number $\chi_a(G)$ is at most*

$$f(\Delta,\kappa) := \left(\frac{1}{\kappa} + \frac{3}{2}\sqrt{\frac{3\kappa}{2}}\right)\Delta^{4/3} + \Delta - \frac{\Delta^{1/3}}{\kappa}.$$

In particular, if $\Delta \geqslant 3$ and $\kappa = \frac{2^{5/3}}{3}$, it follows that $\chi_a(G) \leqslant \frac{9}{2^{5/3}} \cdot \Delta^{4/3} + \Delta <$ $2.83483 \cdot \Delta^{4/3} + \Delta$.

Proof. Fix a graph G with maximum degree Δ. Without loss of generality, let $V(G) = \{1, \ldots, n\}$. The main idea of the proof is as follows.

We first consider a randomized procedure that takes as input a partial acyclic coloring of G using $f(\Delta,\kappa)$ colors and tries to assign a random color from a specifically restricted subset of $f(\Delta,\kappa)$ colors to the smallest (with respect to its number) uncolored vertex v. If the partial coloring extended by the coloring of v is still a partial acyclic coloring of G, then the procedure ends — and thus this extended partial coloring is kept. On the other hand, if the coloring of v creates a two-colored cycle, or if v is assigned the same color as one of its neighbors, then the procedure uncolors a specific subset of colored vertices (which includes v) and then ends. This procedure is called **EXTEND**.

Next, we set up a procedure **LOG** that creates a compact record containing enough information to be able to perform the following. Suppose we have a partial acyclic coloring c of G with $f(\Delta,\kappa)$ colors. We execute EXTEND and obtain a new partial acyclic coloring c' of G. Furthermore, let x be the (randomly chosen) color that EXTEND tried to assign to the smallest uncolored vertex v in c. The record constructed by LOG shall contain enough information that it is possible to reconstruct both c and x from the record and c'. Our aim is to create the record in such a way that, in an amortized sense, its size is smaller than that of the list that EXTEND can choose the color x from.

Finally, we consider the following randomized coloring algorithm. Start with an empty coloring, that is, every vertex is uncolored in the initial partial coloring. Then repeatedly execute the procedures EXTEND and LOG until all the vertices of G are assigned a color in the current partial coloring. One execution of EXTEND followed by one execution of LOG is called a *step* of the algorithm.

Note that the algorithm might never terminate. However, we show that the probability that it actually does terminate, after sufficiently many steps, is positive. This will follow from the fact that after t steps (for a sufficiently large integer t), the number of ways how to t-times choose a color in the procedure EXTEND will be (strictly) greater than the number of all possible records corresponding to the executions that have not terminated in t steps times the number of all possible precolorings (recall our aim to make the amortized size of a record small).

Let us now be precise. For a vertex $v \in V(G)$, let $D(v)$ be the set of vertices $u \in V(G)$ different from v such that the number of common neighbors of u and v is at least $\kappa \cdot \Delta^{2/3}$. By symmetry, $u \in D(v) \iff v \in D(u)$. A vertex $u \in D(v)$ is said to be *dangerous for v*.

If u and v are dangerous for each other, then there are lots of 4-cycles containing both u and v, namely $\Omega(\Delta^{4/3})$. This is why the procedure EXTEND is designed in such a way that it never tries to assign to v a color that is currently assigned to a vertex that is dangerous for v. Similarly, the procedure shall never try to assign to v a color that is currently assigned to one of the neighbors of v. Formally, for a partial acyclic coloring c, we let

- $c[N(v)]$ be the set of colors assigned in c to the neighbors of v;
- $c[D(v)]$ be the set of colors assigned in c to the vertices that are dangerous for v; and
- $L_c(v) := \{1, 2, \ldots, f(\Delta, \kappa)\} \setminus \big(c[N(v)] \cup c[D(v)]\big)$.

Note that $|c[N(v)]| \leqslant \Delta$. Moreover, $|c[D(v)]| \leqslant \left(\Delta^{4/3} - \Delta^{1/3}\right)/\kappa$. Indeed, since the number of edges $\{w, w'\}$ with $w \in N(v)$ and $w' \in V(G) \setminus \{v\}$ is at most $\Delta(\Delta - 1)$, the size of $D(v)$ is at most $\left(\Delta^{4/3} - \Delta^{1/3}\right)/\kappa$.

Therefore,

$$|L_c(v)| \geqslant \frac{3}{2}\sqrt{\frac{3\kappa}{2}} \cdot \Delta^{4/3}.$$

For the simplicity of our analysis, we shall always assume that $|L_c(v)| = \frac{3}{2}\sqrt{\frac{3\kappa}{2}} \cdot \Delta^{4/3}$ (in the case of having a strict inequality for some choice of c and v, we simply remove $|L_c(v)| - \frac{3}{2}\sqrt{\frac{3\kappa}{2}} \cdot \Delta^{4/3}$ colors from $L_c(v)$ arbitrarily).

Next, for a vertex v and an integer k, we give an upper bound on the number of $2k$-cycles incident with v that could become two-colored at some step of the execution of the algorithm.

Assertion 1. *For a vertex $v \in V(G)$ and an integer $k \geqslant 2$, let $\mathcal{C}_{2k}(v)$ be the set of all $2k$-cycles $W = v, w_2, w_3, \ldots, w_{2k}$ incident with v such that no two vertices at distance two on W are dangerous for each other. Then*

$$|\mathcal{C}_{2k}(v)| < \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^{2k-2}.$$

Proof. We actually show that

$$|\mathcal{C}_{2k}(v)| < \frac{\kappa}{2} \cdot \Delta^{2k-4/3}.$$

Since $\kappa \geqslant 2/\Delta^{2/3}$ and $k \geqslant 2$, we have $\frac{\kappa}{2} \cdot \Delta^{2k-4/3} \leqslant \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^{2k-2}$ and the statement then follows. First, there are at most $\binom{\Delta}{2} < \Delta^2/2$ choices of w_2 and w_{2k}. Fix a choice of w_2 and w_{2k}. Next, we fix one by one the vertices $w_3, w_4, \ldots, w_{2k-2}$; for each of them, there are at most $\Delta - 1 < \Delta$ choices. Finally, since w_{2k-2} and w_{2k} are not dangerous for each other, there are less than $\kappa \cdot \Delta^{2/3}$ choices to choose w_{2k-1}. Combining all estimates together, we conclude that

$$|\mathcal{C}_{2k}(v)| < \frac{\kappa}{2} \cdot \Delta^{2+2k-4+2/3} = \frac{\kappa}{2} \cdot \Delta^{2k-4/3}.$$

$\square$

The last bit that we need to describe the procedure EXTEND is to fix linear orderings on the $2k$-cycles in $\mathcal{C}_{2k}(v)$ for every $v \in V(G)$ and $k \geqslant 2$. Fix v and k, and consider a $2k$-cycle $v, w_2, w_3, \ldots, w_{2k}$ containing v. We define the identifier of the cycle as follows: if $w_2 < w_{2k}$, then the identifier is $w_2 w_3 \ldots w_{2k}$; otherwise, it is $w_{2k} w_{2k-1} \ldots w_2$. The linear ordering $\mathcal{O}_{2k}(v)$ of the elements of $\mathcal{C}_{2k}(v)$ is just given by the lexicographical ordering of their identifiers.

Now we are ready to describe the procedure EXTEND. It takes as input a partial acyclic coloring c, and outputs a new partial acyclic coloring c'. The procedure is defined as follows.

- Let v be the smallest uncolored vertex in c.

- Pick a color x uniformly at random from the list $L_c(v)$.
- If the extension of c obtained by assigning the color x to v is a partial acyclic coloring of G, then we set c' to be this extension.
- Otherwise, let $\mathcal{W}$ be the set of all two-colored cycles in the extension of c. Let $W \in \mathcal{W}$ be the $2k$-cycle that has the largest length and, subject to that, the lexicographically smallest identifier $w_2w_3 \dots w_{2k}$. We set c' to be the restriction of c to the vertices $V \setminus \{w_4, w_5, \dots, w_{2k}\}$, i.e., we uncolor the vertex set of W except the two adjacent vertices w_2 and w_3.

We continue with the description of the procedure LOG. At the end of its t-th execution, LOG outputs a record R^t that is based on the previous record R^{t-1} and the coloring and possible uncolorings that happened during the t-th execution of EXTEND. In order to make the analysis easier, we decompose R^t into two parts R_1^t and R_2^t and analyse them separately. A record R_1^t shall be a bit string that keeps track of all colorings and uncolorings that have been performed during the first t executions of EXTEND, and a record R_2^t shall be an integer that stores the information about the $2k$-cycles that have been uncolored.

We thus define R_1^t and R_2^t recursively. For convenience, we let R_1^0 be the empty string and $R_2^0 \coloneqq 0$. Now assume that $t \geqslant 1$. Let v be the smallest uncolored vertex after the $(t-1)$-th execution of EXTEND, so $v = 1$ if $t = 1$. If the t-th execution of EXTEND assigns a color to v and keeps the extended colouring, then we set R_1^t to be R_1^{t-1} to which we append one 0, and $R_2^t \coloneqq R_2^{t-1}$. Otherwise, let W be the $2k$-cycle uncolored during the t-th execution of EXTEND, and let z be the index of W in $\mathcal{C}_{2k}(v)$ ordered according to $\mathcal{O}_{2k}(v)$. Recall that z is always an integer between 1 and $\max\{|\mathcal{C}_{2k}(v)| : v \in V(G)\}$, which is at most $\left\lfloor \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^{2k-2} \right\rfloor$. We let R_1^t be R_1^{t-1} to which we append one 0 and $(2k-2)$ ones, and we set

$$R_2^t \coloneqq R_2^{t-1} \cdot \left\lfloor \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^{2k-2} \right\rfloor + (z-1).$$

Let us realize that the records R_1^{t-1} and R_2^{t-1} can be reconstructed from the records R_1^t and R_2^t. Indeed, let p be the position of the last 0 in R_1^t and let q be the number of ones after this 0, noting that q might be equal to zero. Then R_1^{t-1} is equal to the first $p-1$ elements of R_1^t and R_2^{t-1} is equal to

$$\left\lfloor R_2^t \Big/ \left\lfloor \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^{q} \right\rfloor \right\rfloor.$$

Our next step is to show that the records R_1^t and R_2^t actually also contain enough information to determine the set of uncolored vertices after t steps of the algorithm.

Assertion 2. *For any positive integer t, the records R_1^t and R_2^t determine the set V_t, defined to be the set of uncolored vertices of G after t steps of the algorithm.*

Proof. We prove the statement by induction on the positive integer t. If $t = 1$, then necessarily R_1^1 is the list containing only one zero, $R_2^1 = 0$, and $V_t = \{2, 3, \dots, n\}$. Suppose now that $t > 1$. As we observed above, R_1^t and R_2^t determine the records R_1^{t-1} and R_2^{t-1}. By the induction hypothesis, R_1^{t-1} and R_2^{t-1} determine V_{t-1}. Therefore, we can find the smallest vertex v in V_{t-1}, which is the vertex that EXTEND attempts to color in the t-th step.

If R_1^t is equal to R_1^{t-1} with one 0 appended, then coloring v has not created any two-colored cycle and hence $V_t = V_{t-1} \setminus \{v\}$. On the other hand, if R_1^t is equal to R_1^{t-1} with one 0 and q ones appended, where $q \geqslant 1$, then we set $z := \left(R_2^t \bmod \left\lfloor \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^q \right\rfloor\right) + 1$ and we let $w_2 w_3 \dots w_{q+2}$ be the identifier of the z-th element of $\mathcal{C}_{q+2}(v)$ according to $\mathcal{O}_{q+2}(v)$. Since this was the $(q+2)$-cycle that was uncolored during the t-th execution of EXTEND, we deduce that $V_t = V_{t-1} \setminus \{w_4, w_5, \dots, w_{q+2}\}$. □

Finally, we show that the records R_1^t and R_2^t together with the partial coloring after t steps fully determine the partial coloring after $t-1$ steps of the algorithm.

Assertion 3. *Fix a positive integer t. Let c be the partial coloring of G obtained after $t-1$ steps of the algorithm, c' the partial coloring after t steps, and x the color that was used to color the smallest uncolored vertex during the t-th execution of EXTEND. Then R_1^t, R_2^t and c' determine both x and c.*

Proof. Again, we prove the assertion by induction on the positive integer t. If $t = 1$, then c' contains exactly one colored vertex. Its color is x and c is indeed the empty coloring.

Let $t > 1$. We first use R_1^t and R_2^t to determine the records R_1^{t-1} and R_2^{t-1}. Next, we utilize Assertion 2 and, using R_1^{t-1} and R_2^{t-1}, we determine the smallest uncolored vertex v after the $(t-1)$-th step of the algorithm. Now, as in the proof of Assertion 2, the records $R_1^{t-1}, R_2^{t-1}, R_1^t$ and R_2^t are used to determine if the coloring of v at the t-th execution of EXTEND has created a two-colored cycle or not. In the former case, we also determine, again in the same way as in the proof of Assertion 2, the identifier $w_2 w_3 \dots w_{2k}$ of the two-colored $2k$-cycle incident with v that was uncolored by EXTEND.

If there was no two-colored cycle, then clearly $x = c'(v)$ and c can be obtained from c' by uncoloring the vertex v. On the other hand, if EXTEND uncolored the $2k$-cycle with the identifier $w_2 w_3 \dots w_{2k}$, then we know that $x = c'(w_3)$ and c can be obtained by modifying c' in the following way: we color the vertices $w_4, w_6, \dots, w_{2k}$ with the color $c'(w_2)$, and the vertices $w_5, w_7, \dots, w_{2k-1}$ with the color $c'(w_3)$. $\square$

Before we continue the exposition and present our upper bounds on the number of possible records that the procedure LOG can create, let us introduce some additional notation. Again, we consider the situation just after t steps of the algorithm. For an integer $i \leqslant t$, let u_i be the number of vertex-uncolorings that were performed during the i-th execution of EXTEND. Specifically, if the coloring that was performed at the i-th execution did not create any two-colored cycle, then $u_i = 0$. On the other hand, if during this execution EXTEND uncolored a two-colored $2k$-cycle, then $u_i = 2k - 2$. Next, let $U_i := \sum_{j=1}^{t} u_j$, that is, U_i is the total number of vertex-uncolorings that were performed from the beginning of the first step till the end of the i-th step. Since each execution of EXTEND performs exactly one vertex-coloring, it follows that $U_i \leqslant i$ for every $i \leqslant t$. (In fact, one even sees that $U_i < i$.)

We are now ready to present the following two assertions which, assuming that the algorithm has not colored the whole graph after t steps, give upper bounds on the number of possible records R_1^t and R_2^t, respectively.

Assertion 4. *Let $\mathcal{R}_1^t$ be the set of all possible records R_1^t that can be obtained by performing t steps of the algorithm that do not result in coloring the whole graph G. Then there exists an absolute constant C, depending only on G and not on t, such that*

$$\left|\mathcal{R}_1^t\right| \leqslant C \cdot \frac{\left(3\sqrt{3}/2\right)^t}{t^{3/2}}.$$

Proof. Let c be the partial coloring of G obtained after t steps of the algorithm. Assume that c is not an acyclic coloring of the whole graph G.

By its definition, the record R_1^t contains exactly t zeros, and for each $i \leqslant t$, the i-th zero is followed by exactly u_i ones. Since $U_i \leqslant i$ for all $i \leqslant t$, the record R_1^t is a partial Dyck word. Thus the number of 1's in R_1^t can be written as $t - r$ for some non-negative integer r. Further, the difference between the number of 0's and the number of 1's in R_1^t is equal to the number of colored vertices in c, hence $r \leqslant n - 1$. Therefore, it follows from Lemma 3 that

$$\left|\mathcal{R}_1^t\right| \leqslant \sum_{r=0}^{n-1} C_{\mathrm{DYCK}} \cdot \frac{\left(3\sqrt{3}/2\right)^{t+r}}{(t+r)^{3/2}} \leqslant \left(n \cdot C_{\mathrm{DYCK}} \cdot \left(3\sqrt{3}/2\right)^{n-1}\right) \cdot \frac{\left(3\sqrt{3}/2\right)^t}{t^{3/2}}.$$

$\square$

Assertion 5. *For any positive integer t, the record R_2^t is an integer satisfying*

$$0 \leqslant R_2^t \leqslant \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^{U_t} - 1.$$

Proof. We prove the statement by induction on U_t. If $U_t = 0$, then $R_2^t = 0$. Assume now that $U_t > 0$. Let i be the number of the step where the U_t-th uncoloring occurs. Thus, during the i-th step, the procedure EXTEND attempts to color a vertex v, which creates a two-colored cycle. Let ℓ be the length of this cycle and z its index in $\mathcal{C}_{2k}(v)$ ordered by $\mathcal{O}_{2k}(v)$. Assertion 1 implies that the integer z is at most $\left\lfloor \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^{\ell-2} \right\rfloor$. Moreover, the induction hypothesis ensures that R_2^{i-1} is an integer satisfying

$$0 \leqslant R_2^{i-1} \leqslant \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^{U_t-(\ell-2)} - 1.$$

The conclusion follows, since

$$\begin{aligned} R_2^t &= R_2^{i-1} \cdot \left\lfloor \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^{\ell-2} \right\rfloor + (z-1) \\ &\leqslant \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^{U_t} - \left\lfloor \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^{\ell-2} \right\rfloor + (z-1) \\ &\leqslant \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^{U_t} - 1. \end{aligned}$$

□

Since R_2^t is always an integer and $U_t \leqslant t$, we immediately deduce the following.

Corollary 6. *For any positive integer t, the record R_2^t is an integer between 0 and $\left\lfloor \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^{t} \right\rfloor - 1$.*

The only thing that remains to do in order to finish the proof of Theorem 4 is to combine the assertions together. Let C_{COL} be the number of all possible partial acyclic colorings of G using $f(\Delta,\kappa)$ colors. So $C_{\mathrm{COL}} \leqslant (f(\Delta,\kappa)+1)^n$. Therefore, using Assertion 4 and Corollary 6, we infer that there are at most

$$C_{\mathrm{COL}} \cdot C \cdot \left(\left(3\sqrt{3}/2\right)^t \cdot t^{-3/2}\right) \cdot \left(\Delta^{4/3} \cdot \sqrt{\kappa/2}\right)^t = o(1) \cdot \left(\frac{3}{2}\sqrt{\frac{3\kappa}{2}} \cdot \Delta^{4/3}\right)^t$$

choices for a tuple (c', R_1^t, R_2^t), where the $o(1)$ term tends to 0 as t tends to infinity. On the other hand, by repeatedly applying Assertion 3, a tuple (c', R_1^t, R_2^t)

determines the (randomly chosen) color x at the i-th step for every $i \leqslant t$. Therefore, assuming that the algorithm has not terminated after the t-th step — that is, there are still some uncolored vertices — it had at most $o(1) \cdot \left(\frac{3}{2}\sqrt{\frac{3\kappa}{2}} \cdot \Delta^{4/3}\right)^t$ possible ways how to choose the colors from the corresponding lists. Hence, if t is large enough, the algorithm terminates with a positive probability — in fact, this probability tends to 1 as t tends to infinity.

We conclude that

$$\chi_a(G) \leqslant f(\Delta, \kappa) = \frac{3}{2}\sqrt{\frac{3\kappa}{2}}\Delta^{4/3} + \left(\Delta^{4/3} - \Delta^{1/3}\right)/\kappa + \Delta,$$

which finishes the proof. □

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