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PURE-INJECTIVE HULLS OF MODULES OVER VALUATION RINGS

FRANÇOIS COUCHOT

Abstract. If \( \hat{R} \) is the pure-injective hull of a valuation ring \( R \), it is proved that \( \hat{R} \otimes_R M \) is the pure-injective hull of \( M \), for every finitely generated \( R \)-module \( M \). Moreover \( \hat{R} \otimes_R M \cong \oplus_{1 \leq k \leq n} \hat{R}/A_k \hat{R} \), where \( (A_k)_{1 \leq k \leq n} \) is the annihilator sequence of \( M \). The pure-injective hulls of uniserial or polyserial modules are also investigated. Any two pure-composition series of a countably generated polyserial module are isomorphic.

The aim of this paper is to study pure-injective hulls of modules over valuation rings. If \( R \) is a valuation domain and \( S \) a maximal immediate extension of \( R \), then, in [10], Warfield proved that \( S \) is a pure-injective hull of \( R \). Moreover, for each finitely generated \( R \)-module \( M \), he showed that \( S \otimes_R M \) is a pure-injective hull of \( M \) and a direct sum of gen \( M \) indecomposable pure-injective modules. We extend this last result to every valuation ring \( R \) by replacing \( S \) with the pure-injective hull \( \hat{R} \) of \( R \). As in the domain case \( \hat{R} \) is a faithfully flat module. Moreover, for each \( x \in \hat{R} \) there exist \( r \in R \) and \( y \in 1 + \mathfrak{P} \hat{R} \) such that \( x = ry \). This property allows us to prove most of the main results of this paper. We extend results obtained by Fuchs and Salce on pure-injective hulls of uniserial modules over valuation domains ([5, chapter XIII, section 5]). We show that the length of any pure-composition series of a polyserial module \( M \) is its Malcev rank \( Mr_M \) and its pure-injective hull \( \hat{M} \) is a direct sum of \( p \) indecomposable pure-injective modules, where \( p \leq Mr_M \). But it is possible to have \( p < Mr_M \) and we prove that the equality holds for all \( M \) if and only if \( R \) is maximal (Theorem 4.5). This result is a consequence of the fact that \( R \) is maximal if and only if \( R/N \) and \( R \mathfrak{N} \) are maximal, where \( N \) is the nilradical of \( R \) (Theorem 4.4). If \( U_1, \ldots, U_n \) are the factors of a pure-composition series of a polyserial module \( M \) then the collection \( (\hat{R} \otimes_R U_k)_{1 \leq k \leq n} \) is uniquely determined by \( M \). To prove this, we use the fact that \( \hat{R} \otimes_R U \) is an unshrinkable uniserial \( T \)-module for each uniserial \( R \)-module \( U \), where \( T = \text{End}_R(\hat{R}) \). When \( R \) satisfies a countable condition, the collection of uniserial factors of a polyserial module \( M \) is uniquely determined by \( M \) (Proposition 3.7).

In this paper all rings are associative and commutative with unity and all modules are unital. As in [8] we say that an \( R \)-module \( E \) is divisible if, for every \( r \in R \) and \( x \in E \), \( (0 : r) \subseteq (0 : x) \) implies that \( x \in rE \), and that \( E \) is \textbf{fp-injective} (or \textbf{absolutely pure}) if \( \text{Ext}^1_R(F, E) = 0 \), for every finitely presented \( R \)-module \( F \). A ring \( R \) is called \textbf{self fp-injective} if it is fp-injective as \( R \)-module. An exact sequence \( 0 \to F \to E \to G \to 0 \) is \textbf{pure} if it remains exact when tensoring it with any \( R \)-module. In this case we say that \( F \) is a \textbf{pure} submodule of \( E \). Recall that a module \( E \) is fp-injective if and only if it is a pure submodule of every overmodule. A module is said to be \textbf{uniserial} if its submodules are linearly ordered.
by inclusion and a ring $R$ is a valuation ring if it is uniserial as $R$-module. Recall that every finitely presented module over a valuation ring is a finite direct sum of cyclic modules [3, Theorem 1]. Consequently a module $E$ over a valuation ring $R$ is fp-injective if and only if it is divisible.

An $R$-module $F$ is pure-injective if for every pure exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$$

of $R$-modules, the following sequence

$$0 \rightarrow \text{Hom}_R(L, F) \rightarrow \text{Hom}_R(M, F) \rightarrow \text{Hom}_R(N, F) \rightarrow 0$$

is exact. An $R$-module $B$ is a pure-essential extension of a submodule $A$ if $A$ is a pure submodule of $B$ and, if for each submodule $K$ of $B$, either $K \cap A \neq 0$ or $(A + K)/K$ is not a pure submodule of $B/K$. We say that $B$ is a pure-injective hull of $A$ if $B$ is pure-injective and a pure-essential extension of $A$. By [10, or [3, chapter XIII] each $R$-module $M$ has a pure-injective hull and any two pure-injective hulls of $M$ are isomorphic.

In the sequel $R$ is a valuation ring, $P$ its maximal ideal, $Z$ its subset of zerodivisors and $M$ the pure-injective hull of $M$, for each $R$-module $M$. As in [3, p.69], for every proper ideal $A$, we put $A^* = \{s \in R \mid (A : s) \neq A\}$. Then $A^*/A$ is the set of zerodivisors of $R/A$ whence $A^*$ is a prime ideal. In particular $\{0\}^3 = Z$. When $A^* = P$, we say that $A$ is an archimedean ideal. Then $A$ is archimedean if and only if $R/A$ is self fp-injective.

1. Properties of $\hat{R}$

The first assertion of the following proposition will play a crucial role to prove the main results of this paper.

**Proposition 1.1.** The following assertions hold:

1. For each $x \in \hat{R}$ there exist $a \in R$, $p \in P$ and $y \in \hat{R}$ such that $x = a + py$.
2. For each archimedean ideal $A$ of $R$, $\hat{R}/A\hat{R}$ is an essential extension of $R/A$.
3. $\hat{R}/P\hat{R} \cong R/P$.

**Proof.** The third assertion is an immediate consequence of the first.

We also deduce the second assertion from the first. Since $R$ is a pure submodule of $\hat{R}$, the natural map $R/A \rightarrow \hat{R}/A\hat{R}$ is monic. Let $x \in \hat{R} \setminus R + A\hat{R}$. We have $x = a + py$ for $a \in R$, $p \in P$ and $y \in \hat{R}$. Hence $pa \notin A$. Since $A$ is archimedean, there exists $r \in (A : pa) \setminus (A : a)$. So $rx \in R + A\hat{R} \setminus A\hat{R}$.

We proceed by steps to prove the first assertion.

**Step 1.** Suppose that $R$ is self fp-injective. In this case, $\hat{R} \cong E_R(R)$ by [3, Lemma XIII.2.7]. We may assume that $x \notin R$. Then there exists $d \in \hat{R}$ such that $dx \in R$ and $dx \neq 0$. Since $\hat{R}$ is a pure submodule of $\hat{R}$ we have $dx = db$ for some $b \in R$. By [1, Lemma 2] $(0 : x) = (0 : b)$, whence $x = bz$ for some $z \in R$ since $\hat{R}$ is divisible. In the same way, there exists $c, u \in R$ such that $cz = cu \neq 0$. We get that $(0 : u) = (0 : z) = b(0 : b) = 0$. So $u$ is a unit of $R$. Since $z - u \notin R$, there exists $s, q \in R$ and $y \in \hat{R}$ such that $0 \neq sq = s(z - u) \in R$ and $z - u = qy$. We have $c \in (0 : z - u) = (0 : q)$. So $q \in P$. Now we put $a = bu$ and $p = qu^{-1}$ and we get $x = a + py$.

**Step 2.** Now we prove that $\hat{R}/r\hat{R} \cong E_{R/rR}(R/rR)$ for each $0 \neq r \in P$. If $\cap_{a \neq 0}aR = 0$ then it is an immediate consequence of [3, Theorem 5.6]. Else $P$ is
not faithful, \( R \) is self fp-injective and \( \hat{R} \cong E_p(R) \). By Step 1 and the implication 1 \( \Rightarrow 2 \) the second assertion holds. So it remains to show that \( \hat{R}/r\hat{R} \) is injective over \( \hat{R}/rR \). Let \( J \) be an ideal of \( R \) such that \( \hat{R}r \subset J \) and \( g: J/\hat{R}r \to \hat{R}/r\hat{R} \) be a nonzero homomorphism. For each \( x \in \hat{R} \) we denote by \( \hat{x} \) the image of \( x \) in \( \hat{R}/r\hat{R} \). Let \( a \in J \setminus Rr \) such that \( \hat{y} = g(\hat{a}) \neq 0 \). Then \( (Rr : a) \subseteq (r\hat{R} : y) \). Let \( t \in R \) such that \( r = at \). Thus \( ty = rz \) for some \( z \in \hat{R} \). It follows that \( t(y - az) = 0 \). So, since \( at = r \neq 0 \), we have \( (0 : a) \subset Rr \subseteq (0 : y - az) \). The injectivity of \( \hat{R} \) implies that there exists \( x \in \hat{R} \) such that \( y = a(x + z) \). We put \( x_a = x + z \). If \( b \in J \setminus Ra \) then \( a(x_a - x_b) \in r\hat{R} \). Hence \( x_b \in x_a + (r\hat{R} : a) \). Since \( \hat{R} \) is pure-injective, by [10, Theorem 4] there exists \( x \in \cap a \in J x_a + (r\hat{R} : a) \). It follows that \( g(\hat{a}) = ax \) for each \( a \in J \).

Step 3. Now we prove the first assertion in the general case. If \( \cap a \neq 0 \), then \( R \) is self fp-injective. So the result holds by Step 1. If \( \cap a \neq 0 \), we put \( F = \cap x R \hat{R} \). We will show that \( F = 0 \). Let \( x \in F \cap R \). Then \( x \in R \cap r\hat{R} = rR \) for each \( r \in R \), \( r \neq 0 \). Therefore \( x = 0 \) and \( F \cap R = 0 \). Let \( x \in \hat{R} \), \( r, a \in R \) and \( z \in F \) such that \( rx = a + z \). There exists \( y \in \hat{R} \) such that \( z = ry \). So \( r(x - y) = a \), whence there exists \( b \in R \) such that \( rb = a \). It follows that \( R \) is a pure submodule of \( R/F \). Since \( \hat{R} \) is a pure-essential extension of \( R \) we deduce that \( F = 0 \). Let \( x \in \hat{R} \). We may assume that \( x \notin \hat{R} \). There exists \( 0 \neq r \in R \) such that \( x \notin r\hat{R} \). If \( x \in R + r\hat{R} \) then \( x = a + ry \), with \( a \in R \) and \( y \in \hat{R} \). We have \( a \notin rR \) else \( x \notin r\hat{R} \). So \( r = pa \) for some \( p \in P \). If \( x \notin r + r\hat{R} \) then, since \( R/rR \) is self fp-injective, from Steps 1 and 2 we deduce that \( x - a - paz \in r\hat{R} \) for some \( a \in R \), \( p \in P \) and \( z \in \hat{R} \). It is obvious that \( a \notin rR \). Now it is easy to conclude. \( \square \)

As in the domain case we have:

**Proposition 1.2.** \( \hat{R} \) is a faithfully flat \( R \)-module.

**Proof.** Let \( x \in \hat{R} \) and \( r \in R \) such that \( rx = 0 \). By Proposition 1.1 there exist \( a \in \hat{R} \), \( p \in P \) and \( y \in \hat{R} \) such that \( x = a + pay \). So \( rpay \in \hat{R} \). It follows that there exists \( b \in R \) such that \( ra(1 + pb) = 0 \). Hence \( ra = 0 \) and \( r \otimes x = ra \otimes (1 + py) = 0 \).

## 2. Pure-injective hulls of uniserial modules

The following lemma and Proposition 2.2 will be useful to prove the pure-injectivity of some modules in the sequel.

**Lemma 2.1.** Let \( U \) be a module and \( F \) a flat module. Then, for each \( r, s \in R \), \( F \otimes_R (sU :_U r) \cong (F \otimes_R sU :_{sU} r) \).

**Proof.** We put \( E = F \otimes_R U \). Let \( \phi \) be the composition of the multiplication by \( r \) in \( U \) with the natural map \( U \to U/sU \). Then \( (sU :_U r) = \ker(\phi) \). It follows that \( F \otimes_R (sU :_U r) \) is isomorphic to \( \ker(1_F \otimes \phi) \) since \( F \) is flat. We easily check that \( 1_F \otimes \phi \) is the composition of the multiplication by \( r \) in \( E \) with the natural map \( E \to E/sE \). It follows that \( F \otimes_R (sU :_U r) \cong (sE :_E r) \).

**Proposition 2.2.** Every pure-injective \( R \)-module \( F \) satisfies the following property: if \( (x_i)_{i \in I} \) is a family of elements of \( F \) and \( (A_i)_{i \in I} \) a family of ideals of \( R \) such that the family \( F = (x_i + A_i F)_{i \in I} \) has the finite intersection property, then \( F \) has a non-empty intersection. The converse holds if \( F \) is flat.
Proof. Let \( i \in I \) such that \( A_i \) is not finitely generated. By Lemma 29, either \( A_i = P r_i \) or \( A_i = \bigcap_{r \in R \setminus A_i} c R \). If, \( \forall i \in I \) such that \( A_i \) is not finitely generated, we replace \( x_i + A_i F \) by \( x_i + r_i F \) in the first case, and by the family \( (x_i + c F)_{i \in I} \) in the second case, we deduce from \( \mathcal{F} \) a family \( \mathcal{G} \) which has the finite intersection property. Since \( \mathcal{F} \) is pure-injective, it follows that there exists \( x \in F \) which belongs to each element of the family \( \mathcal{G} \) by Theorem 4. We may assume that the family \( (A_i)_{j \in I} \) has no smallest element. So, if \( A_i \) is not finitely generated, there exists \( j \in I \) such that \( A_j \subset A_i \). Let \( c \in A_i \setminus P A_j \) such that \( x_j + c F \in \mathcal{G} \). Then \( x - x_j \in c F \subseteq A_i F \) and \( x - x_j \in A_i F \). Hence \( x - x_j \in A_i F \) for each \( i \in I \).

Conversely, if \( F \) is flat then by Lemma 2.3, we have \((s F : x) = (s R : x) F \) for each \( s, r \in R \). We use Theorem 4 to conclude. \( \square \)

Proposition 2.3. Let \( U \) be a uniserial module and \( F \) a flat pure-injective module. Then \( F \otimes_R U \) is pure-injective.

Proof. Let \( E = F \otimes_R U \). We use Theorem 4 to prove that \( E \) is pure-injective. Let \( (x_i)_{i \in I} \) be a family of elements of \( F \) such that the family \( \mathcal{F} = (x_i + N_i)_{i \in I} \) has the finite intersection property, where \( N_i = (s_i E : x_i r_i) \) and \( r_i, s_i \in R, \forall i \in I \).

First we assume that \( U = R/A \) where \( A \) is a proper ideal of \( R \). So \( E \cong F/AF \). If \( s_i \notin A \) then \( N_i = (s_i F : x_i r_i)/AF = (Rs_i : r_i)F/AF \). We set \( A_i = (Rs_i : r_i) \) in this case. If \( s_i \in A \) then \( N_i = (AF : x_i r_i)/AF = (A : r_i)F/AF \). We put \( A_i = (A : r_i) \) in this case. For each \( i \in I \), let \( y_i \in F \) such that \( x_i = y_i + AF \). It is obvious that the family \( (y_i + A_i F)_{i \in I} \) has the finite intersection property. By Proposition 2.3, this family has a non-empty intersection. Then \( F \) has a non-empty intersection too.

Now we assume that \( U = F \otimes_R U \) is not finitely generated. It is obvious that \( \mathcal{F} \) has a non-empty intersection if \( x_i + N_i = E, \forall i \in I \). Now assume there exists \( i_0 \in I \) such that \( x_{i_0} + N_{i_0} \notin E \). Let \( I' = \{ i \in I \mid N_i \subseteq N_{i_0} \} \) and \( \mathcal{F}' = (x_i + N_i)_{i \in I'} \). Then \( \mathcal{F} \) and \( \mathcal{F}' \) have the same intersection. By Lemma 2.4, \( N_i = F \otimes_R (s_i U : r_i U) \). It follows that \( (s_i U : r_i U) \subset U \) because \( N_i \notin E \). Hence \( \exists u \in U \) such that \( x_{i_0} + N_{i_0} \subseteq F \otimes_R E \). Then, \( \forall i \in I' \), \( x_i + N_i \subseteq F \otimes_R E \). We have \( F \otimes_R E \cong F/(0 : u) F \). From the first part of the proof \( F/(0 : u) F \) is pure-injective. So we may replace \( R \) with \( R/(0 : u) \) and assume that \( (0 : u) = 0 \). Let \( A_i = ((s_i U : r_i U) : u), \forall i \in I' \). Thus \( N_i = A_i F, \forall i \in I' \). By Proposition 2.3, \( \mathcal{F}' \) has a non-empty intersection. So \( \mathcal{F} \) has a non-empty intersection too. \( \square \)

Let \( U \) be an \( R \)-module. As in [5, p.338] we set \( U_2 = \{ s \in R \mid \exists u \in U, u \neq 0 \text{ and } su = 0 \} \) and \( U^2 = \{ s \in R \mid sU \subset U \} \).

Then \( U_2 \) and \( U^2 \) are prime ideals.

Now it is possible to determine the pure-injective hull of each uniserial module. We get a generalization of [5, Corollary XIII.5.5]

Theorem 2.4. The following assertions hold:

1. Let \( U \) be a uniserial \( R \)-module and \( J = U^2 \cup U_2 \). Then \( \widehat{U} \otimes_R U \) is the pure-injective hull of \( U \). Moreover \( \widehat{U} \) is an essential extension of \( U \) if \( J = U_2 \).

2. For each proper ideal \( A \) of \( R \), \( \widehat{R} / A \widehat{R} \) is the pure-injective hull of \( R \)-\( A \). Moreover \( \widehat{R} / A \widehat{R} \cong E_{R/A}(R/A) \) if \( A \) is archimedean.
Proof. (1) If $s \in R \setminus J$ then multiplication by $s$ in $U$ is bijective. So $U$ is an $R_J$-module. After replacing $R$ with $R_J$, we may assume that $J = P$. We put $\tilde{U} = R_J \otimes_R U$.

Suppose that $P = U^3$. By [10, Proposition 6] $\tilde{U} = \tilde{U} \oplus V$ where $V$ is a submodule of $\tilde{U}$. Let $v \in V$. Then $v = x \otimes u$ where $u \in U$ and $x \in \tilde{R}$. By Proposition 1.4 $x = a + pay$, where $a \in R$, $p \in P$ and $y \in \tilde{R}$. Since $pU \subseteq U$, $\exists u' \in U \setminus (Pu \cup pU')$. Then $v = cau'$ for some $c \in R$ and $v = cau' + pcay \otimes u'$. We have $y \otimes u' = z + w$ where $w \in V$ and $z \in \tilde{U}$. So $cau' + pcaz = 0$. Since $U$ is pure in $\tilde{U}$, there exists $z' \in U$ such that $cau' + pcaz' = 0$. If $v \neq 0$ the equality $v = (1 + py) \otimes cau'$ implies $cau' \neq 0$. By [10, Lemma 5] we get that $u' \in pU$, whence a contradiction. Hence $V = 0$.

Now suppose that $P = U_1$. If $0 \neq z \in \tilde{U}$ then $z = x \otimes u$ where $u \in U$ and $x \in \tilde{R}$. By Proposition 1.4 there exist $a \in R$, $p \in P$ and $y \in \tilde{R}$ such that $x = a + p\text{ay}$. So $z = au + y \otimes pau$. Let $A = (0 : au)$. By [10, Lemma 26], $A^\sharp = P$. So $(0 : pau) = (A : p) \neq A$. Let $r \in (A : p) \setminus A$. Then $0 \neq rz \in U$.

(2) We apply the first assertion by taking $U = R/A$. In this case, $U^3 = P$. The pure-injective hull of $R/A$ is the same over $R$ and over $R/A$. Since $R/A$ is self fp-injective when $A$ is archimedean then we use [3, Lemma XIII.2.7] to prove the last assertion. \hfill $\Box$

In the previous theorem, if $U$ is not cyclic and if $U^3 \subseteq U_1$ then $\tilde{U}$ is not necessarily isomorphic to $E_{R/(0 : U)}(U)$. For instance:

Example 2.5. Assume that $P = Z$ and $P$ is faithful. We choose $U = P$. Then $U^3 = U_1 = P$, $U = P \tilde{R}$ and $E_{R}(U) = \tilde{R}$.

If $U$ is a non-standard uniserial module over a valuation domain $R$ then $\tilde{U}$ is indecomposable by [3, Proposition 5.1] and there exists a standard uniserial module $V$ such that $\tilde{U} \cong V$ by [3, Theorem XIII.5.9]. So, $\tilde{R} \otimes_R U \cong \tilde{R} \otimes_R V$ doesn’t imply $U \cong V$. However, it is possible to get the following proposition:

Proposition 2.6. Let $U$ and $V$ be uniserial modules and $J = U^3 \cup U_2$. Assume that $\tilde{R} \otimes_R U \cong \tilde{R} \otimes_R V$. Then $U$ and $V$ are isomorphic if one of the following conditions is satisfied:

1. $U^3 = J$ and $J \neq J^2$,
2. $U$ is countably generated.

Proof. Let $\phi : \tilde{R} \otimes_R U \rightarrow \tilde{R} \otimes_R V$ be the isomorphism. Let $0 \neq u \in U$. Then $\phi(u) = x \otimes v$ for some $x \in \tilde{R}$ and $v \in V$. By proposition 1.4 we may assume that $x = 1 + py$ for some $p \in P$ and $y \in \tilde{R}$. First we shall prove that $(0 : u) = (0 : v)$. It is obvious that $(0 : v) \subseteq (0 : u)$. Let $r \in (0 : u)$. Then $x \otimes ru = 0$. From the flatness of $\tilde{R}$ we deduce that there exist $s \in R$ and $z \in \tilde{R}$ such that $x = sz$ and $srz = 0$. If $s \in P$ then we get that $1 = qe$ for some $q \in P$ and $e \in \tilde{R}$. Since $R$ is pure in $\tilde{R}$, it follows that $1 \in P$. This is absurd. Hence $s$ is a unit and $r \in (0 : v)$.

Let $v, v'$ be nonzero elements of $V$ and $x, x' \in 1 + P \tilde{R}$ such that $x \otimes v = x' \otimes v'$. There exists $t \in R$ such that $v = tv'$. Now we shall prove that $t$ is a unit of $R$. We get that $(x' - tx) \otimes v' = 0$. If $t \in P$, as above we deduce that $v' = 0$, whence a contradiction.

Let $u \in U$ and $v \in V$ as in the first part of the proof. By [10, Lemma 26] we have $U_1 = (0 : u)^3 = (0 : v)^3 = V_1$. Let $p \in P$. We shall prove that $u \in pU$ if
and only if $v \in pV$. If $v = pw$ for some $w \in V$ then $\phi(u) = px \otimes w = p\phi(z)$ for some $z \in \hat{R} \otimes_R U$. Since $U$ is a pure submodule, then $u = pw'$ for some $u' \in U$. Conversely, if $u = pw'$ for some $u' \in U$ and $\phi(u') = x' \otimes v'$ where $v' \in V$ and $x' \in 1 + P\hat{R}$, we get that $x' \otimes pw' = x \otimes v$. From above, we deduce that $v \in pV$. So, $U^2 = V^4$.

Now we can prove that $U$ and $V$ are isomorphic when the first condition is satisfied. In this case $U$ and $V$ are modules over $R_J$. Since $J \neq J^2$, $JR_J$ is a principal ideal of $R_J$. Since $JU \subset U$ and $Jv \subset V$, $U$ and $V$ are cyclic over $R_J$. Let $u \in U$ and $v \in V$ as in the first part of the proof, and suppose that $U = R_J u$. If $v = rw$ for some $r \in R_J$ and $w \in V$ then we get, as above, that $u = ru'$ for some $u' \in U$. So $r$ is a unit and $U$ and $V$ are isomorphic.

Let $\{u_i\}_{i \in I}$ be a spanning set of $U$. For each $i \in I$, let $v_i \in V$ and $x_i \in 1 + P\hat{R}$ such that $\phi(u_i) = x_i \otimes v_i$. Suppose that $(0 : U) \subset (0 : u), \forall u \in U$. From the first part of proof we deduce that $(0 : v) \subset (0 : v), \forall v \in V$. We have $\cap_{i \in I} (0 : v_i) = (0 : U)$. Thus $\cap_{v \in V} (0 : v) = (0 : v)$. So, for each $v \in V$ there exists $i \in I$ such that $(0 : v_i) \subset (0 : v)$. Hence $v \in Rv_i$. Now, suppose $\exists u \in U$ such that $(0 : u) = (0 : U)$. By [3, Lemma X.14] $J = U^2$. We may assume that $J = J^2$ and $I$ is infinite. Then $JU = U$ and $JV = V$. Let $v \in V$. There exists $p \in J$ such that $v \in pV$. But there exists $i \in I$ such that $u_i \notin pU$. So, $v_i \notin pV$. Hence $v \in R_J v_i$. Now suppose that $I = N$. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of elements of $P$ such that $a_n = a_n a_{n+1}, \forall n \in \mathbb{N}$. We put $\varphi(u_0) = v_0$. Suppose that $\varphi(u_n) = s_n v_n$ where $s_n$ is a unit. Hence $\varphi(u_n+1) = t_n \varphi(u_n)$. Hence we set $\varphi(u_{n+1}) = t_n^{-1} v_{n+1}$. So, by induction on $n$, we get an isomorphism $\varphi : U \rightarrow V$. Hence $U$ and $V$ are isomorphic.

Let $T = \text{End}_R(\hat{R})$. Then $T$ is a local ring by [1, Proposition 5.1] and [1, Theorem XIII.3.10]. For each $R$-module $M$, $\hat{R} \otimes_R M$ is a left $T$-module. As in [1] we say that a left uniserial $T$-module $F$ is shrinkable if there exists two $T$-submodules $G$ and $H$ of $F$ such that $0 \subset H \subset G \subset F$ and $F \cong G/H$. Otherwise $F$ is said to be unshrinkable.

**Proposition 2.7.** Let $U$ be a uniserial $R$-module. Then:

(1) $\hat{R} \otimes_R U$ is a left unshrinkable uniserial $T$-module.

(2) $\text{End}_T(\hat{R} \otimes_R U)$ is a local ring.

**Proof.** (1) Let $x \in 1 + P\hat{R}$. First we prove that $Rx$ is a pure submodule of $\hat{R}$. Let $a, b \in R$ and $y \in \hat{R}$ such that $by = ax$. By Proposition [1, Lemma 1.17] there exists $z \in \hat{R}$ such that $a \notin Rbc$. Then $bc = ra$ for some $r \in R$. If $x = 1 + qx'$ for some $q \in P$ and $x' \in \hat{R}$, we get that $a(1 - r) = a(rpz - qx') = aty'$ for some $t \in P$ and $y' \in \hat{R}$. Since $R$ is a pure submodule of $\hat{R}$ there exists $s \in R$ such that $a(1 - r - ts) = 0$. We deduce that $a = 0$, whence a contradiction. So $a \in Rbc$. By using similar arguments we easily show that $Rx$ is faithful.

Let $z, z' \in \hat{R} \otimes_R U$. We have $z = x \otimes u$ and $z' = x' \otimes u'$ where $x, x' \in 1 + P\hat{R}$ and $u, u' \in U$. Assume that $u' = ru$ for some $r \in R$. The homomorphism $\phi : Rx \rightarrow Rx'$ such that $\phi(x) = rx'$ is well defined and can be extended to $\hat{R}$. We get that $\phi z = z'$. Hence $\hat{R} \otimes_R U$ is uniserial over $T$.

Suppose that $\hat{R} \otimes_R U$ is shrinkable over $T$. By [1, Lemma 1.17] there exists $z \in \hat{R} \otimes_R U$ such that $T z$ is shrinkable. We have $z = x \otimes u$ where $x \in 1 + P\hat{R}$ and
Let \( u \in U \). So \( Tz = \hat{R} \otimes_R Ru \). There exist \( z' \in Tz \) and a non-injective \( T \)-epimorphism \( \alpha : T \to Tz \). Let \( K = \text{Ker } \alpha \). We may assume that \( \alpha(z') = z \). We have \( z' = z' \otimes ru \) where \( z' \in 1 + P \hat{R} \) and \( r \in R \). Let \( y \) be a nonzero element of \( K \). Thus \( y = tz' = ay' \otimes ru \) for some \( t \in T \), \( y' \in 1 + P \hat{R} \) and \( a \in R \). But there exist \( s, s' \in T \) such that \( z' = sy' \) and \( y' = s'x' \). So \( 0 \neq ax' \otimes ru \in K \). Since \( y \neq 0 \) we have \( aru \neq 0 \). On the other hand \( x \otimes aru = \alpha(ax' \otimes ru) = 0 \). It follows that \( aru = 0 \) whence a contradiction. So \( \hat{R} \otimes_R U \) is unshrinkable.

(2) is an immediate of (1) and \( \text{[3]} \) Proposition 9.24.

**Proposition 2.8.** Let \( \epsilon \) be a cardinal. Consider a \( \epsilon \)-generated \( R \)-module \( M \) and \( U \) a pure uniserial \( R \)-submodule of \( M \). Then \( U \) is \( \epsilon \)-generated.

**Proof.** We easily check that \( \hat{R} \otimes_R U \) is a pure submodule of \( \hat{R} \otimes_R M \). By Proposition 2.3 \( \hat{R} \otimes_R U \) is pure-injective. Hence \( \hat{R} \otimes_R U \) is a summand of \( \hat{R} \otimes_R M \).

On the other hand \( \hat{R} \otimes_R M \) is a \( \epsilon \)-generated \( T \)-module. Then \( \hat{R} \otimes_R U \) is also \( \epsilon \)-generated over \( T \). We may assume that \( \hat{R} \otimes_R U \) is generated by \( (1 \otimes u_i) \in I \), where \( I \) is a set whose cardinal is \( \epsilon \) and \( u_i \in U \), \( \forall i \in I \). Let \( V \) be the submodule of \( U \) generated by \( (u_i)_{i \in I} \). Then the inclusion map \( V \to U \) induces an isomorphism \( \hat{R} \otimes_R V \to \hat{R} \otimes_R U \). Since \( \hat{R} \) is faithfully flat it follows that \( V = U \). □

From Theorem 2.3 we deduce the following corollary on the structure of indecomposable injective modules.

**Corollary 2.9.** Let \( E \) be an indecomposable injective module, \( J = E \setminus x \) and \( A(E) = \{(0 : E, x) \mid 0 \neq x \in E \} \). Then:

1. \( \forall A, B \in A(E), A \subseteq B \) there exists a monomorphism \( \varphi_{A,B} : \hat{R}_J \otimes_R R_J / A \hat{R}_J \to \hat{R}_J / A \hat{R}_J \) such that \( \varphi_{A,C} = \varphi_{A,B} \circ \varphi_{B,C} \), \( \forall A, B, C \in A(E) \), \( A \subseteq B \subseteq C \).
2. \( E \cong \varprojlim \{(\hat{R}_J / A \hat{R}_J, \varphi_{A,B}) \mid A, B \in A(E), A \subseteq B \} \).
3. \( E \cong \hat{R}_J \otimes_R U \) if \( (0 : J, e) \in \hat{R}_J \) for some \( e \in E \).
4. Suppose that \( E \) contains a uniserial \( R_J \)-module \( U \) such that \( A(E) = A(U) \).

Then \( E \cong \hat{R}_J \otimes_R U \). Moreover, \( \forall A, B \in A(E), A \subseteq B \), there exists \( r \in R \) such that one can choose \( \varphi_{A, B} = 1_{\hat{R}_J} \otimes \bar{r} \) where \( \bar{r} : R_J / B \to R_J / A \) is defined by \( \bar{r}(a + B) = ar + A \), \( \forall a \in R \).

**Proof.** (1) If \( A \in A(E) \) then \( A = J = \text{[3]} \) Lemma 26. Therefore \( A \) is an archimedean ideal of \( R_J \). By Theorem 2.3 there exists an isomorphism \( \phi_A : \hat{R}_J \otimes_R A \hat{R}_J \to (0 : E, A) \).

Let \( u_{A,B} : (0 : E, B) \to (0 : E, A) \) be the inclusion map , \( \forall A, B \in A(E), A \subseteq B \). We set \( \varphi_{A,B} = \phi_A^{-1} \circ u_{A,B} \circ \phi_B \). It is easy to check the first assertion.

(2) and (3) These assertions are now obvious.

(4) First we prove that \( U \) is fp-injective. Let \( x \in E \) and \( s \in R \) such that \( 0 \neq sx \in U \). We put \( u = sx \). From \( A(E) = A(U) \), it follows that \( \exists v \in U \) such that \( (0 : R_J, v) = (0 : R_J, x) \) and consequently \( u = tv \) for some \( t \in R \). We set \( A = (0 : R_J, x) \). We get that \( (0 : R_J, u) = (A : R_J, t) = (A : R_J, s) \). By \( \text{[3]} \) Lemma 26

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\(^1\) We know that this condition holds if \( R \) satisfies an additional hypothesis: see \( \text{[3]} \) Corollary 22. \( \text{[3]} \) Theorem 5.5| or Remark 5.4.
conclude by Theorem 2.4 and [5, Lemma XIII.2.7] that $E \cong \hat{R}_J \otimes_R U$.

Let $u, v \in U$ such that $(0 :_{R_J} u) = A$ and $(0 :_{R_J} v) = B$. There exists $r \in R$ such that $v = ru$ and $B = (A : r)$ (if $A = B$ we take $v = u$ and $r = 1$). So $\hat{r}$ is a monomorphism.

\section{Pure-injective hulls of polyserial modules}

We say that a module $M$ is \textbf{polyserial} if it has a pure-composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

(i.e. $M_k$ is a pure submodule of $M$, for each $k$, $0 \leq k \leq n$) where $M_k/M_{k-1}$ is uniserial for each $k$, $1 \leq k \leq n$. By [5, Lemma I.7.8], if $M$ is finitely generated, $M$ has a pure-composition series, where $M_k/M_{k-1} \cong R/A_k$ and $A_k$ is a proper ideal, for each $k$, $1 \leq k \leq n$. We denote by $\text{gen} M$ the minimal number of generators of $M$. By [5, Lemma V.5.3] $n = \text{gen} M$. The following sequence $(A_1, \cdots, A_n)$ is called the \textbf{annihilator sequence} of $M$ and is uniquely determined by $M$, up to the order (see [5, Theorem V.5.5]).

Now we can extend the result obtained by Warfield [10] in the domain case for finitely generated modules.

\begin{theorem}
Let $M$ be a finitely generated $R$-module. Then $\hat{R} \otimes_R M \cong \hat{M}$. Moreover, $\hat{M} \cong \hat{R}/A_1 \hat{R} \oplus \cdots \oplus \hat{R}/A_n \hat{R}$ where $(A_1, \cdots, A_n)$ is the annihilator sequence of $M$.
\end{theorem}

\begin{proof}
It is easy to verify that $M$ is a pure submodule of $\hat{R} \otimes_R M$. We have that $\hat{R} \otimes_R M_1$ is a pure submodule of $\hat{R} \otimes_R M$ too. By Proposition 2.3, $\hat{R} \otimes_R M_1$ is pure-injective. It follows that $\hat{R} \otimes_R M \cong (\hat{R} \otimes_R M_1) \oplus (\hat{R} \otimes_R M/M_1)$. By induction on $n$ we get that $\hat{R} \otimes_R M \cong \hat{R}/A_1 \hat{R} \oplus \cdots \oplus \hat{R}/A_n \hat{R}$. So $\hat{R} \otimes_R M$ is pure-injective. By [11, Proposition 6] $\hat{M}$ is a direct summand of $\hat{R} \otimes_R M$. So $\hat{R} \otimes_R M \cong \hat{M} \oplus V$, where $V$ is a submodule of $\hat{R} \otimes_R M$. From Proposition [1] we deduce that, for each $x \in \hat{R} \otimes_R M$, there exist $m \in M$, $p \in P$ and $y \in \hat{R} \otimes_R M$ such that $x = m + py$. Assume that $x \in V$. There exists $z \in \hat{M}$ and $v \in V$ such that $x = m + pz + pv$. It follows that $x = pv$, whence $V = PV$. On the other hand, $\hat{R}/A \hat{R}$ is indecomposable by [3, Proposition 5.1] and $\text{End}_R(\hat{R}/A \hat{R})$ is local by [11, Theorem 9] or [5, Theorem XIII.3.10], for every proper ideal $A$. By Krull-Schmidt Theorem $V \cong \hat{R}/A_{k_1} \hat{R} \oplus \cdots \oplus \hat{R}/A_{k_p} \hat{R}$ where $\{k_1, \cdots, k_p\}$ is a subset of $\{1, \cdots, n\}$. If $V \neq 0$, by Proposition [1] we get $V \neq PV$. This contradiction completes the proof.
\end{proof}

The \textbf{Malcev rank} of a module $N$ is defined as the cardinal number

$$\text{Mr } N = \sup \{\text{gen } M \mid M \subseteq N, \text{ gen } M < \infty\}.$$  

The following proposition is identical to the first part of [5, Proposition XII.1.6]. Here we give a different proof.

\begin{proposition}
The length of any pure-composition series of a polyserial module $M$ equals $\text{Mr } M$.
\end{proposition}
Proof. Let $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ be a pure-composition series of $M$ with uniserial factors. As in [3, Corollary XII.1.5] we prove that $M \subset M$.

Equality holds for $n = 1$. From the pure-composition series of $M$, we deduce a pure-composition series of $M/M_1$ of length $n - 1$. By induction hypothesis $M/M_1$ contains a finitely generated submodule $Y$ with $\text{gen } Y = n - 1$.

Assume that $Y$ is generated by $\{y_2, \ldots, y_n\}$. Let $x_2, \ldots, x_n \in M$ such that $y_k = x_k + M_1$ and $F$ be the submodule of $M$ generated by $x_2, \ldots, x_n$. If $F \cap M_1 = M_1$ then $M_1 \subseteq F$ and $M_1$ is a pure submodule of $F$. In this case $M_1$ is finitely generated by Proposition 2.8. It follows that the following sequence is exact:

$$0 \rightarrow \frac{M_1}{PM_1} \rightarrow F \rightarrow \frac{Y}{PY} \rightarrow 0.$$  

So we have $\text{gen } Y = \text{gen } F - \text{gen } M_1 \leq n - 2$. We get a contradiction since $\text{gen } Y = n - 1$. Hence $F \cap M_1 \neq M_1$. Let $x_1 \in M_1 \setminus F \cap M_1$. Let $X$ be the submodule of $M$ generated by $x_1, \ldots, x_n$. Clearly $Rx_1 = M_1 \cap X$. We will show that $P_{x_1} = Rx_1 \cap PX$. Let $x \in Rx_1 \cap PX$. Then $x = p \sum_{k=1}^n a_k x_k = px_1$ where $p \in P$ and $a_1, \ldots, a_n$ are elements of $R$. It follows that $p \sum_{k=2}^n a_k x_k = (r - pa_1)x_1$. So $(r - pa_1)x_1 \in M_1 \cap F \subset Rx_1$. We deduce that $r - pa_1 \in P$ whence $r \in P$. Hence $x \in Px_1$. Consequently the following sequence is exact:

$$0 \rightarrow \frac{Rx_1}{Px_1} \rightarrow \frac{X}{PX} \rightarrow \frac{Y}{PY} \rightarrow 0.$$  

Then $\text{gen } X = n$. \hfill \qed

Now we study the pure-injective hulls of polyserial modules.

Theorem 3.3. Let $M$ be a polyserial module with the following pure-composition series:

$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$

For each integer $k, 1 \leq k \leq n$ we put $U_k = M_k/M_{k-1}$. Then:

1. There exists a subset $I$ of $\{k \in \mathbb{N} | 1 \leq k \leq n\}$ such that $\tilde{M} \cong \bigoplus_{k \in I} \tilde{U}_k$.
2. $\tilde{R} \otimes R M$ is pure-injective and isomorphic to $\bigoplus_{k=1}^n \tilde{R} \otimes R U_k$.
3. The collection $(\tilde{R} \otimes R U_k)_{1 \leq k \leq n}$ is uniquely determined by $M$.

Proof. (1) Let $N$ be a pure submodule of $M$. The inclusion map $N \rightarrow \tilde{N}$ can be extended to $w : M \rightarrow \tilde{N}$. Let $f : M \rightarrow \tilde{N} \oplus M/N$ defined by $f(x) = (w(x), x + N)$, for each $x \in M$. It is easy to verify that $f$ is a pure monomorphism. It follows that $\tilde{M}$ is a summand of $\tilde{N} \oplus M/N$. So, by induction on $n$, we easily get that $\tilde{M}$ is a summand of $\bigoplus_{k=1}^n \tilde{U}_k$. Since, $\forall k \in \mathbb{N}$, $1 \leq k \leq n$, $\tilde{U}_k$ is indecomposable by [3, Proposition 5.1] and $\text{End}_R(\tilde{U}_k)$ is local by [1, Theorem 9] or [3, Theorem XIII.3.10], we apply Krull-Schmidt Theorem to conclude.

(2) We do as in the proof of Theorem 1.3.

(3) Since $\tilde{R} \otimes R M$ and $\tilde{R} \otimes R U_k$ are $T$-modules, we conclude by Proposition 2.7 and Krull-Schmidt theorem. \hfill \qed

Corollary 3.4. If $M$ is polyserial and countably generated then any two pure-composition series of $M$ are isomorphic.

Proof. By Theorem 3.3 the collection $(\tilde{R} \otimes R U_k)_{1 \leq k \leq n}$ is uniquely determined by $M$. It remains to show that, if $U$ and $V$ are uniserial modules such that $\tilde{R} \otimes R U \cong \tilde{R} \otimes R V$. \hfill \qed
$\hat{R} \otimes_R V$ then $U \cong V$. It is an immediate consequence of Proposition 3.5 and Proposition 2.8.  

Recall that an $R$-module $M$ is finitely (respectively countably cogenerated) if $M$ is a submodule of a product of finitely (respectively countably) many injective hulls of simple modules.

The following proposition completes [1, Corollary 35].

**Proposition 3.5.** The following conditions are equivalent:

1. Every finitely generated $R$-module is countably cogenerated and every ideal of $R$ is countably generated.
2. For each prime ideal $J$ which is the union of the set of primes properly contained in $M$ there is a countable subset whose union is $J$, and for each prime ideal $J$ which is the intersection of the set of primes containing properly $J$ there is a countable subset whose intersection is $J$.
3. Each uniserial module is countably generated.

(1) $\iff$ (2) holds by [1, Corollary 35]

(3) $\Rightarrow$ (2) Let $J$ be a prime ideal. Then $J$ and $R_J$ are uniserial $R$-modules. So they are countably generated. If $R_J$ is generated by $\{t_n^{-1} | n \in \mathbb{N}\}$, where $t_n \notin J \forall n \in \mathbb{N}$, then $J = \cap_{n \in \mathbb{N}} R t_n$. Now it is easy to get the second condition.

(1) $\Rightarrow$ (3) Let $U$ be a uniserial module and $J = U^2 \cup U^2$. Then $U$ is an $R_J$-module. But $R/J$ countably cogenerated is equivalent to $R_J$ countably generated. Hence $U$ is countably generated over $R$ if and only if $U$ is countably generated over $R_J$. So we may assume that $J = P$.

Now suppose that $U^2 = P$. If $P U \subset U$ then $U = Ru$ where $u \in U \setminus P U$. Now suppose that $P U = U$. Let $r, s \in P$ such that $rU \neq 0$. If $rU = rsU$ then by [1, Lemma 5] we have $U = sU$, hence a contradiction. Let $\{p_n | n \in \mathbb{N}\}$ be a spanning set of $P$ such that $p_{n+1} \notin R p_n$. Then $U = \cup_{n \in \mathbb{N}} p_n U$. We may assume that $p_n U \neq 0$, $\forall n \in \mathbb{N}$. So $p_n U \subset p_{n+1} U$ for each $n \in \mathbb{N}$. Let $u_n \in p_{n+1} U \setminus p_n U$ for each $n \in \mathbb{N}$. Then $U$ is generated by $\{u_n | n \in \mathbb{N}\}$.

Now suppose that $U^2 = P$. Assume that $(0 : u) = (0 : U)$ for some $u \in U$. Let $v \in U$ such that $u = av$ for some $a \in R$. By [1, Lemma 2] $(0 : v) = ((0 : v) : a)$. We get that $(0 : v) = ((0 : v) : a) = (0 : U)$. Since $(0 : v)^2 = P$ by [1, Lemma 26] $a$ is a unit, and consequently $U$ is cyclic. Now we assume that $(0 : U) \subset (0 : u)$ for each $u \in U$. We have $(0 : U) = \cap_{u \in U} (0 : u)$. By [1, Lemma 30] there exists a countable family $(u_n)_{n \in \mathbb{N}}$ of elements of $U$ such that $(0 : U) = \cap_{n \in \mathbb{N}} (0 : u_n)$ and $u_{n+1} \notin R u_n$, $\forall n \in \mathbb{N}$. If $u \in U$, since $(0 : u) \neq (0 : U)$, then there exists $n \in \mathbb{N}$ such that $(0 : u) \subset (0 : u)$. Hence $u \in Ru_n$ and $U is generated by $\{u_n | n \in \mathbb{N}\}$.

**Remark 3.6.** In the same way, one can prove that the two first conditions of [1, Proposition 32] (respectively [1, Corollary 34]) are equivalent to the following: each indecomposable injective module $E$ such that $E_J = P$ contains a uniserial pure submodule which is countably generated (respectively each indecomposable injective module contains a uniserial pure submodule which is countably generated).

**Proposition 3.7.** Suppose that $R$ satisfies the equivalent conditions of Proposition 3.5. Then any two pure-composition series of a polyseri al $R$-module are isomorphic.

**Proof.** It is an immediate consequence of Proposition 3.5 and Corollary 3.4. □
4. Two criteria for maximality of $R$

By Theorem 3.1, if $M$ is finitely generated, then $\widehat{M}$ is a direct sum of gen $M$ indecomposable pure-injective modules and gen $M = Mr M$ by Corollary XII.1.7. But Theorem 4.5 proves that, if $M$ is polyserial, then $\widehat{M}$ is not necessarily a direct sum of $Mr M$ indecomposable pure-injective modules.

As in [7], if $x \in \hat{R} \setminus R$, we say that $B(x) = \{ r \in R \mid x \notin R + r\hat{R} \}$ is the breath ideal of $x$. Then Proposition 4.2 is a generalization of Proposition 1.4. The following lemma is useful to prove this proposition.

**Lemma 4.1.** Let $J$ be a proper ideal such that $J = \cap_{c \notin J} c \hat{R}$. Then $\hat{R}/J = \cap_{c \notin J} c \hat{R}$.

**Proof.** By Theorem 2.4 $\hat{R}/J$ is the pure-injective hull of $R/J$. In the proof of Step 3 of Proposition 1.3 it is already shown that $\cap_{a \neq 0} a\hat{R} = 0$ if $\cap_{a \neq 0} aR = 0$. So we apply this result to $R/J$ to get the lemma.

Recall that the ideal topology of $R$ is the linear topology which has as a basis of neighborhoods of 0 the nonzero principal ideals.

**Proposition 4.2.** Let $A$ be a proper ideal. Then $R/A$ is Hausdorff and non-complete in its ideal topology if and only if $A = B(x)$ for some $x$ in $\hat{R} \setminus R$.

**Proof.** To show that $R/B(x)$ is Hausdorff, we do as in [7]. Proposition 1.4, we prove that $a \notin B(x)$ implies that $pa \notin B(x)$ for some $p \in P$. We have $x = r + ay$ where $r \in R$ and $y \in \hat{R}$. By Proposition 1.3, $\hat{R} = R + p\hat{R}$. So $y = s + pz$, for some $s \in R$, $p \in P$ and $z \in \hat{R}$. Therefore we get $x = r + as + paz \in R + pa\hat{R}$. For each $a \notin B(x)$, $x \in r_a + a\hat{R}$ for some $r_a \in R$. If the family $(r_a + a\hat{R})_{a \notin B(x)}$ has a non-empty intersection then, by using Lemma 1.1, we get that $x \in R + B(x)\hat{R}$, whence a contradiction. So $R/B(x)$ is non-complete.

Conversely, assume that $R/A$ is Hausdorff and non-complete. Then there exists a family $(r_a + a\hat{R})_{a \notin A}$ which has the finite intersection and an empty total intersection. Since $\hat{R}$ is pure-injective, the total intersection of the family $(r_a + a\hat{R})_{a \notin A}$ contains an element $x$ which doesn’t belong to $R$. Clearly $B(x) \subseteq A$. If $x = r + b\hat{R}$ for some $r \in R$ and $b \in A$ then $r \in r_a + aR$ for each $a \notin A$, since $R$ is a pure submodule of $\hat{R}$. We get a contradiction. So $A = B(x)$.

The following lemma is a generalization of [7] Lemma 1.3. It will be useful to prove Proposition 4.4.

**Lemma 4.3.** Let $x \in \hat{R}$ such that $x = r + ay$ for some $r, a \in R$ and $y \in \hat{R}$. Then $B(y) = (B(x) : a)$.

**Proof.** Let $t \notin B(y)$. Then $y = s + tz$ for some $s \in R$ and $z \in \hat{R}$. It follows that $x = r + as + aty$. So $t \notin (B(x) : a)$.

Conversely, if $t \notin (B(x) : a)$ then we get the following equalities $x = r + ay = s + tz$ for some $s \in R$ and $z \in \hat{R}$. Since $R$ is a pure submodule of $\hat{R}$ it follows that $a(y - tz - b) = 0$ for some $b \in R$. From the flatness of $\hat{R}$ we deduce that $(y - tz - b) \in (0 : a)\hat{R}$. But $ta \notin B(x)$ implies that $ta \neq 0$, whence $(0 : a) \subset Rt$. Hence $t \notin B(y)$. \[\square\]
Theorem 4.4. Let \( N \) be the nilradical of \( R \). Then \( R \) is maximal if and only if \( R/N \) and \( R_N \) are maximal.

**Proof.** Suppose that \( R \) is maximal. It is obvious that \( R/N \) is maximal. By [1, Lemma 2] \( R_N \) is maximal too.

Conversely assume that \( R/N \) and \( R_N \) are maximal. Let \( K \) be the kernel of the natural map \( R \to R_N \). Let \( r \in K \). Thus there exists \( s \in R \setminus N \) such that \( sr = 0 \). It follows that \( K \subseteq N \subseteq (0 : r) \). Then \( K^2 = 0 \). So \( K \) is a uniserial \( R/K \)-module which is linearly compact if \( R/K \) is maximal. Consequently \( R \) is maximal if and only if \( R/K \) is maximal. In the sequel we may assume that \( K = 0 \). So \( N = Z \) and it is an \( R_N \)-module. It is enough to show that \( N \) is a linearly compact module. Let \( (A_i)_{i \in I} \) be a family of ideals contained in \( N \) and \( (x_i)_{i \in I} \) a family of elements of \( N \) such that the family \( F = (x_i + A_i)_{i \in I} \) has the finite intersection property. We put \( A = \cap_{i \in I} A_i \). We may assume that \( A \subset A_i \), \( \forall i \in I \).

First suppose that \( N \subset A^2 \). Assume that the total intersection of \( F \) is empty. Then \( R/A \) is non-complete in its ideal topology. By Proposition 1.2 there exists \( x \in R \setminus A \) such that \( B(x) = A \). Let \( b \in A^2 \setminus N \). There exists \( a \in (A : b) \setminus A \). Since \( B(x) = A \) we have \( x = r + ay \) for some \( r \in R \) and \( y \in \bar{R} \). By Lemma 1.3 \( B(y) = (A : a) \). Since \( b \in B(y) \) we have \( N \subset B(y) \). By Proposition 1.2 \( R/B(y) \) is non-complete in its ideal topology. This contradicts that \( R/N \) is maximal. So the total intersection of \( F \) is non-empty in this case.

Now we assume that \( N = A^1 \). Then \( A \) is an ideal of \( R_N \). By [1, Lemma 29] either \( A = Na \) for some \( a \in N \) or \( A = \cap_{a \in A} aR_N \).

First we assume that \( A = Na \). We may suppose that \( A_i \subset aR_N \), \( \forall i \in I \). Since \( F \) has the finite intersection property, \( x_i + aR_N = x_j + aR_N \), \( \forall i, j \in I \). Let \( y \in x_i + aR_N \) for each \( i \in I \). Then \( (x_i - y + A_i)_{i \in I} \) is a family of cosets of \( aR_N \) which has the finite intersection property. But \( aR_N/aN \) is a uniserial module over \( R/N \). Then \( aR_N/aN \) is linearly compact since \( R/N \) is maximal. Thus \( \cap_{i \in I} (x_i - y + A_i) \neq 0 \).

Hence the total intersection of \( F \) is non-empty.

Now suppose that \( A = \cap_{a \in A} aR_N \). By Proposition 1.2 and [1, Lemma 30] there exists a countable family \( (a_n)_{n \in \mathbb{N}} \) of elements of \( N \setminus A \) such that \( A = \cap_{n \in \mathbb{N}} a_n R_N \) and \( a_n \notin a_{n+1} R_N \), \( \forall n \in \mathbb{N} \). By induction on \( n \) we get a subfamily \( (A_i)_{i \in I} \) of the family \( (A_i)_{i \in I} \) such that \( A_i \subset a_i R_N \) in the following way: we choose \( i_0 \in I \) such that \( A_{i_0} \subset a_0 R_N \) and, \( \forall n \in \mathbb{N} \), we pick \( i_{n+1} \) such that \( A_{i_{n+1}} \subset A_{i_n} \cap a_{n+1} R_N \). Then the family \( (x_{i_n} + a_n R_N)_{n \in \mathbb{N}} \) has the finite intersection property. Since \( R_N \) is maximal there exists \( x \in x_{i_n} + a_n R_N \), \( \forall n \in \mathbb{N} \). But the equality \( A = \cap_{a \in A} aR_N \) implies that, \( \forall n \in \mathbb{N} \), there exists an integer \( m > n \) such that \( a_m R_N \subseteq A_{i_n} \). Since \( x - x_{i_n} \in a_m R_N \) and \( x_{i_n} - x_{i_n} \in A_{i_n} \) we get that \( x \in x_{i_n} + A_{i_n} \), \( \forall n \in \mathbb{N} \). Hence \( F \) has a non-empty total intersection. The proof is now complete. \( \square \)

Theorem 4.5. Then \( R \) is maximal if and only if, for each polynseral \( R \)-module \( M \), \( \hat{M} \) is direct sum of \( Mr \) indecomposable pure-injective modules.

**Proof.** If \( R \) is maximal, then each polynseral module \( M \) is a direct sum of \( Mr \) pure-injective uniserial modules by [3, Proposition XII.2.4] (even if \( R \) is not a domain, this proposition holds, with the same proof).

If \( R \) is not maximal then \( R/N \) or \( R_N \) is not maximal by Theorem 1.4.

Assume that \( R' = R/N \) is not maximal. Then \( E = R'/R' \) is a nonzero torsion-free \( R' \)-module. Let \( x \in \bar{R}' \setminus R' \), \( \bar{x} \) be its image in \( E \) and \( U \) the submodule of \( E \) such that \( U/R' \bar{x} \) is the torsion submodule of \( E/R' \bar{x} \). Then \( U \) is a pure submodule
of $E$, a rank one torsion-free module and a uniserial module. Let $M$ be the inverse image of $U$ by the natural map $\hat{R}' \to E$. Then $M$ is a pure submodule of $\hat{R}'$ and a non-uniserial polyserial module with the two following (standard) uniserial factors: $R'$ and $U$. We have $\text{Mr } M = 2$. Let $W$ be a submodule of $\hat{R}'$ such that $M \cap W = 0$ and $M \to \hat{R}'/W$ is a pure monomorphism. Thus $R' \cap W = 0$ and $\hat{R}' \to \hat{R}'/W$ is a pure monomorphism too. Since $R'$ is pure-essential in $\hat{R}'$ it follows that $W = 0$. We conclude that $M$ is pure-essential in $\hat{R}'$, so that $\hat{M} = \hat{R}' \subset \hat{R}' \oplus \hat{U}$. (Let us observe that $M$ and $U$ are not finitely generated by Theorem 3.1.)

Suppose that $R' = R_N$ is not maximal. After replacing $R'$ with $R'/rR'$, where $r$ is a non-unit of $R'$, we may assume that $R'$ is coherent and self fp-injective by [1, Theorem 11]. Then $E = \hat{R}'/R'$ is a nonzero fp-injective $R'$-module. By [2, Lemma 6] $E$ contains a pure uniserial submodule $U$. We define $M$ as above. Then $\text{Mr } M = 2$ and $M$ is an essential submodule of $\hat{R}'$. So $\hat{M} = \hat{R}'$. □

References


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