Approximate hedging problem with transaction costs in stochastic volatility markets

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Abstract

We study the problem of option replication in general stochastic volatility markets with transaction costs using a new form for enlarged volatility in Leland's algorithm [25]. The asymptotic results recover the existing works in the Leland spirit and enable us to fix the under-hedging property pointed out by Kabanov and Safarian in [20]. We analyze possible relationships between the present setting and high frequency markets with proportional transaction costs. Possibilities to improve the convergence rate and reduce the option price inclusive transaction costs are also discussed.

Keywords: Leland strategy, transaction costs, stochastic volatility, quantile hedging, pricing option

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1 Introduction

In the theory of hedging options, Leland's strategy provides a simple way to eliminate efficiently risks caused by transaction costs. This prescription is based on the idea that transaction costs can be compensated by enlarging the volatility parameter in the *delta* Black-Scholes strategy. The pioneering work in this field was first given

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in [25], where a discrete approximation was used to study the asymptotic behavior of the *hedging error* (difference of the terminal portfolio value and the payoff) as the number of transactions goes to infinity. It was then shown in [32] that the hedging error vanishes if the transaction cost percentage converges to zero at a power rate. Unfortunately, this property does not hold for the interesting case when the proportional cost is a constant. In [20], the authors found the explicit limit of the hedging error but unexpectedly, it is a negative quantity. It means that the option is actually under-hedged in limit if the investor follows the Leland strategy. The convergence problem was investigated in the paper [15] and then, a complete answer was provided in [36] with the corresponding limit theorem allowing to identify the asymptotic distribution of the hedging error. Recently, a modified strategy with non-uniform revisions has been suggested in [29, 9] and it turns out that the rate of convergence is improved. For related results, see further in [28, 29, 14, 15].

Many empirical studies show that the constant volatility assumption in the classical Black-Scholes is not realistic and the Black-Scholes formula constructed under this assumption has an inaccuracy in anticipating option prices. The discrepancy between Black-Scholes option prices and market-traded ones, known as *smile curve*, can be explained by using stochastic volatility (SV) models which have been used to describe complex markets e.g. when fat-tailed returns are taken into account. It is well-known that modeling SV markets contains some intrinsic difficulties [12]. In fact, the incompleteness property makes the pricing problem more challenging to deal with. Hence, derivatives may not be perfectly hedged with only trading the underlying assets and asymptotic analysis is in general an efficient tool for studying such models. See [12] and the references therein for detailed discussions.

In this work, we study the problem of hedging European style options in SV markets in the presence of transaction costs using a simpler form of *adjusted volatility*. We will show that the payoff can be approximately replicated by establishing limit theorems for both Leland's strategy and Lépinette's one in a general SV setting. In particular, these asymptotic results recover the existing works in [20, 36, 9, 29] and also provide the possibility to improve the rate of convergence. It turns out that superhedging is attainable and both of mentioned strategies are close to the well-known buy-and-hold strategy. We finally point out that the option price can be reduced following the spirit of quantile hedging.

Let us emphasize that the classical form for enlarged volatility $\widehat{\sigma}$ proposed in [25] and then applied in [20, 21, 26, 27, 29] is no longer applicable in SV models from a practical point of view. The reason is that the quantity $\lambda_t = \int_t^1 \widehat{\sigma}_u^2 \mathrm{d}u$ appearing in the Black-Scholes formula is substantially dependent on future realizations of the random process driving the volatility. Therefore, the strategy is not available for investors in this case. To surpass this issue, we suggest to use an adjusted volatility which is independent of the initial volatility and much more simple than the one used in the previous works. In particular, the same asymptotic results are obtained for SV contexts and the rate of convergence can be improved by controlling the model parameter. Furthermore, note that in the existing works, asymptotic analyses are mainly based on moment estimates. This technique does no longer work in general SV models unless some intrinsic conditions are imposed on the model parameters, see [2, 30]. This undesirable property can be avoided by establishing convergences in probability to keep the model setting as general as possible. This can be considered as the main contribution of this note in the literature of discrete hedging with proportional transaction costs.

As discussed in [36], the option price of Leland's strategy is too high because it includes transaction costs. Another practical advantage of our method is that a simple method can be proposed to lower the option price as long as the option seller is willing to take a risk in option replication. This approach is inspired from the theory of quantile hedging [11].

The remainder of the paper is organized as follows. In Section 2, we briefly give a general view of Leland's approach then formulate the problem and present our principal results in Section 3. The new choice of adjusted volatility allows us to propose a reasonable way in Section 4 to fix the underhedging situation (shown in [20]) and reduce the option price in the presence of transaction costs. Section 5 discusses some common SV models for which our condition on volatility function is fulfilled. A numerical result for Hull-White model is also provided for illustration. Section 6 discusses a connection of the present context to high frequency markets with proportional transaction costs. The proofs of Main Results are reported in Section 7 and auxiliary lemmas can be found in the Appendix.

2 Hedging with transaction costs: a review on the Leland approach

In a complete no-arbitrage model (i.e. there exists a unique equivalent martingale measure under which the stock price is a martingale), options can be completely replicated by a self-financing trading strategy. Option price, defined as the replication cost, is the initial capital that the investor must introduce into his portfolio to obtain a complete hedge. It can be computed as the expectation of the discounted claim under the unique equivalent martingale measure.

Let us consider a continuous time model of two-asset financial market on the time interval $[0, 1]$, where the bond price is a constant over the time and equals to one. The stock price dynamics follows the stochastic differential equation

$$\mathrm{d}S_t = \sigma_0 S_t \mathrm{d}W_t \,, \tag{2.1}$$

where $\sigma_0 > 0$ is a positive constant and $(W_t)_{0\le t\le 1}$ is a standard Wiener process. As usual we denote $\mathcal{F}_t = \sigma\{W_u \,,\, 0 \le u \le t\}$. We recall that a financial strategy $(\beta_t, \gamma_t)_{0\le t\le 1}$ (the fractions of wealth invested in bond and stock respectively) is called an *admissible self-financing strategy* if it is $(\mathcal{F}_t)$ - adapted, integrable with $\int_0^t (|\beta_t| + \gamma_t^2)\,\mathrm{d}t < \infty$ a.s. and the portfolio value satisfies the equality

$$V_t = \beta_t + \gamma_t S_t = V_0 + \int_0^t \gamma_u \mathrm{d}S_u, \quad t \in [0,1].$$

The classical hedging problem is to find an admissible self-financing strategy (β_t, γ_t) whose terminal portfolio value exceeds the payoff $h(S_1) = (S_1 - K)_+$; that is

$$V_1 = V_0 + \int_0^1 \gamma_u \mathrm{d}S_u \geq h(S_1) \qquad \text{a.s.},$$

where K is the option strike. For this problem, Black and Scholes [4] proposed a dynamically replicating self-financing strategy with $\gamma_t = C_x(t, S_t)$ (partial derivative with respect to the space variable), where the option price $C(t, S_t)$ reads the famous formula

$$C(t,x) = C(t,x,\sigma_0) = x\Phi(\widetilde{\mathbf{v}}(t,x)) - K\Phi(\widetilde{\mathbf{v}}(t,x) - \sigma_0\sqrt{1-t}), \tag{2.2}$$

where $\widetilde{\mathbf{v}}(t,x) = \mathbf{v}(\sigma_0^2(1-t), x)$ and

$$\mathbf{v}(\lambda, x) = \frac{\ln(x/K)}{\sqrt{\lambda}} + \frac{\sqrt{\lambda}}{2}. \tag{2.3}$$

Here Φ is the standard normal distribution function. In the sequel, we denote by φ the $\mathcal{N}(0,1)$ density, i.e. $\varphi(z) = \Phi'(z)$. One can check directly that

$$C_x(t,x) = \Phi(\widetilde{\mathbf{v}}(t,x)) \quad \text{and} \quad C_{xx}(t,x) = \frac{\varphi(\widetilde{\mathbf{v}}(t,x))}{x\sigma_0\sqrt{1-t}}. \tag{2.4}$$

Clearly, hedging via discrete strategies is especially attractive since dynamically adjusted portfolios are impossible in practice. However, discrete time hedging, in turn, will face to intrinsic problems because of the presence of transaction costs. In particular, transaction costs are random and path-dependent, so they significantly effect the hedging error. Additionally, despite of the fact argued by Black and Scholes that the hedging error may be relatively small if trading activities take place reasonably frequently, transaction costs may increase without limit as portfolio revisions are frequent, so it may lead to an explosion.

2.1 Constant volatility case

The above considerations lead us to the Leland approach [25], which provides an efficient technique to compensate transaction costs. This method is simply based on the intuition that the option price should include transaction costs as a reasonable extra fee necessary for the option seller to cover the option return. In some situations (discussed in the next two sections), this strategy successfully replicates the payoff including transaction costs by simply adjusting the volatility parameter in Black-Scholes's model.

Let us shortly describe the Leland approach in [25, 20]. Suppose that for each trading activity, the investor has to pay a fee directly proportional to the trading volume measured in dollar value. Naturally, we suppose that the proportional transaction cost is given by the law $\kappa_* n^{-\alpha}$, where n is the number of revisions, $0 \le \alpha \le 1/2$ and $\kappa_* > 0$ are two fixed parameters. To compensate transaction costs the investor is suggested to enlarge the volatility as

$$\widehat{\sigma}^2 = \sigma_0^2 + \varrho\, n^{1/2-\alpha} \quad \text{and} \quad \varrho = \kappa_* \sigma_0 \sqrt{8/\pi}\,. \tag{2.5}$$

We assume further that the portfolio is revised discretely at $t_i = \frac{i}{n}$, $i \in \{1, 2, .., n\}$, by following the strategy (which is a piecewise process so-called Leland's strategy)

$$\gamma_t^n = \sum_{i=1}^n \widehat{C}_x(t_{i-1}, S_{t_{i-1}}) \mathbf{1}_{(t_{i-1}, t_i]}(t), \quad \widehat{C}(t,x) = C(t,x,\widehat{\sigma}). \tag{2.6}$$

It means that the number of shares held in the interval $(t_{i-1}, t_i]$ is the delta strategy calculated at the left bound of this interval. Then, the portfolio value takes the following form

$$V_1^n = V_0^n + \int_0^1 \gamma_u^n \mathrm{d}S_u - \kappa_* n^{-\alpha}\, J_n\,, \tag{2.7}$$

where the total trading volume measured in dollar value is given by

$$J_n = \sum_{i=1}^n S_{t_i} |\gamma_{t_i}^n - \gamma_{t_{i-1}}^n|\,. \tag{2.8}$$

The option price is now given by the initial time-value of the solution $\widehat{C}(t,x)$ of the Black-Scholes PDE with the adjusted volatility $\widehat{\sigma}$

$$\begin{cases} \widehat{C}_t(t,x) + \dfrac{1}{2}\widehat{\sigma}^2 x^2 \widehat{C}_{xx}(t,x) = 0\,, \quad 0 \le t < 1, \\ \widehat{C}(1,x) = h(x)\,. \end{cases} \tag{2.9}$$

Using Itô's formula we can represent the hedging error $V_1^n - h(S_1)$ as

$$\int_0^1 \left(\gamma_t^n - \widehat{C}_x(t,S_t) \right) \mathrm{d}S_t + \frac{1}{2}(\widehat{\sigma}^2 - \sigma_0^2) \int_0^1 S_t^2 \widehat{C}_{xx}(t,S_t)\mathrm{d}t - \kappa_* n^{-\alpha} J_n\,. \tag{2.10}$$

Remark 1 (Leland)**.** *The specific form* (2.5) *results from the following intuition: the Lebesgue's integral in* (2.10) *is clearly well-approximated by the Riemann sum of the terms* $\sigma_0 S^2_{t_{i-1}} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})\Delta t$*, while*

$$S_{t_i}|\gamma^n_{t_i} - \gamma^n_{t_{i-1}}| \approx \sigma_0 S^2_{t_{i-1}} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})|\Delta W_{t_i}| \approx \sigma_0 \sqrt{2/(n\pi)}\, S^2_{t_{i-1}} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}),$$

since $\mathbf{E}|\Delta W_{t_i}| = \sqrt{2/\pi}\sqrt{\Delta t} = \sqrt{2/(\pi n)}$*. Hence, it is reasonable to expect that choosing the modified volatility as in* (2.5) *may give an appropriate approximation to compensate transaction costs.*

Leland [25] conjectured that if the proportional transaction cost is a constant, i.e. $\alpha = 0$ then, the portfolio value of strategy (2.6) converges in probability to the payoff $h(S_1)$ as $n \to \infty$. He also gave a remark without proof that this result is still true for the case $\alpha = 1/2$. The latter remark is correct and was completely proved by Lott in [32], where one can find a rigorous explanation why the Leland strategy is important in practice.

Theorem 2.1 (Leland-Lott)**.** *For* $\alpha = 1/2$*, strategy* (2.6) *defines an approximately replicating strategy as the number of revision intervals* n *tends to infinity, i.e.*

$$\mathbf{P} - \lim_{n\to\infty} V^n_1 = h(S_1)\,.$$

This result was then extended by Ahn *et al* in [1] to general diffusion models. Kabanov and Safarian [20] also observed that the Leland approach is still valid as long as the cost proportion converges to zero as $n \to \infty$.

Theorem 2.2 (Kabanov-Safarian)**.** *For* $0 < \alpha \leq 1/2$*,*

$$\mathbf{P} - \lim_{n\to\infty} V^n_1 = h(S_1)\,.$$

It is, of course, possible to study the Leland-Lott approximation in sense of L^2-convergence. A such result[1] was established in [28, 21] for the case $\alpha = 1/2$.

Theorem 2.3 (Kabanov-Lépinette)**.** *Let* $\alpha = 1/2$*. The mean-square approximation error for Leland's strategy with* ϱ *defined in* (2.5) *satisfies the following asymptotic equality*

$$\mathbf{E}\left(V^n_1 - h(S_1)\right)^2 = B(S_1)n^{-1} + o(n^{-1}) \quad as \quad n \to \infty,$$

where B *is some positive function.*

The above result suggests that the normalized replication error $n^{1/2}(V^n_1 - h(S_1))$ converges in law as $n \to \infty$.

[1] Seemingly, mean-square replication may not contain much useful information since gains and losses have different meaning in practice. Clearly, if $\alpha = 1/2$ the modified volatility is independent of n

Theorem 2.4 (Lépinette-Kabanov [21])**.** *Let $\alpha = 1/2$. Then the processes $Y^n = n^{1/2}(V^n - h(S_1))$ converge weakly in the Skorokhod space $\mathcal{D}[0,1]$ to the distribution of the process $Y_\bullet = \int_0^\bullet B(S_t)\mathrm{d}Z_t$, where Z is an independent Wiener process.*

Remark 2. *An interesting connection of this case with the problem of hedging under proportional transaction costs in high frequency markets is discussed in Section 6.*

It is crucial to note that the Leland approximation in Remark 1 is not mathematically accurate and so, his first conjecture is not correct. In fact, as $n \to \infty$, the trading volume J_n may be approximated by the following sum, which converges in probability to $J(S_1, \varrho)$ given in (2.13),

$$-\sum_{i=1}^{n} \lambda_{i-1}^{-1/2} S_{t_{i-1}} \widetilde{\varphi}(\lambda_{i-1}, S_{t_{i-1}}) |\sigma_0 \varrho^{-1} Z_i + q(\lambda_{i-1}, S_{t_{i-1}})| \, \Delta\lambda_i \,, \tag{2.11}$$

where $\lambda_i = \lambda_{t_i} = \widehat{\sigma}^2(1 - t_i)$, $Z_i = \Delta W_{t_i}/\sqrt{\Delta t_i}$ and

$$\widetilde{\varphi}(\lambda, x) = \varphi(\mathbf{v}(\lambda, x)), \quad q(\lambda, x) = \frac{\ln(x/K)}{2\lambda} - \frac{1}{4} \,. \tag{2.12}$$

In approximation procedures, one should also pay attention to the fact that $\widehat{C}(\cdot,\cdot)$ and its derivatives substantially depend upon n. This property leads to the following important result: there is a *non trivial discrepancy* between the limit of the terminal portfolio value and the payoff in the practically interesting case $\alpha = 0$.

Theorem 2.5 (Kabanov-Safarian)**.** *If $\alpha = 0$ then,*

$$\mathbf{P} - \lim_{n\to\infty} V_1^n = h(S_1) + \min(S_1, K) - \kappa_* J(S_1, \varrho) \,,$$

where

$$J(x, \varrho) = x \int_0^{+\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x) \, \mathbf{E} \, |\widetilde{\varrho} Z + q(\lambda, x)| \, \mathrm{d}\lambda \,, \tag{2.13}$$

with $\widetilde{\varrho} = \sigma_0 \varrho^{-1}$ and $Z \sim \mathcal{N}(0,1)$ independent of S_1.

Under-hedging: It is important to observe that the problem of option replicating is not solved in this case. Indeed, taking into account that $\mathbf{E}\,|\widetilde{\varrho} Z| = 1/(2\kappa_*)$ and the identity

$$\min(x, K) = \frac{x}{2} \int_0^{\infty} \lambda^{-1/2} \, \widetilde{\varphi}(\lambda, x) \mathrm{d}\lambda \,, \tag{2.14}$$

we obtain (for the parameter ϱ given in (2.5)) that

$$\min(x, K) - \kappa_* J(x, \varrho) = x\kappa_* \int_0^{+\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x) \, \left(\mathbf{E}\,|\widetilde{\varrho} Z| - \mathbf{E}\,|\widetilde{\varrho} Z + q(\lambda, x)|\right) \mathrm{d}\lambda.$$

Now, Andreson's inequality (see, for example [19], page 155) implies directly that for any $q \in \mathbb{R}$, $\mathbf{E}\,|\widetilde{\varrho}Z + q| \geq \mathbf{E}\,|\widetilde{\varrho}Z|$. Therefore, $\mathbf{P} - \lim_{n\to\infty} (V_1^n - h(S_1)) \leq 0$, i.e. the option is asymptotically underhedged in this case.

Another important point should be noted here is that the coefficient ϱ appearing in (2.5) can be chosen in an arbitrary way. We now state the main result in [36], which also provides the convergence rate for the hedging error.

Theorem 2.6 (Pergamenshchikov)**.** *Consider the Leland strategy* (2.6) *with* $\alpha = 0$ *and let* ϱ *in* (2.5) *be some fixed positive constant. Then, the sequence of random variables*

$$n^{1/4}(V_1^n - h(S_1) - \min(S_1, K) + \kappa_* J(S_1, \varrho)) \tag{2.15}$$

weakly converges to a centered mixed Gaussian variable as $n \to \infty$.

This result is important because it not only gives the asymptotic information of the hedging error but also provides a reasonable way to fix the underhedging issue. More precisely, as discussed in [36], by choosing a suitable value of ϱ the investor can get a portfolio whose terminal value exceeds the option return as desired.

Darses and Lépinette [29] noted that one can modify the Leland strategy to improve the convergence rate in Theorem 2.6. In particular, one can apply a non-uniform revision times $(t_i)_{1\leq i\leq n}$ defined by

$$t_i = g\,(i/n)\ , \quad g(t) = 1 - (1-t)^{\mu} \quad \text{for some} \quad \mu \geq 1 \tag{2.16}$$

and then adjust the volatility as $\widehat{\sigma}_t^2 = \sigma_0^2 + \kappa_* \sigma_0 \sqrt{8/\pi}\sqrt{nf'(t)}$, where f is the inverse function of g. It was also suggested in [29] to use the following modified discrete strategy to release the discrepancy appearing in Theorems 2.5 and Theorem 2.6:

$$\gamma_t^n = \sum_{i=1}^{n} \left(\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \int_0^{t_{i-1}} \widehat{C}_{xt}(u, S_u)\mathrm{d}u \right) \mathbf{1}_{(t_{i-1},t_i]}(t)\,. \tag{2.17}$$

Theorem 2.7. *Let* V_1^n *be the terminal portfolio value of the strategy* (2.17) *with* $\alpha = 0$*. Then, for any* $1 \leq \mu < \mu_{\max}$ *the sequence* $n^{\beta}(V_1^n - h(S_1))$ *weakly converges to a centered mixed Gaussian variable as* $n \to \infty$*, where*

$$\beta = \frac{\mu}{2(\mu+1)} \quad \text{and} \quad \mu_{\max} = \frac{3+\sqrt{57}}{8}\,. \tag{2.18}$$

2.2 Time-depending volatility case

We assume in this subsection that the stock price is driven by $\mathrm{d}S_t = \sigma(t)S_t\mathrm{d}W_t$, where σ is some positive deterministic function. Under the non-uniform rebalancing plan (2.16) the investor should modify the volatility as

$$\widehat{\sigma}_t^2 = \sigma^2(t) + \kappa_*\sigma(t)n^{1/2-\alpha}\sqrt{f'(t)8/\pi} \tag{2.19}$$

to replicate the option with general payoff H, which is a continuous function having continuous derivatives except a finite number of points. We now state the main achievement in time-depending volatility models in [26].

Theorem 2.8 (Lépinette)**.** *Let σ be a strictly positive Lipschitz and bounded function and $H(\cdot)$ be a piecewise twice differentiable function. Suppose furthermore that there exist $x_* \geq 0$ and $\delta \geq 3/2$ such that $\sup_{x \geq x_*} x^\delta |H''(x)| < \infty$. Then, for $\alpha > 0$ the portfolio value of strategy* (2.17) *converges in probability to the payoff $H(S_1)$ as $n \to \infty$. If $\alpha = 0$, then*

$$\mathbf{P} - \lim_{n \to \infty} V_1^n = H(S_1) + H_1(S_1) - \kappa_* H_2(S_1),$$

where $H_1(\cdot)$ and $H_2(\cdot)$ are positive functions depending on the payoff H.

Remark 3. *Theorem 2.7 still holds in the setting of Theorem 2.8 [29].*

It is clear that the Leland algorithm is important for option pricing and hedging thanks to its easy practical implementation. The most interesting case $\alpha = 0$ still needs to be investigated in more general situations, for instance, where volatility depends on other external random factors or jumps in stock prices are taken into account. It is worth noticing that the methodology used in the existing works needs a delicate treatment and seemingly, it is difficult to apply for such models.

2.3 Forms of adjusted volatility

Recall from Remark 1 that choosing the modified volatility as in (2.5) would give an appropriate approximation to compensate transaction costs. However, it is not always the case since the option price inclusive transaction costs $\widehat{C}(t, S_t)$ now depends intrinsically on the rebalancing number n. In more general models, this specific choice can cause to technical issues. For example, in local stochastic models [26], proving the existence of solution to (2.9) requires an effort since now $\widehat{\sigma}$ is computed in terms of the stock price and time. This feature makes the Cauchy problem more challenging to deal with. Nevertheless, it is interesting to point out that the true volatility $\sigma^2(t)$ plays no role in the approximation procedure. In fact, all results reviewed above for the case $\alpha = 0$ can be recovered by using the form $\widehat{\sigma}_t^2 = \kappa_* \sigma(t) n^{1/2} \sqrt{f'(t) 8/\pi}$, where the first term $\sigma^2(t)$ has been removed. More general, we can completely remove $\sigma(t)$ out of the formula of enlarged volatility by taking the new form

$$\widehat{\sigma}_t^2 = \varrho \sqrt{n f'(t)}, \tag{2.20}$$

for some positive constant ϱ. Of course, the limit of transaction costs will slightly change since ϱ is no longer related to the terminal value of volatility, see Theorem 2.6. This important observation follows from the fact which can be proved similarly as Lemma 1.2.8 in [21] (page 16)

$$\int_0^1 \sigma^2(t) S_t^k \frac{\partial^k \widehat{C}}{\partial x^k}(t, S_t)\mathrm{d}t = O(\widehat{\sigma}^{-1}) = O(n^{-1/4}) \quad \text{as } n \to \infty, \tag{2.21}$$

for all $k \geq 2$. The asymptotic representation (2.21) still holds if $\sigma = \sigma(y_t)$ for some extra random process y_t as long as $\mathbf{E}\sup_{0\leq t\leq 1}\sigma^2(y_t) < \infty$. This motivates our assumption $(\mathbf{C}_1)$ in our model (3.1).

Let us emphasize that using the new form (2.20) has two folds of importance. From a technical point of view, it allows us to carry out a much more simple approximation than what have been done in the existing literature. More importantly, when volatility depends on some external factor, says y_t, the Leland strategy is no longer available for practitioners. The reason is that the quantity $\lambda_t = \int_t^1 \widehat{\sigma}_u^2 \mathrm{d}u$, which is substantially dependent on future realizations of y_t (from now, at time t, to the terminal date $t = 1$), is impossible to obtain from practical point of view. In contrast, the simpler form is still helpful in this context since it is a deterministic function of t.

3 Model and Main Results

Let $(\Omega, \mathcal{F}_1, (\mathcal{F}_t)_{0\leq t\leq 1}, \mathbf{P})$ be the standard filtered probability space with two standard independent $(\mathcal{F}_t)_{0\leq t\leq 1}$ adapted Wiener processes $(W_t^{(1)})$ and $(W_t^{(2)})$ taking their values in $\mathbb{R}$. Our financial market consists of one risky asset governed by the following equations on the time interval $[0\,,1]$:

$$\begin{cases} \mathrm{d}S_t = \sigma(y_t) S_t \mathrm{d}W_t^{(1)} \\ \mathrm{d}y_t = F_1(t, y_t)\mathrm{d}t + F_2(t, y_t)(\mathbf{r}\mathrm{d}W_t^{(1)} + \sqrt{1-\mathbf{r}^2}\mathrm{d}W_t^{(2)}), \end{cases} \tag{3.1}$$

where $-1 \leq \mathbf{r} \leq 1$. It is well-known in the literature of SDEs, for example [13, 31], that if $F_1(t, y)$ and $F_2(t, y)$ are measurable in $(t, y) \in [0, T] \times \mathbb{R}$, linearly bounded and locally Lipschitz then, there exists a unique solution y to the last equation of system (3.1), see Theorem 5.1. We assume in this model that the bond interest rate equals to 0, i.e. the non-risky asset is chosen as the *numéraire*.

As discussed in the previous section, we use the adjusted volatility given by

$$\widehat{\sigma}_t^2 = \varrho\sqrt{nf'(t)} = \frac{1}{\sqrt{\mu}}\varrho\sqrt{n}(1-t)^{\frac{1-\mu}{2\mu}}, \quad 1 \leq \mu < 2. \tag{3.2}$$

The parameter $\varrho > 0$ plays an important role in controlling the rate of convergence and it will be specified later. As discussed in details below, the limit of the total trading volume J_n is essentially related to the dependence of ϱ on the number of revisions n. For convenience, recall that $\widehat{C}(t, x)$ is the solution of the Cauchy problem (2.9) with two first derivatives given as in (2.4): $\widehat{C}_x(t,x) = \Phi(\mathbf{v}(\lambda_t, x))$ and $\widehat{C}_{xx}(t,x) = x^{-1}\lambda_t^{-1/2}\,\widetilde{\varphi}\,(\lambda_t, x)$, where

$$\lambda_t = \int_t^1 \widehat{\sigma}_s^2\,\mathrm{d}s = \widetilde{\mu}\,\varrho\sqrt{n}(1-t)^{\frac{1}{4\beta}} \quad \text{and} \quad \widetilde{\mu} = 2\sqrt{\mu}/(\mu+1)\,. \tag{3.3}$$

Remark 4. *We will also see in Section 4.1 that the underhedging situation pointed out in [20] can be fixed by controlling the parameter ϱ.*

We will make use of the following condition on the volatility function.

$(\mathbf{C}_1)$ *Assume that $\sigma(y)$ is a C^2-function and there exists a positive constant $\sigma_{\min}$ such that*

$$0 < \sigma_{\min} \le \sigma(y) \text{ for all } y \in \mathbb{R} \quad \text{and} \quad \mathbf{E}\sup_{0\le t\le 1}\sigma^2(y_t) < \infty.$$

Assumption $(\mathbf{C}_1)$ is not too restrictive and it is indeed fulfilled in almost all popular SV models of the existing literature, see Section 5 and [37].

3.1 Asymptotic results for Leland's strategy

Let us consider the option hedging problem for the model (3.1) in the case of constant proportional cost via Leland's strategy γ_t^n defined in (2.6). This strategy yields a portfolio whose terminal value V_1^n is defined as in (2.7), where rebalancing times (t_i) are given by (2.16). Now, by Itô's formula we obtain

$$h(S_1) = \widehat{C}(1,S_1) = \widehat{C}(0,S_0) + \int_0^1 \widehat{C}_x(t,S_t)\mathrm{d}S_t - \frac{1}{2}I_{1,n}\,, \tag{3.4}$$

where $I_{1,n} = \int_0^1 \left(\widehat{\sigma}_t^2 - \sigma^2(y_t)\right) S_t^2\widehat{C}_{xx}(t,S_t)\mathrm{d}t$. Setting $V_0 = \widehat{C}(0,S_0)$ we can represent the hedging error as

$$V_1^n - h(S_1) = \frac{1}{2}I_{1,n} + I_{2,n} - \kappa_* J_n\,, \tag{3.5}$$

where $I_{2,n} = \int_0^1 \left(\gamma_t^n - \widehat{C}_x(t,S_t)\right)\mathrm{d}S_t$ and J_n is defined in (2.8).

The goal is to find the limit of the hedging error and point out the convergence rate as $n\to\infty$. To this end, we investigate the limit of the terms that contribute in $V_1^n - h(S_1)$ using the essential property $\widehat{\sigma}\to\infty$ as $n\to\infty$. In our setting, $I_{2,n}$ convergences to zero faster than n^β with β defined in (2.18), whereas the gamma error $I_{1,n}$ approaches to $2\min(S_1,K)$ at the same rate. On the other hand, the total trading volume J_n converges in probability to the random variable $J(S_1,y_1,\varrho)$ defined by

$$J(x,y,\varrho) = x\int_0^{+\infty} \lambda^{-1/2}\widetilde{\varphi}(\lambda,x)\,\mathbf{E}\,\left|\sigma(y)\varrho^{-1}Z + q(\lambda,x)\right|\,\mathrm{d}\lambda\,, \tag{3.6}$$

where $Z \sim \mathcal{N}(0,1)$ independent of S_1 and y_1.

In order to determine the asymptotic distribution we need to find the martingale remaining part of the above terms. The most challenging issue in our analysis is that the rest term of total transaction costs naturally takes a discrete form whereas the one obtained by studying $I_{1,n}$ has a continuous form. To combine these two quantities into a unified form that permits one to apply the theory of limit theorem for martingales, we use a special discretization procedure set up in Section 7.

We now state our first asymptotic result for Leland's strategy.

Theorem 3.1. *If condition* $(\mathbf{C}_1)$ *is fulfilled then for any* $\varrho > 0$ *the sequence*

$$n^{\beta}(V_1^n - h(S_1) - \min(S_1, K) + \kappa_* J(S_1, y_1, \varrho))$$

weakly converges to a centered mixed Gaussian variable as $n \to \infty$.

Remark 5. *This theorem is a generalization including an improved convergence rate of the results in [20, 36] where the uniform revision is taken and the volatility is assumed to be a constant.*

Remark 6. *For classical European call option with payoff* $h(x) = (x - K)^+$, *one easily observes that* $h(x) + \min(x, K) = x$. *Then, one deduces from Theorem 3.1 that the wealth process* V_1^n *approaches to* $S_1 - \kappa_* J(S_1, y_1, \varrho)$ *as* $n \to \infty$. *In fact, this is not a big surprise because the option is now sold at high price. The reason is that* $C(0, S_0, \hat{\sigma}) \to S_0$ *as* $\hat{\sigma} \to \infty$. *In other words, Leland's strategy now converges to the well-known buy-and-hold one [24], i.e. to cover the option the seller just takes the trivial strategy: buy a stock share at time* $t = 0$ *for price* S_0 *and keep it until the expiry.*

By letting $\varrho \to \infty$ we observe that

$$\lim_{\varrho \to \infty} J(x, y, \varrho) = x \int_0^{+\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x) |q(\lambda, x)| \mathrm{d}\lambda := J^*(x), \tag{3.7}$$

which is independent of y. This suggests that the rate of convergence in Theorem 3.1 can be improved if ϱ is taken as a function of n. Our next result is established under the following condition on ϱ.

$(\mathbf{C}_2)$ *The parameter* $\varrho = \varrho(n)$ *is a function of* n *such that*

$$\lim_{n \to \infty} \varrho(n) = \infty \quad \text{and} \quad \lim_{n \to \infty} \varrho\, n^{-\frac{\mu}{2(\mu+2)}} = 0\,.$$

The specific choice for ϱ in condition $(\mathbf{C}_2)$ provides the possibility to drop the dependence on volatility in the asymptotic result of the hedging error.

Theorem 3.2. *Under conditions* $(\mathbf{C}_1), (\mathbf{C}_2)$*, the sequence*

$$\theta_n (V_1^n - h(S_1) - \min(S_1, K) + \kappa_* J^*(S_1)) \quad \text{with} \quad \theta_n = n^{\beta} \varrho^{2\beta}$$

weakly converges to a centered mixed Gaussian variable as $n \to \infty$*.*

Remark 7. *The asymptotic distributions in both Theorem 3.1 and Theorem 3.2 are explicitly determined in their proofs in Section 7. Furthermore, these results still hold if* $\widehat{\sigma}_t^2 = \sigma^2(y_t) + \varrho\sigma(y_t)\sqrt{nf'(t)}$ *and the limit of transaction costs is now given by*

$$J'(x, \varrho) = x \int_0^{+\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x)\, \mathbf{E} \left| Z\varrho^{-1} + q(\lambda, x) \right| \mathrm{d}\lambda. \tag{3.8}$$

However, such a use for enlarged volatility is far away from practical significance as discussed in Subsection 2.3.

3.2 Asymptotic result for Lépinette's strategy

Let us consider the modified strategy $\overline{\gamma}_t^n$ defined in (2.17), which produces a portfolio whose terminal values $\overline{V}_1^n$ defined by $\overline{V}_1^n = \overline{V}_0^n + \int_0^1 \overline{\gamma}_t^n \mathrm{d}S_t - \kappa_* \overline{J}_n$, where

$$\overline{J}_n = \sum_{i=1}^{n} S_{t_i} |\overline{\gamma}_{t_i}^n - \overline{\gamma}_{t_{i-1}}^n| \,. \tag{3.9}$$

Now by Itô's formula, one presents the hedging error as

$$\overline{V}_1^n - h(S_1) = \frac{1}{2} I_{1,n} + \overline{I}_{2,n} - \kappa_* \overline{J}_n \,, \tag{3.10}$$

where $\overline{I}_{2,n} = I_{2,n} + \sum_{i \geq 1} (S_{t_i} - S_{t_{i-1}}) \int_0^{t_{i-1}} \widehat{C}_{xt}(u, S_u) \mathrm{d}u$. We obtain the following result using the form (3.2) for enlarged volatility.

Theorem 3.3. *Suppose that* $(\mathbf{C}_1)$ *is fulfilled. Then, for any* $\varrho > 0$*, the sequence*

$$n^{\beta} (\overline{V}_1^n - h(S_1) - \eta \min(S_1, K)) \quad \text{with} \quad \eta = 1 - \kappa_* \sigma(y_1) \varrho^{-1} \sqrt{8/\pi}$$

weakly converges to a centered mixed Gaussian variable as $n \to \infty$*.*

Remark 8. *If volatility is a constant then it is interesting to see that Theorem 2.7 can be recovered from Theorem 3.3 with* $\varrho = \kappa_* \sigma \sqrt{8/\pi}$*. Also note that in our model, the parameter* μ *takes its values in the interval* $[1, 2)$*, that is slightly more general than the condition imposed in Theorem 2.7. Moreover, if the classical form of adjusted volatility is applied for Lepinette's strategy* $\overline{\gamma}_t^n$ *then the option can be completely replicated by taking* $\varrho = \kappa_* \sqrt{8/\pi}$*, even in the SV models and we recover again the result established in [9].*

In the context of condition $(\mathbf{C}_2)$, the cumulated cost $\kappa_* \overline{J}_n$ converges to 0 whereas the hedging error approaches to the terminal value S_1 of the buy-and-hold strategy. Hence, the option is over replicated in this case, see Remark 6.

Corollary 3.1. *Assume that $\varrho \to \infty$ under condition $(\mathbf{C}_2)$ and condition $(\mathbf{C}_1)$ holds. Then, the wealth sequence $\overline{V}_1^n$ converges in probability to $h(S_1) + \min(S_1, K) = S_1$.*

Note that no improved-convergence version of Theorem 3.3 is obtained since $\kappa_* \overline{J}_n$ converges to 0 at order of ϱ.

4 Applications for pricing problems

This section presents some applications of the results in Section 3 for the problem of option pricing with transaction costs. We first emphasize that it is impossible to obtain a *non-trivial* perfect hedge with the presence of transaction costs even in constant volatility models. In other words, to cover completely the option return, the seller can take the *buy-and-hold* strategy, but this makes the option price too expensive. However, once the investor accepts to take a risk in his hedging problem, the option price can be lowered in a way so that the payoff will be covered with a given probability.

4.1 Superhedging with transaction costs

To stand on the safe side, the investor will search for strategies providing the terminal value that exceeds the payoff. Such strategies usually concern solutions to dynamic optimization problems. More precisely, let H be a general contingent claim and denote by $\mathcal{A}(x)$ the set of all *admissible strategies* π with the initial capital x and $V_T^{\pi,x}$ the terminal value of strategy π. Then, the super-replication cost of H is determined as

$$U_0 = \inf \left\{ x \in \mathbb{R} : \exists \pi \in A(x),\, V_T^{\pi,x} \geq H \quad \text{a.s.} \right\}, \tag{4.1}$$

see [24] and the references therein.

In the presence of transaction costs, Cvitanić and Karatzas [8] show that the *buy-and-hold strategy* is the unique choice if one wishes to successfully replicate the option and then S_0 is the super-replication price. In this section, we will show that this property still holds in the sense of approximate superhedging via Leland's spirit. The following observation is just a direct consequence of Theorem 3.2 when ϱ is used as a function of n.

Proposition 4.1. *Under conditions $(\mathbf{C}_1)$ and $(\mathbf{C}_2)$, $\mathbf{P} - \lim_{n\to\infty} V_1^n \geq h(S_1)$. The same property holds for Lépinette's strategy.*

Proof. Note first that $J^*(x) \le \min(x, K)$, for all $x > 0$. Hence, by Theorem 3.2

$$\mathbf{P} - \lim_{n\to\infty} (V_1^n - h(S_1)) \ge (1 - \kappa_*) \min(S_1, K).$$

The term in the left hand side is obviously non negative since $\kappa_* < 1$ hence the conclusion follows. The conclusion for Lépinette strategy directly follows from Theorem 3.3. $\square$

4.2 Asymptotic quantile pricing

As seen above, the superhedging cost is too high from the buyer's point of view though it indeed gives the seller a successful hedge with probability one. More practically, one can ask that how much initial capital can be reduced by accepting a shortfall probability in replication objective. More precisely, the seller may take a risk and look for hedges with the minimal initial cost defined by

$$\inf \left\{ x \in \mathbb{R}, \exists \pi \in A(x) : P\left(V_T^{\pi,x} \ge H\right) \ge 1 - \varepsilon \right\},$$

with a given significant level $0 \le \varepsilon \le 1$. See [11, 35, 5, 7, 6] for discussions in details. Let us adapt this idea to the hedging problem in the presence of transaction costs. As seen above, the super-hedging price is S_0 if Leland's algorithm is used to replicate the option. On the seller's side we propose to sell the option at the price $\delta S_0 < S_0$, (where $0 < \delta < 1$ will be properly chosen) and follow Leland's strategy as before for replication. To be safe at the terminal moment, we need to choose the parameter ϱ such that the probability that the terminal portfolio exceeds the sum of the real objective (i.e. the payoff) and the additional amount $(1-\delta)S_0$ is greater than $1-\varepsilon$, where ε is a significant level predetermined by the seller. We easily observe that this purpose can be achieved by Proposition 4.1. To determine the option price it now remains to choose value δ. We suggest to define it by

$$\delta_\varepsilon = \inf \left\{ a > 0 : \Upsilon(a) \ge 1 - \varepsilon \right\}, \tag{4.2}$$

where $\Upsilon(a) = \mathbf{P}\left((1 - \kappa_*) \min(S_1, K) > (1 - a)S_0\right)$. The quantity δ_ε is called quantile price of the option at level ε and the difference $(1-\delta_\varepsilon)S_0$ is the reduction amount of option price (initial cost for quantile hedging). Clearly, the smaller value of δ_ε is, the cheaper the option is.

We show that the option price is significantly reduced, compared with powers of parameter ε.

Proposition 4.2. *Let δ_ε be Leland price defined by* (4.2) *and assume that $\sigma_{\max} = \sup_{y\in\mathbb{R}} \sigma(y) < \infty$. Then, for any $r > 0$,*

$$\lim_{\varepsilon\to 0} (1 - \delta_\varepsilon)\varepsilon^{-r} = +\infty\,. \tag{4.3}$$

Proof. Observe that $0 < \delta_\varepsilon \le 1$ and δ_ε tends to 1 as $\varepsilon \to 0$. Set $b = 1 - \kappa_*$. Then for sufficiently small ε such that $\delta_\varepsilon > a > 1 - bK/S_0$ one has

$$1 - \varepsilon > \mathbf{P}(b \min(S_1, K) > (1-a)S_0) = 1 - \mathbf{P}(S_1/S_0 \le (1-a)/b).$$

Therefore,

$$\varepsilon < \mathbf{P}\left(S_1/S_0 \le (1-a)/b\right) \le \mathbf{P}\left(X_1 \le -z_a\right), \tag{4.4}$$

where $X_t = \int_0^t \sigma(y_t) \mathrm{d}W_t^{(1)}$ and $z_a = \ln(b/(1-a)) - \sigma^2_{\max}/2$. To estimate this probability we note that for any integer $m \ge 1$,

$$\mathbf{E}\,(X_1)^{2m} \le \sigma^{2m}_{\max}(2m-1)!!$$

(see, for example, [31, Lemma 4.11, page 130]). Setting now $R(\upsilon) = 2\upsilon\sigma^2_{\max}$, we obtain that for any $0 < \upsilon < 1/2\sigma^2_{\max}$,

$$\mathbf{E}\, e^{\upsilon X_1^2} = \sum_{m=0}^{\infty} \frac{\upsilon^m}{m!} \mathbf{E}\,(X_1)^{2m} \le \sum_{m=0}^{\infty} \frac{\upsilon^m}{m!} \sigma^{2m}_{\max}(2m-1)!! \le \frac{1}{1 - R(\upsilon)}\,.$$

Therefore, for $\varepsilon > 0$ sufficiently small one has

$$\varepsilon \le \mathbf{P}(X_1 \le -z_a) = \mathbf{P}(-X_1 \ge z_a) \le e^{-\upsilon z_a^2}\, \mathbf{E}\, e^{\upsilon X_1^2} \le \frac{e^{-\upsilon z_a^2}}{1 - R(\upsilon)}\,,$$

i.e. $1 - a \ge b\, e^{-\iota_\varepsilon(\upsilon)}$, where $\iota_\varepsilon(\upsilon) = \sqrt{|\ln \varepsilon(1 - R(\upsilon))|/\upsilon} + \sigma^2_{\max}/2$. Letting now $a \to \delta_\varepsilon$ one obtains $1 - \delta_\varepsilon \ge b e^{-\iota_\varepsilon(\upsilon)}$, which implies (4.3). $\square$

The boundedness of volatility function is essential for the above comparison proposition. If one wishes to relax this assumption, the price reduction is now less free than in Proposition 4.2.

Proposition 4.3. *Suppose that there exists a positive constant $\alpha > 1/2$ such that $\mathbf{E} \exp\{\alpha \int_0^1 \sigma^2(y_s) ds\} < \infty$. Then, for $r_\alpha = (2\sqrt{2\alpha} + 1)/2\alpha$,*

$$\liminf_{\varepsilon \to 0}\, \varepsilon^{-r_\alpha}\,(1 - \delta_\varepsilon) > 0\,. \tag{4.5}$$

Proof. For any positive constant L we set

$$\tau = \tau_L = \inf\left\{t > 0 : \int_0^t \sigma^2(y_s) \mathrm{d}s \ge L\right\} \wedge 1, \tag{4.6}$$

which is understood as the first time that the log-price's variance passes the level L. Then, one deduces from (4.4) that

$$\varepsilon \le \mathbf{P}\left(\mathcal{E}_1^{-1}(\sigma) \ge u_a, \int_0^1 \sigma^2(y_s)\mathrm{d}s \le L\right) + \mathbf{P}\left(\int_0^1 \sigma^2(y_s)\mathrm{d}s \ge L\right), \tag{4.7}$$

where $\mathcal{E}_t(\sigma) = e^{\int_0^t \sigma(y_s)\mathrm{d}W_s^{(1)} - \frac{1}{2}\int_0^t \sigma^2(y_s)\mathrm{d}s}$, $u_a = (1-\kappa_*)/(1-a)$ and $\delta_\varepsilon > a > 1-bK/S_0$. Note that for any $p > 0$, the process $\chi_t = \mathcal{E}_{\tau\wedge t}(-p\sigma)$ is a martingale, i.e. $\mathbf{E}\chi_t = 1$. Therefore, the first probability in the right side of (4.7) can be estimated as

$$(u_a)^{-p}\,\mathbf{E}\,\mathcal{E}_\tau^{-p}(\sigma) = (u_a)^{-p}\,\mathbf{E}\,\chi_1\, e^{\check{p}\int_0^\tau \sigma^2(y_s)\mathrm{d}s} \le (u_a)^{-p}\, e^{\check{p}L}\,,$$

where $\check{p} = (p^2+p)/2$. By hypothesis and Chebysev's inequality one obtains

$$\mathbf{P}\left(\int_0^1 \sigma^2(y_s)\mathrm{d}s \ge L\right) \le C_\alpha e^{-\alpha L} \quad \text{with} \quad C_\alpha = \mathbf{E}\exp\left\{\alpha\int_0^1 \sigma^2(y_s)\mathrm{d}s\right\}\,.$$

Hence, $\varepsilon \le (u_a)^{-p}\, e^{\check{p}L} + C_\alpha e^{-\alpha L}$. Choosing $L = \alpha^{-1}\ln(2C/\varepsilon)$ and letting $a \to \delta_\varepsilon$, one deduces that for any $p > 0$ and for some positive constant $\tilde{C}_\alpha$,

$$1 - \delta_\varepsilon \ge \tilde{C}_\alpha\,\varepsilon^{\gamma^*(p)}, \quad \text{where} \quad \gamma^*(p) = (p+1)/(2\alpha) + p^{-1}\,.$$

Note that $r_\alpha = \min_{p>0}\gamma^*(p) = \gamma^*(\sqrt{2\alpha})$. Therefore, taking in the last inequality $p = \sqrt{2\alpha}$ we obtain the property (4.5). $\square$

Remark 9. *It is clear that* $r_\alpha < 1$ *if* $\alpha > 3/2 + \sqrt{2}$. *The condition used in Proposition 4.3 holds for such* α *when* σ *is linear bounded and* y_t *follows an Orstein-Uhlenbeck process, see the Appendix D. The same quantile pricing results can be established for Lépinette strategy.*

5 Examples

In this section, we list some well-known SV models for which condition $(\mathbf{C}_1)$ is fulfilled. For this aim, we will need some moment estimates for solutions to general non-linear SDEs

$$\mathrm{d}y_t = F_1(t,y_t)\mathrm{d}t + F_2(t,y_t)\mathrm{d}Z_t, \quad y(0) = y_0, \tag{5.1}$$

with Z is a standard Wiener process and F_1, F_2 are two smooth functions. We first recall the well-known result in theory of SDEs, see for example [13], Th.2.3, p.107.

Theorem 5.1. *Suppose that* $F_1(t,y)$ *and* $F_2(t,y)$ *are measurable in* $(t,y) \in [0,T] \times \mathbb{R}$, *linearly bounded and locally Lipschitz. If* $\mathbf{E}\,|y_0|^{2m} < \infty$ *for some integer* $m \ge 1$ *then, there exists a unique solution* y_t *to* (5.1) *and*

$$\mathbf{E}\,|y_t|^{2m} < (1 + \mathbf{E}\,|y_0|^{2m})e^{\alpha t}, \quad \mathbf{E}\sup_{0\le s\le t}|y_s|^{2m} < M(1 + \mathbf{E}\,|y_0|^{2m}),$$

where α, M *are positive constants depending on* t, m.

We will see that in the context of the previous theorem, condition $(\mathbf{C}_1)$ holds if the volatility function σ satisfies the condition of polynomial growth $|\sigma(y)| \le C(1+|y|^m)$ for some positive constant C and $m \ge 1$.

Hull-White models: Consider the case where y_t follows a geometric Brownian motion

$$\mathrm{d}S_t = (y_t + \sigma_{\min})S_t \mathrm{d}W_t \quad \text{and} \quad \mathrm{d}y_t = y_t(a\mathrm{d}t + b\mathrm{d}Z_t), \tag{5.2}$$

where $\sigma_{\min} > 0$, a and b are some constants and Z is a standard Brownian motion correlated to W_t. Put $y^* = \sup_{0\le t\le 1}|y_t|$. Then, by Theorem 5.1 one has

$$\mathbf{E}\,(y^*)^{2m} \le C(1 + \mathbf{E}|y_0|^{2m}) < \infty$$

as long as $\mathbf{E}|y_0|^{2m} < \infty$. Therefore, condition $(\mathbf{C}_1)$ is clearly fulfilled.

Uniform Elliptic Volatility models: Consider the case where volatility is driven by an Orstein-Uhlenbeck process of mean-reverting

$$\mathrm{d}S_t = (y_t^2 + \sigma_{\min})S_t \mathrm{d}W_t \quad \text{and} \quad \mathrm{d}y_t = (a - by_t)\mathrm{d}t + \mathrm{d}Z. \tag{5.3}$$

In this case $\sigma(y) = y^2 + \sigma_{\min}$ and condition $(\mathbf{C}_1)$ is obviously verified throughout Theorem 5.1.

Stein-Stein models:

$$\mathrm{d}S_t = \sqrt{y_t^2 + \sigma_{\min}}\,S_t \mathrm{d}W_t \quad \text{and} \quad \mathrm{d}y_t = (a - by_t)\mathrm{d}t + \mathrm{d}Z_t. \tag{5.4}$$

We have $\sigma(y) = \sqrt{y^2 + \sigma_{\min}}$ and condition $(\mathbf{C}_1)$ is also verified applying Theorem 5.1.

Heston models: Heston [18] proposed a SV model where volatility is driven by a CIR process, which is also called squared root process. This kind of model can be used in our context. Indeed, assume now that the price dynamics is given by the following

$$\mathrm{d}S_t = \sqrt{y_t + \sigma_{\min}}\,S_t \mathrm{d}W_t \quad \text{and} \quad \mathrm{d}y_t = (a - by_t)\mathrm{d}t + \sqrt{y_t}\,\mathrm{d}Z_t, \quad y_0 \ge 0. \tag{5.5}$$

For any a and $b > 0$, there exists a unique strong solution $y_t > 0$. Note that the Lipschitz condition of diffusion coefficient in Theorem 5.1 is violated but using stopping times method, we can directly show that $\mathbf{E}\,y^* < \infty$ (see the Appendix C) hence, condition $(\mathbf{C}_1)$ is satisfied.

Similarly, one can verify that $(\mathbf{C}_1)$ also holds for Ball-Roma's models [3] or, more generally, for a class of processes of bounded diffusion holding the following condition.

$(\mathbf{A})$ *There exist positive constants* a, b, M *such that*

$$yF_1(t,y) \le a - by^2 \quad \textit{and} \quad |F_2(t,y)| \le M, \quad \textit{for all} \quad t > 0, y \in \mathbb{R}.$$

Proposition 5.1. *Under condition* (**A**)*, there exists a constant* $\alpha > 0$ *such that* $\mathbf{E}\, e^{\alpha |y|_1^2} < \infty$*, where* $|.|_1^2$ *stands for* $\sup_{0\le t\le 1} y_t^2$.

Proof. A proof can be made using the method in Proposition 1.1.2 in [22]. □

Scott models: Let us consider the situation where volatility follows an Orstein-Uhlenbeck as in Stein-Stein's models. Assume now that the function σ takes the exponential form

$$\mathrm{d}S_t = (e^{\delta y_t} + \sigma_{\min}) S_t \mathrm{d}W_t^{(1)} \quad \text{and} \quad \mathrm{d}y_t = (a - b y_t)\mathrm{d}t + \mathrm{d}Z_t, \tag{5.6}$$

where a, b and $\sigma_{\min} > 0$ are constants and $\delta > 0$ is chosen such that $2\delta \le \alpha$ defined as in Proposition 5.1. Here $\sigma(y) = e^{\delta y} + \sigma_{\min}$ and then condition ($\mathbf{C}_1$) is fulfilled since

$$\mathbf{E} \sup_{0\le t\le 1} |\sigma(y)|^2 \le 2\sigma_{\min}^2 + 2\mathbf{E}\,(e^{2\delta}\mathbf{1}_{\{|y_t|\le 1\}} + e^{2\delta|y|_1^2}\mathbf{1}_{\{|y_t|>1\}}) < \infty.$$

Numerical result for Hull-White's model: We provide a numerical example for Lépinette's strategy used for the Hull-White model (5.2) discussed above with correlation coefficient $\alpha = 0.05$. The initial values are $S_0 = K = 1$, $y_0 = 2$, with parameter values $\sigma_{\min} = 2, a = -2, b = 1$. The gain is measured by $V_1^n - \max(S_1 - K, 0)$. By Theorem 3.3, the theoretical discrepancy is defined by $V_1^n - \max(S_1 - K, 0) - (1-\eta)\min(S_1, K)$ including the 95% interval. To see the strategy performance, we also compute the option price including the initial amount of shares to hold in the Lépinette strategy by simulating $N = 500$ trajectories in the crude Monte-Carlo method, see Table 1 and Table 2. It turns out that the hedging strategy converges quite rapidly to the buy-and-hold and the option price approaches to the superhedging price S_0. In constrast, convergence of the corrected replication error to 0 is somehow slow. In fact, increasing values of ϱ can provide a better convergence but this unexpectedly leads to the superhedge more rapidly. This evidence again emphasizes the importance of price reduction discussed in Subsection 4.2.

n	gain	error	lower bound	upper bound	price	strategy
10	0.1523845	-0.2225988	-0.2363122	-0.2088854	0.7914033	0.9013901
50	0.2966983	-0.0596194	-0.0670452	-0.0521936	0.9399330	0.9706068
100	0.3086120	-0.0288526	-0.0350141	-0.0226911	0.9746527	0.9875094
500	0.2955755	0.0032387	-0.0005821	0.0070594	0.9991733	0.9995891
1000	0.2851002	0.0012409	-0.0021596	0.0046415	0.9999300	0.9999652

Table 1: Convergence for Lépinette's strategy for $\kappa_* = 0.01, \varrho = 2$.

n	gain	error	lower bound	upper bound	price	strategy
10	0.2859197	-0.0744180	-0.0813544	-0.0674816	0.9246420	0.9659700
50	0.3172523	-0.0069238	-0.0115426	-0.0023049	0.9921661	0.9962377
100	0.3033519	0.0007474	-0.0030916	0.0045864	0.9984346	0.9992385
500	0.3618707	0.0001296	-0.0024741	0.0027333	0.9999977	0.9999989
1000	0.3334375	0.0003996	-0.0020559	0.0028550	1	1

Table 2: Convergence for Lépinette's strategy for $\kappa_* = 0.001, \varrho = 4$.

6 High frequency markets

We now assume that purchases of the risky asset are carried out at a higher ask price $S_t + \varepsilon_t$ whereas sales only earn a lower bid price $S_t - \varepsilon_t$, where the mid price S_t is given as in model (3.1) and ε_t is the halfwidth of the bid-ask spread. Then, for any trading strategy ψ_t of finite variation the wealth process can be determined by

$$V_t = V_0 + \int_0^t \psi_s \mathrm{d}S_s - \int_0^t \varepsilon_s \mathrm{d}|\psi|_s, \tag{6.1}$$

where $|\psi|$ is the total variation of strategy ψ_t. Here the first two terms are the classical components in frictionless frameworks, which respectively describe the initial capital and gains from trading. The last integral accounts for transaction costs incurred by trading activities by weighting the total variation [2] of the strategy with the halfwidth of the spread.

For problems of optimal investment and consumption with small transaction costs [23], the additional terms should be added in the formulation of V_t. In such cases, approximate solutions are usually determined throughout an asymptotic expansion around 0 of the halfwidth spread ε, where the leading corrections are obtained by collecting the inputs from the frictionless problem.

In this section, we are only interested in the replication purpose using discrete strategies in the Leland spirit. Assume that for his replication aim, the option seller will apply a discrete hedging strategy $\psi_t^{n,\varepsilon}$ that will be executed at n dates defined by $t_i = g(i/n)$ as in Section 3. The corresponding wealth process is now given by

$$V_t^{n,\varepsilon} = V_0^{n,\varepsilon} + \int_0^t \psi_s^{n,\varepsilon} \mathrm{d}S_s - \sum_{i=1}^n \varepsilon_{t_i} |\psi_{t_i}^{n,\varepsilon} - \psi_{t_{i-1}}^{n,\varepsilon}|. \tag{6.2}$$

In order to partially eliminate the influence of transaction costs in the replication error, we intend to apply again the increasing volatility principle for the present context. Note that in high frequency markets, the bid-ask spread is in general of the same order of magnitude as price jumps and hence ε_t is assumed to be of the form

[2] It is important to know that the classical Black-Scholes strategy is not finite variation.

$\kappa_* n^{-1/2} S_t$, for some positive constant κ_*. Then, it is interesting to see that this case corresponds to the Leland-Lott framework with $\alpha = 1/2$. [3]

In our context, it is interesting to see that the enlarged volatility $\widehat{\sigma}_t^2 = \varrho\sqrt{nf'(t)}$ is still helpful if the option seller uses the Leland strategy or the Lépinette one in the place of $\psi_t^{n,\varepsilon}$.

Proposition 6.1. *Assume that* $\varepsilon_t = \kappa_* n^{-1/2} S_t$. *If the option seller uses the enlarged volatility of the form* $\widehat{\sigma}^2 = \varrho\sqrt{nf'(t)}$ *and follows the Leland or the Lépinette strategy then, the sequence of portfolio values* $V_1^{n,\varepsilon}$ *converges in probability to* $h(S_1)+\min(S_1, K) = S_1$. *In particular,* $n^\beta(V_1^{n,\varepsilon} - S_1)$ *converges to a mixed Gaussian variable as* $n \to \infty$.

Proof. . The proof is just a direct consequence of Theorem 3.1 in Section 3 since now the total transaction cost converges to zero. □

To describe more illiquidity markets we can assume that $\varepsilon_t = \kappa_* S_t$. Then, this assumption clearly corresponds to the case $\alpha = 0$ and hence the results in Section 3 are recalled for these situations.

We conclude the section by considering the case where the stock spreads equal after sticks regardless of the stock price, i.e. $\varepsilon_t = \kappa_*$, for some positive constant κ_*. This means that transaction costs are now based on the volume of traded shares instead of on traded amount of money as treated in the literature and in Section 3. Fortunately, our methodology still works for such cases. The following result is just an analogue of Theorem 3.1 with a small modification in the limit of transaction costs, defined by

$$J_0(x, y, \varrho) = \int_0^{+\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x)\, \mathbf{E} \left| \sigma(y)\varrho^{-1} Z + \frac{\ln(x/K)}{2\lambda} - \frac{1}{4} \right| \mathrm{d}\lambda\,, \tag{6.3}$$

where $Z \sim \mathcal{N}(0,1)$ independent of S_1, y_1

Proposition 6.2. *Let* $\varepsilon_t = \kappa_* > 0$ *and* $\widehat{\sigma}^2 = \varrho\sqrt{nf'(t)}$. *For Leland's strategy under condition* $(\mathbf{C}_1)$, *the sequence* $n^\beta\left(V_1^{n,\varepsilon} - h(S_1) - \min(S_1, K) + \kappa_*\, J_0(S_1, y_1\varrho)\right)$ *weakly converges to a centered mixed Gaussian variable as* $n \to \infty$. *Furthemore, if Lépinette's strategy is used then* $n^\beta\left(\overline{V}_1^{n,\varepsilon} - h(S_1) - (1-\eta_0)\min(S_1, K)\right)$ *weakly converges to a centered mixed Gaussian variable, where* $\eta_0 = \sigma(y_1)\varrho^{-1} S_1^{-1}\sqrt{8/\pi}$.

Proof. The proof is similar to that of Theorem 3.1, see Section 7. □

Remark 10. *When* $\varrho \to \infty$ *under condition* $(\mathbf{C}_2)$ *one obtains an improved-rate version of the above results as in Theorem 3.2 and the initial volatility is completely removed out of the limit of transaction costs.*

[3] We would like to thank the anonymous referee for pointing out the correspondence of the case $\alpha = 1/2$ to this setting.

7 Proofs

The limit theorems in Section 3 are proved in the following generic procedure:

Step 1: Determine the principal term of the hedging error. In particular, we will point out that the gamma term $I_{1,n}$ converges to $2\min(S_1, K)$ while the cumulated transaction cost approaches to its limit J defined in (3.6). Both convergences are at order of $\theta_n = n^{\beta}\varrho^{2\beta}$.

Step 2: Represent the residual terms, which are in the form of stochastic intergral, at order of θ_n as martingales. Since the residual terms resulting from the analysic of transaction costs are naturally discrete, we needs to discretize all the stochastic integrals using a special procedure set up below in Subsection 7.2.

Step 3: Determine the limit distribution of the residual using limit theorem results for martingales established in [17]. This result is the key tool but we need in fact some special versions compatible with our context. These will be explicitly constructed in Subsection 7.3.

7.1 Preliminary

Note that $\widehat{C}(t,x)$ and its derivatives can be represented as functions of λ_t and x, where

$$\lambda_t = \lambda_0(1-t)^{\frac{1}{4\beta}} := \lambda_0\nu(t) \quad \text{and} \quad \lambda_0 = \widetilde{\mu}\varrho\sqrt{n}. \tag{7.1}$$

Moreover, the function $\widetilde{\varphi}(\lambda, x)$ (appearing in all k-th ($k \geq 2$) degree derivatives of $\widehat{C}$ with respect to the space variable and also for derivatives in time via the relation (2.9)) is exponentially decreasing to 0 as λ approaches to 0 or ∞. This property motives our analysis in terms of variable λ. In particular, let us fixe two functions l_*, l^* and let $1 \leq m_1 < m_2 \leq n$ be two integers such that $l_* = \lambda_0\nu(g(m_2/n))$ and $l^* = \lambda_0\nu(g(m_1/n))$. Then, all terms corresponding with index $j \notin [m_1, m_2]$ can be ignored in the approximation analysis at a certain order depending on the choice of l_* and l^*. For our purpose, the desired order is $\theta_n \sim \lambda_0^{2\beta}$. Therefore we will take for example $l_* = 1/\ln^3 n,\ l^* = \ln^3 n$ and define

$$m_1 = n - \left[n\left(l^*/\lambda_0\right)^{2/(\mu+1)}\right] \quad \text{and} \quad m_2 = n - \left[n\left(l_*/\lambda_0\right)^{2/(\mu+1)}\right], \tag{7.2}$$

where the notation $[x]$ stands for the integer part of a number x. Below we focus on the subsequence (t_j) of trading times and the corresponding sequence $\left(\lambda_j\right)$ defined as

$$t_j = 1 - (1 - j/n)^{\mu} \quad \text{and} \quad \lambda_j = \lambda_0(1-t_j)^{\frac{1}{4\beta}}, \quad m_1 \leq j \leq m_2. \tag{7.3}$$

Note that $\left(t_j\right)$ is an increasing sequence with values in $[t^*, t_*]$, where $t_* = 1-(l_*/\lambda_0)^{4\beta}$ and $t^* = 1-(l^*/\lambda_0)^{4\beta}$, whereas $\left(\lambda_j\right)$ is decreasing in $[l_*, l^*]$. Therefore, in the sequel

we make use the notation $\Delta t_j = t_j - t_{j-1}$ whereas $\Delta\lambda_j = \lambda_{j-1} - \lambda_j$, for $m_1 \le j \le m_2$ to avoid the negative sign in discrete sums.
Below, Itô integrals will be discretized throughout the following sequences of independent normal random variables

$$Z_{1,j} = \frac{W^{(1)}_{t_j} - W^{(1)}_{t_{j-1}}}{\sqrt{t_j - t_{j-1}}}\,, \qquad Z_{2,j} = \frac{W^{(2)}_{t_j} - W^{(2)}_{t_{j-1}}}{\sqrt{t_j - t_{j-1}}}. \tag{7.4}$$

We set

$$p(\lambda, x, y) = \frac{\varrho}{\sigma(y)}\left(\frac{\ln(x/K)}{2\lambda} - \frac{1}{4}\right). \tag{7.5}$$

and write for short $p_{j-1} = p(\lambda_{j-1}, S_{t_{j-1}}, y_{t_{j-1}})$. This reduced notation is also frequently applied for functions appearing in the approximation procedure. With the sequence of revision times (t_j) in hand, we consider the centered sequences

$$\begin{cases} Z_{3,j} = |Z_{1,j} + p_{j-1}| - \mathbf{E}\left(|Z_{1,j} + p_{j-1}|\,|\,\mathcal{F}_{j-1}\right), \\ Z_{4,j} = |Z_{1,j}| - \mathbf{E}\left(|Z_{1,j}|\,|\,\mathcal{F}_{j-1}\right) = |Z_{1,j}| - \sqrt{2/\pi}. \end{cases} \tag{7.6}$$

The sequences $(Z_{3,j})$ and $(Z_{4,j})$ will serve in finding the Dood decomposition of considered terms. To represent the limit of transaction costs, we introduce the functions

$$\begin{cases} G(a) = \mathbf{E}\left(|Z + a|\right) = 2\varphi(a) + a\left(2\Phi(a) - 1\right), \\ \Lambda(a) = \mathbf{E}\left(|Z + a| - \mathbf{E}\,|Z + a|\right)^2 = 1 + a^2 - G^2(a), \end{cases} \tag{7.7}$$

for $a \in \mathbb{R}$ and $Z \sim \mathcal{N}(0,1)$. We also write $o_{\mathbf{P}}(n^{-r})$ for generic sequences of random variables $(X_n)_{n\ge1}$ satisfying $\mathbf{P} - \lim_{n\to\infty} n^r X_n = 0$.

7.2 Approximation for stochastic integrals

First, we introduce the following condition.
$(\mathbf{H}_1)$ *A is a continuously differentiable $\mathbb{R}_+ \to \mathbb{R}$ function having absolutely integrable derivative A' and*

$$\lim_{n\to\infty} \theta_n \left(\int_0^{l_*} |A(\lambda)|\mathrm{d}\lambda + \int_{l^*}^{+\infty} |A(\lambda)|\mathrm{d}\lambda\right) = 0\,.$$

Now, for any $L > 0$, we consider the stopping time

$$\tau^* = \tau^*_L = \inf\left\{t \ge 0 : \sigma^*_t > L\right\}, \tag{7.8}$$

where $\sigma^*_t = \max\{\sigma(y_t), |\sigma'(y_t)|, |\sigma''(y_t)\}|$ and denote by $S^*_t = S_{\tau_L\wedge t}$ and $y^*_t = y_{\tau_L\wedge t}$ the stopped processes.[4] We present here the approximation procedure for Itô's

[4] In the context of Theorem 3.2, the integrability on derivatives of $\sigma(y_t)$ is not necessary to hold and $\sigma^*_t = \sigma(y_t)$ for this case.

stochastic integrals throughout the sequences $(Z_{1,j})$ and $(Z_{2,j})$. In particular, the discrete approximation concerns the class of functions holding the below technical condition.
$(\mathbf{H}_2)$ *Let A be a $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ continuously differentiable function satisfying the following: there exist $\gamma > 0$ and a positive function U such that*

$$\sup_{\lambda>0} \min(\lambda^\gamma, 1)|A(\lambda, x, y)| \le U(x, y) \quad \textit{and} \quad \sup_{0\le t\le 1} \mathbf{E}\,(S^*_t)^m U^{2r}(S^*_t, y^*_t) < \infty,$$

for any $-\infty < m < +\infty$, $r \ge 0$ and $L > 0$.

Remark 11. *We can check directly that $\partial^k_x \widehat{C}(\lambda, x) = x^{k-1}\lambda^{-k/2}\widetilde{\varphi}(\lambda, x)P(\ln(x/K))$, where P is some polynomial. Therefore, all the functions A appearing in our approximation are of the form $\lambda^{-k/2}x^m\bar{\sigma}(y)P(\ln(x/K))$, where $\bar{\sigma}$ can be σ or its first two derivatives σ', σ''. In constant or bounded volatility settings, it can be shown with some computational effort e.g. [9, 26, 29] that*

$$\sup_{0\le t\le 1} \mathbf{E}S^m_t \ln^{2r} S_t < \infty, \quad \textit{for any} \quad m \in \mathbb{R},\, r \ge 0. \tag{7.9}$$

However, it is not always fulfilled for SV models with unbounded volatility and natural conditions on the correlation and the coefficients of the equation of y_t are of course necessarily required, see for instant [2, 30] for discussion on this interesting direction. This undesirable property prevent us to carry out an asymptotic analysis using L^2 estimates as in the existing works. It is important to note that (7.9) is true for processes stopped at τ^*. Therefore, only convergences in probability can be provided in the below approximation.

For simplicity, in the sequel we use the notation $\check{S} = (S, y)$. The following technique is used frequently in our asymptotic analysis.

Proposition 7.1. *Assume that $A_0 = A_0(\lambda, x, y)$ is a function satisfying $(\mathbf{H}_2)$ and $A(\lambda, x, y) = A_0(\lambda, x, y)\widetilde{\varphi}(\lambda, x)$. Then, for $i = 1, 2$,*

$$\int_0^1 \widehat{\sigma}^2_t \left(\int_t^1 A(\lambda_t, \check{S}_u)\mathrm{d}W^{(i)}_u \right) \mathrm{d}t = \varrho^{-1} \sum_{j=m_1}^{m_2} \overline{A}_{j-1}\, Z_{i,j}\Delta\lambda_j + o(\theta^{-1}_n), \tag{7.10}$$

where $\theta_n = n^\beta \varrho^{2\beta}$, $\overline{A}_j = \overline{A}(\lambda_j, \check{S}_{t_j})$ and $\overline{A}(\lambda, x, y) = \int_\lambda^\infty A(z, x, y)\mathrm{d}z$.

Proof. Making use of the stochastic Fubini theorem one gets

$$\widehat{I}_n = \int_0^1 \widehat{\sigma}^2_t \left(\int_t^1 A(\lambda_t, \check{S}_u)\mathrm{d}W^{(i)}_u \right) \mathrm{d}t = \int_0^1 \left(\int_0^u \widehat{\sigma}^2_t\, A(\lambda_t, \check{S}_u)\mathrm{d}t \right) \mathrm{d}W^{(i)}_u\,.$$

Changing the variables $v = \lambda_t$ for the inner integral, we obtain

$$\int_0^u \widehat{\sigma}^2_t\, A(\lambda_t, \check{S}_u)\mathrm{d}t = \int_{\lambda_u}^{\lambda_0} A(v, \check{S}_u)\mathrm{d}v = \overline{A}(\lambda_u, \check{S}_u) - \overline{A}(\lambda_0, \check{S}_u).$$

In other words, $\widehat{I}_n = \widehat{I}_{1,n} - \widehat{I}_{2,n}$, where $\widehat{I}_{1,n} = \int_0^1 \check{A}_u \, \mathrm{d}W_u^{(i)}$, $\check{A}_u = \overline{A}(\lambda_u, \check{S}_u)$ and $\widehat{I}_{2,n} = \int_0^1 \overline{A}(\lambda_0, \check{S}_u) \, \mathrm{d}W_u^{(i)}$. Moreover, we have

$$\widehat{I}_{1,n} = \int_0^{t^*} \check{A}_u \mathrm{d}W_u^{(i)} + \int_{t^*}^{t_*} \check{A}_u \mathrm{d}W_u^{(i)} + \int_{t_*}^1 \check{A}_u \mathrm{d}W_u^{(i)} := R_{1,n} + R_{2,n} + R_{3,n} \,. \quad (7.11)$$

Let $\varepsilon > 0$ and $b > 0$. One observes that

$$\mathbf{P}(\theta_n |\widehat{I}_{2,n}| > \varepsilon) \le \mathbf{P}(\theta_n |\widehat{I}_{2,n}| > \varepsilon, \, \tau_L^* = 1) + \mathbf{P}(\tau_L^* < 1).$$

Due to condition $(\mathbf{C}_1)$ one gets

$$\limsup_{L \to \infty} \mathbf{P}(\tau_L^* < 1) = 0 \,. \quad (7.12)$$

In view of $(\mathbf{H}_2)$, one has

$$|\overline{A}(\lambda_0, x, y)| \le C\sqrt{K x^{-1}} |U(x,y)| \int_{\lambda_0}^{\infty} e^{-\lambda/8} \mathrm{d}\lambda \le C\sqrt{K} \widetilde{U}(x,y) e^{-\lambda_0/8},$$

where $\widetilde{U}(x,y) = x^{-1/2} U(x,y)$. Now, putting $\check{A}_u^* = \check{A}_{u \wedge \tau^*}$ and $\widehat{I}_{2,n}^* = \int_0^1 \check{A}_u^* \, \mathrm{d}W_u^{(i)}$, one has $\mathbf{P}(\theta_n |\widehat{I}_{2,n}| > \varepsilon, \, \tau_L^* = 1) = \mathbf{P}(\theta_n |\widehat{I}_{2,n}^*| > \varepsilon)$. Using the Chebychev inequality one gets

$$\mathbf{P}(\theta_n |\widehat{I}_{2,n}^*| > \varepsilon) \le \varepsilon^{-2} \theta_n^2 \mathbf{E} \, (\widehat{I}_{2,n}^*)^2 \le C \varepsilon^{-2} \theta_n^2 e^{-\lambda_0/8} \sup_{0 \le t \le 1} \mathbf{E} \, \widetilde{U}^2(\check{S}_t^*).$$

Hence, due to condition $(\mathbf{H}_2)$, the integral $\widehat{I}_{2,n} = o(\theta_n^{-1})$ as $n \to \infty$. Similarly, taking into account that $l^* \le \lambda_u \le \lambda_0$ for $0 \le u \le t^*$, we get $R_{1,n} = o(\theta_n^{-1})$.

Next, let us show the same behavior for the last term in (7.11). Indeed, for some fixed $\eta > 0$ and $L > 0$, one has

$$\mathbf{P}\left(\theta_n |R_{3,n}| > \varepsilon\right) \le \mathbf{P}\left(\theta_n |R_{3,n}| > \varepsilon, \, \Gamma_{1,\eta,L}\right) + \mathbf{P}\left(\Gamma_{1,\eta,L}^c\right), \quad (7.13)$$

where $\Gamma_{1,\eta,L} = \left\{\inf_{t_* \le u \le 1} |\ln(S_u/K)| > \eta, \, \tau_L^* = 1\right\}$. Then, taking into account Lemma A.3, the continuity of σ and its derivatives and the uniformly integrability condition $(\mathbf{C}_1)$, one gets $\overline{\lim}_{\eta \to 0} \overline{\lim}_{L \to \infty} \mathbf{P}(\Gamma_{1,\eta,L}^c) = 0$. On $\Gamma_{1,\eta,L}$, we have $\check{A} = \check{A}^*$ and

$$|\check{A}_u^*| \le U(\check{S}_u^*) \int_{\lambda_u}^{\infty} (1 + z^{-\gamma}) \widetilde{\varphi}(z, S_u^*) \mathrm{d}z \le \widetilde{U}(\check{S}_u^*) \check{f}_u^*,$$

where $\check{f}_u^* = \sqrt{K/(2\pi)} \int_{\lambda_u}^{\infty} (1 + z^{-\gamma}) e^{-\eta^2/(2z) - z/8} \mathrm{d}z$. Set

$$\Gamma_{3,j} = \{|\check{A}_u| \le \widetilde{U}(\check{S}_u^*) \check{f}_u^*\}, \quad \widehat{A}_u^* = \check{A}_u^* \mathbf{1}_{\Gamma_{3,j}} \quad \text{and} \quad \widehat{R}_{3,n} = \int_{t_*}^1 \widehat{A}_u^* \mathrm{d}W_u^{(i)}.$$

By the Chebychev inequality again on obtains

$$\mathbf{P}\left(\theta_n|R_{3,n}|>\varepsilon,\Gamma_{1,\eta,L}\right)=\mathbf{P}\left(\theta_n|\widehat{R}_{3,n}|>\varepsilon\right)\le\theta_n^2\varepsilon^{-2}\int_{t_*}^{1}\mathbf{E}(\widehat{A}_u^*)^2\mathrm{d}u.$$

$$\le\theta_n^2\varepsilon^{-2}\sup_{0\le u\le 1}\mathbf{E}\,\widetilde{U}^2(\check{S}_u^*)\int_{t_*}^{1}(\check{f}_u^*)^2\mathrm{d}u.$$

Taking into account that

$$\int_{t_*}^{1}(\check{f}_u^*)^2\mathrm{d}u=K/(2\pi)\lambda_0^{-4\beta}\int_0^{l_*}\left(\int_\lambda^\infty(1+z^{-\gamma})e^{-\eta^2/(2z)-z/8}\mathrm{d}z\right)^2\mathrm{d}\lambda\le C\lambda_0^{-4\beta}l_*,$$

we conclude that $\lim_{n\to\infty}\mathbf{P}\left(\theta_n|R_{3,n}|>\varepsilon,\,\Gamma_{1,\eta,L}\right)=0$ and hence $R_{3,n}=o(\theta_n^{-1})$.

It remains to discretize the integral term $R_{2,n}$ using the sequence $(Z_{i,j})$. The key steps for this aim are the followings. First, we represent

$$R_{2,n}=\int_{t^*}^{t_*}\check{A}_u\mathrm{d}W_u^{(i)}=\sum_{j=m_1}^{m_2}\int_{t_{j-1}}^{t_j}\check{A}_u\mathrm{d}W_u^{(i)}.$$

and replace the Itô integral in the last sum with $\overline{A}_{j-1}Z_{i,j}\sqrt{\Delta t_j}$. Next, Lemma A.1 allows to substitute $\sqrt{\Delta t_j}=\varrho^{-1}\Delta\lambda_j$ into the last sum to obtain the martingale $\mathcal{M}_{m_2}$ defined by $\mathcal{M}_k=\varrho^{-1}\sum_{j=m_1}^{m_2}\overline{A}_{j-1}Z_{i,j}\Delta\lambda_j$, $m_1\le k\le m_2$. We need to show that $\mathbf{P}-\lim_{n\to\infty}\theta_n|R_{2,n}-\mathcal{M}_{m_2}|=0$. Equivalently, $\sum_{j=m_1}^{m_2}B_{j,n}=o(\theta_n^{-1})$, where

$$B_{j,n}=\int_{t_{j-1}}^{t_j}\widetilde{A}_{u,j}\mathrm{d}W_u^{(i)}\quad\text{and}\quad\widetilde{A}_{u,j}=\bar{A}(\lambda_u,\check{S}_u)-\bar{A}(\lambda_{j-1},\check{S}_{t_{j-1}}).$$

For this aim, let us introduce the set

$$\Gamma_{2,b}=\left\{\sup_{t^*\le u\le 1}\sup_{z\in\mathbb{R}}\left(|A(z,\check{S}_u)|+\left|\partial_x\bar{A}(z,\check{S}_u)\right|+\left|\partial_y\bar{A}(z,\check{S}_u)\right|\right)\le b\right\}.$$

Then, for any $\varepsilon>0$,

$$P\left(\theta_n|\sum_{j=m_1}^{m_2}B_{j,n}|>\varepsilon\right)\le\mathbf{P}\left(\theta_n|\sum_{j=m_1}^{m_2}B_{j,n}|>\varepsilon,\,\Gamma_{2,b},\,\tau^*=1\right)+\mathbf{P}(\Gamma_{2,b}^c)+\mathbf{P}(\tau^*<1).$$

Put

$$\widehat{A}_{u,j}=\widetilde{A}_{u,j}\mathbf{1}_{\{|\widetilde{A}_{u,j}|\le b(|\lambda_u-\lambda_{j-1}|+|S_u^*-S_{t_{j-1}}^*|+|y_u^*-y_{t_{j-1}}^*|)\}},\quad\widehat{B}_{j,n}=\int_{t_{j-1}}^{t_j}\widehat{A}_{u,j}\mathrm{d}W_u^{(i)}.$$

Then, by the Chebychev inequality one has,

$$\mathbf{P}\left(\theta_n|\sum_{j=m_1}^{m_2} B_{j,n}|>\varepsilon,\,\Gamma_{2,b},\,\tau^*=1\right)=\mathbf{P}\left(\theta_n|\sum_{j=m_1}^{m_2}\widehat{B}_{j,n}|>\varepsilon\right)\le\varepsilon^{-2}\theta_n^2\sum_{j=m_1}^{m_2}\mathbf{E}\,\widehat{B}_{j,n}^2.$$

Clearly,

$$\begin{aligned}\mathbf{E}\,\widehat{B}_{j,n}^2&\le 2b^2\left(\int_{t_{j-1}}^{t_j}((\lambda_u-\lambda_{j-1})^2+\mathbf{E}(S_u^*-S_{t_{j-1}}^*)^2+\mathbf{E}(y_u^*-y_{t_{j-1}}^*)^2)\mathrm{d}u\right)\\&\le C\left((\Delta\lambda_j)^3+(\Delta t_j)^2\right).\end{aligned}$$

Consequently, $\theta_n^2\sum_{j=m_1}^{m_2}\mathbf{E}\,\widehat{B}_{j,n}^2\le C\theta_n^2\sum_{j=m_1}^{m_2}\left((\Delta\lambda_j)^3+(\Delta t_j)^2\right)$. Taking into account Lemma A.1 and condition $(\mathbf{C}_2)$, we conclude that the latter sum converges to 0. On the other hand, $\lim_{b\to\infty}\overline{\lim}_{n\to\infty}\mathbf{P}(\Gamma_{2,b}^c)=0$ by Lemma A.4 hence, the proof is completed. □

7.3 Limit theorems for approximations

We first recall the following result in [17], which is extremely useful for studying asymptotic distribution of discrete martingales.

Theorem 7.1. *[Theorem 3.2 and Corollary 3.1, p.58 in [17]] Let $\mathcal{M}_n=\sum_{i=1}^n X_i$ be a zero-mean, square integrable martingale and ς be an a.s. finite random variable. Assume that the following convergences are satisfied in probability: for any $\delta>0$,*

$$\sum_{i=1}^n\mathbf{E}\left(X_i^2\mathbf{1}_{\{|X_i|>\delta\}}|\mathcal{F}_{i-1}\right)\longrightarrow 0\quad\text{and}\quad\sum_{i=1}^n\mathbf{E}\left(X_i^2|\mathcal{F}_{i-1}\right)\longrightarrow\varsigma^2.$$

Then, the sequence $(\mathcal{M}_n)$ converges in law to X whose characteristic function is $\mathbf{E}\exp(-\frac{1}{2}\varsigma^2t^2)$, i.e. X has a Gaussian mixture distribution.

Below we will establish some special versions of Theorem 7.1. In particular, our aim is to study the asymptotic distribution of discrete martingales resulting from approximation (7.10) in Proposition 7.1. More pricisely, consider $(\mathcal{M}_k)$ defined as

$$\mathcal{M}_k=\sum_{j=m_1}^k\upsilon_j,\quad m_1\le k\le m_2,\tag{7.14}$$

where $\upsilon_j=\sum_{i=1}^3 A_{i,j-1}Z_{i,j}\Delta\lambda_j$, $A_{i,j}=A_i(\lambda_j,\check{S}_{t_j})$ and $Z_{i,j}$ are defined as in (7.4) and (7.6). To describe the asymptotic variance of $\mathcal{M}$, let us introduce the following function

$$\begin{aligned}\mathbf{L}(\lambda,x,y)&=A_1^2(\lambda,x,y)+2A_1(\lambda,x,y)A_3(\lambda,x,y)(2\Phi(p)-1)\\&\quad+A_3^2(\lambda,x,y)\,\Lambda(p)+A_2^2(\lambda,x,y)\,,\end{aligned}\tag{7.15}$$

where p is defined in (7.5). Set

$$\check{\mu} = \frac{1}{2}(\mu+1)\widetilde{\mu}^{\frac{2}{\mu+1}} \quad \text{and} \quad \widehat{\mu} = (\mu-1)/(\mu+1). \tag{7.16}$$

Proposition 7.2. *Let $A_i^0 = A_i^0(\lambda, x, y)$, $i = 1, 2, 3$ be functions having property $(\mathbf{H}_2)$ and $A_i(\lambda, x, y) = A_i^0(\lambda, x, y)\widetilde{\varphi}(\lambda, x)$. Then, for any fixed $\varrho > 0$ the sequence $(n^\beta \mathcal{M}_{m_2})_{n\ge1}$ weakly converges to a mixed Gaussian variable with mean zero and variance ς^2 defined as $\varsigma^2 = \varsigma^2(\check{S}_1) = \check{\mu}\varrho^{\frac{2}{\mu+1}} \int_0^{+\infty} \lambda^{\widehat{\mu}} \mathbf{L}(\lambda, \check{S}_1)\mathrm{d}\lambda$. The same property still holds if some (or all) of the functions A_i are of the form $\int_\lambda^\infty A_i^0(z, x, y)\widetilde{\varphi}(z, x)\mathrm{d}z$.*

Proof. Note that the square integrability property is not guaranteed for the random variables (υ_j). To overcome this issue let us take their "stopped version" υ_j^* obtained by substituting $\check{S}_{t_{j-1}}$ by $\check{S}^*_{t_{j-1}}$ in the functions A_i, i.e. $\upsilon_j^* = \sum_{i=1}^3 A_i(\lambda_j, \check{S}^*_{t_j})\, Z_{i,j}\Delta\lambda_j$. In this sense, we denote by $\mathcal{M}_k^* = \sum_{j=m_1}^k \upsilon_j^*$ the corresponding stopped martingale. First, we show throughout Theorem 7.1 that for any $L > 0$ this martingale weakly converges to a mixed Gaussian variable with mean zero and variance $\varsigma^*(L) = \varsigma^2(\check{S}_1^*)$ defined in the proposition. To this end, setting $\Gamma_{1,\eta} = \{\inf_{t^*\le u\le 1} |\ln(S_u^*/K)| > \eta\}$ and $\mathbf{a}_j^* = \mathbf{E}\,(\upsilon_j^{*2}\, \mathbf{1}_{\{|\upsilon_j^*|>\delta\}}|\mathcal{F}_{j-1})$, we obtain

$$\mathbf{P}\left(n^{2\beta}|\sum_{j=m_1}^{m_2} \mathbf{a}_j^*| > \varepsilon\right) \le \mathbf{P}\left(n^{2\beta}|\sum_{j=m_1}^{m_2} \mathbf{a}_j^*| > \varepsilon,\, \Gamma_{1,\eta}\right) + \mathbf{P}(\Gamma^c_{1,\eta}). \tag{7.17}$$

It suffices to show the convergence to 0 of the first probability in the right side of (7.17). Along with the proof of Proposition 7.1, one has on the set $\Gamma_{1,\eta}$ and for $t^* \le u \le t_*$ that

$$\max_{i=1,2,3} \left|A_i(\lambda_u, \check{S}_u^*)\right| \le \widetilde{U}(\check{S}_u^*)(1 + \lambda_u^{-\gamma}) \tag{7.18}$$

for some $\gamma > 0$ and $\widetilde{U}(\check{S}) = S^{-1/2}U(\check{S})$. Set

$$\Gamma_{3,j} = \left\{\max_{1\le i\le 3} \left|A_i(\lambda_u, \check{S}_u^*)\right| \le \widetilde{U}(\check{S}_u^*)(1 + \lambda_u^{-\gamma})\right\}, \quad \widehat{\upsilon}_j^* = \upsilon_j^*\mathbf{1}_{\Gamma_{3,j}}$$

and $\widehat{\mathbf{a}}_j^* = \mathbf{E}\,(\widehat{\upsilon}_j^{*2}\mathbf{1}_{\{|\widehat{\upsilon}_j^*|>\delta\}}|\mathcal{F}_{j-1})$. We observe that

$$\mathbf{P}\left(n^{2\beta}|\sum_{j=m_1}^{m_2} \mathbf{a}_j^*| > \varepsilon,\, \Gamma_{1,\eta,L}\right) = \mathbf{P}\left(n^{2\beta}|\sum_{j=m_1}^{m_2} \widehat{\mathbf{a}}_j^*| > \varepsilon\right) \le \varepsilon^{-1}n^{2\beta}\sum_{j=m_1}^{m_2} \mathbf{E}\,\widehat{\mathbf{a}}_j^*$$

by Markov's inequality. Using the Chebychev inequality and then again the Markov inequality,

$$\mathbf{E}\,\widehat{\mathbf{a}}_j^* = \mathbf{E}\,\left(\widehat{\upsilon}_j^{*2}\mathbf{1}_{\{|\widehat{\upsilon}_j^*|>\delta\}}\right) \le \sqrt{\mathbf{E}\,\widehat{\upsilon}_j^{*4}}\sqrt{\mathbf{P}(|\widehat{\upsilon}_j^*| > \delta)} \le \delta^{-2}\mathbf{E}\,\widehat{\upsilon}_j^{*4}$$

$$\le 9\delta^{-2}(1 + \lambda_u^{-\gamma})^4(\Delta\lambda_j)^4\mathbf{E}\,\widetilde{U}^4(\check{S}_u^*)\sum_{i=1}^3 Z_{i,j}^4.$$

Taking into account that all of $Z_{i,j}$ have bounded moments and using (7.18), we obtain

$$\varepsilon^{-1} n^{2\beta} \sum_{j=m_1}^{m_2} \mathbf{E}\,\widehat{\mathbf{a}}_j^* \le 9\varepsilon^{-1}\delta^{-2} n^{2\beta} \sum_{j=m_1}^{m_2} (1+\lambda_u^{-\gamma})^4 (\Delta\lambda_j)^4,$$

which converges to 0 by Lemma A.1. Let us verify the limit of the sum of conditional variances $\mathbf{E}(\upsilon_j^{*2}|\mathcal{F}_{j-1})$. Setting $\upsilon_{i,j}^* = A_{i,j-1}^* Z_{i,j}\,\Delta\lambda_j$. Since $Z_{1,j}$ and $Z_{2,j}$ are independent, $\mathbf{E}\left(\upsilon_{1,j}^*\upsilon_{3,j}^*|\mathcal{F}_{j-1}\right) = \mathbf{E}\left(\upsilon_{2,j}^*\upsilon_{3,j}^*|\mathcal{F}_{j-1}\right) = 0$. It follows that

$$\mathbf{E}(\upsilon_j^{*2}|\mathcal{F}_{j-1}) = \mathbf{E}(\upsilon_{1,j}^{*2}|\mathcal{F}_{j-1}) + \mathbf{E}(\upsilon_{2,j}^{*2}|\mathcal{F}_{j-1}) + \mathbf{E}(\upsilon_{3,j}^{*2}|\mathcal{F}_{j-1}) + 2\mathbf{E}(\upsilon_{1,j}^*\upsilon_{2,j}^*|\mathcal{F}_{j-1}).$$

Observe that for $Z \sim N(0,1)$ and some constant a, $\mathbf{E}(Z\,|Z+a|) = 2\Phi(a) - 1$ and $\mathbf{E}\,(Z+a)^2 - (\mathbf{E}|Z+a|)^2 = \Lambda(a)$. On the other hand, $\Delta\lambda_j = n^{-2\beta}(1+o(1))\check{\mu}\,\varrho^{\frac{2}{\mu+1}}\lambda_{j-1}^{\widehat{\mu}}$ by Lemma A.1. So,

$$n^{2\beta}\mathbf{E}(\upsilon_j^{*2}|\mathcal{F}_{j-1}) = (1+o(1))\check{\mu}\,\varrho^{\frac{2}{\mu+1}}\,\lambda_{j-1}^{\widehat{\mu}}\,\mathbf{L}(\lambda_{j-1}, \check{S}_{t_{j-1}}^*)\Delta\lambda_j.$$

By Lemma A.5, the sum $n^{2\beta}\sum_{j=m_1}^{m_2}\mathbf{E}(\upsilon_j^{*2}|\mathcal{F}_{j-1})$ converges in probability to $\varsigma^*(L)$. Thus, $n^\beta\mathcal{M}_{m_2}^*$ weakly converges to $\mathcal{N}(0,\varsigma^*(L))$ throughout Theorem 7.1. Moreover, the property (7.12) implies

$$\sup_{\delta>0}\lim_{L\to\infty}\overline{\lim_{n\to\infty}}\mathbf{P}(n^\beta|\mathcal{M}_{m_2} - \mathcal{M}_{m_2}^*| > \delta) = 0\,.$$

Therefore, taking into account that $\varsigma^*(L)$ converges a.s. to ς^2 as $L\to\infty$, we conclude that $n^\beta\mathcal{M}_{m_2}$ converges in law to $\mathcal{N}(0,\varsigma^2)$, which completes the proof. □

Let us consider martingales of the following form resulting from the approximation for Lépinette's strategy,

$$\overline{\mathcal{M}}_k = \sum_{j=m_1}^{k}\left(A_{1,j-1}\,Z_{1,j} + A_{2,j-1}\,Z_{2,j} + A_{4,j-1}\,Z_{4,j}\right)\Delta\lambda_j, \quad m_1 \le k \le m_2. \tag{7.19}$$

Their limiting variance is defined throughout the function

$$\overline{\mathbf{L}}(\lambda,x,y) = A_1^2(\lambda,x,y) + A_2^2(\lambda,x,y) + (1-2/\pi)A_4^2(\lambda,x,y). \tag{7.20}$$

Then, the following result is similar to Proposition 7.2.

Proposition 7.3. *Let $A_i^0 = A_i^0(\lambda,x,y)$, $i=1,2,4$ be functions having property $(\mathbf{H}_2)$ and $A_i(\lambda,x,y) = A_i^0(\lambda,x,y)\widetilde{\varphi}(\lambda,x)$. Then, for any fixed $\varrho>0$ the sequence $(n^\beta\,\overline{\mathcal{M}}_{m_2})_{n\ge1}$ weakly converges to a mixed Gaussian variable with mean zero and variance $\overline{\varsigma}^2$ given by $\overline{\varsigma}^2 = \check{\mu}\,\varrho^{\frac{2}{\mu+1}}\int_0^{+\infty}\lambda^{\widehat{\mu}}\,\overline{\mathbf{L}}(\lambda,\check{S}_1)\mathrm{d}\lambda$. The same property still holds if some (or all) of the functions A_i are of the form $\int_\lambda^\infty A_i^0(z,x,y)\widetilde{\varphi}(z,x)\mathrm{d}z$.*

Proof. The conclusion follows directly from the proof of Proposition 7.2 and the observation that $\mathbf{E}\, Z_{4,j}^2 = \mathbf{E}(|Z_{1,j}| - \sqrt{2/\pi})^2 = 1 - 2/\pi$, and $\mathbf{E}\,(Z_{i,j} Z_{4,j}) = 0$, for $i = 1, 2$ and $m_1 \le j \le m_2$. $\square$

We will see below that the influence of volatility in the limit of the hedging error can be eliminated when ϱ diverges to infinity.

Proposition 7.4. *Let $A_i^0 = A_i^0(\lambda, x)$, $i = 1, 3$ be functions having property ($\mathbf{H}_2$) and $A_i(\lambda, x) = A_i^0(\lambda, x)\widetilde{\varphi}(\lambda, x)$. Consider the martingale*

$$\check{\mathcal{M}}_k = \sum_{j=m_1}^{k} \left(A_{1,j-1}\, Z_{1,j} + A_{3,j-1}\, Z_{3,j} \right) \Delta\lambda_j := \sum_{j=m_1}^{k} \check{v}_j, \quad m_1 \le k \le m_2.$$

Then, under condition ($\mathbf{C}_2$), the sequence $\left(n^{\beta}\, \varrho^{\frac{-1}{\mu+1}}\, \check{\mathcal{M}}_{m_2} \right)_{n \ge 1}$ weakly converges to a mixed Gaussian variable with mean zero and variance $\check{\varsigma}^2 = \check{\mu} \int_0^{+\infty} \lambda^{\widehat{\mu}}\, \check{\mathbf{L}}(\lambda, S_1) \mathrm{d}\lambda$, where $\check{\mathbf{L}}(\lambda, x) = A_1^2(\lambda, x) + 2A_1(\lambda, x)A_3(\lambda, x) + A_3^2(\lambda, x)$. The same property still holds if some (or all) of the functions A_i are of the form $\int_\lambda^\infty A_i^0(z, x)\widetilde{\varphi}(z, x)\mathrm{d}z$.

Proof. Let us determine the limit of conditional variances. We first observe that

$$n^{2\beta} \varrho^{\frac{-2}{\mu+1}}\, \mathbf{E}(\check{v}_j^2 | \mathcal{F}_{j-1}) = \check{\mu}(1 + o(1))\, \lambda_{j-1}^{\widehat{\mu}}\, \check{\mathbf{Q}}(\lambda_{j-1}, \check{S}_{t_{j-1}}) \Delta\lambda_j, \tag{7.21}$$

where $\check{\mathbf{Q}}(\lambda, x, y) = A_1^2(\lambda, x) + A_3^2(\lambda, x)\, \Lambda(p) + 2A_1(\lambda, x)A_3(\lambda, x)\, (2\Phi(|p|) - 1)$. One can check directly that the function $G(\cdot)$ defined in (7.7) satisfies the following inequalities: $|a| \le G(a) \le |a| + 2\varphi(a)$, for any $a \in \mathbb{R}$. This implies that $|\Lambda(a) - 1| \le 4|a|\varphi(a) + \varphi^2(a)$ and hence, $\sup_{a \in \mathbb{R}} |\Lambda(a)| < \infty$. Note also that $\check{\mathbf{Q}} \to \check{\mathbf{L}}$ a.s. as $n \to \infty$ since $p(\lambda, x, y) \to \infty$ as $\varrho = \varrho(n) \to \infty$ for any $x > 0$ and $\lambda \ne 2\ln(x/K)$. Using now Lemma A.5, we claim that the sum in the right hand side of (7.21) converges in probability to $\check{\varsigma}^2$ and the proof is completed by running again the argument in the proof of Proposition 7.2. $\square$

7.4 Proof of Theorem 3.1

The gamma error $I_{1,n}$ approximates to $2\min(S_1, K)$ at order θ_n. In particular, setting $\bar{I}_{1,n} = \int_0^1 \lambda_t^{-1/2} \widehat{\sigma}_t^2\, (S_t \widetilde{\varphi}(\lambda_t, S_t) - S_1 \widetilde{\varphi}(\lambda_t, S_1))\, \mathrm{d}t$ and changing the variable $v = \int_t^1 \widehat{\sigma}_s^2 \mathrm{d}s$ we can represent $I_{1,n}$ as

$$I_{1,n} = S_1 \int_0^{\lambda_0} v^{-1/2} \widetilde{\varphi}(v, S_1)\, \mathrm{d}v + \bar{I}_{1,n} + o_{\mathbf{P}}(\theta_n^{-1}).$$

The first integral in the right side converges a.s. to $2\min(S_1, K)$ by (2.14) while $\bar{I}_{1,n}$ is approximated by $\int_0^1 \widehat{\sigma}_t^2 \left(\int_t^1 \sigma(y_u) S_u H(\lambda_t, S_u) \mathrm{d}W_u^{(1)} \right) \mathrm{d}t$, where $H(\lambda, x) =$

$(2^{-1}\lambda^{-1/2} - \lambda^{-3/2}\ln(x/K))\widetilde{\varphi}(\lambda, x)$. Discretization technique of Proposition 7.1 is applied to replace the latter double integral by $\mathcal{U}_{1,m_2}$ defined as

$$\mathcal{U}_{1,k} = \varrho^{-1} \sum_{j=m_1}^{k} \sigma(y_{t_{j-1}}) S_{t_{j-1}} \check{H}_{j-1} Z_{1,j} \Delta\lambda_j, \quad m_1 \le k \le m_2, \tag{7.22}$$

where $\check{H}(\lambda, x) = \int_\lambda^\infty (z^{-1/2}/2 - z^{-3/2}\ln(x/K))\widetilde{\varphi}(z, x)\mathrm{d}z$. We summarize the asymptotic form of $I_{1,n}$ in the following.

Proposition 7.5. *If ϱ either is constant or satisfies condition* ($\mathbf{C}_2$) *then,*

$$\mathbf{P} - \lim_{n\longrightarrow\infty} \theta_n \left| I_{1,n} - 2\min(S_1, K) - \mathcal{U}_{1,m_2} \right| = 0.$$

Next, we claim that the term $I_{2,n}$ is θ_n - negligible.

Proposition 7.6. *If ϱ either is a positive constant or satisfies condition* ($\mathbf{C}_2$) *then $\theta_n I_{2,n}$ converges to 0 in probability as $n \to \infty$.*

Proof. See the Appendix B. □

Let us study the trading volume J_n. It is easy to check that for $v \ge 0$, $1-\Phi(v) \le Cv^{-1}\varphi(v)$ and $\int_0^{t^*} \widetilde{\varphi}(\lambda_u, S_u)\mathrm{d}u + \int_{t_*}^1 \widetilde{\varphi}(\lambda_u, S_u)\mathrm{d}u$ almost surely converges to 0 more rapidly than any power of n. Therefore, one can truncate the sum and keep only the part corresponding to index $m_1 \le j \le m_2$. In other words, J_n is approximated by $J_{1,n} = \sum_{j=m_1}^{m_2} S_{t_{j-1}} \left|\Delta\Phi_j\right|$. Putting $b_j = \left|\Delta\Phi_j\right| - \widetilde{\varphi}_{j-1}\left|\Delta\mathbf{v}_j\right|$, we can represent $J_{1,n}$ as $J_{1,n} = J'_{1,n} + \varepsilon_{1,n} + \varepsilon_{2,n}$, where

$$J'_{1,n} = \sum_{j=m_1}^{m_2} S_{t_{j-1}}\widetilde{\varphi}_{j-1}\left|\Delta\mathbf{v}_j\right|, \quad \varepsilon_{1,n} = \sum_{j=m_1}^{m_2} \Delta S_{t_{j-1}}\left|\Delta_j\Phi\right|, \quad \varepsilon_{2,n} = \sum_{j=m_1}^{m_2} S_{t_{j-1}} b_j.$$

In view of (A.5) and condition ($\mathbf{C}_2$), we can easily show that $\varepsilon_{1,n} = o(\theta_n^{-1})$ as $n \to \infty$. Furthermore, using the Taylor expansion we obtain $\left|\varepsilon_{2,n}\right| \le CS_{\sup}\sum_{j=m_1}^{m_2} \left|\Delta\mathbf{v}_j\right|^2$ for some constant $C > 0$ and $S_{\sup} = \sup_{0\le t\le 1} S_t$. Taking into account

$$\mathbf{E}\left|\mathbf{v}_{j-1} - \mathbf{v}_j\right|^2 \le C\left(\frac{1}{n\lambda_{j-1}} + \left(\lambda_{j-1}^{1/2} - \lambda_j^{1/2}\right)^2 + \left(\lambda_{j-1}^{-1/2} - \lambda_j^{-1/2}\right)^2\right).$$

and using condition ($\mathbf{C}_2$) and (A.5), we get $\mathbf{P} - \lim_{n\to\infty} \theta_n \left|\varepsilon_{2,n}\right| = 0$. Now using Itô's Lemma and the substitution $\lambda_j = \lambda_0(1-t_j)^{4\beta}$ one replaces $J'_{1,n}$ with

$$J_{2,n} = \sum_{j=m_1}^{m_2} \lambda_{j-1}^{-1/2} S_{t_{j-1}}\widetilde{\varphi}_{j-1}|\varkappa_j|\Delta\lambda_j := \sum_{j=m_1}^{m_2} \zeta_j, \quad \varkappa_j = \varrho^{-1}\sigma(y_{t_{j-1}})Z_{1,j} + q_{j-1}, \tag{7.23}$$

where q is defined in (2.12). We will determine the limit of J_n throughout the Doob's decomposition w.r.t. the filtration $\left(\mathcal{F}_j\right)_{m_1\le j\le m_2}$ of $J_{2,n}$. To this end, note that

$$\mathbf{E}(\zeta_j|\mathcal{F}_{j-1}) = \lambda_{j-1}^{-1/2} S_{t_{j-1}} \widetilde{\varphi}_{j-1}\Delta\lambda_j\,\mathbf{E}(|\varkappa_j||\mathcal{F}_{j-1}),$$

where $\mathbf{E}(\left|\varkappa_j\right|\,|\mathcal{F}_{j-1}) = \varrho^{-1}\sigma(y_{t_{j-1}})G(p_{j-1}) := D_{j-1}$ and $G(p)$ is defined in (7.7). Let

$$B(\lambda,x,y) = \lambda^{-1/2} x\widetilde{\varphi}(\lambda,x)D(\lambda,x,y) \quad \text{and} \quad J_{3,n} = \sum_{j=m_1}^{m_2} B_{j-1}\Delta\lambda_j. \tag{7.24}$$

We observe that $J_{2,n} = J_{3,n} + \mathcal{U}_{2,m_2}$, where

$$\mathcal{U}_{2,k} = \sum_{j=m_1}^{k} \lambda_{j-1}^{-1/2} S_{t_{j-1}}\widetilde{\varphi}_{j-1}\overline{\varkappa}_j\Delta\lambda_j \quad \text{and} \quad \overline{\varkappa}_j := \left|\varkappa_j\right| - D_{j-1}. \tag{7.25}$$

Making use of the substitution $\check{S}_{t_{j-1}}$ by $\check{S}_1$ everywhere in $J_{3,n}$ gives $J_{3,n} = J_{4,n} + J_{5,n}$, where $J_{4,n} = \sum_{j=m_1}^{m_2} B(\lambda_{j-1},\check{S}_1)\Delta\lambda_j$, $J_{5,n} = \sum_{j=m_1}^{m_2} B^*_{j-1}\Delta\lambda_j$ and $B^*_{j-1} = B(\lambda_{j-1},\check{S}_{t_{j-1}}) - B(\lambda_{j-1},\check{S}_1)$. Observe that the sum $J_{4,n}$ converges a.s. to $J(S_1,y_1,\varrho)$ at rate θ_n by Lemma A.2. Now, Itô's Lemma applied for B^*_{j-1} leads to the stochastic integrals with respect to the Wiener processes. The approximation technique in Proposition 7.1 then allows us to approximate the sum of stochastic integrals by $\mathcal{U}_{3,m_2}$, where

$$\mathcal{U}_{3,k} = \varrho^{-1}\sum_{j=m_1}^{k}(Q_{1,j-1}Z_{1,j} + Q_{2,j-1}Z_{2,j})\Delta\lambda_j\,, \quad m_1\le k\le m_2$$

and $Q_1 = \int_\lambda^\infty (x\sigma(y)\partial_x B + \mathbf{r}F_2(t(\lambda),y)\partial_y B)\mathrm{d}z$ and $Q_2 = \sqrt{1-\mathbf{r}^2}F_2(t(\lambda),y)\int_\lambda^\infty \partial_y B\mathrm{d}z$ with $t(\lambda) = 1-(\lambda/\lambda_0)^{4\beta}$. The asymptotic form of J_n is summarized in the following.

Proposition 7.7. *For any fixed $\varrho > 0$, the total trading volume J_n admits the following asymptotic form*

$$\mathbf{P} - \lim_{n\longrightarrow\infty}\theta_n\left|J_n - J(S_1,y_1,\varrho) - (\mathcal{U}_{2,m_2} + \mathcal{U}_{3,m_2})\right| = 0.$$

Now, the martingale part $\mathcal{M}_{m_2}$ of the hedging error is given by

$$\mathcal{M}_k = \frac{1}{2}\mathcal{U}_{1,k} - \kappa_*(\mathcal{U}_{2,k} + \mathcal{U}_{3,k}) = \varrho^{-1}\sum_{j=m_1}^{k}\sum_{i=1}^{3} A_{i,j-1}Z_{i,j}\Delta\lambda_j, \quad m_1\le k\le m_2,$$

where $A_1 = -\sigma(y)\,x\check{H}/2$, $A_2 = \kappa_* Q_2$ and $A_3 = -\kappa_*\sigma(y)\lambda^{-1/2}x\widetilde{\varphi}(\lambda,x)$. It is easy to see that the assumption of Proposition 7.2 is fulfilled for these functions A_i, $i = 1,2,3$ and hence, the sequence $\left(n^\beta\mathcal{M}_{m_2}\right)_{n\ge1}$ converges in law to a mixed Gaussian variable by Proposition 7.2 and Theorem 3.1 is proved. $\square$.

7.5 Proof of Theorem 3.2

When $\varrho \to \infty$ under condition $(\mathbf{C}_2)$ the approximation of J_n is slightly different since the dependence of volatility on the limits can be now removed completely. Observing that $\mathbf{E}\,|aZ+b|$ may be approximated by $b(2\Phi(b/a)-1)$ as $a \to 0$, we replace $J_{3,n}$ in (7.24) with the sum $\widehat{J}_{3,n} = \sum_{j=m_1}^{m_2} \widehat{B}_{j-1}\Delta\lambda_j$, where

$$\widehat{B}(\lambda,x) = \lambda^{-1/2}\, x\, \widetilde{\varphi}(\lambda,x) q(\lambda,x) \widetilde{\Phi}(\varrho q(\lambda,x)), \quad \widetilde{\Phi}(q) = 2\Phi(\varrho\, q) - 1$$

and $q(\lambda,x)$ defined in (2.12). Put $\widehat{J}_{4,n} = \sum_{j=m_1}^{m_2} \widehat{B}(\lambda_{j-1},S_1)\,\Delta\lambda_j$ and $\widehat{J}_{5,n} = \widehat{J}_{3,n} - \widehat{J}_{4,n}$. We present $\widehat{J}_{5,n} = \sum_{j=m_1}^{m_2} \widehat{B}^*_{j-1}\Delta\lambda_j$, where $\widehat{B}^*_{j-1} = \widehat{B}(\lambda_{j-1},S_{t_{j-1}}) - \widehat{B}(\lambda_{j-1},S_1)$. Now, using Lemma A.2 we can show directly that $\lim_{n\to\infty}\theta_n|\widehat{J}_{4,n} - J^*(S_1)| = 0$ a.s. Furthermore, Itô's formula allows to replace $\widehat{B}^*_{j-1}$ by $\int_{t_{j-1}}^1 \partial_x \widehat{B}(\lambda_{j-1},S_u)\mathrm{d}S_u$. Direct calculations give

$$\partial_x \widehat{B} = \lambda^{-1/2}\, \widetilde{\varphi}(\lambda,x)[-2q^2(\lambda,x)\widetilde{\Phi}(\lambda,x) + \frac{1}{2\lambda}\widetilde{\Phi}(\lambda,x) + \frac{\varrho}{\lambda}\varphi(\varrho\, q(\lambda,x))].$$

Clearly, $\widetilde{\Phi}(\varrho q) \to \mathrm{sign}(q)$ and $\varphi(\varrho\, q) \to 0$ as $\varrho \to \infty$. Now, using the technique in Proposition 7.1, we can approximate $\widehat{J}_{5,n}$ by $\widehat{\mathcal{U}}_{3,m_2}$ defined by

$$\widehat{\mathcal{U}}_{3,k} = \varrho^{-1} \sum_{j=m_1}^{k} \sigma(y_{t_{j-1}}) S_{t_{j-1}} N_{j-1}\, Z_{1,j}\Delta\lambda_j, \quad m_1 \le k \le m_2,$$

where $N(\lambda,x) = \int_\lambda^{+\infty} z^{-1/2}\widetilde{\varphi}(z,x)\,(-2q^2(z,x) + 1/(2z))\,\mathrm{sign}(q(z,x))\mathrm{d}z$. The asymptotic representation of trading costs is summerized in the following.

Proposition 7.8. *Under conditions* $(\mathbf{C}_1)$ *and* $(\mathbf{C}_2)$*, the trading volume* J_n *admits the following asymptotic form*

$$\mathbf{P} - \lim_{n\longrightarrow\infty} \theta_n |J_n - J^*(S_1) - (\mathcal{U}_{2,m_2} + \widehat{\mathcal{U}}_{3,m_2})| = 0.$$

Now, the martingale part $\varrho^{-1}\check{\mathcal{M}}_{m_2}$ of the hedging error is determined by

$$\frac{1}{\varrho}\check{\mathcal{M}}_k = \frac{1}{2}\mathcal{U}_{1,k} - \kappa_*(\mathcal{U}_{2,k} + \widehat{\mathcal{U}}_{3,k}) = \frac{1}{\varrho}\sum_{j=m_1}^{k}(\check{A}_{1,j-1}Z_{1,j} + \check{A}_{3,j-1}Z_{3,j})\Delta\lambda_j, \quad m_1 \le k \le m_2$$

with two functions $\check{A}_i$, $i = 1,2$ explicitly determined and satisfying the assumption of Proposition 7.4. Since $\theta_n \varrho^{-1}\check{\mathcal{M}}_{m_2} = n^\beta \varrho^{-\frac{1}{\mu+1}}\check{\mathcal{M}}_{m_2}$, Theorem 3.2 is proved throughout Proposition 7.4. □

7.6 Proof of Theorem 3.3

The key technique in Proposition 7.1 is used to obtain a smart martingale approximation for the sum $\sum_{i\geq 1}\Delta S_{t_i}\int_0^{t_{i-1}}\widehat{C}_{xt}(u,S_u)\mathrm{d}u$.

Proposition 7.9. *If ϱ either is a positive constant or satisfies condition $(\mathbf{C}_2)$, then*

$$\mathbf{P}-\lim_{n\to\infty}\theta_n|\overline{I}_{2,n}-\overline{\mathcal{U}}_{1,m_2}|=0,$$

where $\overline{\mathcal{U}}_{1,k}=\varrho^{-1}\sum_{j=m_1}^{k}\sigma(y_{t_{j-1}})S_{t_{j-1}}\,Y_{j-1}\,Z_{1,j}\,\Delta\lambda_j,\quad m_1\leq k\leq m_2$ *and* $Y(\lambda,x)=\int_\lambda^\infty z^{-3/2}\ln(x/K)\widetilde{\varphi}(z,x)\mathrm{d}z$.

Proof. The proof follows by replacing ΔS_{t_j} by $\varrho^{-1}\sigma(y_{t_{j-1}})S_{t_{j-1}}\Delta\lambda_{t_j}$ as in Proposition 7.1. $\square$

Let us now study the trading volume $\overline{J}_n$ following the procedure in the approximation of J_n. In particular, Itô's Lemma leads to

$$\overline{\gamma}_{t_i}-\overline{\gamma}_{t_{i-1}}=\int_{t_{i-1}}^{t_i}\widehat{C}_{xx}(u,S_u)\mathrm{d}S_u+\frac{1}{2}\int_{t_{i-1}}^{t_i}\widehat{C}_{xxx}(u,S_u)\sigma^2(y_u)S_u^2\mathrm{d}u,$$

where the time-correction which involves the term q_{j-1} in the formula of $\varkappa_j$ defined by (7.23) has been removed. We now approximate $\overline{J}_n$ by $\overline{J}_{1,n}$, where

$$\overline{J}_{1,n}=\varrho^{-1}\sum_{j=m_1}^{m_2}\overline{B}_{j-1}\left|Z_{1,j}\right|\Delta\lambda_j\quad\text{and}\quad\overline{B}(\lambda,x,y)=\sigma(y)\,x\,\lambda^{-1/2}\widetilde{\varphi}(\lambda,x).$$

Since $\mathbf{E}|Z|=\sqrt{2/\pi}$, for $Z\sim\mathcal{N}(0,1)$, the Dood' decomposition of $\overline{J}_{1,n}$ is given by $\overline{J}_{2,n}+\bar{\mathcal{U}}_{2,m_2}$, where

$$\overline{J}_{2,n}=\varrho^{-1}\sqrt{2/\pi}\sum_{j=m_1}^{m_2}\overline{B}_{j-1}\Delta\lambda_j\quad\text{and}\quad\bar{\mathcal{U}}_{2,m_2}=\varrho^{-1}\sum_{j=m_1}^{m_2}\overline{B}_{j-1}Z_{4,j}\Delta\lambda_j.$$

Again, the substitution $\check{S}_{t_{j-1}}$ by $\check{S}_1$ in $\overline{J}_{2,n}$ gives $\overline{J}_{2,n}=\overline{J}_{4,n}+\overline{J}_{3,n}$ where

$$\overline{J}_{4,n}=\varrho^{-1}\sqrt{2/\pi}\sum_{j=m_1}^{m_2}\overline{B}_{j-1}\Delta\lambda_j,\quad\overline{J}_{3,n}=\varrho^{-1}\sqrt{2/\pi}\sum_{j=m_1}^{m_2}\overline{B}^*_{j-1}\Delta\lambda_j$$

and $\overline{B}^*_{j-1}=\overline{B}(\lambda_{j-1},\check{S}_{t_{j-1}})-\overline{B}(\lambda_{j-1},\check{S}_1)$. Observe that $\overline{J}_{4,n}$ converges a.s. to $\eta\min(S_1,K)$ by Lemma A.2 and (2.14). We now find the suitable martingale approximation for $\overline{J}_{3,n}$. By Itô's formula once again, $\overline{B}^*_{j-1}$ can be replaced by $\int_t^1\overline{Q}_1(\lambda_{j-1},\check{S}_u)\mathrm{d}W_u^{(1)}+\int_t^1\overline{Q}_2(\lambda_{j-1},\check{S}_u)\mathrm{d}W_u^{(2)}$, where $\overline{Q}_1=\sigma(y)x\partial_x\overline{B}+\mathbf{r}F_2(t(\lambda),y)\partial_y\overline{B}$ and $\overline{Q}_2=\sqrt{1-\mathbf{r}^2}F_2(t(\lambda),y)\partial_y\overline{B}$. Direct calculations show that $\partial_x\overline{B}=\sigma(y)(2^{-1}\lambda^{-1/2}-$

$\lambda^{-3/2}\ln(X/K))\widetilde{\varphi}(\lambda,x)$ and $\partial_y\overline{B} = \sigma'(y)\lambda^{-1/2}x\widetilde{\varphi}(\lambda,x)$. Now, Proposition 7.1 is applied to approximate $\overline{J}_{3,n}$ by the martingale $\overline{\mathcal{U}}_{3,m_2}$ defined as

$$\overline{\mathcal{U}}_{3,k} = \varrho^{-1}\sum_{j=m_1}^{k}(\overline{A}_{1,j-1}Z_{1,j} + \overline{A}_{2,j-1}Z_{2,j})\Delta\lambda_j, \quad m_1 \le k \le m_2$$

for explicit functions $\overline{A}_i, i = 1,2$. The final asymptotic form of $\overline{J}_n$ is given below.

Proposition 7.10. *If ϱ is a positive constant independent of n then,*

$$\mathbf{P} - \lim_{n\to\infty}\theta_n|\overline{J}_n - \eta\min(S_1,K) - (\overline{\mathcal{U}}_{2,m_2} + \overline{\mathcal{U}}_{3,m_2})| = 0.$$

Hence, the martingale part of the hedging error for Lépinette's strategy is determined by $\overline{\mathcal{M}}_{m_2} = \mathcal{U}_{1,m_2} + \overline{\mathcal{U}}_{1,m_2} - \kappa_*(\overline{\mathcal{U}}_{2,m_2} + \overline{\mathcal{U}}_{3,m_2})$, which can be represented in the form

$$\overline{\mathcal{M}}_k = \varrho^{-1}\sum_{j=m_1}^{k}(A_{1,j-1}Z_{1,j} + A_{4,j}Z_{4,j-1} + A_{2,j-1}Z_{2,j})\Delta\lambda_j, \quad m_1 \le k \le m_2$$

for explicit functions A_i holding the assumption of Proposition 7.3. Then, the convergence in law to a mixed Gaussian variable of the sequence $\left(n^{\beta}\overline{\mathcal{M}}_{m_2}\right)_{n\ge1}$ is guaranteed by Proposition 7.3 and hence, Theorem 3.3 is proved. $\square$.

8 Conclusion

We studied the option replication in Leland's spirit for general stochastic volatility settings using a new form of enlarged volatility, which is simpler than the ones used in the previous works. We established the limit theorems for both Leland' strategy and Lépinette's one, which proved that the influence of transaction costs can be approximately controlled. The setting of model (3.1) is general enough for practice purposes since it includes many famous SV models. A connection of the present framework to high frequency markets with proportional transaction costs was also discussed. In fact, the approach is still applicable for more general settings where the friction rule admits a representation of separate-variable kind [33], which also includes the case where trading costs are based on the number of traded shares instead of trading volume in dollar value.[5] We pointed out that increasing volatility can compensate trading costs and the option price is now expensive and rapidly approaches to the buy-and-hold super-hedging price. This undesirable property can be relatively released in the spirit of quantile hedging. Lastly, in the accompanying paper, we extended the method to multidimensional frameworks for European options with general payoff written on several assets [34].

[5] This extension was presented by the first author at the 7th Colloquium Bachelier on Mathematical Finance and Stochastic Calculus in Metabief, January 2013.

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Appendix

A Auxiliary Lemmas

Lemma A.1. *There exist two positive constants C_1, C_2 such that*

$$C_1\, n^{-2\beta} \varrho^{\frac{2}{\mu+1}}\, \nu_0(l_*) \le \inf_{m_1\le j\le m_2} |\Delta\lambda_j| \le \sup_{m_1\le j\le m_2} |\Delta\lambda_j| \le C_2 n^{-2\beta} \varrho^{\frac{2}{\mu+1}}\, \nu_0(l^*), \quad \text{(A.1)}$$

where $\nu_0(x) = x^{(\mu-1)/(\mu+1)}$. Moreover,

$$\Delta\lambda_j = n^{-2\beta} \varrho^{\frac{2}{\mu+1}}\, \nu_0(\lambda_{j-1})(1+o(1)) \quad \text{and} \quad \Delta\lambda_j\, (\Delta t_j)^{-1/2} = \varrho(1+o(1)). \quad \text{(A.2)}$$

Lemma A.2. *If ϱ either is a positive constant or satisfies condition $(\mathbf{C}_2)$ then, for any function A satisfying condition $(\mathbf{H}_1)$*

$$\lim_{n\to\infty} \theta_n \left| \sum_{j=m_1}^{m_2} \mathbf{1}_{\{\lambda_{j-1}\ge a\}} A(\lambda_{j-1})\Delta\lambda_j - \int_a^\infty A(\lambda)\mathrm{d}\lambda \right| = 0. \quad \text{(A.3)}$$

In particular,

$$\lim_{n\to\infty} \theta_n \left| \sum_{j=m_1}^{m_2} A(\lambda_{j-1})\Delta\lambda_j - \int_0^\infty A(\lambda)\mathrm{d}\lambda \right| = 0.$$

Proof. Recall first that $(\lambda_j)_{m_1\le j\le m_2}$ is decreasing with $l_* = \lambda_{m_2}$ and $l^* = \lambda_{m_1}$. Setting $\varpi_{k,n} = \sum_{j=m_1}^k A(\lambda_{j-1})\Delta\lambda_{j-1} - \int_{\lambda_k}^{l^*} A(\lambda)\mathrm{d}\lambda$ and $\varpi_n^* = \sup_{m_1\le k\le m_2} |\varpi_{k,n}|$, we show that

$$\lim_{n\to\infty} \theta_n\, \varpi_n^* = 0\,. \quad \text{(A.4)}$$

Indeed, $\varpi_{k,n} = \sum_{j=m_1}^k \int_{\lambda_j}^{\lambda_{j-1}} A_1(z)\mathrm{d}z$, with $A_1(z) = \int_z^{\lambda_{j-1}} A'(u)\mathrm{d}u$. Therefore,

$$|\varpi_{k,n}| \le \sum_{j=m_1}^k |\Delta\lambda_j| \int_{\lambda_j}^{\lambda_{j-1}} |A'(u)|\mathrm{d}u \le \max_{m_1\le j\le m_2} |\Delta\lambda_j| \int_0^{+\infty} |A'(u)|\mathrm{d}u\,.$$

On the other hand, by Lemma A.1 one has

$$\max_{m_1\le j\le m_2} |\Delta\lambda_j| \le C n^{-2\beta} \varrho^{\frac{2}{\mu+1}} (l^*)^{\frac{\mu-1}{\mu+1}}. \quad \text{(A.5)}$$

So, $\theta_n \varpi_n^* \le C n^{-\beta} \varrho^{\frac{\mu+2}{\mu+1}} (l^*)^{\frac{\mu-1}{\mu+1}} \int_0^{+\infty} |A'(u)| \mathrm{d}u$, which converges to zero by condition $(\mathbf{C}_2)$. Note that if $a < \lambda_{m_2}$ or $a > \lambda_{m_1}$ then, the left side of the equality (A.3) can be estimated by $\varpi_n^* + \int_0^{l_*} |A(\lambda)| \mathrm{d}\lambda + \int_{l^*}^{+\infty} |A(\lambda)| \mathrm{d}\lambda$. Moreover, if $a \in [\lambda_k, \lambda_{k-1}]$ for some $m_2 \le k \le m_1$ then, the difference in the absolute sign of (A.3) can be represented as $\varpi_{k,n} - \int_a^{\lambda_{k-1}} A(\lambda) \mathrm{d}\lambda - \int_{l^*}^{+\infty} A(\lambda) \mathrm{d}\lambda$. Taking into account that A is bounded (since the derivative A' is absolutely integrable) and the property (A.5) we obtain the limiting equality (A.3). Hence, Lemma A.2. □

Lemma A.3. *For any $\varepsilon > 0$,*

$$\limsup_{v \to 1} \mathbf{P}(\inf_{v \le u \le 1} |\ln(S_u/K)| \le \varepsilon) = 0.$$

Proof. Let η be some positive number. Clearly, $\mathbf{P}(\inf_{v \le u \le 1} |\ln(S_u/K)| \le \varepsilon)$ is bounded by

$$\mathbf{P}(\inf_{v \le u \le 1} |\ln(S_u/K)| \le \varepsilon, |\ln(S_1/K)| > \eta) + \mathbf{P}(|\ln(S_1/K)| \le \eta).$$

Let us show that the first probability is equal to zero for v sufficiently close to 1. Indeed, denoting $\psi_u = \int_u^1 \sigma^2(y_s) \mathrm{d}s - \frac{1}{2} \int_u^1 \sigma(y_s) \mathrm{d}W_s^{(1)}$ we can check that $\psi_v^* \to 0$ a.s. as $v \to 1$, where $\psi_v^* = \sup_{v \le u \le 1} |\psi_u|$. So, if $|\ln(S_1/K)| > \eta$ then for $v \le u \le 1$,

$$|\ln(S_u/K)| = |\ln(S_1/K) - \psi_u| \ge ||\ln(S_1/K)| - \psi_v^*| \ge \frac{1}{2} |\ln(S_1/K)| > \eta/2.$$

Therefore, for $\eta > 2\varepsilon$, one obtains $\inf_{v \le u \le 1} |\ln(S_u/K)| \ge \eta/2 > \varepsilon$ and so,

$$\mathbf{P}(\inf_{v \le u \le 1} |\ln(S_u/K)| \le \varepsilon, |\ln(S_1/K)| > \eta) = 0.$$

Letting now $\eta \to 0$ we get $\mathbf{P}(|\ln(S_1/K)| \le \eta) \to \mathbf{P}(S_1 = K)$. Note that conditioning on σ-field generated by the Wiener process driving y, the log-price process $\ln S_t$ has Gaussian distribution. Hence, $\mathbf{P}(S_1 = K) = 0$ and the proof is completed. □

Lemma A.4. *Suppose that $A_0 = A_0(\lambda, x, y)$ and the derivatives $\partial_x A_0, \partial_y A_0$ verify condition $(\mathbf{H}_2)$. Set $A(\lambda, x, y) = A_0(\lambda, x, y) \widetilde{\varphi}(\lambda, x)$, $\bar{A}(\lambda, x, y) = \int_\lambda^\infty A(z, x, y) \mathrm{d}z$ and define*

$$r_n = \sup_{(z,r,d) \in [l_*, l^*] \times \mathcal{B}} \left(|\partial_\lambda \bar{A}(z, r, d)| + |\partial_x \bar{A}(z, r, d)| + |\partial_y A(z, r, d)| \right),$$

where $\mathcal{B} = [S_{\min}, S_{\max}] \times [y_{\min}, y_{\max}]$ with $S_{\min} = \inf_{t^* \le u \le t_*} S_u$, $S_{\max} = \sup_{t^* \le u \le t_*} S_u$ and $y_{\min} = \inf_{t^* \le u \le t_*} y_u$, $y_{\max} = \sup_{t^* \le u \le t_*} y_u$. Then, $\lim_{b \to \infty} \overline{\lim}_{n \to \infty} \mathbf{P}(r_n > b) = 0$.

Proof. Let $\varepsilon > 0$. On the set $\Gamma_{1,\varepsilon} = \{\inf_{t_* \le u \le 1} |\ln(S_u/K)| \ge \varepsilon\}$,

$$\sup_{S_{\min} \le r \le S_{\max}} \widetilde{\varphi}(q, r) \le (2\pi)^{-1/2} \sqrt{K r^{-1}} \exp\{-\varepsilon^2/(2q) - q/8\}.$$

By condition $(\mathbf{H}_2)$, there exists $\gamma > 0$ such that

$$|\bar{A}_x(z,r,d)| \le C|\widetilde{U}(r,d)| \int_z^\infty (q^{-1/2} + q^\gamma) e^{-\varepsilon^2/(2q) - q/8} \mathrm{d}q \le C_\epsilon \widetilde{U}(r,d),$$

where $\widetilde{U}$ is some function verifying $\sup_{0\le t\le 1} \mathbf{E}\, \widetilde{U}(\check{S}^*_t) < \infty$. For $\eta > 0$ and $N > 0$, let

$$\Gamma_{2,\eta} = \{ \sup_{(r,d)\in\mathcal{B}} |\widetilde{U}(r,d) - \widetilde{U}(\check{S}_1)| < \eta\} \bigcap \{|\widetilde{U}(\check{S}_1)| < N\}.$$

It is clear that $|\widetilde{U}(r,d)| < N + \eta$ on the set $\Gamma_{2,\eta}$. Similarly, taking into account that $\partial_\lambda \bar{A}(z,r,d) = -A(z,r,d)$, $\partial_y \bar{A}(z,r,d) = \int_\lambda^\infty \partial_y A_0(z,x,y)\widetilde{\varphi}(z,x)$ we deduce that both $|\partial_\lambda \bar{A}(z,r,d)|$ and $|\partial_y \bar{A}(z,r,d)|$ are bounded on $\Gamma_{2,\eta}$ by a constant $C_{N,\eta}$ independent of b. Now, for $b > \max(N+\eta, C_{N,\eta})$,

$$\begin{aligned}
\mathbf{P}(r_n > b) &\le \mathbf{P}(r_n > b, \Gamma_{1,\varepsilon}) + \mathbf{P}(\Gamma^c_{1,\varepsilon}) \\
&\le \mathbf{P}(r_n > b, \Gamma_{1,\varepsilon}, \Gamma_{2,\eta}) + \mathbf{P}(\Gamma^c_{1,\varepsilon}) + \mathbf{P}(\Gamma^c_{2,\eta}) = \mathbf{P}(\Gamma^c_{1,\varepsilon}) + \mathbf{P}(\Gamma^c_{2,\eta}) \\
&\le \mathbf{P}(\Gamma^c_{1,\varepsilon}) + \mathbf{P}(\Gamma^c_{2,\eta}, \tau^* = 1) + \mathbf{P}(\tau^* < 1) \\
&= \mathbf{P}(\Gamma^c_{1,\varepsilon}) + \mathbf{P}(\sup_{(r,d)\in\mathcal{B}} |\widetilde{U}(r,d) - \widetilde{U}(\check{S}^*_1)| \ge \eta) \\
&\quad + \mathbf{P}(|\widetilde{U}(\check{S}^*_1)| > N) + \mathbf{P}(\tau^* < 1).
\end{aligned}$$

By Lemma A.3, $\lim_{n\to\infty} \mathbf{P}(\Gamma^c_{1,\varepsilon}) \to 0$ for any $\varepsilon > 0$ given. Thanks to the continuity of the functions S_t and y_t one gets $\lim_{n\to\infty} \mathbf{P}\left(\sup_{(r,d)\in\mathcal{B}} |\widetilde{U}(r,d) - \widetilde{U}(\check{S}^*_1)| \ge \eta\right) = 0$. Moreover, the integrability of $\widetilde{U}(\check{S}^*_1)$ implies that $\mathbf{P}(|\widetilde{U}(\check{S}^*_1)| > N)$ converges to 0 as $N \to \infty$. By (7.12), $\mathbf{P}(\tau^* < 1)$ converges to 0 as $L \to \infty$ and the proof is completed. $\square$

Lemma A.5. *Let $A^0 = A^0_i(\lambda,x,y)$ be a function having property $(\mathbf{H}_2)$ and $\overline{A}(\lambda,x,y) = \int_\lambda A^0(z,x,y)\widetilde{\varphi}(z,x)\mathrm{d}z$, $\widetilde{A}(\lambda,x,y) = \overline{A}^2(\lambda,x,y)$. Then, for any $\gamma > 0$,*

$$\mathbf{P} - \lim_{n\to\infty} \left| \sum_{j=m_1}^{m_2} \lambda^\gamma_{j-1} \widetilde{A}(\lambda_{j-1}, \check{S}_{t_{j-1}}) \Delta\lambda_j - \int_0^\infty \lambda^\gamma \widetilde{A}(\lambda, \check{S}_1) \mathrm{d}\lambda \right| = 0,$$

where $\check{S}_t = (S_t, y_t)$. The same property still holds if $\overline{A}(\lambda,x,y) = A^0(\lambda,x,y)\widetilde{\varphi}(x,y)$ or the product of these above kinds.

Proof. . We just prove for the first case $\overline{A}(\lambda,x,y) = \int_\lambda A^0(z,x,y)\widetilde{\varphi}(z,x)\mathrm{d}z$ since the same argument can be made for the other cases. First, we split the expression under the absolute sign as $\sum_{j=m_1}^{m_2} \lambda^\gamma_{j-1}\widetilde{A}(\lambda_{j-1}, \check{S}_1)\Delta\lambda_j + \sum_{j=m_1}^{m_2} \Delta_{j,n}\Delta\lambda_j$, where $\Delta_{j,n} = \widehat{A}(\lambda_{j-1}, \check{S}_{t_{j-1}}) - \widehat{A}(\lambda_{j-1}, \check{S}_1)$ and $\widehat{A}(\lambda,x,y) = \lambda^\gamma \widetilde{A}(\lambda,x,y)$. It is clear that for any x, y, the function $\widehat{A}(\cdot,x,y)$ satisfies condition $(\mathbf{H}_1)$ hence, the sum $\sum_{j=m_1}^{m_2}\widehat{A}(\lambda_{j-1},\check{S}_1)\Delta\lambda_j$ converges a.s. to $\int_0^\infty \widehat{A}(\lambda,\check{S}_1)\mathrm{d}\lambda = 0$ by Lemma A.2. It remains to show that $\mathbf{P}(|\Delta_n|>\varepsilon)\to 0$ for any $\varepsilon>0$ given. The technique in Lemma A.3 is still helpful for this purpose. Let $S_{\min}=\inf_{t^*\le u\le 1} S_u$, $S_{\max}=\sup_{t^*\le u\le 1} S_u$, $y_{\min}=\inf_{t^*\le u\le 1} y_u$ and $y_{\max}=\sup_{t^*\le u\le 1} y_u$ and define

$$\widehat{a}_n=\sup_{(z,r,d)\in[l_*,l^*]\times\mathcal{B}}\left(|\partial_x\widehat{A}(z,r,d)|+|\partial_y\widehat{A}(z,r,d)|\right),$$

where $\mathcal{B}=[S_{\min},S_{\max}]\times[y_{\min},y_{\max}]$. Clearly,

$$\mathbf{P}(|\Delta_n|>\varepsilon)\le \mathbf{P}(|\Delta_n|>\varepsilon,\ \widehat{a}_n\le b,\ \tau^*=1)+\mathbf{P}(\widehat{a}_n>b)+\mathbf{P}(\tau^*<1).$$

By (7.12), $\overline{\lim}_{L\to\infty}\mathbf{P}(\tau^*<1)=0$. Moreover, in view of Lemma A.4, one gets $\lim_{b\to\infty}\overline{\lim}_{n\to\infty}\mathbf{P}(\widehat{a}_n>b)=0$. Therefore, the proof is completed if $P(|\Delta_n|>\varepsilon,\ \widehat{a}_n\le b,\tau^*=1)\to 0$. Note that on the set $\{\widehat{a}_n\le b,\tau^*=1\}$, one has

$$|\Delta_{j,n}|\le b(|y_1^*-y_{t_{j-1}}^*|+|S_1^*-S_{t_{j-1}}^*|),\quad m_1\le j\le m_2.$$

Denote $\widehat{\Delta}_{j,n}=\Delta_{j,n}\mathbf{1}_{\left\{|\Delta_{j,n}|\le b\left(|y_1^*-y_{t_{j-1}}^*|+|S_1^*-S_{t_{j-1}}^*|\right)\right\}}$. It follows that

$$\mathbf{P}(|\Delta_n|>\varepsilon,\widehat{a}_n\le b,\tau^*=1)=\mathbf{P}\left(|\sum_{j=m_1}^{m_2}\widehat{\Delta}_{j,n}\Delta\lambda_j|>\varepsilon\right).$$

By the Markov inequality, the probability in the right side is bounded by

$$\begin{aligned}\varepsilon^{-1}\sum_{j=m_1}^{m_2}\mathbf{E}|\widehat{\Delta}_{j,n}|\Delta\lambda_j &\le b\varepsilon^{-1}\sum_{j=m_1}^{m_2}(\mathbf{E}|y_1^*-y_{t_{j-1}}^*|+\mathbf{E}|S_1^*-S_{t_{j-1}}^*|)\Delta\lambda_j\\ &\le C\varepsilon^{-1}\lambda_0^{-2\beta}\sum_{j=m_1}^{m_2}\lambda_{j-1}^{2\beta}\Delta\lambda_j\to 0.\quad\square\end{aligned}$$

B Proof of Proposition 7.6

The singularity of $\widehat{C}$ at the maturity $T=1$ requires a separate treatment. For this aim, let $\varepsilon_n=n^{-2\beta}\varrho^{-4\beta}l_*$. We then represent $I_{2,n}$ as $I_{2,n}=\int_0^{1-\varepsilon_n}\varpi_n(t)\mathrm{d}W_t^{(1)}+\int_{1-\varepsilon_n}^1\varpi_n(t)\mathrm{d}W_t^{(1)}$, where $\varpi_n(t)=(\gamma_t^n-\widehat{C}_x(t,S_t))\sigma(y_t)S_t$. Since $|\gamma_t^n-\widehat{C}_x(t,S_t)|\le 1$, we obtain $\lim_{n\to\infty}\theta_n^2\,\mathbf{E}\int_{1-\varepsilon_n}^1\varpi_n^2(t)\mathrm{d}t=0$. Thus, it remains to prove that

$$\lim_{n\longrightarrow\infty}\theta_n^2\sum_{j=1}^n\int_{\widehat{t}_{j-1}}^{\widehat{t}_j}\mathbf{E}(\gamma_t^n-\widehat{C}_x(t,S_t))^2\mathrm{d}t=0,\quad \widehat{t}_j=\min(t_j,1-\varepsilon_n).$$

Let us introduce the discrete sums $w_1(t) = \sum_{j=1}^{n} \lambda_t^{-1}(x_t - x_{\widehat{t}_{j-1}})^2 \xi_j(t)$, $w_2(t) = \sum_{j=1}^{n} x_t^2 \,(\lambda_t^{-1/2} - \lambda_{\widehat{t}_{j-1}}^{-1/2})^2 \,\xi_j(t)$ and $w_3(t) = \sum_{j=1}^{n} (\lambda_t^{1/2} - \lambda_{\widehat{t}_{j-1}}^{1/2})^2 \,\xi_j(t)$, where $\xi_j(t) = \mathbf{1}_{(\widehat{t}_{j-1},\widehat{t}_j]}(t)$ and $x_t = \ln(S_t/K)$. Clearly, $|\gamma_t^n - \widehat{C}_x(t, S_t)|^2 \le w_1(t) + w_2(t) + w_3(t)$. Taking into account that

$$\sup_{n,\,1\le j\le n} \; n \sup_{0\le t\le 1} \mathbf{E}(x_t - x_{\widehat{t}_{j-1}})^2 \,\xi_j(t) < \infty \quad \text{and} \quad \sup_{0\le t\le 1} \mathbf{E}\, x_t^2 < \infty,$$

one gets $\theta_n^2 \mathbf{E} \int_0^{1-\varepsilon_n} w_1(t) \mathrm{d}t \le C n^{2\beta - 3/2} \varrho^{4\beta-1}$, which converges to 0 by $(\mathbf{C}_2)$. The particular choice of ε ensures that $\theta_n^2 \mathbf{E} \int_0^{1-\varepsilon_n} w_2(t) \mathrm{d}t \le C\theta_n^2 n^{-2} (\varepsilon_n)^{-(4\beta+1)/4\beta} \lambda_0^{-1}$, which tends to 0. The convergence for $w_3(t)$ can be shown in the same way. $\square$

C A moment property used in Heston's models

This section gives the explanation for the applicability of Heston's model (5.1) when σ is linearly bounded. In particular, we prove that $\mathbf{E}\, y^* < \infty$, where $y^* = \sup_{0\le t\le 1} |y_t|$. For this aim, we introduce the process $\psi_t = y_{t\wedge\tau_N}$, where τ_N is a stopping time defined by $\tau_N = \inf\{t \ge 0 : |y_t| \ge N\} \wedge 1$ for $N > 0$. Note that ψ_t is the solution of the following SDE

$$d\psi_t = A_t \mathrm{d}t + B_t \mathrm{d}Z_t,$$

where $A_t = (a - b y_t)\, \mathbf{1}_{t\le\tau_N}$, $\quad B_t = \sqrt{y_t}\, \mathbf{1}_{t\le\tau_N}$. Below, $c_1, c_2, ...$ stand for positive constants. First, it is easy to see that $|A_s| \le c_1 + c_2 \psi_t^*$, where $\psi_t^* = \sup_{0\le s\le t} |\psi_s|$. Consequently, for $0 \le u \le t \le 1$,

$$|\psi_u| \le c_1 + c_2 \int_0^t \psi_s^* \mathrm{d}s + \sup_{0\le u\le t} \left| \int_0^u B_s \mathrm{d}Z_s \right|.$$

Therefore,

$$\mathbf{E}|\psi_t^*| \le c_1 + c_2 \int_0^t \mathbf{E}\psi_s^* \mathrm{d}s + \sqrt{\mathbf{E} \sup_{0\le u\le t} \left| \int_0^u B_s \mathrm{d}Z_s \right|^2}.$$

Taking into account that $B_s^2 = \psi_s \mathbf{1}_{s\le\tau_N} \le \psi_s^*$ and using Dood's inequality, we get

$$\mathbf{E}\psi_t^* \le c_1 + c_2 \int_0^t \mathbf{E}\psi_s^* \mathrm{d}s + c_3 \sqrt{\int_0^t \mathbf{E}\psi_s^* \mathrm{d}s} \le c_4 + c_5 \int_0^t \mathbf{E}\psi_s^* \mathrm{d}s.$$

Thanks to Gronwall-Bellman's inequality, one claims that $\mathbf{E}\psi_t^*$ is bounded by some positive constant independent of N. Hence, $\mathbf{E} \sup_{0\le t\le 1} |y_{t\wedge\tau_N}| = \mathbf{E}\psi^* < \infty$. Note that $\sup_{0\le t\le 1} |y_{t\wedge\tau_N}|$ converges to y^* as $N \to \infty$. Then, Fatou's Lemma allows us to conclude that $\mathbf{E}\, y^* \le \liminf_{N\to\infty} \mathbf{E} \sup_{0\le t\le 1} |y_{t\wedge\tau_N}| < \infty$ and model (5.5) enjoys condition $(\mathbf{C}_1)$. $\square$

D Moments of Orstein-Uhlenbeck's processes

Lemma D.1. *Suppose that* $\sigma(z) \leq \gamma(1+|z|)$ *for all* z *with some constant* $\gamma > 0$ *and let* y_t *be an Orstein-Uhlenbeck process defined by* $\mathrm{d}y_t = (a - by_t)\mathrm{d}t + \mathrm{d}Z_t$ *with some constants* a *and* $b > 0$. *Put* $X_\alpha = \exp\left\{2\alpha\gamma^2 \int_0^1 y_s^2 \mathrm{d}s\right\}$ *and* $\alpha_* = b^2(2\gamma^2(2b + a^2))^{-1}$. *Then,* $\mathbf{E}X_\alpha < \infty$ *for* $0 < \alpha < \alpha_*$.

Proof. Remark that $(a - by)y \leq a^2/(2b) - by^2/2$. Then, by adapting Proposition 1.1.5 in [22], p.24, we can show that $\mathbf{E}\,|y_t|^{2m} \leq m!\,(2/b + a^2/b^2)^m$, $m \geq 1$. It follows that

$$\mathbf{E}X_\alpha \leq \sum_{m=0} (\alpha 2\gamma^2)^m (m!)^{-1} \mathbf{E}\,|y_t|^{2m} \leq \sum_{m=0} \left(2/b + a^2/b^2\right)^m (\alpha 2\gamma^2)^m < \infty$$

if $0 < \alpha < \alpha_*$. If y_t is mean-reverting then b takes very big values and it is possible to choose $\alpha > 3/2 + \sqrt{2}$ as discussed in Remark 9. □

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