The invariant of Turaev-Viro from Group category

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Abstract. A Group category is a spherical category whose simple objects are invertible. The invariant of Turaev-Viro with this particular category is in fact the invariant of Dijkgraaf-Witten whose the group and the 3-cocycle is given by the simple objects and the associativity constraint of the category.

Contents

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Introduction

In 1992 M. Wakui [15] reformulated the invariant of Dijkgraaf-Witten [4] and he proved the topological invariance in a rigorous way. The invariance is based upon the triangulation and the Pachner moves. Once given a finite group and a 3-cocycle the Dijkgraaf-Witten invariant is defined combinatorially. Moreover in this paper he built a topological quantum field theory (TQFT) from this invariant. The same year V. Turaev and O. Viro [12] built an invariant of 3 manifold thanks to 6-j symbol to prove the topological invariance they showed a relative version of a theorem of Alexander [1] on equivalence of triangulation. This invariant was reformulated in a categorical languages [11] and the TQFT was built. In the same spirit of [11] J.W. Barret and B.W. Westburry [2] have built a 3-manifold invariant using spherical categories. In this construction the topological invariance puts back down the triangulation and the Pachner moves. Independently I. Gelfand and D. Kazhdan [6] have built a 3-manifold using spherical categories and in 1993 D.N. Yetter has studied an untwisted version of the invariant of Dijkgraaf-Witten in [16] the Turaev-Viro style in . In fact these constructions are reformulations of the Turaev-Viro invariant. In the rest of the paper we will call such kind of invariant the invariant of Turaev-Viro and it will be denoted : $TV_{\mathcal{C}}$ where $\mathcal{C}$ is the category used to build the invariant.
The main goal of this paper is to give a relation between this two approaches based on triangulation. That's why we utilize a "special" spherical category. Roughly speaking, it is a spherical category such that every simple object is invertible and has a dimension equal to one. The dimension is given by the spherical structure. In [10], F. Quinn called this category : "Group category". In [7] invertible objects are called simples currents and the tensor category whose every simple objects are invertible is denoted Pointed category. The authors have denoted Picard category of $\mathcal{C}$ the full tensor category of $\mathcal{C}$ whose objects are direct sum of invertible objects of $\mathcal{C}$. Thus if there is a finite number of simple object and if every object is finite direct sum of simple object then a pointed category is equal to its Picard category. In this paper we will use the terminology of F. Quinn [10]. L. Crane and D.N. Yetter have studied group cocycle to describe monoidal category with duals in [3]. Here is an outline of the paper. In Section 1 we recall the definition of the Dijkgraaf-Invariant [15]. In Section 2 we give the definition and we recall some facts on the Group category. In Section 3 we give the definition of the Turaev-Viro invariant of 3-manifold. In Section 4, we compute the Turaev-Viro invariant in the case of Group-category with other conditions and we show the main theorem (4.2) . In section 5 we give a topological interpretation of the admissible colorings. In Section 6 we give the construction the TQFT which arises from this invariant. We end the paper by discussing a few examples.

1. The invariant of Dijkgraaf-Witten

Throughout this paper k will be a commutative field such that $car(k) = 0$ and $\overline{k} = k$.
We use the description of [15]. Let G be a finite group, this group will be always a multiplicative group. Moreover k is a representation of G with the trivial action. Then we can define $Z^3(G, k^\star)$ the set of 3-cocycle of G with coefficients in $k^\star$ and we fix $\alpha \in Z^3(G, k^\star)$. Let T be a n-simplex with $n \geq 1$, a color of T is the following

data :

$$\gamma : \{\text{oriented edges of T}\} \to G, \tag{1.1}$$

which satisfies the conditions :

(i) for any oriented edge e : $\gamma(\overline{e}) = \gamma(e)^{-1}$, where $\overline{e}$ is the oriented edge with the opposite orientation.

(ii) For any oriented 2-simplex (012) of T we have :

$$\gamma(01)\gamma(12)\gamma(20) = 1.$$

We denote $Col(T)$ the set of all colors of T, if T is a triangulation of a n-manifold M, with $n \geq 2$, we denote $Col(M,T)$ the set of all colors of M given by T. When there is no ambiguity on the choice of a triangulation, we denote $Col(M)$ the set of colors of M. If M is a manifold with boundary : ∂M, then ∂M is endowed with a triangulation which comes from the triangulation of M. If τ is a color of ∂M then the set of all colors of M which extend τ, is denoted $Col(M,\tau)$. We give an order to the set of vertices of a triangulation of M, then each 3-simplex has an orientation given by the ascending order. Then for $\gamma \in Col(M)$ and for the 3-simplex (0123) we put :

$$\alpha(\Delta,\gamma) = \alpha(\gamma(01),\gamma(12),\gamma(20)),$$

with $\alpha \in Z^3(G,k^\star)$.

Theorem 1.1 (Wakui (92)[15]). *Let G be a finite group, we fix a 3-cocycle $\alpha \in Z^3(G,k^\star)$. Let M be a compact oriented triangulated 3-manifold, T is a triangulation of M. We denote the number of vertices of T by n_0 and T^3 the set of 3-simplex in T. All the 3-simplex are oriented by a numbering of the vertices. Given $\tau \in Col(\partial M)$, we define the Dijkgraaf-Witten invariant by :*

$$Z_M(\tau) = \mid G \mid^{-n_0} \sum_{\gamma \in Col(M,\tau)} \prod_{\Delta \in T^3} \alpha(\Delta,\gamma)^{\epsilon_\Delta},$$

where

$$\epsilon_\Delta = \begin{cases} 1 & if \Delta \text{ and } M \text{ have the same orientation,} \\ -1 & \text{otherwise.} \end{cases}$$

Then $Z_M(\tau)$ does not depend on the choice of triangulation of M and the choice of order of vertices in M whenever we fix a triangulation of ∂M and τ.

Thanks to the independence of the choice of numbering, we can consider a numbering of the triangulation such that the 3-simplex have the same orientation of M. Then the invariant is :

$$Z_M(\tau) = \mid G \mid^{-n_0} \sum_{\gamma \in Col(M,\tau)} \prod_{\Delta \in T^3} \alpha(\Delta,\gamma),$$

where all 3-simplex Δ have the same orientation of M.

Remark 1.2. *If we consider M without boundary, then $Z_M(\emptyset)$ is a 3-manifold invariant and we denote it : Z_M.*

2. Group category

In this section, we review some basics facts on Group category.

2.1. **Definition.** Let $\mathcal{C}$ be a monoidal category, by a scalar object [14] of $\mathcal{C}$ we shall mean an object of $\mathcal{C}$ such that : $End(X) = k$. If $\mathcal{C}$ is abelian and k is algebraically closed then an object is scalar iff it is an simple object. We denote the set of isomorphism classes of scalar objects of $\mathcal{C}$ by $\Lambda_{\mathcal{C}}$.

Definition 2.1. *A finitely semisimple monoidal category is a monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$ such that :*

(a) *$\mathcal{C}$ is an abelian k-category and $\otimes$ is a bifunctor k-linear,*

(b) *every object of $\mathcal{C}$ is a finite direct sum of scalar objects of $\mathcal{C}$,*

(c) *$\sharp\Lambda_{\mathcal{C}} < \infty$ and I is a scalar object,*

(d) *$\mathcal{C}$ is sovereign*[1]

If $\mathcal{C}$ is a finitely semisimple monoidal category, then every object X of $\mathcal{C}$ admits a right duality : $(X, X^\vee, e_X, h_X)$ and a left duality : $(X^\vee, X, \epsilon_X, \eta_X)$, we can take the same object because $\mathcal{C}$ is sovereign. By definition of duality :

$$\begin{aligned} e_X &: X \otimes X^\vee \to I \\ \epsilon_X &: X^\vee \otimes X \to I \\ \eta_X &: I \to X \otimes X^\vee \\ h_X &: I \to X^\vee \otimes X. \end{aligned}$$

and we have the following equalities :

$$\begin{aligned} (e_X \otimes id_X)(id_X \otimes h_X) &= id_X \\ (id_X \otimes \epsilon_X)(\eta_X \otimes id_X) &= id_X \\ (id_{X^\vee} \otimes e_X)(h_X \otimes id_{X^\vee}) &= id_{X^\vee} \\ (\epsilon_X \otimes id_{X^\vee})(id_{X^\vee} \otimes \eta_X) &= id_{X^\vee}. \end{aligned}$$

The left quantum trace of an endomorphism $f \in End_{\mathcal{C}}(X)$ is defined by :

$$tr_l(f) = e_X(f \otimes id_{X^\vee})\eta_X,$$

the right quantum trace of an endomorphism $f \in End_{\mathcal{C}}(X)$ is defined by :

$$tr_r(f) = \epsilon_X(id_{X^\vee} \otimes f)h_X.$$

for any endomorphisms f, g in $\mathcal{C}$ we have :

$$\begin{aligned} tr_l(f \otimes g) &= tr_l(f)tr_l(g), \\ tr_r(f \otimes g) &= tr_r(f)tr_r(g), \\ tr_r(f) &= tr_l(f^\vee), \end{aligned}$$

the multiplication is given by the multiplication of $k = End(I)$.

Definition 2.2. *A spherical category is a finitely semisimple monoidal category such that, for all endomorphism f in $\mathcal{C}$ we have : $tr_l(f) = tr_r(f)$.*

[1] $\mathcal{C}$ admits a right and a left duality which are isomorphic as monoidal functor.

In a spherical category we denote the left trace by tr and so we have $tr = tr_l = tr_r$. The quantum dimension of an object X in a spherical category $\mathcal{C}$ is defined by :

$$dim(X) = tr(id_X),$$

so we have $dim(X) = dim(X^\vee)$.

Definition 2.3.

(i) *An object X of a monoidal category $\mathcal{C}$ is called invertible iff there exists an object Y such that $X \otimes Y \cong I$, where I is the tensor unit of $\mathcal{C}$.*

(ii) *A monoidal category is called pointed iff every scalar object is invertible.*

(iii) *The Group category Pic($\mathcal{C}$) of the monoidal category $\mathcal{C}$ is the full monoidal subcategory of $\mathcal{C}$ whose objects are direct sums of invertible objects of $\mathcal{C}$.*

(iv) *A Group category is a pointed spherical category.*

(v) *A θ-category is a braided, pointed finitely semisimple monoidal category.*

2.1.1. *Example of Group category.* G is a finite group, we denote $k[G]$ the category whose objects are G-graded finite dimensional k-vector spaces[2] and whose morphisms are k-linear morphism that preserves the grading. If V and W are objects of $k[G]$ the monoidal structure of $k[G]$ is given by :

$$(V \otimes W)_g = \sum_{\substack{h,k \\ hk=g}} V_h \otimes W_k.$$

the associativity is the identity and the isomorphism classes of scalar objects are in bijection with G : $g \leftrightarrow \delta_g$ where δ_g is defined in the following way :

$$(\delta_g)_h = \begin{cases} k & \text{if } g = h, \\ 0 & otherwise. \end{cases}$$

and every scalar object is invertible, thus $k[G]$ is a Group category. $k[G]$ is a θ-category iff G is an abelian group.

2.2. Some results on Group category.

Whenever $\mathcal{C}$ is a Group category, it follows from the definition of a Group category and the quantum dimension that for all $X \in \Lambda_{\mathcal{C}}$: $dim(X)^2 = 1$. The Grothendieck ring of $\mathcal{C}$ is isomorphic to the group algebra of the finite group $\Lambda_{\mathcal{C}}$, it is denoted $\mathcal{K}_0(\mathcal{C}) \cong \mathbb{Z}[\Lambda_{\mathcal{C}}]$.

Proposition 2.4. *If $\mathcal{C}$ is a Group category then :*

(i) *all invertible objects are in $\Lambda_{\mathcal{C}}$,*

(ii) $(\Lambda_{\mathcal{C}}, \otimes, I)$ *is a finite group.*

Proof (i) : If X is invertible in $\mathcal{C}$ then there exists an object Y in $\mathcal{C}$ such that : $X \otimes Y \cong I$, thus we have :

$$\sum_{Z \in \Lambda_{\mathcal{C}}} \mu_Z(X) Z \otimes Y = I,$$

where $\mu_Z(X) = dim_k(Hom_{\mathcal{C}}(X, Z))$ and so we have :

$$\sum_{Z', Z \in \Lambda_{\mathcal{C}}} \mu_Z(X)\mu_{Z'}(Z \otimes Y) Z' = I,$$

[2]we can define a similar category, using G-graded free A-modules, with A a commutative ring

since I is a scalar object

$$\sum_{Z\in\Lambda_{\mathcal{C}}} \mu_Z(X)\mu_{Z'}(Z\otimes Y) = \begin{cases} 1 & \text{if } Z' = I, \\ 0 & \textit{otherwise}. \end{cases}$$

Moreover $Hom_{\mathcal{C}}(X,Y)$ is finite for all objects in $\mathcal{C}$ and so $\mu_Z(X) \in \mathbb{N} \hookrightarrow k$, thus there is only one $Z_0 \in \Lambda_{\mathcal{C}}$ such that $\mu_{Z_0}(X) \neq 0$ and moreover $\mu_{Z_0}(X) = 1$ so $X = Z_0 \in \Lambda_{\mathcal{C}}$. We can notice that if Y is the inverse of X then $X \otimes Y \cong I$ and so $Y \cong X^\vee$. □

Proof (ii) : If X is scalar then by definition of a Group category X is invertible and so there is Y an object of $\mathcal{C}$ such that $X \otimes Y \cong I$ we have seen that $Y = X^\vee \in \Lambda_{\mathcal{C}}$. In a finitely scalar monoidal category we have : $X^\vee \otimes X = I \oplus Z$ where Z is an object of $\mathcal{C}$, thus we have :

$$\begin{aligned} X^\vee &\cong X^\vee \otimes I \\ &\cong X^\vee \otimes (X \otimes X^\vee) \\ &\cong (X^\vee \otimes X) \otimes X^\vee \\ &\cong X^\vee \oplus Z \otimes X^\vee \end{aligned}$$

If X is scalar then $X^\vee$ is scalar thus $Z \otimes X^\vee = 0$ and $X^\vee \neq 0$. So it follows that $Z = 0$ and $X^\vee \otimes X \cong I$, then $X^\vee$ is the left and right inverse of X. If X and Y are scalar objects then $X \otimes Y$ is an object of $\mathcal{C}$ and :

$$End_{\mathcal{C}}(X \otimes Y) \cong Hom_{\mathcal{C}}(X, X \otimes Y \otimes Y^\vee) \cong End_{\mathcal{C}}(X) \cong k,$$

then $X \otimes Y$ is a scalar object thus $(\Lambda_{\mathcal{C}}, \otimes, I)$ is a finite group. □

Theorem 2.5 ([5], section 7.5)**.** *Suppose G is a finite group, then : Group categories with underlying group G correspond to $H^3(G, k^\star)$.*

In fact $H^3(G, k^\star)$ classifies all the associativity constraint(up to monoidal equivalences). The group G gives the set of isomorphic classes of scalar objects and an element $\alpha \in H^3(G, k^\star)$ gives the associativity constraint of the Group category. If we take $\alpha, \alpha' \in Z^3(G, k^\star)$ such that $[\alpha] = [\alpha'] \in H^3(G, k^\star)$ then we obtain two Group categories denoted by $\mathcal{C}(G,\alpha)$ and $\mathcal{C}(G,\alpha')$ such that : $\mathcal{C}(G,\alpha) \cong^{\otimes} \mathcal{C}(G,\alpha')$ (monoidal equivalence).

2.3. **6j-symbol.** We fix $\mathcal{D}$ a finitely monoidal category then for all object X in $\mathcal{D}$ we have : $X = X_1 \oplus ... \oplus X_n$ with $X_i \in \Lambda_{\mathcal{D}}$ then for all $1 \geq j \geq n$ there are morphisms $i_j \in Hom_{\mathcal{D}}(X_j, X)$ and $p_j \in Hom_{\mathcal{D}}(X, X_j)$ such that $p_j i_j = id_{X_j}$ and $\sum_j i_j p_j = id_X$.

Lemma 2.6. *We fix $a, b, c, d, e, f \in \Lambda_{\mathcal{D}}$ then the following application*

$$\begin{aligned} \Psi : Hom(a, e\otimes d) \otimes_k Hom(e, b\otimes c &\to Hom(a, (b\otimes c)\otimes d) \\ v \otimes w &\mapsto (w \otimes id_d)v \end{aligned}$$

induces an isomorphism between $Hom_{\mathcal{D}}(a, (b\otimes c)\otimes d)$ and $\bigoplus_{e\in\Lambda} Hom_{\mathcal{D}}(a, e\otimes d) \otimes_k Hom_{\mathcal{D}}(e, b\otimes c)$. In the same vein we have : $Hom_{\mathcal{D}}(a, b\otimes(c\otimes d)) \cong \bigoplus_{f\in\Lambda} Hom_{\mathcal{D}}(a, b\otimes f) \otimes_k Hom_{\mathcal{D}}(f, c\otimes d)$

Proof : By definition of $\mathcal{D}$ we have : $b\otimes c=\oplus_{e\in\lambda_{\mathcal{D}}}\mu_e(b\otimes c)e$ with $\mu_e(b\otimes c)=dim_k(Hom(e,b\otimes c))$. Then for all $f\in Hom_{\mathcal{D}}(a,(b\otimes c)\otimes d)$ we have :

$$\begin{aligned} f &= id_{b\otimes c}\otimes id_d f \\ &= \sum_{e\in\Lambda}(i_e p_e\otimes id_d)f \\ &= \sum_{e\in\Lambda}(i_e\otimes id_d)(p_e\otimes id_d)f, \end{aligned}$$

and so Ψ is surjective. Moreover the vector spaces are finite dimensional and they have the same dimension thus we get the isomorphism. The second isomorphism is obtained in the same way. $\square$

a, the associativity constraint of $\mathcal{D}$, induces a natural isomorphism : $(X\otimes Y)\otimes Z\cong X\otimes(Y\otimes Z)$, for all $X,Y,Z\in ob(\mathcal{D})$. Then we have the following commutative square :

$$\begin{tikzcd} \bigoplus_{e\in\Lambda} Hom_{\mathcal{D}}(a,e\otimes d)\otimes_k Hom_{\mathcal{D}}(e,b\otimes c) \arrow[r] \arrow[d, "\cong"'] & \bigoplus_{f\in\Lambda} Hom_{\mathcal{D}}(a,b\otimes f)\otimes_k Hom_{\mathcal{D}}(f,c\otimes d) \arrow[d, "\cong"] \\ Hom_{\mathcal{D}}(a,(b\otimes c)\otimes d) \arrow[r, "\cong"'] & Hom_{\mathcal{D}}(a,b\otimes(c\otimes d)) \end{tikzcd}$$

the previous commutative square induces two linear applications :

$$\left\{\begin{array}{ccc} a & b & c \\ d & e & f \end{array}\right\} : Hom_{\mathcal{D}}(a,e\otimes d)\otimes_k Hom_{\mathcal{D}}(e,b\otimes c)\to Hom_{\mathcal{D}}(a,b\otimes f)\otimes_k Hom_{\mathcal{D}}(f,c\otimes d)$$

$$\left\{\begin{array}{ccc} a & b & c \\ d & e & f \end{array}\right\}_{inv} : Hom_{\mathcal{D}}(a,b\otimes f)\otimes_k Hom_{\mathcal{D}}(f,c\otimes d)\to Hom_{\mathcal{D}}(a,e\otimes d)\otimes_k Hom_{\mathcal{D}}(e,b\otimes c),$$

these are the 6j-symbol of $\mathcal{D}$.
We define a bilinear form in the following way : for all objects X,Y,

$$\begin{aligned} \omega_{X,Y} : Hom_{\mathcal{D}}(X,Y)\otimes Hom_{\mathcal{D}}(Y,X) &\to k \\ f\otimes g &\mapsto tr_g(fg). \end{aligned}$$

By definition $\mathcal{D}$ doesn't admit negligible morphism so $\omega_{\_,\_}$ is a non-degenerate bilinear form and it defines an adjoint of $\left\{\begin{array}{ccc} a & b & c \\ d & e & f \end{array}\right\}$, for all $(a,b,c,de,f)\in\Lambda_{\mathcal{D}}$, this adjoint is denoted by :

$\lambda(a,b,c,d,e,f)\in(Hom_{\mathcal{D}}(e\otimes d,a)\otimes Hom_{\mathcal{D}}(b\otimes c,e)\otimes Hom_{\mathcal{D}}(a,b\otimes f)\otimes Hom_{\mathcal{D}}(f,c\otimes d))^\star$.

2.4. **6j-symbol from Group category.** If $\mathcal{C}$ is a Group category then for all X,Y scalar objects $X\otimes Y$ is a scalar object. Thus if X,Y,Z are scalar objects then :

$$Hom(Z,X\otimes Y)\cong\left\{\begin{array}{ll} k & ,\text{if } X\otimes Y\cong Z \\ 0 & ,otherwise \end{array}\right. \tag{2.1}$$

In the case of Group category the isomorphisms (lemma 2.6) become :

Lemma 2.7. *For all scalar objects (a,b,c,d,e,f) we have :*

$$Hom_{\mathcal{C}}(a,e\otimes d)\otimes_k Hom_{\mathcal{C}}(e,b\otimes c)\cong Hom_{\mathcal{C}}(a,(b\otimes c)\otimes d) \tag{2.2}$$

$$Hom_{\mathcal{C}}(a,b\otimes f)\otimes_k Hom_{\mathcal{C}}(f,c\otimes d)\cong Hom_{\mathcal{C}}(a,b\otimes(c\otimes d)) \tag{2.3}$$

(i) $\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\} \neq 0$ *iff* $e \cong b \otimes c$, $a \cong (b \otimes c) \otimes d$ *and* $f \cong c \otimes d$

(ii) $\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\}_{inv} \neq 0$ *iff* $e \cong b \otimes c$, $a \cong b \otimes (c \otimes d)$ *and* $f \cong c \otimes d$

Proof : The assertions (i), (ii) and the isomorphisms (2.2), (2.3) come from (2.1). □Thus in the case of the Group category the 6j-symbol $\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\}$ only depends on b, c, d. For all scalar objects b, c, d we put $\alpha(b, c, d) = \left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\}$.
We can define α in $\Lambda_{\mathcal{C}}$. $\Lambda_{\mathcal{C}}$ is a group and for $g \in \Lambda_{\mathcal{C}}$ we denote X_g a representation of this isomorphism class, and so for all $g, h \in \Lambda_{\mathcal{C}}$ we have : $X_g \otimes X_h \cong X_{gh}$. By (2.1) we know that $Hom_{\mathcal{C}}(X_{gh}, X_g \otimes X_h)$ is a one dimensional vector space. For all $g, h \in \Lambda_{\mathcal{C}}$ we put $\phi(g, h)$ a basis of $Hom_{\mathcal{C}}(X_{gh}, X_g \otimes X_h)$.
We put $g, h, k \in \Lambda_{\mathcal{C}}$ and we denote X_g, X_h, X_k their representations. By construction $\alpha(X_g, X_h, X_k)$ is an isomorphism of one dimensional vector spaces thus in the basis ϕ we have :

$$\alpha(X_g, X_h, X_k)(\phi(gh, k) \otimes \phi(g, h)) = \overline{\alpha}(g, h, k)(\phi(g, hk) \otimes \phi(h, k)), \tag{2.4}$$

with $\overline{\alpha} : \Lambda_{\mathcal{C}} \times \Lambda_{\mathcal{C}} \times \Lambda_{\mathcal{C}} \to k^{\star}$. With the same notations the commutative square which defines α induces the following equality :

$$\overline{\alpha}(g, h, k)(id_{X_g} \otimes \phi(h, k))\phi(g, hk) = a(X_g, X_h, X_h)(\phi(g, h) \otimes id_{X_k})(\phi(gh, k)).$$

Thus $\overline{\alpha}$ determine the following isomorphism :

$$\begin{aligned} Hom_{\mathcal{C}}(X_{ghk}, (X_g \otimes X_h) \otimes X_k) &\cong Hom_{\mathcal{C}}(X_{ghk}, X_g \otimes (X_h \otimes X_k)) \\ v &\mapsto a(X_g, X_h, X_h)v, \end{aligned}$$

a is the associativity constraint of $\mathcal{C}$, thus a satisfies the Maclane's pentagon : with X_g, X_h, X_k, X_l scalar objects

$$\begin{tikzcd}
 & (X_g \otimes (X_h \otimes X_i)) \otimes X_j \arrow[dr, "a(g,hi,j)"] & \\
((X_g \otimes X_h) \otimes X_i) \otimes X_j \arrow[ur, "a(g,h,i)\otimes id"] \arrow[d, "a(gh,i,j)"'] & & X_g \otimes ((X_h \otimes X_i) \otimes X_j) \arrow[d, "id\otimes a(h,i,j)"] \\
(X_g \otimes X_h) \otimes (X_i \otimes X_j) \arrow[rr, "a(g,h,ij)"'] & & X_g \otimes (X_h \otimes (X_i \otimes X_j))
\end{tikzcd}$$

If we apply the last equality in the basis ϕ we have :

$$\overline{\alpha}(g, h, kl)\overline{\alpha}(gh, k, l) = \overline{\alpha}(h, k, l)\overline{\alpha}(g, hk, l)\overline{\alpha}(g, h, k).$$

Proposition 2.8. *If $\mathcal{C}$ is a Group category then the 6j-symbol is determined by a 3-cocycle on $Z^3(\Lambda_{\mathcal{C}}, k^{\star})$ and a basis of $Hom(X_{gh}, X_g \otimes X_h)$.*

The relation of a with the identity constraint (r, l) induces that :
$l(h)\overline{\alpha}(g, 1, h) = r(g)$, for all $g, h \in \Lambda_{\mathcal{C}}$ thus $r(g) = \overline{\alpha}(g, 1, 1)$ and $(g) = \overline{\alpha}(1, 1, h)^{-1}$.

We can change $\overline{\alpha}$ such that $\overline{\alpha}$ is normalized[3] without changed the cohomologous class of $\overline{\alpha}$. In term of basis ϕ, it is a change of basis.

3. The invariant of Turaev-Viro

We adopt the approach of [6] rather then [12], but we use a spherical category. Because if we consider a sovereign category there is a problem in the construction. The problem occurs at the level of independence of the numbering of the 3-simplex. Let T be a n-simplex with $n \geq 1$, then a Turaev-Viro color of T (Turaev-Viro point of view) is the following data : $\gamma : \{\text{oriented edges of T }\} \to \Lambda_{\mathcal{C}}$ which satisfies the conditions :

(i) for any oriented edge e : $\gamma(\overline{e}) = \gamma(e)^{\vee}$, where $\overline{e}$ is the oriented edge with the opposite orientation.

The set of all Turaev-Viro color of T is denoted $Col_{TV}(T)$. let T be a n-simplex and we fix a numbering of the vertices of F, every faces of T has an orientation given by the ascending order : (012). For every faces (012) we define the following vector space : $V_{\mathcal{C}}((012),\gamma) = Hom_{\mathcal{C}}(I, \gamma(01)\otimes\gamma(12)\otimes\gamma(20))$.

Lemma 3.1.

$$V_{\mathcal{C}}((012),\gamma) \cong V_{\mathcal{C}}((201),\gamma) \cong V_{\mathcal{C}}((120),\gamma) \tag{3.1}$$

$$V_{\mathcal{C}}((012),\gamma) \cong V_{\mathcal{C}}((021),\gamma)^{\star} \tag{3.2}$$

Proof (3.1) : It comes from the sovereign structure of $\mathcal{C}$.
For all $X, Y, Z \in ob(\mathcal{C})$ we have :

$$\begin{aligned} &Hom_{\mathcal{C}}(I, X\otimes Y\otimes Z) \leftrightarrow Hom_{\mathcal{C}}(I, Y\otimes Z\otimes X)\\ &\qquad f \mapsto (\epsilon_X \otimes id_{Y\otimes Z\otimes X})(id_{X^{\vee}}\otimes f\otimes id_X)(h_x) \quad \square\end{aligned}$$

Proof (3.2) : It comes from the fact that the category $\mathcal{C}$ doesn't admit negligible morphism and so the following bilinear form is non-degenerate :

$$\begin{aligned} &contr : V_{\mathcal{C}}((012),\gamma)\otimes V_{\mathcal{C}}((021),\gamma) \to k\\ &\qquad f\otimes g \mapsto f^{\vee}g = tr(f^{\vee}g) = tr(g^{\vee}f) \quad \square\end{aligned}$$

Thus the vector space $V_{\mathcal{C}}((012),\gamma)$ is independent of the starting point and if we change the orientation of the 2-simplex we obtain the dual vector space. Moreover this dual vector space can be obtain by a change of color in fact :

$$V_{\mathcal{C}}((012),\gamma) \cong V_{\mathcal{C}}((021),\gamma)^{\star} \cong V_{\mathcal{C}}((021),\gamma'),$$

with $\gamma'(02) = \gamma(01)$, $\gamma'(21) = \gamma(12)$, $\gamma'(10) = \gamma(20)$. Let T be the triangulation of a compact oriented surface Σ and T^2 the set of 2-simplex of T, then we define

$$V_{\mathcal{C}}(\Sigma, T) = \bigoplus_{\gamma\in Col(T)} \bigotimes_{f\in T^2} V_{\mathcal{C}}(f,\gamma),$$

and this space is independent of the choice of a numbering of T.
Let Δ be a 3-simplex, a numbering of the vertices of Δ gives an orientation of Δ, with this orientation Δ is denoted (0123). We take $\gamma \in Col(\Delta)$ and we put :

$$\begin{aligned} &V_{\mathcal{C}}((132),\gamma)\otimes V_{\mathcal{C}}((023),\gamma)\otimes V_{\mathcal{C}}((031),\gamma)\otimes V_{\mathcal{C}}((012),\gamma) \stackrel{L((0123),\gamma)}{\longrightarrow} k\\ &\quad v_0\otimes v_1\otimes v_2\otimes v_3 \mapsto dim(\gamma(13))^{-1}\lambda(\gamma(03),\gamma(01),\gamma(12),\gamma(23),\gamma(02),\gamma(13)).\end{aligned} \tag{3.3}$$

[3] r=l=1

This application defines, with duality given by ω, an element
$\tilde{L}((0123),\gamma) \in V_{\mathcal{C}}((132),\gamma)^{\star} \otimes V_{\mathcal{C}}((023),\gamma)^{\star} \otimes V_{\mathcal{C}}((031),\gamma)^{\star} \otimes V_{\mathcal{C}}((012),\gamma)^{\star}$. If T is a triangulation of M, which is an oriented and closed 3-manifold, then we denote T^3 the set of oriented 3-simplex of T and we define the following element :

$$\bigotimes_{\sigma \in T^3} \tilde{L}(\sigma,\gamma).$$

But M is a closed 3-manifold so every 2-simplex is a face of exactly two 3-simplex with opposite orientation. The 3-simplex are oriented such that their orientations correspond to the orientation of M. We denote f the common face of σ_1 and σ_2, so the elements can be written in the following way : $\tilde{L}(\sigma_1,\gamma) \in W \otimes V_{\mathcal{C}}(f,\gamma)$, where W is the tensor product of the three other faces and $\tilde{L}(\sigma_2,\gamma) \in W' \otimes V_{\mathcal{C}}(\overline{f},\gamma)$ with W' the tensor product of the three other faces. The sovereign structure of $\mathcal{C}$ defines a bilinear non-degenerate form on this two vector spaces :

$$\begin{aligned} contr : V_{\mathcal{C}}((012),\gamma) \otimes V_{\mathcal{C}}((021),\gamma) &\to k \\ f \otimes g &\mapsto tr(f^{\vee} g) = tr(g^{\vee} f), \end{aligned}$$

the equality comes from the fact that $I = I^{\vee}$ and the semi-simplicity of $\mathcal{C}$ implies the non degeneracy of $contr$. Since the 3-manifold is closed we can contract every 2-simplex and then

$$Z(M,\gamma) = contr(\otimes_{\sigma \in T^3} \tilde{L}(\sigma,\gamma)) \in k.$$

We fix as in 1, n_0 the number of vertices of a given triangulation (we don't call it $n_0(T)$ for two reasons, the first is historical [12], [11], [15], [6], [2] and the second reason comes from the fact that we use n_0 to describe an object which doesn't depend on the triangulation). T_o is the triangulation with the orientation given by the numberings of the vertices such that the orientaton is the orientation of the manifold M. The invariant of Turaev-Viro is :

$$TV(M) = (\sum_{X \in \Lambda_{\mathcal{C}}} dim(X)^2)^{-n_0} \sum_{\gamma \in Col_{TV}(T)} \prod_{e \in T_o^1} dim(\gamma(e)) Z(M,\gamma), \tag{3.4}$$

in the rest of the paper $\sum_{X \in \Lambda_{\mathcal{C}}} dim(X)^2$ will be denoted by $dim(\mathcal{C})$.

4. The equality

The invariant (3.4), we make a sum over the Turaev-Viro coloring of T and we compute $L(\Delta,\gamma)$ for each Turaev-Viro color γ and each 3-simplex Δ. The linear $L(\Delta,\gamma)$ is computed over vector spaces which are : $V(f,\gamma)$ for each face f of the 3-simplex Δ. But in a monoidal semisimple category we have the following result for all scalar objects a, b, c of $\mathcal{C}$:

$$Hom_{\mathcal{C}}(I, a \otimes b \otimes c) \begin{cases} \ncong 0 & , a^{\vee} \hookrightarrow b \otimes c \\ \cong 0 & , otherwise \end{cases}$$

If $\mathcal{C}$ is Group category then we have the following relation :

$$Hom_{\mathcal{C}}(I, a \otimes b \otimes c) \begin{cases} \cong k & , a^{\vee} \cong b \otimes c \\ \cong 0 & , otherwise \end{cases}$$

Definition 4.1. *Let $\mathcal{C}$ a Group category an T an n-simplex with $n \geq 1$. An admissible colouring of T is the set of application γ from oriented edges of T to $\Lambda_{\mathcal{C}}$ which satisfy :*

(i) $\gamma(\overline{e}) = \gamma(e)^{\vee}$, *where* $\overline{e}$ *is the edge* e *with the opposite orientation*

(ii) *for any oriented 2-simplex of* T *we have :*

$$\gamma(01) \otimes \gamma(12) \otimes \gamma(20) \cong I$$

If $\mathcal{C}$ is a Group category, then an admissible coloring is nothing else than a color in a sense of Wakui (1.1). That's why we denote it $Col(T)$. If $\gamma \in Col_{TV}$ and $\gamma \notin Col(T)$, there is at least one oriented face (012) in T such that : $\gamma(01)\gamma(12) \neq \gamma(02)$. It result that : $V(012, \gamma) = 0$. In a closed manifold every face is in the boundary of exactly two 3-simplex with opposite orientation. the value of $L(\_, \gamma)$ on these 3-simplex is equal to 0. Thus in the sum 3 we have $Z(T, \gamma) = 0$ for every $\gamma \notin Col(T)$. That's why can consider the sum 3 only on the admissible coloring (or coloring in Wakui sense).

Theorem 4.2. *Let* $\mathcal{C}$ *a Group category such that for all* $X \in \Lambda_{\mathcal{C}}$ *we have :* $dim(X) = 1$.
If G *is the underlying group of* $\mathcal{C}$ *and if* $\alpha \in Z^3(G, k^\star)$ *is the associativity constraint of* $\mathcal{C}$ *then for all closed and oriented 3-manifold* M *:*

$$DW_{G,\alpha}(M) = TV_{\mathcal{C}}(M).$$

Proof : $\mathcal{C}$ is Group category The condition $dim(X) = 1$, for all $X \in \Lambda_{\mathcal{C}}$ implies : $dim(\mathcal{C}) = \sharp\Lambda_{\mathcal{C}}$ and so :

$$TV_{\mathcal{C}}(M) = (\sharp\Lambda_{\mathcal{C}})^{-n_0} \sum_{\gamma \in Col_{TV}(T)} Z(T, \gamma),$$

it remains to compute $Z(T, \gamma)$ for an admissible coloring.

Lemma 4.3. *If* $\mathcal{C}$ *is a Group category such that for all* $X \in \Lambda_{\mathcal{C}}$ $dim(X) = 1$ *then :*

$$Z(T, \gamma) = \prod_{(0123) \in T^3} \alpha(\gamma(01), \gamma(12), \gamma(23)),$$

with T *a triangulation of a closed and oriented 3-manifold and* γ *an admissible coloring of* T.

Proof : If T is a triangulation of a closed and oriented 3-manifold and γ is a coloring of T then :

$$Z(T, \gamma) = contr(\otimes_{(0123) \in T^3} \tilde{L}((0123), \gamma) \in k.$$

Moreover if $dim(X) = 1$ for all $X \in \Lambda_{\mathcal{C}}$, by definition of L (3.3) we have $L((0123), \gamma) = \lambda(\gamma(03), \gamma(01), \gamma(12), \gamma(23), \gamma(13), \gamma(02))$. $\mathcal{C}$ is a Group category then $V((012), \gamma) \cong Hom(\gamma(02), \gamma(01) \otimes \gamma(12)) \cong k$. We fix $\Phi(\gamma(01), \gamma(12))$ a basis of this vector space. If we consider (021), the same face with the opposite orientation, then we have, thanks to the contraction : $V((012), \gamma)^\star \cong V((021), \gamma)$ and so we can take the dual basis, it induces a basis $\Phi'(\gamma(02), \gamma(21))$ of $Hom_{\mathcal{C}}(\gamma(02) \otimes \gamma(21), \gamma(01)) \cong V(021, \gamma)$

$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\} (\Phi(b \otimes c, d) \otimes \Phi(b, c)) = \alpha(b, c, d) \Phi(b, c \otimes d) \otimes \Phi(c, d),$$

and the duality given by the non-degenerate bilinear form ω gives a basis of $Hom(e\otimes d, a)$ which is the dual of $Hom(a, e\otimes d)$:

$$\begin{aligned}\omega\, :\, Hom(e\otimes d, a)\otimes Hom(a, e\otimes d) &\to k\\ f\otimes\Phi(e,d) &\mapsto tr(f\Phi(e,d))\end{aligned}$$

and so $\Phi'(e,d)=\Phi(e,d)^{-1}$. Thus we have :

$$\begin{aligned}&\tilde{L}((0123),\gamma)=\tilde{\lambda}(\gamma(03),\gamma(01),\gamma(12),\gamma(23),\gamma(13),\gamma(02))\\ &=\alpha(\gamma(01),\gamma(12),\gamma(23))\Phi(\gamma(01)\otimes\gamma(12),\gamma(23))^{-1}\otimes\Phi(\gamma(01),\gamma(12))^{-1}\\ &\otimes\Phi(\gamma(01),\gamma(12)\otimes\gamma(23))\otimes\Phi(\gamma(12),\gamma(23))\end{aligned}$$

[6] asserts that $L(0123,\gamma)$ doesn't depend on the choice of the numbering which preserve the orientation of (0123) [4] so we can choose any numbering (0123) which preserve the orientation of the 3-simplex. The contraction on $V((012),\gamma)$ induces a contraction on $Hom(\gamma(01)\otimes\gamma(12),\gamma(02))$ given by the isomorphism : $V((012),\gamma)\cong Hom(\gamma(01)\otimes\gamma(12),\gamma(02))$:

$$\begin{aligned}contr\, :\, Hom(\gamma(01)\otimes\gamma(12),\gamma(02))\otimes Hom(\gamma(02),\gamma(01)\otimes\gamma(12)) &\to k\\ f\otimes g &\mapsto tr(fg)\end{aligned}$$

If we consider two 3-simplexes (0123) and (0214) with the same orientation, they have a common face. We change the numbering of (0214) without changing the orientation : (4012) (the cycle (042) has an even signature).

$$\begin{aligned}&\tilde{L}((0123),\gamma)=\tilde{\lambda}(\gamma(03),\gamma(01),\gamma(12),\gamma(23),\gamma(13),\gamma(02))\\ &=\alpha(\gamma(01),\gamma(12),\gamma(23))\Phi(\gamma(01)\otimes\gamma(12),\gamma(23))^{-1}\otimes\Phi(\gamma(01),\gamma(12))^{-1}\\ &\otimes\Phi(\gamma(01),\gamma(12)\otimes\gamma(23))\otimes\Phi(\gamma(12),\gamma(23))\\ &\tilde{L}((4012),\gamma)=\tilde{\lambda}(\gamma(42),\gamma(40),\gamma(01),\gamma(12),\gamma(02),\gamma(41))\\ &=\alpha(\gamma(40),\gamma(01),\gamma(12))\Phi(\gamma(40)\otimes\gamma(01),\gamma(12))^{-1}\otimes\Phi(\gamma(40),\gamma(01))^{-1}\\ &\otimes\Phi(\gamma(40),\gamma(01)\otimes\gamma(12))\otimes\Phi(\gamma(01),\gamma(12))\end{aligned}$$

$$contr(\Phi(\gamma(01),\gamma(12))^{-1}\otimes\Phi(\gamma(01),\gamma(12))=dim(\gamma(01)\otimes\gamma(12))=1$$

$Z(T,\gamma)=\prod_{(0123)\in T^3}\alpha(\gamma(01),\gamma(12),\gamma(23))$. □

5. Topological interpretation of admissible coloring

In this section, we will give a topological interpretation of the admissible coloring of a triangulation.

5.1. The fundamental groupoïd of T. Let T be a n-simplex, we denote $\Pi_1(T)$ the following category :

$Ob(\Pi_1(T))=T^0$

Arrows of $\Pi_1(T)$ are the oriented edges of T and the 0-simplex modulo the relation of 2-simplex, that is if (012) is an oriented 2-simplex then $(01).(12)=(02)$.

[4] In [6] the authors asserts this result for a sovereign category, but there is a problem with this condition. If we use spherical category the result is true.

The composition is given by the concatenation of the edge, and the inverse of an edge is the same edge with the opposite orientation. The identity is given by the 0-simplex himself. We can define the pointed fundamental groupoïd $\Pi_1(T,x)$ in the same way. There is only one object which is x and the set of arrows are loops in x and a loop is a concatenation of edges.

Remark 5.1. *If T is the triangulation of a connected manifold M, then there is an equivalence of category between $\Pi_1(T)$ and the pointed category $\Pi_1(T,x)$ where x is 0-simplex of T. Moreover the set of arrows of $\Pi_1(T,x)$ is $\Pi_1(M,x)$.*

If G is a group then the groupoïd obtained thanks to G will be denoted $\mathcal{G}$. If $\mathcal{C}$ is a Group category we can define the following application :

$$\begin{aligned}\Psi : Col(T) &\to Fun(\Pi_1(T), \Lambda_{\mathcal{C}})\\ \gamma &\mapsto F_\gamma,\end{aligned}$$

the functor F_γ is defined in the following way : for all $x \in T^0$ we have $F_\gamma(x) = \star$, which is the object of $\mathcal{G}$. And $F_\gamma(01) = \gamma(01)$. γ respects the 2-simplex condition, thus F_γ is well defined.

Lemma 5.2. *Ψ is bijective.*

Proof : If $F_\gamma = F_\theta$ then for all oriented edges e we have : $\gamma(e) = \theta(e)$ and so $\gamma = \theta$.
If $F \in Fun(\Pi_1(T), \Lambda_{\mathcal{C}})$ then for all oriented edge e we have $F(e) \in \Lambda_{\mathcal{C}}$ and $F(\overline{e}) = F(e)^\vee$. Thus we can define a coloring of T : $\gamma(e) = F(e)$. We have to check the 2-simplex condition. If (012) is a 2-simplex then : $F((02)) = F((01).(12)) = F((01)) \otimes F((12))$, thus $\gamma \in Col(T)$ and $\Psi(\gamma) = F_\gamma$ and for object $F_\gamma = F$ and for all arrow e : $F_\gamma(e) = \gamma(e) = F(e)$ thus Ψ is bijective. □

5.2. **The gauge action.** We note $\Lambda_{\mathcal{C}}^{T^0}$ the set of application from the 0-simplex of T to $\Lambda_{\mathcal{C}}$. We can define an action of $\Lambda_{\mathcal{C}}^{T^0}$ on $Col(T)$ in the following way :

$$\begin{aligned}\Lambda_{\mathcal{C}}^{T^0} \times Col(T) &\to Col(T)\\ (\delta,\gamma) &\mapsto \gamma^\delta,\end{aligned}$$

such that for all oriented edge (01) : $\gamma^\delta(01) = \delta(0) \otimes \gamma(01) \otimes \delta(1)^\vee$. By a straightforward computation we show that it is an action.
We define the following equivalence relation on $Col(T)$:

$$(\gamma \sim \gamma') \Leftrightarrow (\exists \delta \in \Lambda_{\mathcal{C}}^{T^0} \text{ such that } \gamma^\delta = \gamma'),$$

we can construct the following application :

$$\Theta : \frac{Col(T)}{\sim} \to \frac{Fun(\Pi_1(T), \Lambda_{\mathcal{C}})}{iso} \tag{5.1}$$

$$[\gamma] \mapsto [\psi(\gamma)] \tag{5.2}$$

This application is well defined because if $\gamma' = \gamma^\delta$ then : $\beta(x) = \delta(x)^\vee : F_\gamma(x) \to F_{\gamma'}(x)$ is an in isomorphism in the groupoïd $\Lambda_\mathcal{C}$. For all oriented edge (01) we have :

$$\begin{aligned} \beta(1)F_\gamma(01) &= \gamma(01) \otimes \delta(1)^\vee \\ &= \delta(0)^\vee \otimes \delta(0) \otimes \gamma(01) \otimes \delta(1)^\vee \\ &= \delta(0)^\vee \gamma^\delta(01) \\ &= F_{\gamma^\delta}(01)\beta(0). \end{aligned}$$

Thus β is a natural isomorphism between F_γ and F_{γ^δ}.

Proposition 5.3. *Θ is a bijection.*

Proof : Ψ is surjective thus it follows that Θ is surjective.
Let γ and γ' two admissible colorings of T, if $\Theta(\gamma) = \Theta(\gamma')$ then there is a natural isomorphism between F_γ and $F_{\gamma'}$. We note β this isomorphism, for all 0-simplex x we have : $\beta(x) \in \Lambda_\mathcal{C}$ and $\beta(x) : F_\gamma(x) \cong F_{\gamma'}(x)$. So $\beta \in \Lambda_\mathcal{C}^{T^0}$ and for all oriented edges (xy) we have the following commutative square :

$$\begin{tikzcd} F_\gamma(x) \arrow[r, "\gamma(xy)"] \arrow[d, "\beta(x)"] & F_\gamma(y) \arrow[d, "\beta(y)"] \\ F_{\gamma'(x)} \arrow[r, "\gamma'(xy)"'] & F_{\gamma'(y)} \end{tikzcd}$$

thus $\gamma'^\beta = \gamma$. □

6. Construction of TQFT

6.1. **Triangulate TQFT.** We recall some definitions on TQFT and some results of [9]. We denote Cob the category of 1+2 cobordism : $Ob(Cob)$ is the set of oriented and closed surface, the morphism of Cob are the class of oriented compact 3-manifold, i.e : the classes of diffeomorphisms preserving the boundary. The disjoint union and $\emptyset$ give a strict monoidal structure to Cob.

Definition 6.1. *A is a commutative ring with unit, a TQFT is a monoidal and A-linear functor from Cob to A-mod.*

In [9], there is a way of obtaining a TQFT from a functor which is not monoidal.

Definition 6.2. *Let $\mathcal{C}$ a monoidal category, a monoidal, non unitary functor F is the data*
$(F, \Phi_2, \Phi_0) : \mathcal{C} \to \mathcal{D}$. (F, Φ_2, Φ_0) verify all the axioms of a monoidal functor expected the following : there is at least one $X \in Ob(\mathcal{C})$ such that : $F(id_X) \neq id_{F(X)}$

Proposition 6.3 ([9])**.** *Let $(F, \Phi_2, \Phi_0) : \mathcal{C} \to \mathcal{D}$ a monoidal non unitary functor, with $\mathcal{D}$ a monoidal, A-linear and abelian category, then there is a monoidal functor $\tilde{F} : \mathcal{C} \to \mathcal{D}$ such as : for all $X \in Ob(\mathcal{C})$ $\tilde{F}(X)$ is an sub object of $F(X)$.*

6.2. **Construction of Turaev-Viro.** We recall the construction of Turaev-Viro [12] and we apply the result of [9].
First step
For every object (Σ, T) we assign finite vector space :

$$V(\Sigma, T) = \bigoplus_{c \in Col(T)} \bigotimes_{f \in T^2} V(f, \gamma) = k[Col(T)],$$

it is the vector space spanned by the admissible coloration of T.
For every 3-manifold M whose boundary is $(-\Sigma, T)\coprod(\Sigma', T')$ and for every admissible coloring c,c' of T and T' : $TV_M(c,c') \in k$. Thus we can define :

$$V(M) : V(\Sigma,T) \to V(\Sigma',T') \tag{6.1}$$
$$c \mapsto \sum_{c' \in Col(T')} TV_M(c,c')c'$$

By construction :

$$\begin{aligned} V(M)V(N)(c) &= \sum_{c_1,c_2} TV_N(c,c_1)TV_M(c_1,c_2)c_2 \\ &= (\sharp\Lambda_{\mathcal{C}})^{n_0(\partial N_+)} \sum_{c_2} TV_{M\circ N}(c,c_2)c_2 \\ &= (\sharp\Lambda_{\mathcal{C}})^{n_0(\partial N_+)} V(M\circ N)(c). \end{aligned} \tag{6.2}$$

There are at least three ways of erasing the anomaly. Here are the normalization, with M a 3-manifold whose the boundary is $\partial M = \overline{M}_- \coprod M_+$ and $c \in Col(M_-)$, $c' \in Col(M_+)$.

$$TV_i(M)(c,c') = \Lambda_{\mathcal{C}}^{-n_0(M_-)} TV(M)(c,c') \tag{6.3}$$

$$TV_o(M)(c,c') = \Lambda_{\mathcal{C}}^{-n_0(M_+)} TV(M)(c,c') \tag{6.4}$$

$$TV_m(M)(c,c') = \Lambda_{\mathcal{C}}^{\frac{-n_0(M_-)-n_0(M_+)}{2}} TV(M)(c,c') \tag{6.5}$$

Lemma 6.4.
 (i) *TV_i, TV_o and TV_m are invariants of 3-manifold with boundary.*
 (ii) *TV_i,TV_o and TV_m define the same monoidal non unitary functor (up to monoidal equivalence).*

Proof (i) : The theorem of Pachner define invariant of 3-manifold whose the boundary is fix, thus the triangulation of the boundary remains unchanged. That's why we obtain an invariant of 3 -manifold with boundary.
Proof (ii) : Let V_i (resp. V_o, V_m) the non unitary functor defines from the invariant TV_i (resp. TV_o, TV_m). The natural transformation :

$$\beta : V_o(\Sigma,T) = k[Col(T)] \to V_i(\Sigma,T) = k[Col(T)]$$
$$c \mapsto (\sharp\Lambda_{\mathcal{C}})^{n_0((\Sigma,T))}c$$

is an isomorphism. It remains to show that β is monoidal. Since V_o and V_i are strict we the following square :

$$\begin{tikzcd} V_o((\Sigma,T)\otimes(\Sigma',T')) \arrow[r, "\beta((\Sigma,T)\otimes(\Sigma',T'))"] \arrow[d, "="'] & V_i((\Sigma,T)\otimes(\Sigma',T')) \arrow[d, "="] \\ V_o(\Sigma_1,T_1)\otimes V_o(\Sigma_1',T_1') \arrow[r, "\beta(\Sigma,T)\otimes\beta(\Sigma',T')"] & V_i(\Sigma,T)\otimes V_i(\Sigma',T') \end{tikzcd}$$

and for all $c \in Col(T \coprod T') = Col(T) \otimes Col(T')$,

$$\begin{aligned}\beta((\Sigma,T)\otimes(\Sigma',T'))(c) &= (\sharp\Lambda_{\mathcal{C}})^{n_0((\Sigma,T)\coprod(\Sigma',T'))}c\\ &= (\sharp\Lambda_{\mathcal{C}})^{n_0((\Sigma,T))+n_0((\Sigma',T'))}c\\ &= (\sharp\Lambda_{\mathcal{C}})^{n_0((\Sigma,T))+n_0((\Sigma',T'))}c_1\otimes c_2,\end{aligned}$$

with $c = c_1 \otimes c_2$. And for $c \in Col(T), c' \in Col(T')$, we have :

$$\beta((\Sigma,T))\otimes\beta((\Sigma',T'))(c\otimes c') = (\sharp\Lambda_{\mathcal{C}})^{n_0((\Sigma,T))}(\sharp\Lambda_{\mathcal{C}})^{n_0((\Sigma',T'))}c\otimes c'.$$

Thus the square commutes.
We prove in the way the monoidal isomorphism between V_o and V_m, the isomorphism is given by :

$$\begin{aligned}\kappa((\Sigma,T)) :\ V(\Sigma,T) &\to V(\Sigma,T)\\ c &\mapsto (\sharp\Lambda_{\mathcal{C}})^{-\frac{n_0(\Sigma,T)}{2}}c\end{aligned}$$

□In [15] and [11], the authors used TV_m, here we will TV_i because in this case we don't have to compute the square of $Dim(\mathcal{C})$. We replace TV by TV_i in the definition of (6.1), and we still denote it by V.

Proposition 6.5 ([11], [15])**.** *V is a monoidal non unitary functor from Cob_{tri} to the category of finite dimensional vector spaces.*

We fix $V(\Sigma\times I,T)$ the linear application given by the identity on (Σ,T) in Cob_{tri}. [15] and [11], or in a general framework [9], the TQFT is the following functor :

$$\begin{aligned}\mathcal{V}Cob &\to k-vect\\ \Sigma &\mapsto \mathcal{V}(\Sigma) = im(V(\Sigma\times I,T))\\ M\in Hom(\Sigma,\Sigma') &\mapsto \mathcal{V}(M) = V(M)_{|im(V(\Sigma\times I,T))}\end{aligned}$$

The functor is well defined because for all triangulations T and T' of Σ we have : $im(V(\Sigma\times I,T)) \cong im(V(\Sigma\times I,T'))$. The isomorphism is given by $\Sigma\times I$ with T the triangulation of $\Sigma\times\{0\}$ and T' the triangulation $\Sigma\times\{1\}$.

7. Examples

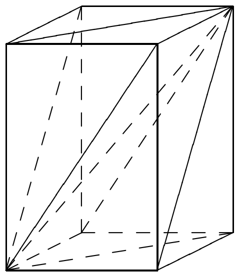
7.1. $\alpha = 1$**.** If $\alpha = 1$ then $\mathcal{C}$ is a strict Group category and $\mathcal{C}$ is equivalent to $k[G]$. Thanks to the gauge action (5.2) and the isomorphism (5.1), we have for all closed, oriented and connected 3-manifold M :

$$TV_{k[G]}(M) = \#G^{-n_0}\#Col(T) = \#\frac{Col(T)}{\sim} = \#\frac{Fun(\Pi_1(T,x),\Lambda_{\mathcal{C}})}{iso},$$

the groupoïd has only one object and so a functor is only defined by the applications from the $\Pi_1(M,x)$ to $\Lambda_{\mathcal{C}}$ and an isomorphism between two functors implies that the applications are conjugate. So we have :

$$TV_{k[G]}(M) = \#\frac{Hom(\Pi_1(M),\Lambda_{\mathcal{C}})}{conj}.$$

7.2. $G = \mathbb{Z}_n$. If G is the cyclic group of order n then we have : $H^3(G, k^\star) = \mathbb{Z}_n$, and α given by (A.2) is a 3-cocycle.
A triangulation of $S^1 \times S^1 \times S^1$ is given in [4] :



There are six 3-simplex. And so the invariant is the following :

$$TV_{\mathbb{Z}_n,\alpha}(S^1 \times S^1 \times S^1) = \frac{1}{n} \sum_{\substack{g,h,k \in \mathbb{Z}_n, \\ [g,h]=[g,k]=[h,k]=1}} \frac{\alpha(g,h,k)\alpha(h,k,g)\alpha(k,g,h)}{\alpha(g,k,h)\alpha(h,g,k)\alpha(k,h,g)}$$
$$= \frac{1}{n} \sum_{g,h,k \in \mathbb{Z}_n} \frac{\alpha(g,h,k)\alpha(h,k,g)\alpha(k,g,h)}{\alpha(g,k,h)\alpha(h,g,k)\alpha(k,h,g)}$$

For every finite group G and for every 3-cocycle $\alpha \in Z^3(G, k^\star)$, we can define :

$$\beta(g,h,k) = \frac{\alpha(g,h,k)\alpha(h,k,g)\alpha(k,g,h)}{\alpha(g,k,h)\alpha(h,g,k)\alpha(k,h,g)},$$

it verifies some properties :

Lemma 7.1.
(i) *For every finite group G and for every $g, h, k \in G$, there is an action of S_3 over β by permutating the terms and if $\sigma \in S_3$: $\sigma.\beta(g,h,k) = \beta(g,h,k)^{\epsilon(\sigma)}$, ϵ is the signature.*
(ii) *if G is abelian and α is A.1 then for all $g, h, k \in G$: $\beta(g,h,k) = 1$,*

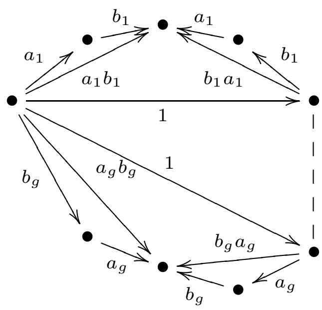
Proof (i) : It is straightforward from the definition of β, we give only the calculation for the permutation (12) and for the cycle (123) :

$$\beta(h,g,k) = \frac{\alpha(h,g,k)\alpha(g,k,h)\alpha(k,h,g)}{\alpha(h,k,g)\alpha(g,h,k)\alpha(k,g,h)}$$
$$= \frac{1}{\frac{\alpha(g,h,k)\alpha(h,k,g)\alpha(k,g,h)}{\alpha(h,g,k)\alpha(g,k,h)\alpha(k,h,g)}}$$
$$= \beta(g,h,k)^{-1}$$
$$\beta(h,k,g) = \frac{\alpha(h,k,g)\alpha(k,g,h)\alpha(g,h,k)}{\alpha(h,g,k)\alpha(k,h,g)\alpha(g,k,h)}$$
$$= \beta(g,h,k) \quad \square$$

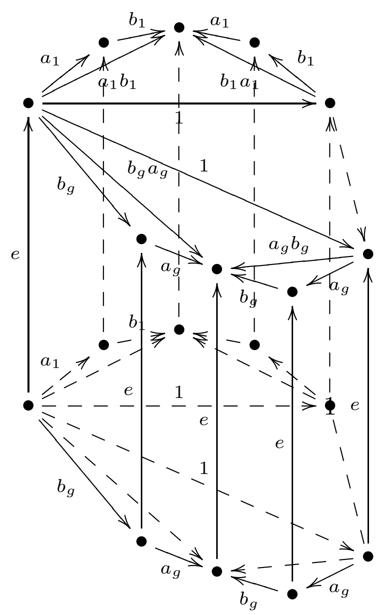
Proof (ii) : $\square$
Thus for $G = \mathbb{Z}_n$ and for α defined by (A.2), we have :

$$TV_{\mathbb{Z}_n,\alpha}(S^1 \times S^1 \times S^1) = \frac{1}{n} \sum_{g,h,k \in \mathbb{Z}_n} 1 = n^2.$$

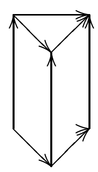
In the same way, we can compute $TV_{\mathbb{Z}_n,\alpha}(\Sigma_g \times S^1)$, where Σ_g is the closed surface of genus g, using the following triangulation of Σ_g :



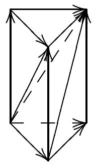
The edges with the same labels are identified, we have 1 on the edges inside the polygon by commutativity of the group $\mathbb{Z}_n$ and the condition (ii) of the admissible coloration. We denote T_g the previous triangulation of Σ_g : $Col(T_g) = k[\mathbb{Z}_n^{2g}]$. Thus $c = (a_1, b_1, ..., a_g, b_g) \in \mathbb{Z}_n \times ... \times \mathbb{Z}_n$ is a coloring of T_g We can define a triangulation of $\Sigma_g \times S^1$:



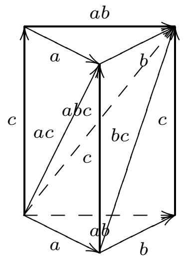
To avoid problems of reading this triangulation, some edges are not coloring. We can recover the corresponding coloring thanks to the condition (ii) of coloring. So by definition of the invariant : $TV_{\mathbb{Z}_n,\alpha} = \frac{1}{n}\sum_{\gamma\in Col(T_g')}\prod_{\Delta\in T_g^3}\alpha(\Delta,\gamma)$. The previous triangulation can be divided into prisms :



Then we have to decompose this prism into 3-simplex. Below we give such a decomposition :



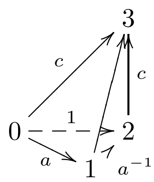
If we assigns a coloring γ to this prism we can compute the term $\alpha(\Delta, \gamma)$ of the prism and by construction we will have the invariant $TV(\Sigma_g \times I)$.



(a, b, c) is a coloring of the prism, the scalar assigns to the prism is then : $\frac{\alpha(a,b,c)\alpha(c,a,b)}{\alpha(a,c,b)}$.
In the triangulation of $\Sigma_g \times S^1$ for a given coloring, we have prism such that the coloring is : (a, a^{-1}, c), where $a, c \in \mathbb{Z}_n$ and so there are at least two edges whose the associated coloring is equal to I. The value of the scalar associated to the prism equip with a coloration (a, a^{-1}, c) is given by the following lemma :

Lemma 7.2. *If G is a finite group and γ is a coloring of a 3-simplex (oriented)Δ such that the coloring of one edge (at least) is* 1 *then for every normalized 3-cocycle in the group cohomology $H^3(G, k^\star)$, $\alpha(\Delta, \gamma) = 1$*

Proof : If we fix γ and Δ we have the following figure :



The numbering of the vertices gives an orientation of Δ, for an orientation given by the numbering $0 < 1 < 2 < 3$ Δ will be noted : (0123). In this case we have : $\alpha(a, a^{-1}, c) = 1$. It follows the definition of $H^3(G, k^\star)$ which isomorphic to : $H^3(BG, k^\star)$. If we change the orientation (presevering or not) we will have : $\alpha(a, a^{-1}, c)$ (preserving the orientation) or $\alpha(a, a^{-1}, c)^{-1}$ [15]. □Thus the invariant is equal to :

$$\begin{aligned} TV_{\mathbb{Z}_n,\alpha}(\Sigma_g \times S^1) &= \frac{1}{n} \sum_{(a_1,b_1,\ldots,a_g,b_g,e)\in\mathbb{Z}_n} \beta(a_1, b_1, e) \times \ldots. \times \beta(a_g, b_g, e) \\ &= \frac{1}{n} \sum_{(a_1,b_1,\ldots,a_g,b_g,e)\in\mathbb{Z}_n} 1 \\ &= n^{2g} \qquad (7.1) \end{aligned}$$

7.3. $G = S_3$. In the Annex B, we give a way of building 3-cocycle, and this construction leads to the 3-cocycle given in [15] and [8] for $\mathbb{Z}_n$. And for S_3 we have $\alpha(x,y,z) = exp(\frac{2i\pi}{4}tr(s(x))(tr(s(y)s(z)s(yz)^{-1})))$. We denote $\mathcal{Z}(x) = \{g \in S_3 \mid gxg^{-1} = x\}$ the center of x we have :

$$\begin{aligned}
TV_{S_3,\alpha}(S^1\times S^1\times S^1) &= \frac{1}{6}\sum_{g,h,k\in S_3,[g,h]=[g,k]=[h,k]=1}\frac{\alpha(g,h,k)\alpha(h,k,g)\alpha(k,g,h)}{\alpha(g,k,h)\alpha(h,g,k)\alpha(k,h,g)}\\
&= \frac{1}{6}(\sum_{h\in S_3}\sharp Z(h) + \sum_{h\neq 1}(\sharp Z(h))^2)\\
&= \frac{1}{\sharp S_3}\sum_{g,h,k\in S_3,[g,h]=[g,k]=[h,k]=1} 1\\
&= \frac{\sharp Col(T_0)}{\sharp S_3}\\
&= \sharp\frac{Hom(\Pi_1(S^1\times S^1\times S^1),S_3)}{conj}
\end{aligned}$$

In fact, for every 3-cocycle $\alpha \in Z^3(S_3,k^\star)$ and thanks to (7.1) :

$$\begin{aligned}
\beta(g,g,k) &= \frac{\alpha(g,g,k)\alpha(g,k,g)\alpha(k,g,g)}{\alpha(g,k,g)\alpha(g,g,k)\alpha(k,g,g)}\\
&= 1\\
\beta(g,h,h) &= \frac{\alpha(g,h,h)\alpha(h,h,g)\alpha(h,g,h)}{\alpha(g,h,h)\alpha(h,g,h)\alpha(h,h,g)}\\
&= 1\\
\beta(g,h,g) &= \frac{\alpha(g,h,g)\alpha(h,g,g)\alpha(g,g,h)}{\alpha(g,g,h)\alpha(h,g,g)\alpha(g,h,g)}\\
&= 1
\end{aligned}$$

Moreover if α is normalized then β becomes normalized. Thus for every $\alpha \in Z^3(G,k^\star)$:

$$\begin{aligned}
TV_{S_3}(S^1\times S^1\times S^1) &= \frac{1}{6}\sum_{g,h,k\in S_3,[g,h]=[g,k]=[h,k]=1}\beta(g,h,k)\\
&= \frac{1}{6}\sum_{g\in S_3}\sum_{h\in\mathcal{Z}(g)}\sum_{k\in Z(g)\cap Z(h)}\beta(g,h,k)\\
&= \frac{1}{6}(\sum_{h\in S_3}\sum_{k\in Z(h)}\beta(1,h,k) + \sum_{g\neq 1}(\sharp\mathcal{Z}(g))^2)\\
&= \frac{1}{6}(\sum_{h\in S_3}\sharp\mathcal{Z}(h) + \sum_{g\neq 1}(\sharp\mathcal{Z}(g))^2)\\
&= \sharp\frac{Hom(\Pi_1(S^1\times S^1\times S^1),S_3)}{conj}
\end{aligned}$$

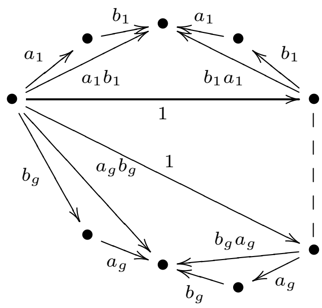
7.4. **Examples of TQFTs.**

7.4.1. $\alpha = 1$. Let Σ a closed and connected surface and T a triangulation of Σ, for all $c \in Col(T)$:

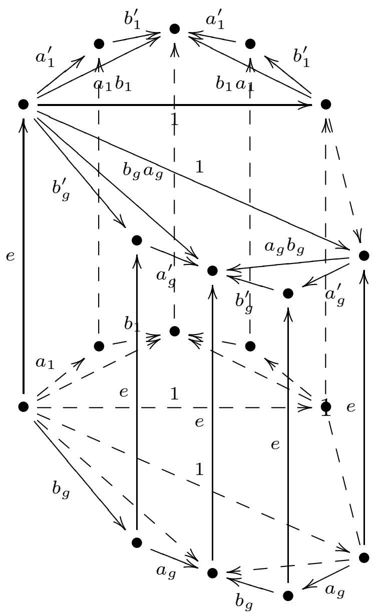
$$\begin{aligned} V(\Sigma \times I, T)(c) &= \sum_{c' \in Col(T)} TV^n((\Sigma, T) \times I)_{c,c'} c' \\ &= \sum_{c' \cong c} \frac{\sharp stab(c)}{\sharp \Lambda_{\mathcal{C}}^{n_0}} c' \\ &= \frac{1}{\sharp \mathcal{O}_c} \sum_{c' \cong c} c' \end{aligned}$$

Thus : $\mathcal{V}(\Sigma) = \frac{k[Col(T)]}{\cong} \simeq \frac{Hom(\Pi_1(\Sigma), \Lambda_{\mathcal{C}})}{conj}$.

7.4.2. *$G = \mathbb{Z}_n$, and α is given by (1).* The vector space $\mathcal{V}(\Sigma)$ doesn't depend on the choice of the triangulation. Thus we can consider the following triangulation :



where $(a_1, b_1, ..., a_g, b_g)$ is an admissible coloring. Thus for the cylinder $\Sigma_g \times I$ we have the following triangulation :



where $(a_1, b_1, ..., a_g, b_g)$ is an admissible coloring of the inward surface, $(a_1', b_1', ..., a_g', b_g')$ is an admissible coloring of the outward surface and e is a gauge. In this case we have :

$$\begin{aligned} V_{\mathbb{Z}_n,\alpha}(\Sigma_g \times I, T)(c) &= \sum_{c' \in Col(T)} TV_i(\Sigma_g \times I)_{c,c'} c' \\ &= \sum_{c' \cong c} n^{-1} n c' \\ &= c \end{aligned}$$

Thus $\mathcal{V}(\Sigma_g) = k[Col(T)] = k[\Lambda_{\mathcal{C}}^{2g}]$

Appendix A. Some computation of $H^3(g, k^\star)$

Let G a finite group and G' a subgroup such that :

$$G' \xrightarrow{p} G \to \{1\}$$

and s is a section of p. Let A an abelian group and G acts trivially on A, then for every application : $< \ \ >$: $G' \times G' \to A$ which verify :

(i) $<$xy,z$>$=$<$x,z$>$+$<$y,z$>$
(ii) $<$x,yz$>$=$<$x,y$>$+$<$x,z$>$

we can define the following application :

$$\alpha : G \times G \times G \to A$$
$$(x, y, z) \mapsto \alpha(x, y, z) =< s(x), s(y)s(z)s(yz)^{-1} > \quad \text{(A.1)}$$

Proposition A.1. *If $< ker(p), ker(p) >= 0$ then $\alpha \in Z^3(G, A)$.*

Proof : For all $x, y, z, t \in G$ we have :

$$\delta(\alpha)(x, y, z, t) = \alpha(y, z, t) - \alpha(xy, z, t) + \alpha(x, yz, t) - \alpha(x, y, zt) + \alpha(x, y, z)$$
$$=< s(y), s(z)s(t)s(zt)^{-1} > - < s(xy), s(z)s(t)s(zt)^{-1} > + < s(x), s(yz)s(t)s(yzt)^{-1} >$$
$$- < s(x), s(y)s(zt)s(yzt)^{-1} > + < s(x), s(y)s(z)s(yz)^{-1} >$$
$$=< s(y)s(xy)^{-1}, s(z)s(t)s(zt)^{-1} > + < s(x), s(z) > + < s(x), s(t) > - < s(x), s(zt) >$$
$$=< s(y), s(z)s(t)s(zt)^{-1} > + < s(x), s(z)s(t)s(zt)^{-1} >$$
$$=< s(x)s(y)s(xy)^{-1}, s(z)s(t)s(zt)^{-1} >$$
$$= 0 \quad \square$$

A.1. **Example.**

A.1.1. $G = \mathbb{Z}_n$. If we consider the following application :

$$\mathbb{Z} \to \mathbb{Z}_n$$
$$x \mapsto \overline{x}$$

$s : \mathbb{Z}_n \to$ which assigns for all $\overline{x} \in \mathbb{Z}_n$ its representative element in $\{0, ..., n-1\}$ is a section of p. We define :

$$< \ \ > \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$$
$$(x, y) \mapsto exp(\frac{2i\pi}{n^2} s(x)s(y)),$$

$< \quad >$ verifies (i) and (ii) and if $x \in ker(p)$ then $s(x) = 0$ thus if x and y are elements of $ker(p)$ then $< x, y >= 0$. It defines a 3-cocycle :

$$\alpha(x, y, z) = exp(\frac{2i\pi}{n^2} s(x)(s(y) + s(z) - s(y + z))), \tag{A.2}$$

we recover the 3-cocycle defines in [15].

A.1.2. $G = S_n$.

$$\begin{aligned} p \,:\, B_n &\to S_n \\ \tilde{\sigma} &\mapsto \sigma \end{aligned}$$

and the section s which assigns for all permutation in S_n an elementary braid

$$\begin{aligned} < \quad > \,:\, B_n \times B_n &\to \mathbb{C} \\ (\sigma, \tau) &\mapsto exp(\frac{2i\pi}{4} tr(x)tr(y)) \end{aligned}$$

with $tr(x) = \sharp$(positive crossing of x) $- \sharp$(negative crossing of x).

Lemma A.2. (1) *If $x \in ker(p)$ then $tr(x)$ is an even number.*

(2) $tr(xy) = tr(x) + tr(y)$.

Proof : We recall that $ker(p) = P_n$ the pure braids group. The first assertion is then a consequence of the presentation of P_n due to Markov (see [13]).
The last assertion is a consequence of the braid relations which preserve the number of signed crossings.
Thus $< \quad >$ defines a 3-cocycle α on S_n. □

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