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Formal verification in Coq of program properties involving the global state effect

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Abstract

The syntax of an imperative language does not mention explicitly the state, while its denotational semantics has to mention it. In this paper we present a framework for the verification in Coq of properties of programs manipulating the global state effect. These properties are expressed in a proof system which is close to the syntax, as in effect systems, in the sense that the state does not appear explicitly in the type of expressions which manipulate it. Rather, the state appears via decorations added to terms and to equations. In this system, proofs of programs thus present two aspects: properties can be verified up to effects or the effects can be taken into account. The design of our Coq library consequently reflects these two aspects: our framework is centered around the construction of two inductive and dependent types, one for terms up to effects and one for the manipulation of decorations.

1 Introduction

The evolution of the state of the memory in an imperative program is a computational effect: the state is never mentioned as an argument or a result of a command, whereas in general it is used and modified during the execution of commands. Thus, the syntax of an imperative language does not mention explicitly the state, while its denotational semantics has to mention it. This means that the state is encapsulated: its interface, which is made of the functions for looking up and updating the values of the locations, is separated from its implementation; the state cannot be accessed in any other way than through its interface.

It turns out that equational proofs in an imperative language may also encapsulate the state: proofs can be performed without any knowledge of the implementation of the state. This is made possible by adding decorations to terms, as in effect systems [8, 13], or by adding decorations to both the terms and the equations [3]. The latter approach uses categorical constructions to model the denotational semantics of the state effect and prove some properties of programs involving this effect. Strong monads, introduced by Moggi [9], were the first categorical approach to computational effects, while Power et al. [11] then proposed the premonoidal categories. Next Hughes [7] extended Haskell with arrows that share some properties with the approach of cartesian effect categories of Dumas et al. [5].

The goal of this paper is to propose a Coq environment where proofs, written in the latter decorated framework for the state effect, can be mechanised.

Proving properties of programs involving the state effect is important when the order of evaluation of the arguments is not specified or more generally when parallelization comes into play [8]. Indeed, pure computations, i.e. those not having any side-effects (or in other words not modifying the state), are independent and could thus be run in parallel. Differently, computations depending on or modifying the state should be handled with more care.

Now, proofs involving side-effects can become quite complex in order to be fully rigorous. We will for instance look at the following property in details: recovering the value of a variable and setting up the value of another variable can be performed in any order. Such properties have been formalized for instance by Plotkin et al. [10] but the full mathematical proof of such properties can be quite large. The decorated approach of [3] helps since it enables a

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verification of such proofs in two steps: a first step checks the syntax up to effects by dropping the decorations; a second step then takes the effects into account.

To some extent, our work looks quite similar to [2] in the sense that we also define our own programming language and verify its properties by using axiomatic semantics. We construct our system on categorical notions (e.g. monads) as done in [1]. In brief, we first declare our system components including their properties and then prove some related propositions. In that manner, the overall idea is also quite close to [12], even though technical details completely differ.

In this paper, we show that the decorated proof system can be developed in Coq thus enabling a mechanised verification of decorated proofs for side-effect systems. We recall in Section 2 the logical environment for decorated equational proofs involving the state effect. Then in Section 3 we present the translation of the categorical rules into Coq as well as their resulting derivations and the necessary additions. The resulting Coq code has been integrated into a library, available there: http://coqeffects.forge.imag.fr. Finally, in Section 4 we give the full details of the proof of the property above and its verification in Coq, as an example of the capabilities of our library. Appendix A is added for the sake of completeness and readability in order to give the logical counterparts of the rules verified in our Coq library.

2 The Logical Environment for Equational Proofs

2.1 Motivation

Basically, in a purely functional programming language, an operation or a term \( f \) with an argument of type \( X \) and a result of type \( Y \), which may be written \( f : X \rightarrow Y \) (in the syntax), is interpreted (in the denotational semantics) as a function \( [f] \) between the sets \([X]\) and \([Y]\), interpretations of \( X \) and \( Y \). It follows that, when an operation has several arguments, these arguments can be evaluated in parallel, or in any order. It is possible to interpret a purely functional programming language via a categorical semantics based on cartesian closed categories; the word “cartesian” here refers to the categorical products, which are interpreted as cartesian products of sets, and which are used for dealing with pairs (or tuples) of arguments. The logical semantics of the language defines a set of rules that may be used for proving properties of programs.

But non-functional programming languages such as C or Java do include computational effects. For instance a C function may modify the state structure and a Java function may throw an exception during the computation. Such operations are examples of computational effects. In this paper we focus on the states effect. We consider the lookup and update operations for modeling the behavior of imperative programs: namely an update operation assigns a value to a location (or variable) and a lookup operation recovers the value of a location. There are many ways to handle computational effects in programming languages. Here we focus on the categorical treatment of [5], adapted to the state effect [3]: this provides a logical semantics relying on decorations, or annotations, of terms and equations.

2.2 Decorated functions and equations for the states effect

The functions in our language are classified according to the way they interact with the state. The classification takes the form of annotations, or decorations, written as superscripts. A function can be a modifier, an accessor or a pure function.

- As the name suggests, a modifier may modify or use the state: it is a read-write function. We will use the keyword \( rw \) as an annotation for modifiers.
- An accessor may use the state structure but never modifies it: it is a read-only function. We will use the keyword \( ro \) for accessors.
- A pure function never interacts with the state. We will use the keyword pure for pure functions.

The denotational semantics of this language is given in terms of the set of states \( S \) and the cartesian product operator ‘\( \times \)’. For all types \( X \) and \( Y \), interpreted as sets \([X]\) and \([Y]\), a modifier function \( f : X \rightarrow Y \) is interpreted as a function \([f] : [X] \times S \rightarrow [Y] \times S\) (it can access the state and modify it); an accessor \( g \) as \([g] : [X] \times S \rightarrow [Y]\) (it can access the state but not modify it); and a pure function \( h \) as \([h] : [X] \rightarrow [Y]\) (it can neither access nor modify the state). There is a
hierarchy among those functions. Indeed any pure function can be seen as both an accessor or a modifier even though it will actually do not make use of its argument $S$. Similarly an accessor can be seen as a modifier.

The state is made of memory locations, or variables; each location has a value which can be updated. For each location $i$, let $V_i$ be the type of the values that can be stored in the location $i$, and let $\text{Val}_i = [V_i]$ be the interpretation of $V_i$. In addition, the unit type is denoted by $\mathbb{1}$; its interpretation is a singleton, it will also be denoted by $\mathbb{1}$.

The assignment of a value of type $V_i$ to a variable $i$ takes an argument of type $V_i$. It does not return any result but it modifies the state: given a value $a \in \text{Val}_i$, the assignment of $a$ to $i$ sets the value of location $i$ to $a$ and keeps the value of the other locations unchanged. Thus, this operation is a modifier from $V_i$ to $\mathbb{1}$. It is denoted by $\text{update}^m_i : V_i \rightarrow \mathbb{1}$ and it is interpreted as $\llbracket \text{update}_i \rrbracket : \text{Val}_i \times S \rightarrow S$.

The recovery of the value stored in a location $i$ takes no argument an returns a value of type $V_i$. It does not modify the state but it observes the value stored at location $i$. Thus, this operation is an accessor from $\mathbb{1}$ to $V_i$. It is denoted by $\text{lookup}^a_i : \mathbb{1} \rightarrow V_i$ and it is interpreted (since $\mathbb{1} \times S$ is in bijection with $S$) as $\llbracket \text{lookup}_i \rrbracket : S \rightarrow \text{Val}_i$.

For each type $X$, the identity operation $id_X : X \rightarrow X$, which is interpreted by mapping each element of $\llbracket X \rrbracket$ to itself, is pure.

Similarly, the final operation $\langle \rangle_X : X \rightarrow \mathbb{1}$, which is interpreted by mapping each element of $\llbracket X \rrbracket$ to the unique element of the singleton $\mathbb{1}$, is pure. In order to lighten the notations we will often use $id_i$ and $\langle \rangle_i$ instead of respectively $id_{\llbracket i \rrbracket}$ and $\langle \rangle_{\llbracket i \rrbracket}$.

In addition, decorations are also added to equations.

- Two functions $f, g : X \rightarrow Y$ are strongly equal if they return the same result and have the same effect on the state structure. This is denoted $f == g$.

- Two functions $f, g : X \rightarrow Y$ are weakly equal if they return the same result but may have different effects on the state. This is denoted $f \sim g$.

The state can be observed thanks to the lookup functions. For each location $i$, the interpretation of the $\text{update}_i$ operation is characterized by the following equalities, for each state $s \in S$ and each $x \in \text{Val}_i$:

\[
\begin{align*}
\llbracket \text{lookup}_i \rrbracket (\llbracket \text{update}_i \rrbracket (s,x)) &= x \\
\llbracket \text{lookup}_j \rrbracket (\llbracket \text{update}_i \rrbracket (s,x)) &= \llbracket \text{lookup}_j \rrbracket (s) \text{ for every } j \in \text{Loc}, \ j \neq i
\end{align*}
\]

According to the previous definitions, these equalities are the interpretations of the following weak equations:

\[
\begin{align*}
\text{lookup}^a_i \circ \text{update}^m_i &\sim id^p_i : V_i \rightarrow V_i \\
\text{lookup}^a_i \circ \text{update}^m_i &\sim \text{lookup}^a_j \circ \langle \rangle^p_i \text{ for every } j \in \text{Loc}, \ j \neq i : V_i \rightarrow V_j
\end{align*}
\]

### 2.3 Sequential products

In functional programming, the product of functions allows to model operations with several arguments. But when side-effects occur (typically, updates of the state), the result of evaluating the arguments may depend on the order in which they are evaluated. Therefore, we use sequential products of functions, as introduced in [5], which impose some order of evaluation of the arguments: a sequential product is obtained as the sequential composition of two semi-pure products. A semi-pure product, as far as we are concerned in this paper, is a kind of product of an identity function (which is pure) with another function (which may be any modifier).

For each types $X$ and $Y$, we introduce a product type $X \times Y$ with projections $\pi^p_{1,X,Y} : X_1 \times X_2 \rightarrow X$ and $\pi^p_{2,X,Y} : X_1 \times X_2 \rightarrow X_2$, which will be denoted simply by $\pi^p_1$ and $\pi^p_2$. This is interpreted as the cartesian product with its projections. Pairs and products of pure functions are built as usual. In the special case of a product with the unit type, it can easily be proved, as usual, that $\pi^p_1 : X \times \mathbb{1} \rightarrow X$ is invertible with inverse the pair $(\pi^{-1}_1)\text{pure} = (id^p_X, \langle \rangle^p_X) : X \rightarrow X \times \mathbb{1}$, and that $\pi^p_2 = \langle \rangle^p_X : X \times \mathbb{1} \rightarrow \mathbb{1}$. The permutation operation $\text{perm}_{X \times Y} : X \times Y \rightarrow Y \times X$ is also pure: it is interpreted as the function which exchanges its two arguments.

Given a pure function $f^p_1 : X \rightarrow Y_1$, interpreted as $[f_1] : [X] \rightarrow [Y_1]$, and a modifier $f^m_2 : X \rightarrow Y_2$ with its interpretation $[f_2] : [X] \times S \rightarrow [Y_2] \times S$, the left semi-pure pair $\langle f_1, f_2 \rangle^m : X \rightarrow Y_1 \times Y_2$ is the modifier interpreted by
The left semi-pure pair \( (f_1, f_2)^{\text{pw}} \) is characterized, up to strong equations, by a weak and a strong equation:

\[
P_1^\text{pure} \circ (f_1, f_2)^{\text{rw}} \sim f_1^{\text{pure}} \quad \text{and} \quad P_2^\text{pure} \circ (f_1, f_2)^{\text{rw}} \equiv f_2^{\text{rw}}
\]

The right semi-pure pair \( (f_1, f_2)^{\text{rw}} : X \to Y_1 \times Y_2 \) where \( f_1^{\text{rw}} : X \to Y_1 \) and \( f_2^{\text{pure}} : X \to Y_2 \) is defined in the symmetric way:

\[
\begin{array}{c}
\begin{array}{c}
\text{Note.} \quad \text{In all diagrams, the decorations are expressed by shapes of arrows: waving arrows for pure functions, dotted arrows for accessors and straight arrows for modifiers.}
\end{array}
\end{array}
\]

The left semi-pure product is defined in the usual way from the left semi-pure pair: given \( f_1^{\text{pw}} : X_1 \to Y_1 \) and \( f_2^{\text{pw}} : X_2 \to Y_2 \), the left semi-pure product of \( f_1 \) and \( f_2 \) is \( (f_1 \times f_2)^{\text{pw}} = (f_1 \circ P_1_{X_1,X_2}, f_2 \circ P_2_{X_1,X_2})^{\text{pw}} : X_1 \times X_2 \to Y_1 \times Y_2 \). It is characterized, up to strong equations, by a weak and a strong equation:

\[
P_1^\text{pure} \circ (f_1 \times f_2)^{\text{pw}} \sim f_1^{\text{pure}} \circ (f_1 \times f_2)^{\text{pw}} \quad \text{and} \quad P_2^\text{pure} \circ (f_1 \times f_2)^{\text{pw}} \equiv f_2^{\text{pure}} \circ (f_1 \times f_2)^{\text{pw}}
\]

The right semi-pure product \( (f_1 \times f_2)^{\text{rw}} : X_1 \times X_2 \to Y_1 \times Y_2 \) is defined in the symmetric way:

\[
\begin{array}{c}
\begin{array}{c}
\text{Now, it is easy to define the left sequential product of two modifiers \( f_1^{\text{rw}} : X_1 \to Y_1 \) and \( f_2^{\text{rw}} : X_2 \to Y_2 \) by composing a right semi-pure product with a left semi-pure one and using \text{id} function as the pure component:}
\end{array}
\end{array}
\]

\[
(f_1 \times f_2)^{\text{rw}} = (\text{id}_{Y_1} \times f_2)^{\text{rw}} \circ (f_1 \times \text{id}_{X_2})^{\text{rw}} : X_1 \times X_2 \to Y_1 \times Y_2
\]

In a symmetric way, the right sequential product of \( f_1^{\text{rw}} : X_1 \to Y_1 \) and \( f_2^{\text{rw}} : X_2 \to Y_2 \) is defined as:

\[
(f_1 \times f_2)^{\text{rw}} = (f_1 \times \text{id}_{X_2})^{\text{rw}} \circ (\text{id}_{Y_1} \times f_2)^{\text{rw}} : X_1 \times X_2 \to Y_1 \times Y_2
\]

The left sequential product models the fact of executing \( f_1 \) before \( f_2 \), while the right sequential product models the fact of executing \( f_2 \) before \( f_1 \); in general they return different results and they modify the state in a different way.
2.4 A property of states

In [10] an equational presentation of states is given, with seven equations. These equations are expressed as decorated equations in [3]. They are the archetype of the properties of the proofs we want to verify. For instance, the fact that modifying a location \(i\) and observing the value of another location \(j\) can be done in any order is called the commutation update-lookup property. This property can be expressed as an equation relating the functions \([\text{update}_i]\) and \([\text{lookup}_j]\). For this purpose, let \([\text{lookup}_j]' : S \times \text{Val}_j \times S\) be defined by

\[
[\text{lookup}_j]'(s) = (s, [\text{lookup}_j](s))
\]

Thus, given a state \(s\) and a value \(a \in \text{Val}_i\), assigning \(a\) to \(i\) and then observing the value of \(j\) is performed by the function:

\[
[\text{lookup}_j]' \circ [\text{update}_i] : \text{Val}_i \times S \to \text{Val}_j \times S
\]

Observing the value of \(j\) and then assigning \(a\) to \(i\) also corresponds to a function from \(\text{Val}_i \times S\) to \(\text{Val}_j \times S\) built from \([\text{update}_i]\) and \([\text{lookup}_j]\)'. This function first performs \([\text{lookup}_j]'\)(s) while keeping \(a\) unchanged, then it performs \([\text{update}_i]((s,a)\) while keeping \(b\) unchanged (where \(b\) denotes the value of \(j\) in \(s\) which has been returned by \([\text{lookup}_j]'(s))\). The first step is \(\text{id}_{\text{Val}_i} \times [\text{lookup}_j]' : \text{Val}_i \times S \to \text{Val}_i \times (\text{Val}_j \times S)\) and the second step is \(\text{id}_{\text{Val}_j} \times [\text{update}_i] : \text{Val}_j \times (\text{Val}_i \times S) \to \text{Val}_j \times S\). An intermediate permutation step is required, it is called \(\text{perm}_{i,j} : \text{Val}_i \times (\text{Val}_j \times S) \to \text{Val}_j \times (\text{Val}_i \times S)\) such that \(\text{perm}_{i,j}(a, (b,s)) = (b, (a,s))\).

Altogether, observing the value of \(j\) and then assigning \(a\) to \(i\) corresponds to the function:

\[
(\text{id}_{\text{Val}_i} \times [\text{update}_i]) \circ \text{perm}_{i,j} \circ (\text{id}_{\text{Val}_i} \times [\text{lookup}_j]'') : \text{Val}_i \times S \to \text{Val}_j \times S
\]

Thus, the commutation update-lookup property means that:

\[
[\text{lookup}_j]' \circ [\text{update}_i] = (\text{id}_{\text{Val}_i} \times [\text{update}_i]) \circ \text{perm}_{i,j} \circ (\text{id}_{\text{Val}_i} \times [\text{lookup}_j]''')
\]

According to Section 2.2, this is the interpretation of the following strong equation, which can also be expressed as a diagram:

\[
\text{lookup}_j'' \circ \text{update}_i'' = \pi_2^\text{pure} \circ (\text{update}_i'' \times \text{id}_j^\text{pure}) \circ (\text{id}_i^\text{pure} \times \text{lookup}_j'') \circ (\text{π}_i^{-1})^\text{pure} : V_i \to V_j
\]

\[
\begin{array}{c}
\text{V}_i \xrightarrow{\text{update}_i} \text{V}_j \\
\xrightarrow{\text{lookup}_j} \text{V}_j \\
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{\text{id}_{\text{Val}_i} \times [\text{update}_i]} \\
\xrightarrow{\text{perm}_{i,j}} \\
\xrightarrow{\text{id}_{\text{Val}_j} \times [\text{lookup}_j]'''} \\
\text{V}_i \times \text{V}_j \xrightarrow{\text{update}_i \times \text{lookup}_j''} \text{V}_j \\
\end{array}
\]

Remark. Using the right sequential product, the right hand-side of the commutation update-lookup equation can be written as \(\pi_2^\text{pure} \circ (\text{update}_i'' \times \text{lookup}_j'') \circ (\pi_i^{-1})^\text{pure} \). In addition, using the left sequential product, it is easy to check that the left hand-side of this equation can be written as \(\pi_2^\text{pure} \circ (\text{update}_i'' \times \text{lookup}_j'') \circ (\pi_i^{-1})^\text{pure} \). Since \(\pi_1^\text{pure} : V_i \times 1 \to V_i\) and \(\pi_2^\text{pure} : 1 \times V_j \to V_j\) are invertible, we get a symmetric expression for the equation which corresponds nicely to the description of the commutation update-lookup property as “the fact that modifying a location \(i\) and observing the value of another location \(j\) can be done in any order”:

\[
\text{update}_i'' \times \text{lookup}_j'' = \text{update}_i'' \times \text{lookup}_j''
\]

3 The Environment in Coq

In this Section we present the core of this paper, namely the implementation in the Coq proof assistant of the rules for reasoning with decorated operations and equations and the proof of the update-lookup commutation property using these rules.

In the preceding section, we have shown proofs of propositions involving effects. We now present the construction of a Coq framework enabling one to formalize such proofs. This framework has been released as STATES-0.5 library and is available in the following web-site: \url{http://coqeffects.forge.imag.fr}. 

5
In order to construct this framework, we need to define data structures, terms, decorations and basic rules as axioms. Those give rise to derived rules and finally to proofs. This organization is reflected in the library with corresponding Coq modules, as shown in the following diagram:

<table>
<thead>
<tr>
<th>BASES:</th>
<th>Memory → Terms → Decorations → Axioms</th>
</tr>
</thead>
<tbody>
<tr>
<td>DERIVED:</td>
<td>D.Terms → D.Pairs → D.Products → D.Rules</td>
</tr>
<tr>
<td>PROOFS:</td>
<td>Proofs</td>
</tr>
</tbody>
</table>

The memory module uses declarations of *locations*. A *location* represents a field on the memory to store and observe data. Then terms are defined in steps. First we give the definitions of *non-decorated terms*: they constitute the main part of the design with the inclusion of all the required functions. For instance, the `lookup` function which observes the current state is defined from `void` (1, the terminal object of the underlying category) to the set of values that could be stored in that specified location.

The next step is to decorate those functions with respect to their manipulation abilities on the state structure. For instance, the `update` function is defined as a *modifier*. The *modifier* status is represented by a `rw` label in the library. All the rules related to decorated functions are stated in the module called `Axioms`.

Then, based on the ones already defined, some other terms are derived. For example, the derived `perm` function takes projections as the basis and replaces the orders of input objects in a *categorical product*. Similarly, by using the already defined rules (given in the `axioms` section), some additional rules are derived concerning *categorical pairs*, *products* and *others* pointing the rules constructed over the ones from different sources.

In the following subsection we detail the system sub-modules. The order of enumeration gives the dependency among sub-modules as shown in the above diagram.

For instance, the module `decorations` requires definitions from the `memory` and the `terms` modules. Then, as an example, we give the full proof, in Coq, of the update-lookup commutation property of [10].

### 3.1 Proof System for States

In this section we give the Coq definitions of our proof system and explore them module by module.

The major ideas in the construction of this Coq framework are:

- All the features of the proof system, that are given in the previous Sections 2.2, 2.3, 2.4 and in the appendix A, definitely constitute the basis for the Coq implementation. In brief, we first declare all the terms without decorations, then we decorate them and after all we end up with the rules involving decorated terms. We also confirm that if one removes all the decorations (hence transforming every operations into pure terms), the proof system remains valid.

- The terms `pair` and `perm_pair` express the construction of pairs of two functions. As shown in Section 2.3, in the presence of effects the order of evaluation matters. Therefore, we first define the left pair in Coq and simply call it `pair` (see Section 3.3). Then, the right pair can be derived using the permutation rule and it is called `perm_pair` (see Section 3.6).

- The most challenging part of the design is the proof implementations of the propositions by [10], since they are quite tricky and long. We assert implementations tricky, because to see the main schema (or flowchart) of the proofs at first sight and coding them in Coq with this reasoning is quite difficult. To do so, we first sketch the related diagrams with marked equalities (strong or weak), then we convert them into some line equations, representing the main propositions to be shown. In order to do so, we use a fractional notation together with the exploited rules for each step. Eventually, Coq implementations are done by coding each step which took part in the fractional notation. From this aspect, without the fractional correspondences, proofs might be seen a little tough to follow. In order to increase the readability score, we divided those implementations into sub-steps and gave the associated relevant explanations. See Section 4 for an example.
• Considering the entire design, we benefit from an important aspect provided by Coq environment, namely dependent types. They provide a unified formalism in the creation of new data types and allow us to deal in a simple manner with most of the typing issues. More precisely, the type term is not a Type, but rather a Type → Type → Type. The domain/codomain information of term is embedded into Coq type system, so that we do not need to talk about ill-typed terms. For instance, \( \text{pi1} \circ \text{final} \) is ill-typed since final is defined from any object \( X : \text{Type} \) to unit where \( \text{pi1} \) if from \( Y : \text{Type} \) to \( Z : \text{Type} \). Therefore, the latter composition cannot be seen as a term.

3.2 Memory

We represent the set of memory locations by a Coq parameter \( \text{Loc} : \text{Type} \). Since memory locations may contain different types of values, we also assume a function \( \text{Val} : \text{Loc} \to \text{Type} \) that indicates the type of values contained in each location.

3.3 Terms

Non-decorated operators, using the monadic equational logic and categorical products, are represented by an inductive Coq data type named term. It basically gets two Coq types, that are corresponding either to objects or to mappings in the given categorical structure, and returns a function type. Those function types are the representations of the homomorphisms of the category. We summarize these non-decorated constructions below:

\[
\text{Inductive} \quad \text{term} : \text{Type} \to \text{Type} \to \text{Type} := \\
| \text{id} : \forall \{X : \text{Type}\}, \text{term} X X \\
| \text{comp} : \forall \{X Y Z : \text{Type}\}, \text{term} X Y \to \text{term} Y Z \to \text{term} X Z \\
| \text{final} : \forall \{X : \text{Type}\}, \text{term} \text{unit} X \\
| \text{pair} : \forall \{X Y Z : \text{Type}\}, \text{term} X Z \to \text{term} Y Z \to \text{term} (X \times Y) Z \\
| \text{pi1} : \forall \{X Y : \text{Type}\}, \text{term} X (X \times Y) \\
| \text{pi2} : \forall \{X Y : \text{Type}\}, \text{term} Y (X \times Y) \\
| \text{lookup} : \forall i : \text{Loc}, \text{term} (\text{Val} i) \text{unit} \\
| \text{update} : \forall i : \text{Loc}, \text{term} \text{unit} (\text{Val} i).
\]

Infix "\( \circ \)" := \( \text{comp} \) (at level 70).

Note that a term of type \( \text{term} X Y \) is interpreted as a function from the set \( Y \) to the set \( X \) (the co-domain, \( X \), is given first.)

The constructor id denotes the identity function: for any type \( X \), id \( X \) has type \( \text{term} X X \). The term comp composes two given compatible function types and returns another one. The term pair represents the categorical product type of two given objects. For instance, if \( \text{term} X Z \) corresponds to a mapping defined from an object \( Z \) to another one denoted as \( X \), then \( \text{pair} \) with input types \( \text{term} X Z \) and \( \text{term} Y Z \), agreeing on domains, returns a new function type of form \( \text{term} (X \times Y) Z \). The terms pi1 and pi2 are projections of products while final maps any object to the terminal object (the singleton set, denoted by 1) of the Cartesian effect category in question. lookup takes a location identifier and denotes the lookup operation for the relevant location. It is mathematically defined from the terminal object of the category. As the name suggests, the update operator updates the value in the specified location.

3.4 Decorations

In order to keep the semantics of state close to syntax, all the operations are decorated with respect to their manipulation abilities on the state structure. In Coq, we define another inductive data type, called kind, to represent these decorations. Its constructors are pure (decorated by pure), ro (for read-only and decoration ro), and rw (for read-write and decoration rw). It should be recalled that if a function is pure, then it could be seen both as ro (accessor) and rw (modifier), due to the hierarchy rule among decorated functions:

\[
\text{Inductive} \quad \text{kind} := \text{pure} | \text{ro} | \text{rw}.
\]
In Coq, we had to define the decorations of terms via the separate inductive data type called is. The latter takes a term and a kind and returns a Prop. In other words, is indicates whether the given term is allowed to be decorated by the given kind or not. For instance, the term id is pure, since it cannot use nor modify the state. Therefore it is by definition decorated with the keyword pure. This decoration is checked by a constructor is_id. To illustrate this, if one (by using apply tactic of Coq) asks whether id is pure, then the returned result would be have to be True. In order to check whether id is an accessor or a modifier, the constructors is_pure_ro and is_ro_rw should be applied beforehand to convert both statements into is pure id. The incidence of decorations upon the terms is summarized below together with their related rules (detailed in Appendix A):

\[
\text{Inductive is: kind } \rightarrow \forall X Y, \text{term} X Y \rightarrow \text{Prop} := \\
| \text{id: } \forall X, \text{is pure (@id X)} \\
| \text{comp: } \forall k X Y Z (f; \text{term} X Y) (g; \text{term} Y Z), \text{is k f } \rightarrow \text{is k g } \rightarrow \text{is k (f o g)} \\
| \text{final: } \forall X, \text{is pure (@final X)} \\
| \text{pair: } \forall k X Y Z (f; \text{term} X Z) (g; \text{term} Y Z), \text{is k f } \rightarrow \text{is k g } \rightarrow \text{is k (pair f g)} \\
| \text{pi1: } \forall X Y, \text{is pure (@pi1 X Y)} \\
| \text{pi2: } \forall X Y, \text{is pure (@pi2 X Y)} \\
| \text{lookup: } \forall i, \text{is ro (lookup i)} \\
| \text{update: } \forall i, \text{is rw (update i)} \\
| \text{ro: } \forall X Y (f; \text{term} X Y), \text{is pure f } \rightarrow \text{is ro f} \\
| \text{rw: } \forall X Y (f; \text{term} X Y), \text{is ro f } \rightarrow \text{is rw f}
\]

The decorated functions stated above are classified into four different manners:

- terms specific to states effect: is_lookup and is_update
- categorical terms: is_id, is_comp and is_final
- terms related to categorical products: is_pair, is_pi1 and is_pi2.
- term decoration conversions based on the operation hierarchy: is_pure_ro and is_ro_rw.

The term comp enables one to compose two compatible functions while the constructor is_comp enables one to compose functions and to preserve their common decoration. For instance, if a ro function is composed with another ro, then the composite function becomes ro as well. For the case of the pair, the same idea is used. Indeed, the constructor is_pair takes two terms agreeing on domains such as term Y1 X, say an ro, and term Y2 X, which is ro as well. is_pair then asserts that the pair of these terms is another ro. It is also possible to create both compositions and pairs of functions with different decorations via the hierarchy rule stated among decoration types. This hierarchy is build via the last two constructors, is_pure_ro and is_rp_rw. The constructor is_pure_ro indicates the fact that if a term is pure, then it can be seen as ro. Lastly is_ro_rw states that if a term is ro, then it can be seen as rw as well.

Note that the details of building pairs with different decorations can be found in the derived pairs module (Pairs.v in the library).

The terms final, pi1 and pi2 are all pure functions since they do not manipulate the state. final forgets its input argument(s) and returns nothing. Although this property could make one think that it generates a sort of side-effect, this is actually not the case. Indeed, it is the only pure function whose co-domain is the terminal object (\(\bot\)) and it is therefore used to simulate the execution of a program: successive, possibly incompatible, functions can then be composed with this intermediate forgetfulness of results.

The lookup functions are decorated with the keyword ro, as accessors. The constructor is_lookup is used to check the validity of the lookup’ decoration. The different update functions are rw and decorated with the keyword rw. Similarly, the constructor is_update is thus used to check the validity of the update’ decoration.

### 3.5 Axioms

We can now detail the Coq implementations of the axioms used in the proof constructions. They use the given monadic equational logic and categorical products. The idea is to decorate also the equations. On the one hand,
the weak equality between parallel morphisms models the fact that those morphisms return the same value but may perform different manipulations of the state. On the other hand, if both the returned results and the state manipulations are identical, then the equality becomes strong.

In order to define these decorations of equations in Coq, we again use inductive terms and preserve the naming strategy of Section 2.

Below are given the reserved notations for strong and weak equalities, respectively.

Reserved Notation " x == y " (at level 80). Reserved Notation " x ∼ y " (at level 80).

We have some number of rules stated w. r. t. strong and weak equalities. The ones used in the proof given in Section 4 are detailed below. It is also worth to note that for each constructor of the given inductive types strong and weak, corresponding rules are shown in Appendix A, see Figures 1, 2 and 5.

Inductive strong: ∀ X Y, relation (term X Y) :=

• The rules strong_refl, strong_sym and strong_trans state that strong equality is reflexive, symmetric and transitive, respectively. Obviously, it is an equivalence relation. See (s-refl), (s-sym) and (s-trans) rules in Figure 1.

• Both id_src and id_tgt state that the composition of any arbitrarily selected function with id is itself regardless of the composition order. See (dec-id-sr) and (dec-id-tg) rules in Figure 1.

• strong_subs states that strong equality obeys the substitution rule. That means that for a pair of parallel functions that are strongly equal, the compositions of the same source compatible function with those functions are still strongly equal. strong_repl states that for those parallel and strongly equal function pairs, their compositions with the same target compatible function are still strongly equal. See (s-subs) and (s-repl) rules in Figure 1.

• ro_weak_to_strong is the rule saying that all weakly equal ro terms are also strongly equal. Intuitively, from the given weak equality, they must have the same results. Now, since they are not modifiers, they cannot modify the state. That means that effect equality requirement is also met. Therefore, they are strongly equal. See (ro-w-to-s) rule in Figure 1.

• comp_final_unique ensures that two parallel rw functions (say f and g) are the same (strongly equal) if they return the same result f ∼ g together with the same effect final ◦ f == final ◦ g. See Figure 2.

with weak: ∀ X Y, relation (term X Y) :=

• pure_weak_repl states that weak equality obeys the substitution rule stating that for a pair of parallel functions that are weakly equal, the compositions of those functions with the same target compatible and pure function are still weakly equal. See (pure-w-repl) rule in Figure 1.

• strong_to_weak states that strong equality could be converted into weak one, free of charge. Indeed, the definition of strong equality encapsulates the one for weak equality. See (s-to-w) rule in Figure 1.

• axiom_2 states that first updating a location i and then implementing an observation to another location k is weakly equal to the operation which first forgets the value stored in the location i and observes location k. See (axiom-rw) rule in Figure 5.

Please note that weak equality is an equivalence relation and obeys the substitution rule such as the strong one.

3.6 Derived Terms

Additional to those explained in Section 3.3, some extra terms are derived via the definitions of already existing ones:

Definition inv_pi1 {X Y}: term (X×unit) (X):= pair id unit.
Definition permut {X Y}: term (X×Y) (Y×X):= pair pi2 pi1.
Definition perm_pair {X Y Z} (f : term Y X) (g : term Z X) : term (Y × Z) X
:= permut o pair g f.
Definition prod {X Y X’ Y’} (f : term X X’) (g : term Y Y’) : term (X × Y) (X’ × Y’)
:= pair (f o pi1) (g o pi2).
Definition perm_prod {X Y X’ Y’} (f : term X X’) (g : term Y Y’) : term (X × Y) (X’ × Y’)
:= perm_pair (f o pi1) (g o pi2).

Val_1 and Val_1 × 1 are isomorphic. Indeed, on the one hand, let us form the left semi-pure pair \( h = (\langle id_{Val_1}, \langle \rangle \rangle_{Val_1}) \). As \( \langle \rangle_{Val_1} \) is pure, then so is also \( h \). Now, from the definitions of semi-pure products the projections yields \( \pi_1 \circ h \sim id_{Val_1} \), which is also \( \pi_1 \circ h = id_{Val_1} \) since all the terms are pure. On the other hand, \( \pi_1 \circ (h \circ \pi_1) = id_{Val_1} \circ \pi_1 = \pi_1 \), \( \pi_1 \circ (id_{Val_1} \times 1) \) and \( \pi_1 \circ (h \circ \pi_1) = \langle id_{Val_1} \times 1 \rangle \), but the latter weak equivalences are strong since all the terms are pure. Therefore \( \pi_1 \circ (h \circ \pi_1) = \pi_1 \circ (id_{Val_1} \times 1) \) and \( \pi_2 \circ (h \circ \pi_1) = \pi_2 \circ (id_{Val_1} \times 1) \) so that \( h \circ \pi_1 = id_{Val_1} \times 1 \). Overall we have that \( \pi_1 \) is invertible and \( \pi_1^{-1} = h = (\langle id_{Val_1}, \langle \rangle \rangle_{Val_1})_r \) as defined above.

We also have the \text{perm} \text{ut} \text{erm} \text{ } \text{term}. \text{It } \text{takes } \text{a } \text{product, \text{switches } \text{the } \text{order } \text{of } \text{arguments} \text{involved } \text{in } \text{the } \text{input } \text{product} \text{cone} \text{and } \text{returns } \text{the } \text{new } \text{product: } \text{its } \text{signature is } \text{term} (Y \times X) (X \times Y). \text{The term } \text{perm}\_\text{pair } f g \text{ is } \text{handled via the composition of } \text{pair } g f \text{ with } \text{perm} \text{ut}. \text{The definition } \text{prod} \text{ is } \text{based } \text{on } \text{the } \text{definition } \text{of } \text{pair } \text{with a } \text{difference that both input functions are taking a product object and returning another one while } \text{perm}\_\text{prod} \text{ is } \text{the } \text{permuted version of } \text{prod } \text{which is } \text{built } \text{on } \text{perm}_\text{pairs}. \text{The decorations of } \text{perm}\_\text{pair}, \text{prod} \text{ and } \text{perm}\_\text{prod} \text{, depend } \text{on } \text{the } \text{decorations of } \text{their } \text{input arguments. For instance, a } \text{perm}\_\text{pair} \text{of } \text{two } \text{pure} \text{ functions is also } \text{pure } \text{while the } \text{prod} \text{ and } \text{perm}\_\text{prod} \text{of } \text{two } \text{rw} \text{s is a } \text{rw}. \text{These properties are provided by } \text{is}\_\text{perm}\_\text{pair}, \text{is}_\text{prod} \text{and } \text{is}_\text{perm}\_\text{prod}. \text{More details can be found in the associated module of the library (Derived}_\text{Terms}.v. \text{Note that it is } \text{also } \text{possible to } \text{create } \text{perm}_\text{pairs}, \text{prods} \text{ and } \text{perm}_\text{prods} \text{of } \text{functions with } \text{different decorations } \text{via the } \text{hierarchy rule } \text{stated } \text{among } \text{decoration types (is}_\text{pure}_\text{ro} \text{and } \text{is}_\text{rp}_\text{rw}. \text{Existence proofs } \text{together } \text{with projection } \text{rules, can also } \text{be } \text{found } \text{in } \text{their respective modules in the library (Decorated}_\text{Pairs}.v \text{and Decorated}_\text{Products}.v).

3.7 Decorated Pairs

In this section we present some of the derived rules, related to \text{pairs} and \text{projections}. \text{In Section 2.3 we have defined the left } \text{semi-pure pair } \langle id_X, f \rangle_r^n : X \rightarrow X \times Y \text{ of the identity } id_X^{pure} \text{ with a modifier } f^{rw} : X \rightarrow Y. \text{In Coq this construction will be called simply the } \text{pair of } id_X^{pure} \text{ and } f^{rw}. \text{The right } \text{semi-pure pair } \langle f, id_X \rangle_r^n : X \rightarrow Y \times X \text{ of } f^{rw} \text{ and } id_X^{pure} \text{ can be obtained as } \langle f, id_X \rangle_r^n \text{ followed by the permutation } \text{perm}_X: X \times Y \rightarrow Y \times X, \text{it will be called the } \text{perm}\_\text{pair} \text{of } f^{rw} \text{ and } id_X^{pure}. \text{Then, the } \text{pair} \text{ and } \text{perm}\_\text{pair} \text{ definitions, together with the hierarchy rules among function classes (is}_\text{pure}_\text{ro} \text{and } \text{is}_\text{ro}_\text{rw}), \text{are used to } \text{derive some other rules related to existences and projections.}

- \text{dec}_\text{pair}_\text{exists}_\text{pur}\text{ev} \text{is the rule that ensures that a } \text{pair} \text{with a } \text{rw} \text{and a } \text{pure} \text{arguments also exists and is } \text{rw} \text{too. weak}_\text{proj}_\text{pi1}_\text{pur}\text{ev} \text{is the first projection rule stating that the first result of the pair is equal to the result of its first coefficient function. In our terms it is given as follows: } \text{pi1} \circ \text{pair } f_1 f_2 \sim f_1. \text{The given equality is weak since its left hand side is } \text{rw}, \text{while its right hand side is } \text{pure}. \text{strong}_\text{proj}_\text{pi1}_\text{pur}\text{ev} \text{is the second projection rule of the semi-pure pair. It states that the second result of the pair and its effect are equal to the result and effect of its second coefficient function. In our terms it is given as follows: } \text{pi2} \circ \text{pair } f_1 f_2 == f_2. \text{dec}_\text{perm}_\text{pair}_\text{exists}_\text{rwpure} \text{is similar with pure and modifier inverted.}

- \text{dec}_\text{pair}_\text{exists}_\text{pur}\text{ero} \text{is similar but with one coefficient function } \text{pure} \text{and the other } \text{ro}. \text{Thus it must be an accessor by itself and its projections must be strongly equal to its coefficient functions, since there is no modifiers involved. These properties are stated via } \text{strong}_\text{proj}_\text{pi1}_\text{pur}\text{ero} \text{(pi1} \circ \text{pair } f_1 f_2 == f_1) \text{and } \text{strong}_\text{proj}_\text{pi2}_\text{pur}\text{ero} \text{(pi2} \circ \text{pair } f_1 f_2 == f_2). \text{dec}_\text{perm}_\text{pair}_\text{exists}_\text{rwpure} \text{is similar with pure and accessor inverted.}

More details can be found in the Decorated_Pairs.v source file.
3.8 Decorated Products

Semi-pure products are actually specific types of semi-pure pairs, as explained in Section 2.3. In the same way in Coq, the pair and perm_pair definitions give rise to the prod and perm_prod ones.

- `dec_prod_exists_purerw` ensures that a `prod` with a `pure` and a `rw` arguments exists and is `rw. weak_proj_pi1_purerw_rect` is the first projection rule and states that `pi1 ◦ (prod f g) ~ f ◦ pi1`. `strong_proj_pi2_purerw_rect` is the second projection rule and assures that `pi2 ◦ (prod f g) == f ◦ pi2`. Similarly, the rule `dec_perm_prod_exists_rwp` with projections: `strong_perm_proj_pi1_rwpure_rect(pi1 ◦ (perm_prod f g) == f ◦ pi1)` and `weak_perm_proj_pi2_rwpure_rect(pi2 ◦ (perm_prod f g) ~ g ◦ pi2)` relate to permuted products.

- `dec_prod_exists_purero` ensures that a `prod` with a `pure` and a `ro` arguments exists and is `ro. weak_proj_pi1_purero_rect` is the first projection rule and states that `pi1 ◦ (prod f g) == f ◦ pi1`. `strong_proj_pi2_purero_rect` is the second projection rule and assures that `pi2 ◦ (prod f g) == f ◦ pi2`. Similarly, permutation rule could be applied to get `dec_prod_exists_ropure_rule` with its projections: `strong_perm_proj_pi1_ropure_rect(pi1 ◦ (perm_prod f g) == f ◦ pi1)` and `weak_perm_proj_pi2_ropure_rect(pi2 ◦ (perm_prod f g) ~ g ◦ pi2)`

For further explanation of each derivation with Coq implementation, refer to the `Decorated_Products.v` source file.

3.9 Derived Rules

The library also provides derived rules which can be just simple shortcuts for frequently used combinations of rules or more involved results. For instance:

- `weak_refl` describes the reflexivity property of the weak equality: It is derived from the reflexivity of the strong equality.

Two pure functions having the same codomain `1` must be strongly equal (no result and state unchanged). Therefore `E_0_3` extends this to a composed function `f ◦ g`, for two pure compatible functions `f` and `g`, and another function `h`, provided that `g` and `h` have `1` as codomain.

- In the same manner, `E_1_4` states that the composition of any `ro` function `h: 1 → X`, with `final` is strongly equal to the `id` function on `1`. Indeed, both have no result and do not modify the state.

More similar derived rules can be found in the `Derived_Rules.v` source file.

4 Implementation of a proof: update-lookup commutation

We now have all the ingredients required to prove the update-lookup commutation property of Section 2.4. The order of operations between updating a location `i` and retrieving the value at another location `j` does not matter. The formal statement is given in Equation (1).

The value intended to be stored into the location `i` is an element of `Val_i` set while the lookup operation to the location `j` takes nothing (apart from `j`), and returns a value read from the set `Val_j`. If the order of operations is reversed, then the element of `Val_i` has to be preserved while the other location is examined. Thus we need to form a pair with the identity and create a product `Val_i × 1`, via `inv_pi1`. Similarly, the value recovered by the lookup operation has to be preserved and returned after the update operation. Then a pair with the identity is also created with update and a last projection is used to separate their results. The full Coq proof thus uses the following steps:

1. `assume i, j:Loc`
2. `lookup j ◦ update i == pi2 ◦ (perm_prod (update i) id)`
3. $(\prod \text{id} \text{ (lookup j)}) \circ \text{inv}_{\pi1}$
   \hspace{1cm} by $\text{comp}_{\text{final, unique}}$

4. step 1

5. final $\circ$ lookup $j$ $\circ$ update $i$ $\Rightarrow$ final

6. $\circ \pi2 \circ (\text{perm}_\prod \text{ (update i) id})$
   \hspace{1cm} by $\text{strong_sym}$

7. $(\text{prod id} \text{ (lookup j)}) \circ \text{inv}_{\pi1}$

8. final $\circ \pi2 \circ (\text{perm}_\prod \text{ (update i) id})$

9. $(\text{prod id} \text{ (lookup j)}) \circ \text{inv}_{\pi1}$

10. $\Rightarrow$ final $\circ$ lookup $j$ $\circ$ update $i$
    \hspace{1cm} by $\text{strong_trans}$

substep 1.1

11. final $\circ \pi2 \circ (\text{perm}_\prod \text{ (update i) id})$

12. $(\text{prod id} \text{ (lookup j)}) \circ \text{inv}_{\pi1}$

13. $\Rightarrow$ pi1 $\circ (\text{perm}_\prod \text{ (update i) id})$

14. $(\text{prod id} \text{ (lookup j)}) \circ \text{inv}_{\pi1}$
    \hspace{1cm} by $E_{0, 3}$

substep 1.2

15. pi1 $\circ (\text{perm}_\prod \text{ (update i) id})$

16. $(\text{prod id} \text{ (lookup j)}) \circ \text{inv}_{\pi1}$

17. $\Rightarrow$ update $i$ $\circ$ pi1 $\circ (\text{prod id} \text{ (lookup j)}) \circ \text{inv}_{\pi1}$
    \hspace{1cm} by $\text{strong_perm_proj}$

substep 1.3

18. update $i$ $\circ$ pi1 $\circ (\text{prod id} \text{ (lookup j)}) \circ \text{inv}_{\pi1}$

19. $\Rightarrow$ update $i$ $\circ$ pi1 $\circ$ inv_pi1
    \hspace{1cm} by $\text{strong_proj}$

substep 1.4

20. update $i$ $\circ$ pi1 $\circ$ inv_pi1 $\Rightarrow$ update $i$ $\circ$ id
    \hspace{1cm} by $\text{id_tgt}$

substep 1.5

21. final $\circ$ lookup $j$ $\circ$ update $i$ $\Rightarrow$ update $i$ $\circ$ id
    \hspace{1cm} by $E_{1, 4}$

step 2

22. lookup $j$ $\circ$ update $i$ $\sim$ pi2 $\circ (\text{perm}_\prod \text{ (update i) id})$

23. $(\text{prod id} \text{ (lookup j)}) \circ \text{inv}_{\pi1}$
    \hspace{1cm} by $\text{weak_trans}$

substep 2.1

24. lookup $j$ $\circ$ update $i$ $\sim$ lookup $j$ $\circ$ final
    \hspace{1cm} by $\text{axiom}_2$

substep 2.2

25. lookup $j$ $\circ$ final $\sim$ lookup $j$ $\circ$ pi2 $\circ$ inv_pi1
    \hspace{1cm} see § 3.7

substep 2.3

26. pi2 $\circ (\text{perm}_\prod \text{ (update i) id})$

27. $(\text{prod id} \text{ (lookup j)}) \circ \text{inv}_{\pi1}$

28. $\sim$ pi2 $\circ (\text{prod id} \text{ (lookup j)}) \circ \text{inv}_{\pi1}$
    \hspace{1cm} by $\text{strong_perm_proj}$

substep 2.4
39. \[ \pi_2 \circ (\text{prod id } (\text{lookup } j)) \circ \text{inv}_{\pi_1} \]

40. \[ \sim \text{lookup } j \circ \pi_2 \circ \text{inv}_{\pi_1} \quad \text{by strong_proj} \]

41. \[ \text{lookup } j \circ \text{update } i = \pi_2 \circ (\text{perm_prod } (\text{update } i) \text{id}) \]

42. \[ \circ (\text{prod id } (\text{lookup } j)) \circ \text{inv}_{\pi_1} \]

To prove such a proposition, the comp_final_unique rule is applied first and results in two sub-goals to be proven:

final \circ \text{lookup } j \circ \text{update } i = \pi_2 \circ (\text{perm_prod } (\text{update } i) \text{id}) \circ (\text{prod id } (\text{lookup } j))

inv_{\pi_1} (to check if both hand sides have the same effect or not) and lookup \ j \circ \text{update } i \sim \pi_2 \circ (\text{perm_prod } (\text{update } i) \text{id}) \circ (\text{prod id } (\text{lookup } j)) \circ \text{inv}_{\pi_1} (to see whether they return the same result or not). Proofs of those sub-goals are given in step 1 and step 2, respectively.

**Step 1.** final \circ \text{lookup } j \circ \text{update } i = \pi_2 \circ (\text{perm_prod } (\text{update } i) \text{id})

\[ \circ (\text{prod id } (\text{lookup } j)) \circ \text{inv}_{\pi_1} : \]

(1.1) The left hand side final \circ \pi_2 \circ (\text{perm_prod } (\text{update } i) \text{id}) \circ (\text{prod id } (\text{lookup } j)) \circ \text{inv}_{\pi_1},

(after the strong_sym rule application) is reduced into: \pi_1 \circ (\text{perm_prod } (\text{update } i) \text{id}) \circ (\text{prod id } (\text{lookup } j)) \circ \text{inv}_{\pi_1}. The base point is an application of E_0.3, stating that final \circ \pi_2 \circ \pi_1 and followed by strong_subs applied to (\text{perm_prod } (\text{update } i) \text{id}), (\text{prod id } (\text{lookup } j)) and \text{inv}_{\pi_1}.

(1.2) In the second sub-step, strong_perm_proj_pi_1_rwpure_rect rule is applied to indicate the strong equality between \pi_1 \circ \text{perm_prod } (\text{update } i) \text{id} and \text{update } i \circ \pi_1. After the applications of strong_subs with arguments \text{perm_prod } (\text{lookup } j) and \text{inv}_{\pi_1}, we get: \pi_1 \circ (\text{perm_prod } (\text{update } i) \text{id}) \circ (\text{prod id } (\text{lookup } j)) \circ \text{inv}_{\pi_1} \circ \text{update } i \circ \pi_1 \circ (\text{prod id } (\text{lookup } j)). Therefore, the left hand side of the equation can now be stated as: \text{update } i \circ \pi_1 \circ (\text{prod id } (\text{lookup } j)) \circ \text{inv}_{\pi_1}.

(1.3) Then, the third sub-step starts with the application of weak_proj_pi_1_purerw_rect rule in order to express the following weak equality: \pi_1 \circ \text{prod id } (\text{lookup } j) \sim \text{id} \circ \pi_1. The next step is converting the existing weak equality into a strong one by the application of ro_weak_to_strong, since none of the components are modifiers. Therefore we get: \pi_1 \circ \text{prod id } (\text{lookup } j) \circ \text{id} \circ \pi_1. Now, using \text{id_tgt}, we remove \text{id} from the right hand side. The subsequent applications of strong_subs with arguments \text{inv}_{\pi_1} and strong_repl enables us to relate \text{update } i \circ \pi_1 \circ (\text{prod id } (\text{lookup } j)) \circ \text{inv}_{\pi_1} with update \text{i} \circ \pi_1 \circ \text{inv}_{\pi_1} via a strong equality.

(1.4) In this sub-step, \text{update } i \circ \pi_1 \circ \text{inv}_{\pi_1} is simplified into update \text{i} \circ \text{id}. To do so, we start with strong_proj_pi_1_purerp so that \text{pi_1} \circ \text{pair id final} \circ \text{id} \circ \text{final defines inv}_{\pi_1}=\text{pair id final}. Then, the application of strong_repl to update \text{i} provides: update \text{i} \circ \text{pi_1} \circ \text{pair id final} = \text{update i} \circ \text{id}.

(1.5) In the last sub-step, the right hand side of the equation, final \circ \text{lookup } j \circ \text{update i}, is reduced into update \text{i} \circ \text{id}. To do so, we use E_1.4 which states that final \circ \text{lookup j} \circ \text{id}. Then, using strong_subs on update i, we get: final \circ \text{lookup j} \circ \text{update i} = \text{id} \circ \text{update i}. By using \text{id_tgt} again we remove \text{id} on the right hand side and id_src rewrites final \circ \text{lookup j} \circ \text{update i} as update \text{i} \circ \text{id}.

At the end of the third step, the left hand side of the equation is reduced into the following form: update \text{i} \circ \text{id} via a strong equality. Thus, in the fourth step, it was sufficient to show final \circ \text{lookup j} \circ \text{update i} = update \text{i} \circ \text{id} to prove final \circ \pi_2 \circ (\text{perm_prod } (\text{update } i) \text{id}) \circ (\text{prod id } (\text{lookup } j)) \circ \text{inv}_{\pi_1} = final \circ \text{lookup j} \circ \text{update i}. This shows that both sides have the same effect on the state structure.

**Step 2.** We now turn to the second step of the proof, namely: \text{lookup j} \circ \text{update i} \sim \pi_2 \circ (\text{perm_prod } (\text{update } i) \text{id}) \circ (\text{prod id } (\text{lookup } j)) \circ \text{inv}_{\pi_1}. The results returned by both input composed functions are examined. Indeed, from step 1 we know that they have the same effect and thus if they also return the same results, then they we will be strongly equivalent.
(2.1) Therefore, the first sub-step starts with the conversion of the left hand side of the equation, lookup \(j\circ\text{update }i\), into lookup \(j\circ\text{final}\) via a weak equality. This is done by the application of the axiom_2 stating that lookup \(j\circ\text{update }i\sim\text{lookup \(j\circ\text{final}\)}\) for \(j \neq i\).

(2.2) The second sub-step starts with the application of strong_proj_pi2_purepure which states \(\text{pi2}\circ(\text{pair id final}) = \text{final}\) still with (pair id final) = inv_pi1. Then, via the applications of strong_repl, with argument lookup \(j\), strong_to_weak and strong_sym, we get: lookup \(j\circ\text{final}\sim\text{lookup \(j\circ\text{pi2}\circ\text{inv_pi1}.

(2.3) In the third sub-step, the right hand side of the equation, \(\text{pi2}\circ(\text{perm\_prod\_update\_i})\circ\text{id}\circ(\text{prod id (lookup \(j\)))} = \text{inv\_pi1}\), is simplified by weak equality: we start with the application of weak_perm_proj_pi2_rwpun since \(\text{pi2}\circ(\text{perm\_prod\_update\_i})\circ\text{id}\circ\text{pi2}\). Then, we once again use id_tgt to remove the identity and the applications of weak_subs with arguments prod id (lookup \(j\)) and inv_pi1 yields the following equation: \(\text{pi2}\circ(\text{perm\_prod\_update\_i})\circ\text{prod id (lookup \(j\))}\circ\text{inv\_pi1}\sim\text{pi2}\circ(\text{prod id (lookup \(j\))})\circ\text{inv\_pi1}.

(2.4) In the last sub-step, \(\text{pi2}\circ\text{prod id (lookup \(j\))}\circ\text{inv\_pi1}\) is reduced into lookup \(j\circ\text{pi2}\circ\text{inv\_pi1}\) via a weak equality using strong_proj_pi2_purew_rect so that \(\text{pi2}\circ(\text{prod id (lookup \(j\))}) = \text{lookup \(j\circ\text{pi2}\). Then, strong_repl is applied with argument inv_pi1. Finally, the strong_to_weak rule is used to convert the strong equality into a weak one.

Both hand side operations return the same results so that the statement lookup \(j\circ\text{update }i\sim\text{pi2}\circ(\text{perm\_prod\_update\_i})\circ\text{prod id (lookup \(j\))}\circ\text{inv\_pi1}\) is proven.

Merging the two steps (same effect and same result) yields the proposition that both sides are strongly equal. The full Coq development can be found in the library in the source file Proofs.v.

\[ \Box \]

5 Conclusion

In this paper, we introduce a framework for the Coq proof assistant. The main goal of this framework is to enable programmers to verify properties of programs involving the global states effect. We use a presentation of these properties which is close to syntax. In other words, the state structure itself is not explicitly mentioned in the verification progress. Instead, it is represented by the term decorations that are used to declare program properties. We then used this framework to verify several well known properties of states as the ones of [10]. In order to verify these properties we first expressed them in the mathematical environment of [3] where the effect of any operation (function) is defined as the distance from being pure and is denoted as \((\gamma) f\) for any \(f: X \rightarrow Y\). Therefore, for the specific case of the global state effect, to check for instance the strong equality between any parallel morphisms \(f, g: X \rightarrow Y\), we first check whether they have the same effect (this is expressed via an equality \((\gamma) f = (\gamma) g\) and then monitor if they return the same result (this is expressed via a weak equation \(f \sim g\)). This scheme has been integrally developed in Coq and Section 4 illustrate the behavior of the resulting proofs on one of the checked proofs of [10]. It is worth noting also that the framework has been successfully used to check a more involved proof, namely that of Hilbert-Post completeness of the global state effect in a decorated setting [6]. The process of writing this proof in our Coq environment (now more than 16 Coq pages) for instance helped discovering at least one non obvious flaw in a preliminary version of the proof.

Future work includes extending this framework to deal with the exception effect: we know from [4] that the core part of exceptions is dual to the global state effect. Then the extension would focus on the pattern matching of the handling of exceptions. We also plan to enable the verification of the composition of effects and to extend the framework to other effects: for monadic or comonadic effects the generic patterns of [6] could then be of help.

References

A Decorated rules for states

In order to prove properties of states, we introduce a set of rules which can be classified as follows: decorated monadic equational logic, decorated categorical products and observational products. The full inference system can be found in [3]. We give here a subset of these rules, enclosing those required for the proof of Section 4.

A.1 Rules for the decorated monadic equational logic

From the usual categorical point of view, the rules of the monadic equational logic are the rules for defining categories “up to equations”: identities are terms, terms are closed under composition, the axioms for identities and associativity of composition are stated only up to equations, and the equations form a congruence.

For dealing with states, we use the decorated version of the rules of the monadic equational logic which is provided in Figure 1. These rules involve three kinds of terms (pure, ro and rw) and two kinds of equations (== and ~); the meaning of these decorations is given in Section 2.2. The decoration d stands for “any decoration”.

For instance, the rule (pure-to-ro) says that if a function is pure, then it can be treated as an accessor, while the rule (ro-to-rw) says that an accessor can be treated as a modifier. The rule (pure-w-repl) says that the replacement rule for ~ holds for pure terms, but there is no general replacement rule for ~: if \( f_1^{\text{rw}} \sim f_2^{\text{rw}} : X \to Y \) and \( g^{\text{rw}} : Y \to Z \) or \( g^{\text{ro}} : Y \to Z \), then in general it cannot be proved that \( g \circ f_1 \sim g \circ f_2 \): indeed, this property does not hold in the intended models.
A.2 Rules for the decorated finite categorical products

The rules of the usual *equational logic* are made of the rules of the monadic equational logic together with the rules for all finite categorical products "up to equations", or equivalently, the rules for a terminal object (or empty product) and for binary products “up to equations”. When dealing with states, we use the *decorated* version of these rules, as described in Figures 2, 3 and 4.

Figure 1: Rules of the decorated monadic equational logic for states

<table>
<thead>
<tr>
<th>(pure-id)</th>
<th>(dec-comp)</th>
<th>(pure-to-ro)</th>
<th>(ro-to-rw)</th>
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<tbody>
<tr>
<td>( \frac{X \to Y}{f \circ \pi_1 : X \to X} )</td>
<td>( \frac{f \circ \phi : X \to Y}{g \circ \phi : Y \to Z} )</td>
<td>( \frac{g \circ \phi : Y \to Z}{f \circ \phi : X \to Y} )</td>
<td>( \frac{g \circ \phi : Y \to Z}{f \circ \phi : X \to Y} )</td>
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<th>(s-trans)</th>
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<tbody>
<tr>
<td>( \frac{f \circ \phi : X \to Y}{f = g} )</td>
<td>( \frac{g = f}{f = g} )</td>
<td>( \frac{f = g}{f = h} )</td>
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<table>
<thead>
<tr>
<th>(dec-assoc)</th>
<th>(dec-id-src)</th>
<th>(dec-id-tgt)</th>
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<tbody>
<tr>
<td>( \frac{f \circ \phi : X \to Y}{f \circ \phi : X \to Y} )</td>
<td>( \frac{h \circ (g \circ \phi) \circ \phi = (h \circ g) \circ \phi}{f \circ \phi : X \to Y} )</td>
<td>( \frac{f \circ \phi : X \to Y}{(f \circ h) \circ \phi = f \circ \phi : X \to Y} )</td>
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<tr>
<th>(s-sub)</th>
<th>(w-sub)</th>
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<tbody>
<tr>
<td>( \frac{f \circ \phi : X \to Y}{f \circ \phi : X \to Y} )</td>
<td>( \frac{f \circ \phi : X \to Y}{f \circ \phi : X \to Y} )</td>
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<th>(ro-w-to-s)</th>
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<td>( \frac{f \circ \phi : X \to Y}{f \circ \phi : X \to Y} )</td>
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<table>
<thead>
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<th>(pure-w-repl)</th>
<th>(w-trans)</th>
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<tbody>
<tr>
<td>( \frac{f \circ \phi : X \to Y}{f \circ \phi : X \to Y} )</td>
<td>( \frac{g \circ \phi : Y \to Z}{g \circ \phi : Y \to Z} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(prod-exists)</th>
<th>(pure-proj-rc)</th>
<th>(pure-proj-rc)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{X_1 \times X_2}{X_1 \times X_2} )</td>
<td>( \frac{\pi_1 \times \pi_2 : X_1 \times X_2 \to X_1}{\pi_1 \times \pi_2 : X_1 \times X_2 \to X_1} )</td>
<td>( \frac{\pi_1 \times \pi_2 : X_1 \times X_2 \to X_2}{\pi_1 \times \pi_2 : X_1 \times X_2 \to X_2} )</td>
</tr>
</tbody>
</table>

Figure 2: Rules of the decorated empty product for states

One of the most important rules given in this context is (dec-comp-final-unique) in Figure 2, which compares both the effects of two given parallel functions (\( \phi \) and \( g \)) and their results. If they have the same effect (\( \phi \circ f = = \phi \circ g \)) and the same result (\( f \sim g \)), then the rule says that they are strongly equal (\( f = = g \)).

The rule (w-pair-unique) in Figure 4 is another important rule for parallel functions (\( f \) and \( g \)) returning a pair of results. It compares the first and the second result of both functions with respect to weak equality (\( \pi_1 \circ f \sim \pi_1 \circ g \) stands for the first comparison and \( \pi_2 \circ f \sim \pi_2 \circ g \) for the second one); if both weak equalities hold, then the rule says that the functions \( f \) and \( g \) are weakly equal (\( f \sim g \)).

Figure 3: Rules of the decorated binary products for states: Existence
### A.3 Rules for the observational products

The rules in Figure 5 are dedicated to the operations for dealing with states: the *lookup* operations for observing the state and the *update* operations for modifying it. Let \( \text{Loc} \) denote the set of locations, for each \( i \in \text{Loc} \) the type \( V_i \) represents the set of possible values that can be stored in the location \( i \), while \( \text{lookup}_i \) and \( \text{update}_i \) correspond to the basic operations that can be performed on this location.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>(axiom-ro)</strong></td>
<td>( \text{lookup}_i \circ \text{update}_i \sim \text{id}_i ) for each ( i \in \text{Loc} )</td>
</tr>
<tr>
<td><strong>(axiom-rw)</strong></td>
<td>( \text{lookup}_i \circ \text{update}_k \sim \text{lookup}_k \circ \text{lookup}_i \circ \text{update}_k )</td>
</tr>
<tr>
<td><strong>(dec-local-to-global)</strong></td>
<td>( f^w, g^w : X \rightarrow Y ) for each ( k \in \text{Loc} ), ( \text{lookup}_k^w \circ f^w \sim \text{lookup}_k^w \circ g^w )</td>
</tr>
</tbody>
</table>

The rule (axiom-ro) states that by updating a location \( i \) and then reading the value that is stored in the same location \( i \), one gets the input value. The equation is weak: indeed, the left hand side returns the same result as the right hand side but they have different state effects: \( \text{lookup}_i \circ \text{update}_i \) is a modifier while \( \text{id}_i \) is pure.

The rule (axiom-rw) indicates that by updating a location \( i \) and then reading the value that is stored in another location \( k \), one gets the value stored in the location \( i \). Besides, forgetting the value stored in the location \( k \) and reading the one located in \( i \), one gets as well the value stored in \( i \). The equation is weak, since both hand side return the same result but they have different state effects: \( \text{lookup}_i \circ \text{update}_i \) is a modifier while \( \text{lookup}_i \circ \text{lookup}_k \circ \text{update}_k \) is an accessor.

The rule (dec-local-to-global) will be used for proving the strong equality of two parallel functions \( f \) and \( g \) (without result) by checking that the observed value at each location is the same after modifying the state according to \( f \) or according to \( g \). Thus, many local observations yield a global result.