Spectral gap and coercivity estimates for linearized Boltzmann collision operators without angular cutoff

Clément Mouhot, Robert Strain

SPECTRAL GAP AND COERCIVITY ESTIMATES FOR LINEARIZED BOLTZMANN COLLISION OPERATORS WITHOUT ANGULAR CUTOFF

CLÉMENT MOUHOT & ROBERT M. STRAIN

Abstract. In this paper we prove new constructive coercivity estimates for the Boltzmann collision operator without cutoff, that is for *long-range* interactions. In particular we give a generalized sufficient condition for the existence of a spectral gap which involves both the growth behavior of the collision kernel at large relative velocities and its singular behavior at grazing and frontal collisions. It provides in particular existence of a spectral gap and estimates on it for interactions deriving from the hard potentials $\phi(r) = r^{-(s-1)}$, $s \geq 5$ or the so-called moderately soft potentials $\phi(r) = r^{-(s-1)}$, $3 < s < 5$, (without angular cutoff). In particular this paper recovers (by constructive means), improves and extends previous results of Pao [46]. We also obtain constructive coercivity estimates for the Landau collision operator for the optimal coercivity norm pointed out in [34] and we formulate a conjecture about a unified necessary and sufficient condition for the existence of a spectral gap for Boltzmann and Landau linearized collision operators.

Mathematics Subject Classification (2000): 76P05 Rarefied gas flows, Boltzmann equation [See also 82B40, 82C40, 82D05].

Keywords: coercivity estimates, linearized Boltzmann operator, linearized Landau operator, quantitative, long-range interaction, non-cutoff, soft potentials, spectral gap.

Contents

Date: July 21, 2006.

1. Introduction

This paper deals with the question of spectral gap and coercivity estimates for the Boltzmann and Landau integro-differential collision operators. This is motivated by the question of obtaining explicit constant in the recent new energy methods in [33, 34, 35, 36, 37, 47, 48, 45, 49, 50]. The starting points are the recent new constructive tools of works [11, 44]. As we shall explain this work recovers, improves, makes explicit and clarifies the works of Pao [46] three decades ago. Before entering into the details, let us introduce the mathematical objects.

1.1. **The Boltzmann equation.**

The Boltzmann equation (Cf. [17, 19]) describes the behavior of a dilute gas when the only interactions taken into account are binary collisions. It reads in some space domain $\Omega \subset \mathbb{R}^N$:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q_{\mathcal{B}}(f,f), \qquad x \in \Omega, \quad v \in \mathbb{R}^N, \quad t \geq 0, \tag{1.1}$$

where $N \geq 2$ is the dimension. In equation (1.1), Q is the quadratic *Boltzmann collision operator*, defined by

$$Q_{\mathcal{B}}(f,f) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \left[f(v')\, f(v'_*) \, - f(v)\, f(v_*) \right] B(|v-v_*|, \cos\theta)\, dv_*\, d\sigma$$

in the so-called "σ-representation" (see [55, Chapter 1, Section 4.6]. In this representation the parametrization of the collision is

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \qquad \sigma \in \mathbb{S}^{N-1}$$

and the *deviation angle* is defined by $\cos\theta = (v'_* - v') \cdot (v_* - v)/|v_* - v|^2$. Up to a jacobian factor $2^{N-2} \sin^{N-2}(\theta/2)$ (see again [55, Chapter 1, Section 4.6]), one can also define the alternative so-called "ω-representation"

$$Q_{\mathcal{B}}(f,f) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \left[f(v')\, f(v'_*) - f(v)\, f(v_*) \right] 2^{N-2} \sin^{N-2}(\theta/2)\, B(|v-v_*|, \cos\theta)\, dv_*\, d\omega$$

with the formula

$$v' = v + ((v_* - v)\cdot\omega)\,\omega, \quad v'_* = v_* - ((v_* - v)\cdot\omega)\,\omega, \qquad \omega \in \mathbb{S}^{N-1}.$$

Remark that this operator is local in x, and therefore all its functional study in the sequel shall be done with no space variable x.

Boltzmann's collision operator has the fundamental properties of conserving mass, momentum and energy

$$\int_{\mathbb{R}^N} Q_{\mathcal{B}}(f,f)\, \phi(v)\, dv = 0, \qquad \phi(v) = 1, v, |v|^2$$

and satisfies well-known Boltzmann's H theorem

$$-\frac{d}{dt}\int_{\mathbb{R}^N} f\log f\,dv = -\int_{\mathbb{R}^N} Q(f,f)\log(f)\,dv \geq 0.$$

The functional $-\int f\log f$ is the *entropy* of the solution. Boltzmann's H theorem implies that at some point $x\in\Omega$, any equilibrium distribution function, *i.e.*, any function which is a maximum of the entropy, has the form of a locally Maxwellian distribution

$$M(\rho,u,T)(v) = \frac{\rho}{(2\pi T)^{N/2}}\exp\left(-\frac{|u-v|^2}{2T}\right), \tag{1.2}$$

where ρ, u, T are the *density*, *mean velocity* and *temperature* of the gas at the point x, defined by

$$\rho = \int_{\mathbb{R}^N} f(v)\,dv, \quad u = \frac{1}{\rho}\int_{\mathbb{R}^N} v\,f(v)\,dv, \quad T = \frac{1}{N\rho}\int_{\mathbb{R}^N}|u-v|^2\,f(v)\,dv. \tag{1.3}$$

For further details on the physical background and derivation of the Boltzmann equation we refer to [17, 19] and [55].

We consider collision kernels of the form

$$B(|v-v_*|,\cos\theta) = |v-v_*|^\gamma\,b(\cos\theta), \quad \gamma\in(-N,+\infty) \tag{1.4}$$

with

$$b(\cos\theta)\sim_{\theta\sim 0} b^*(\theta)\ (\sin\theta/2)^{-(N-1)-\alpha}, \quad \alpha\in[0,2), \tag{1.5}$$

where $b^*(\theta)$ is non-negative, bounded and non-zero near $\theta\sim 0$. When $\alpha\geq 0$ the angular singularity is not integrable, the operator is said to be *non-cutoff.*

An important remark on the quadratic collision operator is: by using the change of variable $\sigma\to-\sigma$ (which changes θ into $\pi-\theta$) one can replace b by

$$\tilde{b}(\cos\theta) = \frac{1}{2}\,\mathbf{1}_{\theta\in[0,\pi/2]}\ [b(\cos\theta)+b(\cos(\pi-\theta))]$$

(where $\mathbf{1}_E$ denotes the usual characteristic function of a set E. Therefore we shall consider without restriction in the sequel that b satisfies the singularity condition (1.5) above, and is 0 on $[\pi/2,\pi]$.

For particles interacting according to some spherical repulsive potential

$$\phi(r) = r^{-(s-1)}, \quad s\in[2,+\infty),$$

the collision kernel is not explicit but it can be shown that for the dimension $N=3$, B satisfies (1.4) with $\gamma=(s-5)/(s-1)$ and (1.5) with $\alpha=2/(s-1)$ (see for instance [17, 19, 5]). Therefore as a convention one shall speak in (1.4) of *hard potentials* when $s\geq 5$, *maxwellian potential* when $s=5$, *soft potentials* when $2<$

$s < 5$. Moreover among soft potentials we shall denote by *moderately soft potentials* the case when $3 \le s < 0$.

Let us mention also for the sake of completeness that in the case of contact interactions (the so-called *hard spheres* model), the collision kernel is locally integrable and explicit: it takes the form (in dimension $N = 3$) $B(|v - v_*|, \cos\theta) = |v - v_*|$ (up to a normalization constant).

1.2. **Linearized Boltzmann collision operator.** We denote by

$$\mu = \mu(v) := (2\pi)^{-N/2} e^{-|v|^2/2}$$

the normalized unique equilibrium with mass 1, momentum 0 and temperature 1. We consider fluctuations around this equilibrium of the form

$$f = \mu + \mu^{1/2}\, g$$

which results the following linearized collision operator (note the sign convention):

$$L_{\mathcal{B}}(g) = -\mu^{-1/2}\, \left[Q_{\mathcal{B}}(\mu, \mu^{1/2} g) + Q_{\mathcal{B}}(\mu^{1/2} g, \mu) \right].$$

For the sake of simplification we shall always consider in the sequel the linearized collision operator around this normalized equilibrium. This is no restriction: a detailed discussion of the dependence of the spectral gap and coercivity estimates on this operator in terms of the mass, mean velocity and temperature can be found in [44].

It is well-known (see [17] for instance) that $L_{\mathcal{B}}$ (acting in the velocity space) is an unbounded symmetric operator on L^2, such that its Dirichlet form satisfies

$$D_{\mathcal{B}}(g) := \langle L_{\mathcal{B}} g, g \rangle \ge 0,$$

and that $D_{\mathcal{B}}(g) = 0$ if and only if $g = \mathbf{P} g$ where

$$\mathbf{P} g = \left(a + b \cdot v + c|v|^2 \right) \mu^{1/2}$$

(with $a, c \in \mathbb{R}$ and $b \in \mathbb{R}^N$) is the L^2 orthogonal projection onto the space of the so-called "collisional invariants"

$$\mathrm{Span}\left\{ \mu^{1/2}, v_1 \mu^{1/2}, \dots, v_N \mu^{1/2}, |v|^2 \mu^{1/2} \right\}.$$

1.3. **The Landau equation.** The Landau equation was written by Landau in 1936 ([42]). It is similar to the Boltzmann equation (1.1) but with a different collision operator $Q_{\mathcal{L}}$. Indeed in the case of long-distance interactions, collisions occur mostly for very small θ. When all collisions become concentrated on $\theta = 0$, one obtains by the so-called *grazing collision limit* asymptotic (see for instance [10, 21, 22, 52, 7] for a detailed discussion) the *Landau collision operator*

$$Q_{\mathcal{L}}(f, f) = \nabla \cdot \left(\int_{\mathbb{R}^N} \mathbf{a}(v - v_*) [f_* (\nabla f) - f (\nabla f)_*] \, dv_* \right), \tag{1.6}$$

where $\nabla = (\partial_{v_1}, \dots, \partial_{v_N})$, $f = f(v)$, $f_* = f(v_*)$ etc. The non-negative symmetric $N \times N$ matrix $\mathbf{a} = \mathbf{a}(z)$ is

$$a_{ij}(z) = \left\{ \delta_{ij} - \frac{z_i z_j}{|z|^2} \right\} |z|^{\gamma+2}. \tag{1.7}$$

This operator is used for instance in plasma physics in the case of a Coulomb potential where $\Phi(|z|) = |z|^{-3}$ in dimension $N = 3$ (for more details see [55, Chapter 1, Section 1.7] and the references therein). Indeed let us mention that for Coulomb interactions the Boltzmann collision operator does not make sense anymore (see [54, Annex I, Appendix]). By analogy with the Boltzmann's case, and even if the physical meaning of such collision kernel is not clear for the Landau collision operator, one shall speak of *hard potentials* when $\gamma > 0$, *maxwellian potentials* when $\gamma = 0$, and *soft potentials* when $\gamma < 0$. Moreover among soft potentials we shall denote by *moderately soft potentials* the case when $-2 \le \gamma < 0$.

1.4. **Linearized Landau collision operator.** Consider again fluctuation around the equilibrium of the form

$$f = \mu + \mu^{1/2}\, g.$$

Then the linearized Landau collision operator is defined by

$$L_{\mathcal{L}} g = -\mu^{-1/2} \left[Q_{\mathcal{L}}(\mu, \mu^{1/2} g) + Q_{\mathcal{L}}(\mu^{1/2} g, \mu) \right].$$

It was proved in [34, 39, 20] that that $L_{\mathcal{L}}$ (acting on the velocity space) is an unbounded symmetric operator on L^2, such that its Dirichlet form satisfies

$$D_{\mathcal{L}}(g) := \langle L_{\mathcal{L}} g, g \rangle \ge 0$$

and $D_{\mathcal{L}}(g) = 0$ if and only if $g = \mathbf{P} g$.

1.5. **Previous results and motivations.** The study of spectral gap estimates for the linearized Boltzmann and Landau collision operators has a long history, see for instance [38, 57, 15, 31, 32, 58, 14, 12] and we refer to [11, 44] for a more detailed discussion of it. Let us just emphasize some references directly related to this paper. For linearized Boltzmann collision operators with locally integrable collision kernel (which is satisfied for instance under the so-called *Grad's angular cutoff*), the existence of a spectral gap is equivalent to $\gamma \ge 0$ (see [32, 14, 19] for non-constructive proofs). The question of obtaining polynomial or exponential rate of relaxation of the form $O(e^{-\lambda t^\beta})$ for $0 < \beta < 1$ and for soft potentials with cutoff was studied in [14, 51, 48, 49], and explicit spectral gap and generalized coercivity estimates were obtained in [11, 44].

For the sake of completeness let us also mention that Cercignani introduced an alternative "potential cutoff" to Grad's angular cutoff in [16] for which results on the spectrum of the linearized collision operator were obtained in [18].

While the cutoff theory of the linearized collision operator (without space variable) is quite developed, for non locally integrable collision kernels, there are quite few works. First for the Boltzmann collision operator at the linearized level there are essentially the two papers by Pao [46]: he proved that the resolvent is compact for inverse power-laws interaction potentials with $s > 3$ (in our notation). His proof was using tools from pseudo-differential operators theory: the idea was to compute the symbol of the linearized operator $L_{\mathcal{B}}$, and then to search for adequate condition for $L_{\mathcal{B}} + C$ (for some constant $C > 0$) to be invertible with compact inverse, by reducing to the Maxwell case and decomposing along spherical harmonics. This fundamental work was critically reviewed in particular for the use of techniques from pseudo-differential operator theory in [40].

The critics of [40] on [46], together with the fact that the proof in [46] was highly technical, were probably the reasons for which the results in [46] were not considered as completely reliable, and this paper was somehow forgotten in the following decades.

One of our goal in this work is to clarify the discussion about the validity of the results [46], by recovering and improving strongly these results. Moreover we replace the question treated in [46] into the unified framework of quantitative coercivity estimates for the linearized collision operators, which is related to the works [11, 44] (for the Landau collision operator at the linearized level, let us mention the key works [20, 34] and the approach developed in [44] which shall be used here). A general motivation for this framework is to provide explicit rate of convergence to equilibrium in the energy methods which have emerged recently in the collisional kinetic theory: see [33, 34, 35, 36, 37, 47, 48, 45, 49, 50].

At the non-linear level, let us mention some breakthroughs related to non-cutoff interactions: in the spatially homogeneous Cauchy theory: [8, 9, 28, 52, 30], in the grazing collision limit from Boltzmann to Landau equation: [10, 22, 21], in the parabolic-like regularizing property in the velocity variable: [23, 24, 25, 41, 53, 5], in the construction of renormalized solutions (in the spirit of [27]): [6, 7].

In the spirit of the method of Pao, a systematic approach by Fourier transform and pseudo-differential operators for the study of linear and non-linear Boltzmann non-cutoff collision operators has been developed by Alexandre [1, 3, 4]. The applications of this technical tools are not clear at now, but they seem to have contributed to the understanding of the Boltzmann equation in two ways: in the development of a renormalization process adapted to the non-cutoff case ([2, 6]), and by providing some sharp estimates from above on the collision operator.

1.6. **Main theorems.** First we state our coercivity results for the linearized Boltzmann collision operator:

Theorem 1.1. *Let B be a collision kernel satisfying (1.4,1.5). Then*

- *For any $\varepsilon > 0$ there is a constant $C_{B,\varepsilon}$, constructive from our proof and depending on B and ε, such that the Dirichlet form $D_{\mathcal{B}}$ of the linearized Boltzmann collision operator associated to B satisfies*
$$D_{\mathcal{B}}(g) \geq C_{B,\varepsilon} \left\| [g - \mathbf{P}g] \, (1+|v|^2)^{(\gamma+\alpha-\varepsilon)/4} \right\|^2_{L^2(\mathbb{R}^N)}.$$
- *There is a constant $C_{B,0}$ (obtained by non-constructive means in our proof) such that*
$$D_{\mathcal{B}}(g) \geq C_{B,0} \left\| [g - \mathbf{P}g] \, (1+|v|^2)^{(\gamma+\alpha)/4} \right\|^2_{L^2(\mathbb{R}^N)}.$$

Second we state the constructive version of the coercivity result in [34] for the linearized Landau collision operator:

Theorem 1.2. *Let $\gamma \in [-N, +\infty)$. Then there is some constant C_γ, constructive from our proof and depending on γ, such that the Dirichlet form $D_{\mathcal{L}}$ of the associated linearized Landau collision operator satisfies:*
$$D_{\mathcal{L}}(g) \geq C_\gamma \left\| [g - \mathbf{P}g] \right\|^2_\sigma.$$
where $\| \cdot \|_\sigma$ is the following anisotropic norm:
$$\|g\|^2_\sigma := \left\| (1+|v|^2)^{\gamma/4} \, \Pi_v \nabla_v g \right\|^2_{L^2(\mathbb{R}^N)} \\ + \left\| (1+|v|^2)^{(\gamma+2)/4} \, [I - \Pi_v] \nabla_v g \right\|^2_{L^2(\mathbb{R}^N)} + \left\| (1+|v|^2)^{(\gamma+2)/4} \, g \right\|^2_{L^2(\mathbb{R}^N)}$$
with
$$\Pi_v \nabla_v g = \left(\frac{v}{|v|} \cdot \nabla_v g \right) \frac{v}{|v|}.$$

1.7. **Consequences on the spectrum.** In a previous paper the first author has proved:

Theorem 1.3 ([44])**.** *Let B be a collision kernel satisfying (1.4,1.5) with $\alpha > 0$. Then there is a constant C_B (constructive from the proof and depending on B) such that the Dirichlet form $D_{\mathcal{B}}$ of the linearized Boltzmann collision operator associated to B satisfies*
$$D_{\mathcal{B}}(g) \geq C_B \left\| [g - \mathbf{P}g] \right\|^2_{H^{\alpha/2}_{\mathrm{loc}}(\mathbb{R}^N)}.$$

Therefore as soon as $\gamma + \alpha > 0$ and $\alpha > 0$ it is straightforward by gathering Theorems 1.1 and 1.3 that the resolvent of $L_{\mathcal{B}}$ is compact and so that the spectrum is purely discrete and the eigenvectors basis is complete in L^2. In dimension $N = 3$ for interactions deriving from inverse-power laws potentials $\phi(r) = r^{-(s-1)}$, the corresponding condition on s is: $(s-5)/(s-1) + 2/(s-1) > 0$, that is $s > 3$.

Hence we can recover completely the results of Pao [46], by constructive means and without tools of pseudo-differential operators.

Moreover Theorem 1.1 yields the first estimates on this spectral gap in the case $\gamma + \alpha > 0$ and answers to the question of the existence of a spectral gap in the limit case $\gamma + \alpha = 0$.

For the linearized Landau collision operator, it had been already shown in [34] that

$$C_1 \left\| [g - \mathbf{P}g] \right\|_\sigma^2 \leq D_{\mathcal{L}}(g) \leq C_2 \left\| [g - \mathbf{P}g] \right\|_\sigma^2$$

by non-constructive means (in fact the constant C_2 could be made explicit from the proof whereas the constant C_1 was obtained by a compactness argument). This implies straightforwardly that the resolvent is compact as soon as $\gamma > -2$ (which implies in this case that the spectrum is purely discrete and the eigenvectors basis is complete in L^2). Moreover spectral gap exists if and only if $\gamma \geq -2$.

In this context, Theorem 1.2 provides the first estimates on it for this whole region (as well as an explicit version of the coercivity estimate).

1.8. **Conjecture and perspectives.** For the linearized Boltzmann collision operator with angular cutoff, the existence of a spectral gap is equivalent to $\gamma \geq 0$ (see [32, 14, 19]). This situation can be loosely thought of as part of the limit case "$\alpha = 0$". For the linearized Landau collision operator, the existence of a spectral gap is equivalent to $\gamma \geq -2$ as discussed above. This situation can be thought of as the limit case "$\alpha = 2$".

From the necessary and sufficient condition in these two limit cases, and the sufficient condition of Theorem 1.1 in the intermediate cases, it is natural to state the following conjecture:

Conjecture. *Let B be a collision kernel satisfying (1.4,1.5) with $\gamma \in (-N, +\infty)$ and $\alpha \in [0, 2)$. Then the linearized Boltzmann collision operator associated to B admits a spectral gap if and only if $\gamma + \alpha \geq 0$. Moreover this statement is still valid if one includes formally the case of angular cutoff in "$\alpha = 0$", and add the linearized Landau collision operator as the limit case "$\alpha = 2$".*

Let us remark that in the limit cases of the linearized Boltzmann collision operator with angular cutoff and the linearized Landau collision operator the conjecture is proved as discussed before. And Theorem 1.1 proves that the condition $\gamma + \alpha \geq 0$ is sufficient for the existence of a spectral gap.

Here are now some open questions linked with this conjecture:

(1) In order to show that the condition is necessary, is it possible to find some particular sequence of functions g_n contradicting the spectral gap estimate when $\gamma + \alpha < 0$?

(2) A related and more difficult question is to identify and understand the coercivity norm for the non-cutoff linearized Boltzmann collision operator. This norm is likely to be intricate and anisotropic for the weight on the fractional derivative part (as for the linearized Landau collision operator). This point is related to the open question of constructing smooth perturbative solutions near equilibrium for the Boltzmann equation without angular cutoff (see the discussion in [45]).

(3) The spectrum is purely discrete in the case $\gamma + \alpha > 0$, $\alpha > 0$ (and we conjecture that this is also true on the non-cutoff borderline case $\gamma + \alpha > 0$, $\alpha = 0$). In the angular cutoff case the geometry of the spectrum can be obtained by perturbation arguments (Cf. [32, 14, 19]). Hence there remains the region $\gamma + \alpha \le 0$ for which there is no information on the geometry of the spectrum at now (including the linearized Landau collision operator when $\alpha = 2$ and $\gamma < -2$).

Finally we shall put in perspective this conjecture with the non-linear case. A non-linear analogous of a spectral gap is provided by Cercignani's conjecture:

$$\mathcal{D}_{\mathcal{B}}(f) \ge C_f \, H(f|M_f)$$

where

$$H(f|M_f) = H(f) - H(M_f) := \int_{\mathbb{R}^N} f \, \log \frac{f}{M_f} \, dv$$

is the relative entropy between f and its associated Maxwellian equilibrium distribution, and

$$\mathcal{D}_{\mathcal{B}}(f) := - \int_{\mathbb{R}^N} Q_{\mathcal{B}}(f, f) \, \log f \, dv \quad (\ge 0)$$

denotes the entropy production functional (for more details we refer to [56]). It was shown in [26] that an equivalent version of this conjecture is true for the Landau collision operator when $\gamma \ge 0$, and it was shown in [56] that this conjecture for the Boltzmann collision operator for $B(v - v_*, \cos\theta) \ge K_B \, (1 + |v - v_*|)^\gamma$ with $\gamma \ge 2$. From this two limit cases Villani "interpolated" the following conjecture:

Conjecture (Villani [56]**).** *Cercignani's conjecture is satisfied if and only if* $\gamma + \alpha \ge 2$.

(The other part of the conjecture on the question of the existence of a spectral gap for the linearized collision operator was not correct –as shown by Theorem 1.1–, which is likely to be explained by the confusion in the field about Pao's results). Obviously this conjecture leaves more room than the one we made on the spectral gap, since it is not stated in which functional space the distribution f lives, but from the results in [56] and [13], it is reasonable to try to prove that Villani's conjecture holds in spaces of nonnegative functions in $L^1(1 + |v|^q) \cap H^k$ for any $q, k \ge 0$.

Still an interesting question is to know whether Cercignani's conjecture could still be true for $\gamma + \alpha < 2$ in some functional space with "stretched exponential decay" (see the discussions in [13, 43]). This is important since the condition $\gamma + \alpha \geq 2$ rules out all physical interactions.

Another important question, if Villani's and our conjectures hold, would be to understand the reasons for this "gap" between the non-linear and linearized behavior of the entropy production for the Boltzmann equation.

1.9. **Methods of the proof and plan of the paper.** Section 2 is devoted to a technical estimate of decay on the kernel of the non-local part of the linearized Boltzmann collision operator, under the angular cutoff assumption. We show that its mixing effects can be quantified into a gain of polynomial weights (this phenomenon was noticed in [32] and is here fully developed). This estimate has been isolated from the rest of the proof since it is the key step, and also since it can be of independent interest for researchers in the field.

Section 3 is devoted to the (constructive) proof of the coercivity estimate for the linearized Boltzmann collision operator when $\varepsilon > 0$ (first point in Theorem 1.1). The idea is to estimate from below the Dirichlet form by truncating the angular part b of the collision kernel on the angles $\theta \in [0, \theta_0(|v - v_*|)]$ with $\theta_0(|v - v_*|) \sim |v - v_*|^{-k}$ for some suitable $k > 0$, and then balance the lower bound on the local part (in which the polynomial weight results from a competing effect between the fact the lower bound on b is big in this region, and the fact that the size of this angular region is small), and the upper bound on the non-local part, for which the previous technical estimate plays a crucial role. The constructive coercivity estimates from [44] are also used.

Section 4 is devoted to the (non-constructive) proof of the coercivity estimate for the linearized Boltzmann collision operator when $\varepsilon = 0$ (second point in Theorem 1.1). The idea is to reduce to a cutoff-like linearized Boltzmann collision operator with a different collision kernel, and then apply a strategy based on Weyl's Theorem about compact perturbation of essential spectrum, in the spirit of [32, 19].

Finally Section 5 is devoted to the (constructive) proof of the coercivity estimate for the linearized Landau collision operator (Theorem 1.2). The idea is to combine the estimates in [34] on the different parts of the decomposition of the operator as "diffusive part + bounded part" (which involves the optimal coercivity norms), with the constructive coercivity estimates of [44] obtained on the global operator, but with non-optimal coercivity norms.

2. A technical estimate on $L_{\mathcal{B}}$

We assume in this section that the collision kernel B takes the particular form (sometimes named "variable hard spheres collision kernels")

$$B_q(|v - v_*|, \cos\theta) = |v - v_*|^q, \quad q \in (-N, +\infty).$$

These collision kernels are non-physical except in the case $q = 1$ (hard spheres) but they shall play an important role in intermediate steps of our proof in the next sections. They satisfy the angular cutoff assumption, that is B_q is locally integrable in terms of v, v_*, σ. It is well-known that under this assumption the collision operator can be split into "gain" and "loss" parts.

Therefore one can decompose the linearized collision operator $L_{\mathcal{B}}$ corresponding to B_q as follows

$$L_{\mathcal{B}} g = \nu_{\mathcal{B}}\, g - K_{\mathcal{B}} g$$

with the non-local part

$$K_{\mathcal{B}} g := \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Big[g(v')\, \mu^{1/2}(v'_*) + g(v'_*)\, \mu^{1/2}(v') - g(v_*)\, \mu^{1/2}(v) \Big]\, \mu^{1/2}(v_*)\, B_q(|v - v_*|)\, dv_*\, d\sigma$$

and the following multiplicative function for the local part

$$\nu_{\mathcal{B}}(v) := \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \mu(v)\, B_q(|v - v_*|)\, dv_*\, d\sigma = \left|\mathbb{S}^{N-1}\right| \big(|\cdot|^q * \mu \big)(v).$$

The non-local part $K_{\mathcal{B}}$ itself splits into a pure convolution part

$$K^c_{\mathcal{B}} g := \left|\mathbb{S}^{N-1}\right| \left[|\cdot|^q * (\mu^{1/2}\, g) \right](v)\, \mu^{1/2}(v)$$

and a "gain" part (denoted so since it flows from the gain part of the original non-linear operator)

$$K^+_{\mathcal{B}}\, g := \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Big[g(v')\, \mu^{1/2}(v'_*) + g(v'_*)\, \mu^{1/2}(v') \Big]\, \mu^{1/2}(v_*)\, B_q(|v - v_*|)\, dv_*\, d\sigma.$$

Note that using the change of variable $\sigma \to -\sigma$, the latter writes also

$$K^+_{\mathcal{B}}\, g := 2 \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} g(v')\, \mu^{1/2}(v'_*)\, \mu^{1/2}(v_*)\, B_q(|v - v_*|)\, dv_*\, d\sigma.$$

We shall focus on the kernel of the $K^+_{\mathcal{B}}$ part. For any locally integrable collision kernel B we can define $k_B = k_B(v, v')$ such that

$$\forall\, v \in \mathbb{R}^N, \quad K^+_{\mathcal{B}} g(v) = \int_{\mathbb{R}^N} g(v')\, k_B(v, v')\, dv'$$

and we denote by $k_q := k_{B_q}$ the kernel obtained for the particular collision kernel B_q above. Note that this definition is independent of the representation we use for the collision operator, either using the "σ" or "ω" parametrization of the unit vector on the sphere.

Then we have the

Proposition 2.1. *The kernel k_q is symmetric:*

$$\forall\, v, v' \in \mathbb{R}^N, \quad k_q(v, v') = k_q(v', v),$$

and for $q > -1$ and for any $s \in \mathbb{R}$ it satisfies the control

$$\forall\, v \in \mathbb{R}^N, \quad \int_{\mathbb{R}^N} k_q(v, v')\,(1 + |v'|)^s\, dv' \le C_{q,s}\,(1 + |v|)^{q+s-(N-1)}$$

for some explicit constant $C_{q,s}$ depending only on q, s.

Let us first compute the kernel k_q using well-known changes of variable (see for instance [32]).

Lemma 2.2. *We have, for $q > -1$, the explicit formula (let us recall that $\omega = (v'-v)/|v'-v|$):*

$$k_q(v, v') = \frac{2^N}{|v'-v|\,(2\pi)^{N/2}} \exp\left\{-\frac{|v'-v|^2}{8} - \frac{|v'-v+2(v\cdot\omega)\omega|^2}{8}\right\}$$
$$\times \left(\int_{\omega^\perp} |v'-v+z|^{q-(N-2)} \exp\left\{-\frac{|z+(v-(v\cdot\omega)\omega)\,|^2}{2}\right\}\, dz\right).$$

Proof of Lemma 2.2. We start from the ω-representation of $K^+_{\mathcal{B}}$:

$$\begin{aligned}
I &:= 2 \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} g(v')\,(\mu_* \mu'_*)^{1/2}\, B_q\, dv_*\, d\sigma \\
&= 2^{N-1} \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} g(v')\,(\mu_* \mu'_*)^{1/2}\, |v-v_*|^q\, \sin^{N-2}(\theta/2)\, dv_*\, d\omega \\
&= 2^{N-1} \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} g(v')\,(\mu_* \mu'_*)^{1/2}\, |v-v_*|^{q-(N-2)}\, |v'-v|^{N-2}\, dv_*\, d\omega.
\end{aligned}$$

Then we perform the change variable $v_* \to V = v_* - v$ (with ω fixed), and the change of variable $V = r\,\omega + z$ with $z \in \omega^\perp$ (with ω fixed). These two changes of variables have jacobian equal to 1. We obtain

$$I = 2^{N-1} \int_{\mathbb{R}\times\mathbb{S}^{N-1}} g(v + r\,\omega)\,|r|^{N-2} \left(\int_{\omega^\perp} (\mu_* \mu'_*)^{1/2}\, |r\omega + z|^{q-(N-2)}\, dz\right) dr\, d\omega.$$

Finally we make the spherical change of variable $(r, \omega) \in \mathbb{R} \times \mathbb{S}^{N-1} \to W = r\omega \in \mathbb{R}^N$ with jacobian $2\, r^{-(N-1)}$ (a factor 2 comes from the fact that $r \in \mathbb{R}$), which yields

$$I = 2^N \int_{\mathbb{R}^N} g(v+W)\, |W|^{-1} \left(\int_{W^\perp} (\mu_* \mu'_*)^{1/2}\, |W+z|^{q-(N-2)}\, dz \right) dW.$$

Let us rewrite the argument of the exponential term:

$$\begin{aligned} |v_*|^2 + |v'_*|^2 = |v+W+z|^2 + |v+z|^2 &= \frac{1}{2}|W + 2(v+z)|^2 + \frac{1}{2}|W|^2 \\ &= \frac{1}{2}|W + 2(v\cdot\omega)\omega|^2 + 2|z + (v - (v\cdot\omega)\omega)\,|^2 + \frac{1}{2}|W|^2 \end{aligned}$$

where $\omega = W/|W|$ (in the last equality we have used that $z \perp W$). We deduce that

$$(\mu_* \mu'_*)^{1/2} = \frac{1}{(2\pi)^{N/2}} \exp\left\{ -\frac{|W|^2}{8} - \frac{|z + (v-(v\cdot\omega)\omega)\,|^2}{2} - \frac{|W + 2(v\cdot\omega)\omega|^2}{8} \right\}.$$

We deduce the formula. One checks that the integral on z is well-defined for $q > -1$ since in this case $q - (N-2) > -(N-1)$. □

Proof of Proposition 2.1. The fact that k_q is symmetric is easily checked from the formula. Let us turn to the bound from above.

First let us assume that $q - (N-2) \geq 0$. We estimate the integral over $z \in \omega^\perp$ in terms of v:

$$\begin{aligned} &\int_{\omega^\perp} |v' - v + z|^{q-(N-2)} \exp\left\{ -\frac{|z + (v-(v\cdot\omega)\omega)\,|^2}{2} \right\} dz \\ &\qquad = \int_{\omega^\perp} |v' - v + \bar{z} - (v - (v\cdot\omega)\omega)\,|^{q-(N-2)} \exp\left\{ -\frac{|\bar{z}|^2}{2} \right\} d\bar{z} \\ &\qquad \leq C\,(1 + |v - (v\cdot\omega)\omega|)^{q-(N-2)}\,(1 + |v'-v|)^{q-(N-2)} \\ &\qquad\qquad \leq C\,(1+|v|)^{q-(N-2)}\,(1+|v'-v|)^{q-(N-2)}. \end{aligned}$$

We deduce that

$$\begin{aligned} &\int_{\mathbb{R}^N} k_q(v,v')\,(1+|v'|)^s\, dv' \\ &\qquad \leq C\,(1+|v|)^{q-(N-2)} \int_{\mathbb{R}^N} |v'-v|^{-1}\,(1+|v'|)^s\,(1+|v'-v|)^{q-(N-2)} \\ &\qquad\qquad \exp\left\{ -\frac{|v'-v|^2}{8} - \frac{|v'-v+2(v\cdot\omega)\omega|^2}{8} \right\} dv' \end{aligned}$$

and we use the elementary inequality

$$(1+|v'|)^s \leq C\,(1+|v|)^s\,(1+|v'-v|)^{|s|}$$

to get

$$\int_{\mathbb{R}^N} k_q(v,v')\,(1+|v'|)^s\,dv' \le C\,(1+|v|)^{q+s-(N-2)}\,J$$

with

$$J := \int_{\mathbb{R}^N} |v'-v|^{-1}\,(1+|v'-v|)^{|s|+q-(N-2)} \exp\left\{-\frac{|v'-v|^2}{8} - \frac{|v'-v+2(v\cdot\omega)\omega|^2}{8}\right\}\,dv'.$$

Now in the term J we perform the change of variable $v' \to u = v'-v$ and then the spherical change of variables $u = r\,\omega$, $r \in \mathbb{R}_+$, $\omega \in \mathbb{S}^{N-1}$, choosing v as the north pole vector in the angle parametrization (note that this is not the same parametrization as the one used to define the deviation angle). It yields

$$J = |\mathbb{S}^{N-2}| \int_0^{+\infty} r^{N-2}\,(1+r)^{|s|+q-(N-2)}\,e^{-\frac{r^2}{8}} \int_0^{\pi} e^{-\frac{(r+2|v|\cos\varphi)^2}{8}}\,\sin^{N-2}\varphi\,d\varphi\,dr.$$

Then for technical reasons we treat separately the case $N=2$ and $N \ge 3$.

First for $N \ge 3$, we have $\sin^{N-2}\varphi \le \sin\varphi$, and thus

$$J \le C \int_0^{+\infty} r^{N-2}\,(1+r)^{|s|+q-(N-2)}\,e^{-\frac{r^2}{8}} \int_0^{\pi} e^{-\frac{(r+2|v|\cos\varphi)^2}{8}}\,\sin\varphi\,d\varphi\,dr.$$

Then we do the change of variable $y = (r+2|v|\cos\varphi)$ in the φ integral, to get

$$J \le C\,|v|^{-1} \int_0^{+\infty} r^{N-2}\,(1+r)^{|s|+q-(N-2)}\,e^{-\frac{r^2}{8}}\,dr \int_{-\infty}^{+\infty} e^{-\frac{y^2}{8}}\,dy \le C\,|v|^{-1}$$

which yields the conclusion for large v (the estimate for small v is immediate).

Second for $N=2$, we perform the same changes of variable which yields

$$J \le C\,|v|^{-1} \int_0^{+\infty} (1+r)^{|s|+q-(N-2)}\,e^{-\frac{r^2}{8}} \int_{(r-2|v|)}^{(r+2|v|)} e^{-\frac{y^2}{8}} \left(1-\left(\frac{y-r}{2\,|v|}\right)^2\right)^{-1/2} dy\,dr$$

$$\le C \int_0^{+\infty} (1+r)^{|s|+q-(N-2)}\,e^{-\frac{r^2}{8}} \int_{(r-2|v|)}^{(r+2|v|)} e^{-\frac{y^2}{8}} \left(4\,|v|^2-(y-r)^2\right)^{-1/2} dy\,dr.$$

Finally we split into two parts: $|y-r| \le |v|$ and $|y-r| \ge |v|$. On the first part we have $(4\,|v|^2-(y-r)^2)^{-1/2} \le |v|^{-1}$ which yields the result. On the second part we have either $r \ge |v|/2$ or $|y| \ge |v|/2$ which gives an exponential decay in v thanks to the terms $e^{-\frac{r^2}{8}}$ and $e^{-\frac{y^2}{8}}$ in the integrand. This concludes the proof in dimension $N = 2$.

Now let us come back to the case $q - (N-2) < 0$. Then we have

$$\int_{\omega^\perp} |v' - v + z|^{q-(N-2)} \exp\left\{ -\frac{|z + (v - (v\cdot\omega)\omega)\,|^2}{2} \right\} dz$$
$$= \int_{\omega^\perp} |v' - v + \bar z - (v - (v\cdot\omega)\omega)\,|^{q-(N-2)} \exp\left\{ -\frac{|\bar z|^2}{2} \right\} d\bar z$$
$$\le C\,(1 + |v - (v\cdot\omega)\omega|)^{q-(N-2)}\,(1 + |v' - v|)^{|q-(N-2)|}.$$

The additional difficulty will be therefore to obtain decay in v since the weight in this formula only involves the projection of v on $\omega^\perp$.

Let us follow the computations as before: we have

$$\int_{\mathbb{R}^N} k_q(v,v')\,(1+|v'|)^s\,dv'$$
$$\le C \int_{\mathbb{R}^N} |v'-v|^{-1}\,(1+|v'|)^s\,(1+|v-(v\cdot\omega)\omega|)^{q-(N-2)}\,(1+|v'-v|)^{|q-(N-2)|}$$
$$\times \exp\left\{ -\frac{|v'-v|^2}{8} - \frac{|v'-v+2(v\cdot\omega)\omega|^2}{8} \right\} dv'.$$

We use again the inequality

$$(1+|v'|)^s \le C\,(1+|v|)^s\,(1+|v'-v|)^{|s|}$$

to get

$$\int_{\mathbb{R}^N} k_q(v,v')\,(1+|v'|)^s\,dv' \le C\,(1+|v|)^s\,J'$$

with

$$J' := \int_{\mathbb{R}^N} |v'-v|^{-1}\,(1+|v'-v|)^{|s|+|q-(N-2)|}\,(1+|v-(v\cdot\omega)\omega|)^{q-(N-2)}$$
$$\exp\left\{ -\frac{|v'-v|^2}{8} - \frac{|v'-v+2(v\cdot\omega)\omega|^2}{8} \right\} dv'.$$

Again we perform the change of variable $v' \to u = v' - v$ and then the spherical change of variables $u = r\,\omega$, $r \in \mathbb{R}_+$, $\omega \in \mathbb{S}^{N-1}$, choosing v as the north pole vector in the angle parametrization. It yields

$$J' = |\mathbb{S}^{N-2}| \int_0^{+\infty} r^{N-2}\,(1+r)^{|s|+|q-(N-2)|}\, e^{-\frac{r^2}{8}} \int_0^{\pi} (1+|v|\,\sin\varphi)^{q-(N-2)}\, e^{-\frac{(r+2|v|\cos\varphi)^2}{8}}\, \sin^{N-2}\theta\, d\varphi\, dr.$$

Then we split between $|\cos\varphi| \le 1/\sqrt{2}$ and $|\cos\varphi| \ge 1/\sqrt{2}$. In the first case we have $\sin\varphi \ge 1/\sqrt{2}$ and thus

$$(1+|v|\sin\varphi)^{q-(N-2)} \le C\,(1+|v|)^{q-(N-2)}$$

and the end of the proof is strictly similar to above. In the second case, we have

$$\frac{(r+2|v|\cos\varphi)^2}{8} \ge \frac{|v|^2}{12} - \frac{r^2}{16}$$

which implies exponential decay in v for J'. □

3. Proof of Theorem 1.1 for $\varepsilon > 0$

Let us first consider a (non locally integrable) collision kernel of the form (in σ-representation)

$$B_{\gamma,\alpha}(|v-v_*|,\cos\theta) = |v-v_*|^{\gamma}\, \sin^{-(N-1)-\alpha}(\theta/2)$$

with $\gamma \in (-N,+\infty)$ and $\alpha \in [0,2)$.

The *Dirichlet form* of the corresponding linearized collision operator $L_{\mathcal{B}}$ is

$$\begin{aligned} D_{\gamma,\alpha}(g) &:= \langle L_{\mathcal{B}} g, g\rangle_{L^2(\mathbb{R}^N)} \\ &= \frac{1}{4}\int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} \left[\left(\frac{g(v')}{\mu(v')^{1/2}}\right) + \left(\frac{g(v'_*)}{\mu(v'_*)^{1/2}}\right) - \left(\frac{g(v)}{\mu(v)^{1/2}}\right) - \left(\frac{g(v_*)}{\mu(v_*)^{1/2}}\right)\right]^2 \\ &\qquad \times |v-v_*|^{\gamma}\, \sin^{-(N-1)-\alpha}(\theta/2)\, \mu(v)\,\mu(v_*)\, dv\, dv_*\, d\sigma. \end{aligned}$$

We want to prove the

Proposition 3.1. *For any $\varepsilon > 0$, there is an explicit constant $C_{\gamma,\alpha,\varepsilon} > 0$ such that*

$$D_{\gamma,\alpha}(g) \ge C_{\gamma,\alpha,\varepsilon} \int_{\mathbb{R}^N} [g - \mathbf{P}g]^2\, (1+|v|)^{\gamma+\alpha-\varepsilon}\, dv.$$

Proof of Proposition 3.1. For any $\beta \in (0,(N-1)+\alpha)$, we introduce the following angular truncation domain (which depends on $|v-v_*|$):

$$\mathcal{C}_\beta = \left\{\sigma \in \mathbb{S}^{N-1}\,;\ \sin^{-(N-1)-\alpha}(\theta/2) \ge |v-v_*|^{\beta}\right\}.$$

One checks easily that $\mathcal{C}_\beta$ is invariant under the pre-post collisional change of variables and the change of variable $(v, v_*, \sigma) \to (v_*, v, -\sigma)$ (for the different classical changes of variable we refer to [55, Chapter 1, Section 4]).

Hence we have

$$D_{\gamma,\alpha}(g) \geq \frac{1}{4} \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathcal{C}_\beta} \left[\left(\frac{g(v')}{\mu(v')^{1/2}} \right) + \left(\frac{g(v'_*)}{\mu(v'_*)^{1/2}} \right) - \left(\frac{g(v)}{\mu(v)^{1/2}} \right) - \left(\frac{g(v_*)}{\mu(v_*)^{1/2}} \right) \right]^2 \times |v - v_*|^{\gamma+\beta} \, \mu(v) \, \mu(v_*) \, dv \, dv_* \, d\sigma$$

$$= \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathcal{C}_\beta} \left[- \left(\frac{g(v')}{\mu(v')^{1/2}} \right) - \left(\frac{g(v'_*)}{\mu(v'_*)^{1/2}} \right) + \left(\frac{g(v)}{\mu(v)^{1/2}} \right) + \left(\frac{g(v_*)}{\mu(v_*)^{1/2}} \right) \right] \left(\frac{g(v)}{\mu(v)^{1/2}} \right) \times |v - v_*|^{\gamma+\beta} \, \mu(v) \, \mu(v_*) \, dv \, dv_* \, d\sigma$$

$$\geq \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathcal{C}_\beta} g^2 \, |v - v_*|^{\gamma+\beta} \, \mu(v_*) \, dv \, dv_* \, d\sigma$$

$$+ \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathcal{C}_\beta} g \, g_* \, |v - v_*|^{\gamma+\beta} \, \mu^{1/2}(v) \mu^{1/2}(v_*) \, dv \, dv_* \, d\sigma$$

$$- \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathcal{C}_\beta} g \, g' \, |v - v_*|^{\gamma+\beta} \, \mu^{1/2}(v_*) \mu^{1/2}(v'_*) \, dv \, dv_* \, d\sigma$$

$$- \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathcal{C}_\beta} g \, g'_* \, |v - v_*|^{\gamma+\beta} \, \mu^{1/2}(v_*) \mu^{1/2}(v'_*) \, dv \, dv_* \, d\sigma$$

$$=: D_1 + D_2 + D_3 + D_4.$$

In the following we shall bound D_1 from below, and D_2, D_3, D_4 from above.

For D_1 we have

$$D_1 \geq \int_{\mathbb{R}^N \times \mathbb{R}^N} g^2 \, |v - v_*|^{\gamma+\beta} \, \mu(v_*) \left(\int_{\sigma \in \mathcal{C}_\beta} d\sigma \right) dv \, dv_*.$$

An easy computation yields

$$\int_{\sigma \in \mathcal{C}_\beta} d\sigma = C \int_0^a \sin^{N-2} \theta \, d\theta$$

with $a = 2 \arcsin \left(|v - v_*|^{-\beta/((N-1)+\alpha)} \right) \geq C \, |v - v_*|^{-\beta/((N-1)+\alpha)}$, and thus

$$\int_{\sigma \in \mathcal{C}_\beta} d\sigma \geq C \int_0^a \theta^{N-2} \, d\theta = C \, a^{N-1} \geq C \, |v - v_*|^{-\frac{\beta \, (N-1)}{(N-1)+\alpha}}.$$

Hence we deduce

$$D_1 \geq C \int_{\mathbb{R}^N} g^2 \left(\int_{\mathbb{R}^N} |v - v_*|^{\gamma + \frac{\beta\alpha}{(N-1)+\alpha}} \mu(v_*) \, dv_* \right) dv$$
$$\geq C \int_{\mathbb{R}^N} g^2 \, (1 + |v|)^{\gamma + \frac{\beta\alpha}{(N-1)+\alpha}} \, dv.$$

This completes the estimate for the first term D_1.

Let us consider the second term D_2.

$$D_2 = \int_{\mathbb{R}^N \times \mathbb{R}^N} g \, g_* \, |v - v_*|^{\gamma+\beta} \, \mu^{1/2}(v) \mu^{1/2}(v_*) \, dv \, dv_*$$
$$\leq \int_{\mathbb{R}^N \times \mathbb{R}^N} g^2 \, \mu^{1/2}(v) \mu^{1/2}(v_*) \, |v - v_*|^{\gamma+\beta} \, dv \, dv_*$$
$$\leq C \left(\int_{\mathbb{R}^N} g^2 \, (1 + |v|)^{\gamma} \, dv \right).$$

Let us consider the terms D_3 and D_4. We remove the truncation $\mathcal{C}_\beta$ by bounding from above by the collision kernel without truncation, and we have

$$|D_3| + |D_4| \ \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} |g(v)| \, |g(v')| \, k_{\gamma+\beta}(v, v') \, dv \, dv'$$

(once the truncation is removed, the kernel is invariant under the change of variable $(v, v_*, \sigma) \to (v, v_*, -\sigma)$ which allows to reduce to the same term).

Hence we can write

$$|D_3| + |D_4| \ \leq C \left(\int_{\mathbb{R}^N} g^2 \, (1 + |v|)^{\gamma+\beta-(N-1)} \, dv \right)^{1/2}$$
$$\times \left[\int_{\mathbb{R}^N} (1 + |v|)^{-(\gamma+\beta-(N-1))} \left(\int_{\mathbb{R}^N} k_{\gamma+\beta}(v, v') \, |g(v')| \, dv' \right)^2 dv \right]^{1/2}$$
$$\leq C \left(\int_{\mathbb{R}^N} g^2 \, (1 + |v|)^{\gamma+\beta-(N-1)} \, dv \right)^{1/2}$$
$$\times \left[\int_{\mathbb{R}^N} (1 + |v|)^{-(\gamma+\beta-(N-1))} \left(\int_{\mathbb{R}^N} k_{\gamma+\beta}(v, v') \, dv' \right) \left(\int_{\mathbb{R}^N} k_{\gamma+\beta}(v, v'') \, g(v'')^2 \, dv'' \right) dv \right]^{1/2}.$$

Then we want to use Proposition 2.1 to get

$$(1 + |v|)^{-(\gamma+\beta-(N-1))} \left(\int_{\mathbb{R}^N} k_{\gamma+\beta}(v, v') \, dv' \right) \leq C. \tag{3.1}$$

It is possible as soon as $\gamma+\beta>-1$, *i.e.*, $\beta>-\gamma-1$. Since $\gamma>-N$ it is enough that $\beta>(N-1)$. Since $\alpha>0$ by assumption, it is always possible to pick β such that

$$(N-1)<\beta<(N-1)+\alpha.$$

For this choice of β, one can apply Proposition 2.1 to get (3.1).

Thus

$$\begin{aligned}|D_3|+|D_4| \ &\leq C\left(\int_{\mathbb{R}^N} g^2\,(1+|v|)^{\gamma+\beta-(N-1)}\,dv\right)^{1/2}\\ &\qquad\times\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} k_{\gamma+\beta}(v,v'')\,g(v'')^2\,dv''\right)\,dv\right]^{1/2}\\ &\leq C\left(\int_{\mathbb{R}^N} g^2\,(1+|v|)^{\gamma+\beta-(N-1)}\,dv\right)^{1/2}\\ &\qquad\qquad\times\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} k_{\gamma+\beta}(v,v'')\,dv\right)\,g(v'')^2\,dv''\right]^{1/2}.\end{aligned}$$

Using again Proposition 2.1 we have

$$\left(\int_{\mathbb{R}^N} k_{\gamma+\beta}(v,v'')\,dv\right)\leq C\,(1+|v''|)^{\gamma+\beta-(N-1)}$$

which yields finally

$$|D_3|+|D_4| \ \leq C\left(\int_{\mathbb{R}^N} g^2\,(1+|v|)^{\gamma+\beta-(N-1)}\,dv\right).$$

But the choice $\beta\in((N-1),(N-1)+\alpha)$ implies straightforwardly

$$\gamma<\gamma+\beta-(N-1)<\gamma+\frac{\beta\alpha}{(N-1)+\alpha}.$$

Hence by trivial interpolation we get for any $\delta>0$ there is an explicit constant $C_\delta>0$ such that

$$|D_3|+|D_4| \ \leq C_\delta\left(\int_{\mathbb{R}^N} g^2\,(1+|v|)^{\gamma}\,dv\right)+\delta\left(\int_{\mathbb{R}^N} g^2\,(1+|v|)^{\gamma+\frac{\beta\alpha}{(N-1)+\alpha}}\,dv\right).$$

Then by taking δ smaller than the constant in the bound from below for D_1 we deduce that

$$D_{\gamma,\alpha}(g)\geq C_+\left(\int_{\mathbb{R}^N} g^2\,(1+|v|)^{\gamma+\frac{\beta\alpha}{(N-1)+\alpha}}\,dv\right)-K_-\left(\int_{\mathbb{R}^N} g^2\,(1+|v|)^{\gamma}\,dv\right).$$

To conclude the proof we finally use the coercivity estimates in [44]. In this paper it is proved that (under our assumption on the collision kernel)

$$D_{\gamma,\alpha}(g) \geq C_0 \left(\int_{\mathbb{R}^N} \left[g - \mathbf{P}g \right]^2 (1+|v|)^\gamma \, dv \right)$$

for some explicit constant $C_0 > 0$.

By combining these two last inequalities, one deduces that

$$D_{\gamma,\alpha}(g) \geq C_1 \left(\int_{\mathbb{R}^N} \left[g - \mathbf{P}g \right]^2 (1+|v|)^{\gamma + \frac{\beta\alpha}{(N-1)+\alpha}} \, dv \right)$$

for some explicit constant $C_1 > 0$ (depending on β). Since β can be taken as close as wanted to $(N-1)+\alpha$, the weight exponent $\gamma + \beta\alpha/((N-1)+\alpha)$ can be taken as close as wanted to $\gamma + \alpha$. This concludes the proof. □

Proof of the first point in Theorem 1.1. To conclude the proof of the first point in Theorem 1.1, it is enough to remark that the proof of Proposition 3.1 can be modified easily in order to start from a collision kernel B such that

$$B \geq K \, B_{\gamma,\alpha} \, \mathbf{1}_{\theta \in [0,\theta_0]}$$

for some constants $K > 0$, $\theta_0 \in (0,\pi]$ and

$$B_{\gamma,\alpha}(|v-v_*|, \cos\theta) = |v-v_*|^\gamma \, \sin^{-(N-1)-\alpha}(\theta/2), \quad \gamma \in (-N,+\infty), \ \alpha \in [0,2),$$

defined as in the beginning of this section. The assumptions of Theorem 1.1 imply such a control.

Indeed one first reduces to the collision kernel $K \, B_{\gamma,\alpha} \, \mathbf{1}_{\theta \in [0,\theta_0]}$ by monotonicity of the Dirichlet form. Then the bound from above on D_2, D_3, D_4 are unchanged, and the bound from below on D_1 is still valid since one gets straightforwardly

$$\int_{\sigma \in \mathcal{C}_\beta} \mathbf{1}_{\theta \in [0,\theta_0]} \, d\sigma \geq \min \left\{ C_1 \, |v-v_*|^{-\frac{\beta(N-1)}{(N-1)+\alpha}} ; C_2 \right\}$$

for some constants $C_1, C_2 > 0$. Finally the coercivity estimates of [44] used in the proof are also still valid for a collision kernel $K \, B_{\gamma,\alpha} \, \mathbf{1}_{\theta \in [0,\theta_0]}$. □

4. Proof of Theorem 1.1 for $\varepsilon = 0$

We start again from some collision kernel which satisfies

$$B \geq K \, B_{\gamma,\alpha} \, \mathbf{1}_{\theta \in [0,\theta_0]}$$

for some constants $K > 0$, $\theta_0 \in (0,\pi]$ and

$$B_{\gamma,\alpha}(|v-v_*|, \cos\theta) = |v-v_*|^\gamma \, \sin^{-(N-1)-\alpha}(\theta/2), \quad \gamma \in (-N,+\infty), \ \alpha \in [0,2).$$

We shall prove the

Proposition 4.1. *There is some constant $C_{B,0} > 0$ (obtained in our proof by compactness argument) such that*

$$D_B(g) \geq C_{B,0} \int_{\mathbb{R}^N} \left[g - \mathbf{P}g\right]^2 (1+|v|)^{\gamma+\alpha}\, dv.$$

The second point in Theorem 1.1 follows immediately from Proposition 4.1.

Proof of Proposition 4.1. We introduce the following angular truncation domain (which depends on $|v - v_*|$):

$$\bar{\mathcal{C}} = \left\{\sigma \in \mathbb{S}^{N-1};\ |v - v'| \leq 1\right\}.$$

One checks easily that $\bar{\mathcal{C}}$ is invariant under the pre-post collisional change of variables and the change of variable $(v, v_*, \sigma) \to (v_*, v, -\sigma)$. Remark that this truncation domain $\bar{\mathcal{C}}$ corresponds to the limit case $\beta = (N-1) + \alpha$ in the truncation domain $\mathcal{C}_\beta$ previously introduced.

Hence we have

$$D_{\gamma,\alpha}(g) \geq \frac{K}{4} \int_{\mathbb{R}^N \times \mathbb{R}^N \times \bar{\mathcal{C}}} \left[\left(\frac{g(v')}{\mu(v')^{1/2}}\right) + \left(\frac{g(v'_*)}{\mu(v'_*)^{1/2}}\right) - \left(\frac{g(v)}{\mu(v)^{1/2}}\right) - \left(\frac{g(v_*)}{\mu(v_*)^{1/2}}\right)\right]^2$$
$$\times |v - v_*|^{\gamma+\alpha+(N-1)}\, \mathbf{1}_{\theta\in[0,\theta_0]}\, \mu(v)\, \mu(v_*)\, dv\, dv_*\, d\sigma$$
$$= K \int_{\mathbb{R}^N \times \mathbb{R}^N \times \bar{\mathcal{C}}} \left[-\left(\frac{g(v')}{\mu(v')^{1/2}}\right) - \left(\frac{g(v'_*)}{\mu(v'_*)^{1/2}}\right) + \left(\frac{g(v)}{\mu(v)^{1/2}}\right) + \left(\frac{g(v_*)}{\mu(v_*)^{1/2}}\right)\right] \left(\frac{g(v)}{\mu(v)^{1/2}}\right)$$
$$\times |v - v_*|^{\gamma+\alpha+(N-1)}\, \mathbf{1}_{\theta\in[0,\theta_0]}\, \mu(v)\, \mu(v_*)\, dv\, dv_*\, d\sigma$$
$$= K\, \langle \hat{L} g, g\rangle$$

with a fictious self-adjoint operator $\hat{L}$ on L^2 defined by

$$\hat{L} g(v) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \left[-\left(\frac{g(v')}{\mu(v')^{1/2}}\right) - \left(\frac{g(v'_*)}{\mu(v'_*)^{1/2}}\right) + \left(\frac{g(v)}{\mu(v)^{1/2}}\right) + \left(\frac{g(v_*)}{\mu(v_*)^{1/2}}\right)\right]$$
$$|v - v_*|^{\gamma+\alpha+(N-1)}\, \mathbf{1}_{|v-v'|\leq 1}\, \mathbf{1}_{\theta\in[0,\theta_0]}\, \mu^{1/2}(v)\, \mu(v_*)\, dv_*\, d\sigma.$$

Using the same decomposition as in Section 2, this operator can be split as

$$\hat{L} = \hat{\nu} - \hat{K}^+ + \hat{K}^c$$

where the multiplicative function $\hat{\nu}$ is

$$\hat{\nu}(v) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} |v - v_*|^{\gamma+\alpha+(N-1)}\, \mathbf{1}_{|v-v'|\leq 1}\, \mathbf{1}_{\theta\in[0,\theta_0]}\, \mu(v_*)\, dv_*\, d\sigma$$

which satisfies by similar computations as above

$$\hat{\nu}(v) \geq C\, (1+|v|)^{\gamma+\alpha}, \quad C > 0.$$

Then we shall show that the remaining terms satisfies some compactness property.

Let us assume first for the sake of clarity that $\gamma + \alpha = 0$. Then the multiplication function $\hat\nu$ is bounded from below by a positive constant $\hat\nu_0 > 0$, and it is straightforward that $\hat K^c$ is a Hilbert-Schmidt operator (we leave these computations to the reader). Let us show that the part $\hat K^+$ can be written as a limit of Hilbert-Schmidt operators (hence showing that it is compact in L^2). The kernel of $\hat K^+$ is by inspection

$$\begin{aligned}\hat k &:= k_{(N-1)}(v,v')\,\mathbf{1}_{|v-v'|\le 1}\,\mathbf{1}_{\theta\in[0,\theta_0]}\\ &\quad = \frac{2^N}{|v'-v|\,(2\pi)^{3/2}}\exp\left\{-\frac{|v'-v|^2}{8}-\frac{|v'-v+2(v\cdot\omega)\omega|^2}{8}\right\}\\ &\quad \times\left(\int_{\omega^\perp}|v'-v+z|\,\exp\left\{-\frac{|z+(v-(v\cdot\omega)\omega)\,|^2}{2}\right\}\,dz\right)\mathbf{1}_{|v-v'|\le 1}\,\mathbf{1}_{\theta\in[0,\theta_0]}\end{aligned}$$

(where k_q, $q\in(-N,1]$ was the kernel computed in Section 2). We approximate this kernel (and correspondingly the operator $\hat K^+$) as follows:

$$\hat k = \hat k^c_\varepsilon + \hat k^r_\varepsilon$$

with

$$\hat k^c_\varepsilon = \left(\mathbf{1}_{|v-v'|\ge\varepsilon}\times\mathbf{1}_{\left|\frac{v}{|v|}\cdot\frac{(v-v')}{|v-v'|}\right|\ge\varepsilon}\right)\,\hat k$$

and

$$\hat k^r_\varepsilon = \hat k - \hat k^c_\varepsilon.$$

By similar straightforward computations as in the proof of Proposition 2.1 we get that $\hat k^r_\varepsilon$ is symmetric in v, v' and

$$\sup_{v\in\mathbb{R}^N}\int_{\mathbb{R}^N}|\hat k^r_\varepsilon|\,dv' \xrightarrow{\varepsilon\to 0} 0$$

and therefore $\hat K^{+,c}_\varepsilon \to \hat K^+$ in L^2 as $\varepsilon\to 0$. Hence it is enough to show that $\hat K^{+,c}$ is compact. But the kernel $\hat k^c_\varepsilon$ satisfies (using the same representation as in

Proposition 2.1)

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \left(\hat{k}^c_\varepsilon\right)^2 dv\,dv' \le C \int_{v\in\mathbb{R}^N} \Bigg(\int_\varepsilon^1 r^{N-3}\,(1+r)^2\,(1+|v|\,\sin\theta)^2\,e^{-\frac{r^2}{4}}$$

$$\int_0^\pi e^{-\frac{(r+2|v|\cos\theta)^2}{4}}\,\sin^{N-2}\theta\,\mathbf{1}_{|\cos\theta|\ge\varepsilon}\,d\theta\Bigg)$$

$$\le C\int_{v\in\mathbb{R}^N}(1+|v|)^2\,e^{-\varepsilon^2|v|^2}\,dv<+\infty.$$

Therefore $\hat{K}^{+,c}$ is a Hilbert-Schmidt operator and the result is proved. By applying Weyl's theorem (exactly as in [32, 19]), we deduce that $\hat{L}$ and $\hat{\nu}$ have the same essential spectrum, which is included in $[\hat{\nu}_0,+\infty)$, and therefore, since $\hat{L}\ge 0$, that 0 is an isolated eigenvalue, which concludes the proof.

When $\gamma+\alpha$ is different from 0, one considers (in the spirit of [29]) the following symmetric weighted modification of $\hat{L}$:

$$\tilde{L}=(1+|\cdot|)^{-(\gamma+\alpha)/2}\,\hat{L}\left((1+|\cdot|)^{-(\gamma+\alpha)/2}\,\cdot\right)$$

and the corresponding splitting $\tilde{L}=\tilde{\nu}-\tilde{K}^++\tilde{K}^c$. Then $\tilde{\nu}$ is strictly positive uniformly (and bounded from above) and it is straightforward again that $\tilde{K}^c$ is a Hilbert-Schmidt operator. Let us focus on the term $\tilde{K}^+$. Its kernel is

$$(1+|v|)^{-(\gamma+\alpha)/2}\,k_{\gamma+\alpha+(N-1)}(v,v')\,(1+|v'|)^{-(\gamma+\alpha)/2}\,\mathbf{1}_{|v-v'|\le 1}\,\mathbf{1}_{\theta\in[0,\theta_0]}$$

and similar computations as above show again that $\tilde{K}^+$ can be written as a limit of Hilbert-Schmidt operators (remark that thanks to the truncation $|v-v'|\le 1$, weights on v and v' can be interchanged up to a constant).

We then conclude by applying Weyl's Theorem to $\tilde{L}$ (as in [29]). We deduce thus that 0 is an isolated eigenvalue of $\tilde{L}\ge 0$, and therefore we obtain the existence of the inequality in Proposition 4.1. $\square$

5. Proof of Theorem 1.2

In [34] it was shown *via* compactness arguments that there is a constant $C^1_\gamma>0$ such that

$$\langle L^{\mathcal{L}}g,g\rangle \ge C^1_\gamma\,\big\|[g-\mathbf{P}g]\big\|^2_\sigma. \tag{5.1}$$

where $\|\cdot\|_\sigma$ is the following anisotropic norm:

$$\|g\|^2_\sigma := \int_{\mathbb{R}^N}\left((1+|v|)^\gamma\,\big|\Pi_v\nabla_v g\big|^2+(1+|v|)^{\gamma+2}\,\big|[\mathrm{Id}-\Pi_v]\nabla_v g\big|^2+(1+|v|)^{\gamma+2}\,g^2\right)dv$$

with

$$\Pi_v \nabla_v g = \left(\frac{v}{|v|} \cdot \nabla_v g \right) \frac{v}{|v|}.$$

In particular this lower bound implies a spectral gap for the linearized Landau collision operator as soon as $\gamma \geq -2$.

On the other hand, in [44] the first author derived the following explicit coercivity estimate:

$$\langle L^{\mathcal{L}}[\mu^{1/2}h], [\mu^{1/2}h] \rangle \geq C_\gamma^2 \int_{\mathbb{R}^N} (1+|v|^2)^{\gamma/2} \Big(|\nabla_v h|^2 + h^2 \Big) \mu(v)\, dv = C_\gamma^2 \, \|h\|_{H^1_\gamma(\mu)}. \tag{5.2}$$

This holds for all $h = h - \hat{\mathbf{P}}h$, where $\hat{\mathbf{P}}$ is the orthogonal projection in $L^2(\mu)$ given by

$$\hat{\mathbf{P}}h = a + b \cdot v + c|v|^2$$

with $a, c \in \mathbb{R}$ and $b \in \mathbb{R}^N$. This estimate originates from the alternate (but equivalent) linearization

$$f = \mu(1+h).$$

Proof of Theorem 1.2. Letting $h = \mu^{-1/2}g$, we first translate the estimate of [44] into a coercivity estimate for g: the r.h.s. of (5.2) is given by

$$\begin{aligned} \|\mu^{-1/2} g\|_{H^1_\gamma(\mu)} &= \int_{\mathbb{R}^N} (1+|v|^2)^{\gamma/2} \left(\left| \nabla_v g + \frac{1}{2} v g \right|^2 + g^2 \right) dv \\ &= \int_{\mathbb{R}^N} (1+|v|^2)^{\gamma/2} \left(|\nabla_v g|^2 + \frac{1}{4} |v|^2 \, |g|^2 + (v \cdot \nabla_v g)\, g + g^2 \right) dv. \end{aligned}$$

We focus our attention on the term without a definite sign:

$$\begin{aligned} \int_{\mathbb{R}^N} (1+|v|^2)^{\gamma/2} (v \cdot \nabla_v g) g \, dv &= \frac{1}{2} \int_{\mathbb{R}^N} (1+|v|^2)^{\gamma/2} v \cdot \nabla_v (g^2)\, dv \\ &= -\frac{1}{2} \int_{\mathbb{R}^N} \nabla \cdot \left(v (1+|v|^2)^{\gamma/2} \right) g^2 \, dv. \end{aligned}$$

Now we look at the derivative of the polynomial

$$\nabla \cdot \left(v(1+|v|^2)^{\gamma/2} \right) = (1+|v|^2)^{\gamma/2} \left(N + \frac{\gamma\, |v|^2}{1+|v|^2} \right).$$

Further notice that

$$\left| N + \frac{\gamma\, |v|^2}{1+|v|^2} \right| \leq N + |\gamma|.$$

Plug in these last few computations to obtain

$$\|\mu^{-1/2}g\|_{H^1_\gamma(\mu)} \geq \frac{1}{4}\int_{\mathbb{R}^N} \left((1+|v|^2)^{\frac{\gamma}{2}}\ |\nabla_v g|^2 + (1+|v|^2)^{\frac{\gamma+2}{2}}\ g^2\right)\, dv$$
$$- \frac{1}{2}(N+|\gamma|)\int_{\mathbb{R}^N}(1+|v|^2)^{\frac{\gamma}{2}} g^2\, dv.$$

On the other hand

$$\|\mu^{-1/2}g\|_{H^1_\gamma(\mu)} \geq \int_{\mathbb{R}^N}(1+|v|^2)^{\frac{\gamma}{2}}\ g^2\, dv.$$

Combining the two last inequalities we thus conclude

$$\|\mu^{-1/2}g\|_{H^1_\gamma(\mu)} \geq C \int_{\mathbb{R}^N} \left[(1+|v|^2)^{\frac{\gamma}{2}}\ |\nabla_v g|^2 + (1+|v|^2)^{\frac{\gamma+2}{2}}\ g^2\right] dv.$$

Combine this with (5.2) to obtain an explicitly computable constant $C > 0$ such that for any $g = g - \mathbf{P}g$, we have

$$\langle L^{\mathcal{L}} g, g\rangle \geq C \int_{\mathbb{R}^N} \left((1+|v|^2)^{\frac{\gamma}{2}} |\nabla_v g|^2 + (1+|v|^2)^{\frac{\gamma+2}{2}}\ g^2\right)\, dv. \tag{5.3}$$

While the power is the same as in (5.1) for the term with no derivative, the derivative term still has a better power in (5.1).

Guo [34] computes that

$$\langle L^{\mathcal{L}} g, g\rangle = \|g\|_\sigma^2 - \langle \partial_i \sigma^i g, g\rangle - \langle Kg, g\rangle.$$

Here we will not give precise definitions of $\partial_i \sigma^i$ and K, we will only use an estimate with explicitly computable constants for these terms to get an explicit lower bound in the σ norm.

The [34, Lemma 5] implies that for any $m > 1$, there is an (explicit) $0 < C(m) < \infty$ such that

$$|\langle \partial_i \sigma^i g, g\rangle| + |\langle Kg_1, g_2\rangle| \leq \frac{1}{m}\, \|g\|_\sigma^2 + C(m)\ \left\|\mathbf{1}_{\{|\cdot|\leq C(m)\}} g\right\|_{L^2}^2.$$

We deduce

$$|\langle \partial_i \sigma^i g, g\rangle| + |\langle Kg_1, g_2\rangle| \leq \frac{1}{m}\, \|g\|_\sigma^2 + C'(m)\, \langle L^{\mathcal{L}} g, g\rangle$$

for another explicit constant $C'(m) > 0$ thanks to (5.3). Taking for instance $m = 2$ we deduce

$$\|g\|_\sigma^2 = \langle L^{\mathcal{L}} g, g\rangle + \langle \partial_i \sigma^i g, g\rangle + \langle Kg, g\rangle \leq (1 + C'(2))\, \langle L^{\mathcal{L}} g, g\rangle + \frac{1}{2}\ \|g\|_\sigma^2,$$

which concludes the proof. $\square$

Acknowledgments: This work was initiated while the first author was visiting Brown University, and he wishes to thank its Department of Applied Mathematics, in particular Yan Guo and Robert Strain for their hospitality. The second author, while finishing this work, was partially supported by a National Science Foundation Mathematical Sciences Postdoctoral Research Fellowship.

References

[1] Alexandre, R. Remarks on 3D Boltzmann linear equation without cutoff. *Transport Theory Statist. Phys. 28*, 5 (1999), 433–473.

[2] Alexandre, R. Une définition des solutions renormalisées pour l'équation de Boltzmann sans troncature angulaire. *C. R. Acad. Sci. Paris Sér. I Math. 328*, 11 (1999), 987–991.

[3] Alexandre, R. Around 3D Boltzmann non linear operator without angular cutoff, a new formulation. Math. Model. Numer. Anal. 34, 3 (2000), 575–590.

[4] Alexandre, R. and El Safadi, M. Littlewood-Paley theory and regularity issues in Boltzmann homogeneous equations. I. Non-cutoff case and Maxwellian molecules. *Math. Models Methods Appl. Sci. 15*, 6 (2005), 907–920.

[5] Alexandre, R., Desvillettes, L., Villani, C., Wennberg, B. Entropy dissipation and long-range interactions. *Arch. Rational Mech. Anal. 152* (2000), 327–355.

[6] Alexandre, R. and Villani, C. On the Boltzmann equation for long-range interactions. *Comm. Pure Appl. Math. 55*, 1 (2002), 30–70.

[7] Alexandre, R., Villani, C., On the Landau approximation in plasma physics. *Ann. Inst. H. Poincaré Anal. Non Linéaire 21*, 1 (2004), 61–95.

[8] Arkeryd, L. Intermolecular forces of infinite range and the Boltzmann equation. *Arch. Rational Mech. Anal. 77*, 1 (1981), 11–21.

[9] Arkeryd, L. Asymptotic behaviour of the Boltzmann equation with infinite range forces. *Comm. Math. Phys. 86*, 4 (1982), 475–484.

[10] Arsen′ev, A. A., Buryak, O. E. On a connection between the solution of the Boltzmann equation and the solution of the Landau-Fokker-Planck equation. *Math. USSR Sbornik 69*, 2 (1991), 465–478.

[11] Baranger, C. and Mouhot, C. Explicit spectral gap estimates for the linearized Boltzmann and Landau operators with hard potentials. *Rev. Matem. Iberoam. 21*(2005), 819–841.

[12] Bobylëv, A. V., The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules. *Mathematical physics reviews, Vol. 7* (1988), Soviet Sci. Rev. Sect. C Math. Phys. Rev., 111–233.

[13] Bobylev, A. V. and Cercignani, C. On the rate of entropy production for the Boltzmann equation. *J. Statist. Phys. 94* (1999), 603–618.

[14] Caflisch, R. E. The Boltzmann equation with a soft potential. I. Linear, spatially-homogeneous. *Comm. Math. Phys. 74* (1980), 71–95.

[15] Carleman, T. Problèmes mathématiques dans la théorie cinétique des gaz. Almqvist and Wiksells Boktryckeri Ab, Uppsala 1957.

[16] Cercignani, C. On Boltzmann Equation with Cutoff Potentials. *Phys. Fluids 10* (1967), 2097–2104.

[17] Cercignani, C. The Boltzmann equation and its applications. Applied Mathematical Sciences, 67. Springer-Verlag, New York, 1988.

[18] Czechowski, Z. and Palczewski, A. Spectrum of the Boltzmann collision operator for radial cut-off potentials. *Bull. Acad. Polon. Sci. Sér. Sci. Tech. 28*, 9–10 (1980), 387–396.

[19] Cercignani, C. and Illner, R. and Pulvirenti, M. The mathematical theory of dilute gases. Springer-Verlag, New York, 1994.

[20] Degond, P. and Lemou, M. Dispersion relations for the linearized Fokker-Planck equation. *Arch. Rational Mech. Anal.* **138** (1997), no. 2, 137–167.

[21] Degond, P. and Lucquin-Desreux, B., The Fokker-Planck asymptotics of the Boltzmann collision operator in the Coulomb case. *Math. Models Methods Appl. Sci. 2*, 2 (1992), 167–182.

[22] Desvillettes, L., On asymptotics of the Boltzmann equation when the collisions become grazing. *Transport Theory Statist. Phys. 21*, 3 (1992), 259–276.

[23] Desvillettes, L. About the regularizing properties of the non-cut-off Kac equation. *Comm. Math. Phys. 168*, 2 (1995), 417–440.

[24] Desvillettes, L. Regularization for the non-cutoff 2D radially symmetric Boltzmann equation with a velocity dependent cross section. Proceedings of the Second International Workshop on Nonlinear Kinetic Theories and Mathematical Aspects of Hyperbolic Systems (Sanremo, 1994). *Transport Theory Statist. Phys. 25*, 3–5 (1996), 383–394.

[25] Desvillettes, L. Regularization properties of the 2-dimensional non-radially symmetric non-cutoff spatially homogeneous Boltzmann equation for Maxwellian molecules. *Transport Theory Statist. Phys. 26*, 3 (1997), 341–357.

[26] Desvillettes, L. and Villani, C. On the spatially homogeneous Landau equation for hard potentials. II. H-theorem and applications. *Comm. Partial Differential Equations 25*, 1–2 (2000), 261–298.

[27] DiPerna, R. J. and Lions, P.-L. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2) 130*, 2 (1989), 321–366.

[28] Elmroth, T. Global boundedness of moments of solutions of the Boltzmann equation for forces of infinite range. *Arch. Rational Mech. Anal. 82*, 1 (1983), 1–12.

[29] Golse, F. and Poupaud, F. Un résultat de compacité pour l'équation de Boltzmann avec potentiel mou. Application au problème de demi-espace. *C. R. Acad. Sci. Paris Sér. I Math. 303* (1986), 585–586.

[30] Goudon, T. On Boltzmann equations and Fokker-Planck asymptotics: influence of grazing collisions. *J. Statist. Phys. 89*, 3–4 (1997), 751–776.

[31] Grad, H. Principles of the kinetic theory of gases. In *Flügge's Handbuch des Physik*, vol. XII, Springer-Verlag (1958), pp. 205–294.

[32] Grad, H. Asymptotic theory of the Boltzmann equation. II. *Rarefied Gas Dynamics (Proc. 3rd Internat. Sympos., Palais de l'UNESCO, Paris, 1962)*, Vol. I (1963), pp. 26–59.

[33] Guo, Y. The Vlasov-Poisson-Boltzmann system near Maxwellians. *Comm. Pure Appl. Math. 55* (2002), 1104–1135.

[34] Guo, Y. The Landau equation in a periodic box. *Comm. Math. Phys. 231* (2002), 391–434.

[35] Guo, Y. The Vlasov-Maxwell-Boltzmann system near Maxwellians. *Invent. Math. 153* (2003), 593–630.

[36] Guo, Y. Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Rational Mech. Anal. 169* (2003), 305–353.

[37] Guo, Y. The Boltzmann equation in the whole space. *Indiana Univ. Math. J. 53* (2004), 1081–1094.

[38] Hilbert, D. Grundzüge einer Allgemeinen Theorie der Linearen Integralgleichungen. *Math. Ann. 72*, (1912), Chelsea Publ., New York, (1953).

[39] Hinton, F. L. Collisional transport in plasma. *Handbook of Plasma Physics, Vol. 1*, North-Holland, Amsterdam, 1983.

[40] Klaus, M. Boltzmann collision operator without cut-off. *Helv. Phys. Acta 50* (1977), 893–903.

[41] Lions, P.-L. Régularité et compacité pour des noyaux de collision de Boltzmann sans troncature angulaire. *C. R. Acad. Sci. Paris Sér. I Math. 326*, 1 (1998), 37–41.

[42] Landau, L. D. Die kinetische Gleichung für den Fall Coulombscher Wechselwirkung. *Phys. Z. Sowjet. 10* (1936), 154. Translation: The transport equation in the case of Coulomb interactions. In D. ter Haar, ed., *Collected papers of L. D. Landau*, pp. 163–170. Pergamon Press, Oxford, 1981.

[43] Mouhot, C. Rate of convergence to equilibrium for the spatially homogeneous Boltzmann equation with hard potentials. *Comm. Math. Phys. 261* (2006), 629–672.

[44] Mouhot, C. Coercivity estimates for the Boltzmann and Landau operators. To appear in *Comm. Partial Diff. Equations*.

[45] Mouhot, C. and Neumann, L. Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus. *Nonlinearity 19* (2006), 969–998.

[46] Pao, Y. P. Boltzmann collision operator with inverse-power intermolecular potentials. I, II. *Comm. Pure Appl. Math. 27* (1974), 407–428, 559–581.

[47] Strain, R. M., Guo, Y. Stability of the relativistic Maxwellian in a collisional plasma. *Comm. Math. Phys. 251* (2004), 263–320.

[48] Strain, R. M., Guo, Y. Almost Exponential Decay Near Maxwellian. *Comm. Partial Diff. Equations 31* (2006), 417–429.

[49] Strain, R. M., Guo, Y. Exponential Decay for Soft Potentials Near Maxwellian. Accepted for publication in *Arch. Rational Mech. Anal.*.

[50] Strain, R. M. The Vlasov-Maxwell-Boltzmann System in the Whole Space. Accepted for publication in *Comm. Math. Phys.*

[51] Ukai, S. and Asano, K. On the Cauchy problem of the Boltzmann equation with a soft potential. *Publ. Res. Inst. Math. Sci. 18*, 2 (1982), 477–519 (57–99).

[52] Villani, C. On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. *Arch. Rational Mech. Anal. 143*, 3 (1998), 273–307.

[53] Villani, C. Regularity estimates via the entropy dissipation for the spatially homogeneous Boltzmann equation without cut-off. *Rev. Mat. Iberoamericana 15*, 2 (1999), 335–352.

[54] Villani, C. Contribution à l'étude mathématique des collisions en théorie cinétique. Univ. Paris-Dauphine France (2000).

[55] Villani, C. A review of mathematical topics in collisional kinetic theory. *Handbook of mathematical fluid dynamics, Vol. I*, 71–305, North-Holland, Amsterdam, 2002.

[56] Villani, C. Cercignani's conjecture is sometimes true and always almost true. *Comm. Math. Phys. 243* (2003), 455–490.

[57] Wang Chang, C. S. and Uhlenbeck, G. E. On the propagation of Sound in Monoatomic Gases. Univ. of Michigan Press, Ann Arbor, Michigan (quoted in [32]).

[58] Wang Chang, C. S. and Uhlenbeck, G. E. and de Boer, J. Studies in Statistical Mechanics, Vol. V. North-Holland, Amsterdam, 1970.

C. Mouhot

CEREMADE, Univ. Paris IX Dauphine
Place du Maréchal de Lattres de Tassigny
75775 Paris cedex 16
FRANCE

e-mail: cmouhot@ceremade.dauphine.fr

R. M. Strain

Harvard University
Department of Mathematics
One Oxford Street
Cambridge, MA 02138
USA

e-mail: strain@math.harvard.edu