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# On the expectation of normalized Brownian functionals up to first hitting times

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## Abstract

Let  $B$  be a Brownian motion and  $T_1$  its first hitting time of the level 1. For  $U$  a uniform random variable independent of  $B$ , we study in depth the distribution of  $B_{UT_1}/\sqrt{T_1}$ , that is the rescaled Brownian motion sampled at uniform time. In particular, we show that this variable is centered.

**Keywords:** Brownian motion, hitting times, scaling, random sampling, Bessel process, Brownian meander, Ray-Knight theorem, Feynman-Kac formula.

## 1 Introduction

In this paper, we study the expectations of the random variables  $A_a^{(m)}$  and  $\tilde{A}_a^{(m)}$  defined for  $a > 0$  and  $m \geq 0$  by

$$A_a^{(m)} = \frac{1}{T_a^{1+m/2}} \int_0^{T_a} |B_s|^m \operatorname{sgn}(B_s) ds, \quad \tilde{A}_a^{(m)} = \frac{1}{T_a^{1+m/2}} \int_0^{T_a} |B_s|^m ds,$$

where  $B$  is a Brownian motion and  $T_a$  denotes the first hitting time of the level  $a$  by  $B$ . First, remark that  $T_a/a^2$  is the first hitting time of  $a$  by  $(B_{a^2s})$ . Therefore, the scaling property of the Brownian motion implies that the laws of  $A_a^{(m)}$  and  $\tilde{A}_a^{(m)}$  do not depend on  $a$ .

To fix ideas, let us now focus in this introduction on the variables  $A_a^{(m)}$ . These variables are clearly asymmetric functionals of the Brownian motion. Nevertheless, we may wonder whether there exist values of  $m$  such that  $A_a^{(m)}$  is centered (we will show later that these variables have moments of all orders). Indeed, consider for example the case where  $m$  is an odd integer: using a symmetry argument, it is clear that

$$\mathbb{E}[A_a^{(m)}] = -\mathbb{E}[A_{-a}^{(m)}],$$

where  $A_{-a}^{(m)}$  is obviously defined. Since these two quantities do not depend on  $a$ , we get a given value, say  $v_m$ , for the expectations when the barrier is positive and  $-v_m$  when it is negative. This somewhat suggests that  $v_m$  may be equal to zero.

In fact, it turns out that the random variable  $A_a^{(m)}$  is centered only for  $m = 1$ . This result has several interesting consequences. In particular, we show that it can be very simply interpreted in terms of the Brownian meander. Moreover, we prove that the expectation of  $A_a^{(m)}$  is negative for  $m < 1$  and positive for  $m > 1$ .

Finally, note that these expectations are closely connected with the random variable  $\alpha$  defined by

$$\alpha = \frac{B_{UT_1}}{\sqrt{T_1}},$$

where  $U$  is a uniform random variable, independent of  $B$ . For example, for  $m$  an odd integer, the  $m$ -th moment of  $\alpha$  is the expectation of  $A_1^{(m)}$ . This led us to give in Theorem 3.2 the law of  $\alpha$ .

The paper is organized as follows. The specific case  $m = 1$  is treated in Section 2. Our main theorem which provides the expectations of  $A_1^{(m)}$  for any  $m \geq 0$  is given in Section 3 together with its proof and some related results. The proofs of several technical results together with additional remarks are relegated to the four Appendices A, B, C and D.

## 2 The case $m = 1$

In this section, we state the nullity of the expectation of  $A_1^{(1)}$ , together with some associated results.

### 2.1 Centering property in the case $m = 1$

**Theorem 2.1.** *The random variable  $A_1^{(1)}$  admits moments of all orders and is centered.*

Theorem 2.1 states that, as far as the expectation is concerned, between 0 and  $T_1$ , the time spent by the Brownian motion in  $(-\infty, 0)$  is balanced by that spent in  $[0, 1]$ . Again, it is tempting to deduce this result from the scaling and symmetry properties of  $A_1^{(1)}$ . However, Theorem 3.1 will formalize that such intuition is wrong. Indeed, we will for example show that the expectation of  $A_1^{(3)}$  is non zero, although it satisfies the same scaling and symmetry properties as  $A_1^{(1)}$ . In fact, we will see that the expectation of  $A_1^{(m)}$  is strictly positive for  $m > 1$  and strictly negative for  $m < 1$ .

Theorem 2.1 can in fact be interpreted as a corollary of the general result given in Theorem 3.1 below. However, using Williams time reversal theorem and some absolute continuity results for Bessel processes, a specific, elegant proof can be written for Theorem 2.1. So we give this proof in Appendix A.

### 2.2 More integrability properties for $A_1^{(1)}$ and connection with Knight's identity

Let  $(L_t)_{t \geq 0}$  be the local time process at 0 of the Brownian motion  $B$  and set

$$\tau_l = \inf\{t \geq 0, L_t > l\},$$

for  $l > 0$ . Recall that Lévy's equivalence result, gives the following equality:

$$(|B_t|, L_t)_{t \geq 0} \stackrel{\mathcal{L}}{=} (S_t - B_t, S_t)_{t \geq 0},$$

with  $S_t = \sup_{s \leq t} B_s$  and  $\stackrel{\mathcal{L}}{=}$  denotes equality in law, see [10]. Thus, we obtain that  $A_1^{(1)}$  has the same law as the random variable  $\zeta$  defined by

$$\zeta = \frac{1}{\tau_1^{3/2}} \int_0^{\tau_1} (L_u - |B_u|) du.$$

We obviously have

$$|\zeta| \leq \frac{1}{\sqrt{\tau_1}} + \frac{1}{\sqrt{\tau_1}} \sup_{u \leq \tau_1} |B_u|.$$

On the one hand, it is well known that  $1/\sqrt{\tau_1}$  follows the law of the absolute value of a standard Gaussian random variable. On the other hand, the celebrated Knight's identity states the following equality:

$$\frac{\tau_1}{(\sup_{u \leq \tau_1} |B_u|)^2} \stackrel{\mathcal{L}}{=} T_2^3,$$

where  $T_2^3 = \inf\{t, R_t = 2\}$ , with  $R$  a three dimensional Bessel process, see [7]. Using the scaling property of the three dimensional Bessel process, we easily get the equality:

$$T_2^3 \stackrel{\mathcal{L}}{=} \frac{4}{(\sup_{u \leq 1} R_u)^2}.$$

Therefore, we deduce that

$$\frac{1}{\sqrt{\tau_1}} \sup_{u \leq \tau_1} |B_u| \stackrel{\mathcal{L}}{=} \frac{1}{2} \sup_{u \leq 1} R_u.$$

Hence, we easily deduce the following proposition:

**Proposition 2.1.** *There exists  $\varepsilon > 0$  such that*

$$\mathbb{E}[\exp(\varepsilon (A_1^{(1)})^2)] < +\infty.$$

We note that the same arguments yield that  $A_1^{(m)}$  and  $\tilde{A}_1^{(m)}$  admit moments of all orders.

### 2.3 Consequences of Theorem 2.1 for the Bessel process, Brownian meander and Brownian bridge

We give in this subsection some corollaries of Theorem 2.1 involving very classical processes, namely the three dimensional Bessel process, the Brownian meander, and the Brownian bridge. We start with a result about the three dimensional process, whose proof is given within the proof of Theorem 2.1 in Appendix A.

**Corollary 2.1.** *Let  $(R_t)_{t \geq 0}$  denote a three dimensional Bessel process. We have*

$$\mathbb{E}\left[\frac{1}{R_1^2} \int_0^1 R_u du\right] = \sqrt{\frac{2}{\pi}}.$$

Thanks to Imhof's relation, see [3, 6], we immediately deduce the following result from Corollary 2.1:

**Corollary 2.2.** *Let  $(m_t)_{t \leq 1}$  be the Brownian meander. We have*

$$\mathbb{E}\left[\frac{1}{m_1} \int_0^1 m_u du\right] = 1.$$

We now give a corollary involving the Brownian bridge.

**Corollary 2.3.** *Let  $(b_t)_{t \leq 1}$  denote the Brownian bridge and  $(l_t)_{t \leq 1}$  its local time at zero. We have*

$$\mathbb{E}\left[\frac{1}{l_1} \int_0^1 |b_u| du\right] = \mathbb{E}\left[\frac{1}{l_1} \int_0^1 l_u du\right] = \frac{1}{2}.$$

*Proof.* From [4], we get the following equality:

$$(m_t, t \leq 1) \stackrel{\mathcal{L}}{=} (|b_t| + l_t, t \leq 1).$$

Thus, using Corollary 2.2, we get

$$\mathbb{E}\left[\frac{1}{l_1} \int_0^1 (|b_u| + l_u) du\right] = 1. \tag{1}$$

Now remark that the process  $(\hat{b}_t) = (b_{1-t})$  is also a Brownian bridge whose local time at time  $t$ , denoted by  $\hat{l}_t$ , satisfies

$$\hat{l}_t = l_1 - l_{1-t}.$$

Consequently,

$$\mathbb{E}\left[\frac{1}{l_1} \int_0^1 l_u du\right] = \mathbb{E}\left[\frac{1}{l_1} \int_0^1 \hat{l}_u du\right] = \mathbb{E}\left[\frac{1}{l_1} \int_0^1 (l_1 - l_u) du\right].$$

This implies

$$\mathbb{E}\left[\frac{1}{l_1} \int_0^1 l_u du\right] = \frac{1}{2}$$

and therefore Equation (1) provides

$$\mathbb{E}\left[\frac{1}{l_1} \int_0^1 |b_u| du\right] = \frac{1}{2}.$$

□

Finally, using a pathwise transformation between the meander and the Brownian excursion, see [2], Corollary 2.2 also enables to show the following result:

**Corollary 2.4.** *Let  $(e_t)_{t \leq 1}$  denote the standard Brownian excursion. We have*

$$\mathbb{E}\left[\int_0^1 e_t dt \int_0^1 \frac{1}{e_u} du\right] = \frac{3}{2}.$$

## 2.4 The case of two barriers

After the striking result given in Theorem 2.1, it is natural to wonder whether the expectation remains equal to zero if  $T_a$  is replaced by  $T_{a,b}$ , where  $T_{a,b}$  is the first exit time of the interval  $(-b, a)$ , with  $a > 0$  and  $b > 0$ . Indeed, remark that the random variable  $A_{a,b}^{(1)}$  defined by

$$A_{a,b}^{(1)} = \frac{1}{T_{a,b}^{3/2}} \int_0^{T_{a,b}} B_s ds,$$

still enjoys a scaling property in the sense that its law only depends on the ratio  $b/a$ . In fact, the following theorem states that the expectation is no longer zero in this case<sup>1</sup>:

**Theorem 2.2.** *Let  $\lambda = b/a$ . We have*

$$\mathbb{E}[A_{a,b}^{(1)}] = \frac{1}{\sqrt{2\pi}}(1 + \lambda) \int_0^\infty \frac{\delta}{\text{sh}(\delta(1 + \lambda))^2} (\lambda \text{sh}(\delta) - \text{sh}(\delta\lambda)) d\delta.$$

*In particular,  $\mathbb{E}[A_{a,b}^{(1)}] \neq 0$  if  $\lambda \notin \{0, 1\}$ .*

The proof of this result is given in Appendix B. In fact a general formula for

$$\mathbb{E}\left[\frac{1}{T_{a,b}^\theta} \int_0^{T_{a,b}} B_s ds\right],$$

with  $\theta > 0$  is given within this proof. Eventually, note that Theorem 2.1 can also be recovered from Theorem 2.2 letting the downward barrier tend to  $-\infty$ .

## 3 The general case

### 3.1 Computation of the expectations

For  $x \in \mathbb{R}$ , we set  $x^+ = \max(x, 0)$  and  $x^- = \max(-x, 0)$ . For  $m \geq 0$ , we define

$$A_+^{(m)} = \frac{1}{T_1^{1+m/2}} \int_0^{T_1} (B_s^+)^m ds, \quad A_-^{(m)} = \frac{1}{T_1^{1+m/2}} \int_0^{T_1} (B_s^-)^m ds,$$

with the convention  $0^0 = 0$ . We also write

$$I_+^{(m)} = \mathbb{E}[A_+^{(m)}], \quad I_-^{(m)} = \mathbb{E}[A_-^{(m)}].$$

and

$$I^{(m)} = I_+^{(m)} - I_-^{(m)}.$$

Furthermore, we note that  $I_\pm^{(m)}$  is the moment of order  $m$  of the random variable  $\alpha^\pm$  where

$$\alpha = \frac{B_{UT_1}}{\sqrt{T_1}},$$

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<sup>1</sup>However, note that from a numerical point of view, the obtained values for  $\mathbb{E}[A_{a,b}^{(1)}]$  are systematically very small, for any  $a$  and  $b$ .

with  $U$  a uniform random variable independent of the Brownian motion  $B$ . We study the variable  $\alpha$  in more details in Section 3.3.

For  $m \geq 0$ , let

$$c_m = \frac{\Gamma(1+m)}{2^{m/2}\Gamma(1+m/2)} = \frac{1}{\sqrt{\pi}} 2^{m/2} \Gamma\left(\frac{1+m}{2}\right) = \mathbb{E}[|N|^m],$$

where  $N$  is a standard Gaussian random variable and  $\Gamma$  denotes the Gamma function. We have the following theorem:

**Theorem 3.1.** *Let  $m \geq 0$  and introduce*

$$\phi(m) = \int_0^2 \frac{y^{m+1}}{1+y} dy.$$

*The following formulas hold:*

$$I_+^{(m)} = \frac{c_m}{2^{m+1}} \phi(m), \quad I_-^{(m)} = \frac{c_m}{2^{m+1}} \log(3).$$

In particular, we note that  $\phi(0) = 2 - \log(3)$ ,  $\phi(1) = \log(3)$ ,  $\phi(2) = 8/3 - \log(3)$  and  $\phi(3) = 4/3 + \log(3)$ . We give the proof of Theorem 3.1 in Section 3.6.

### 3.2 Comments about Theorem 3.1

- The function  $\phi$  is well defined for  $m \in (-2, +\infty)$  and satisfies  $\phi(-1) = \phi(1) = \log(3)$ . Thus, we retrieve in Theorem 3.1 the fact that  $\mathbb{E}[A_1^{(1)}] = I_+^{(1)} - I_-^{(1)} = 0$ .

- We easily get that  $\phi$  is twice differentiable and, for  $m \geq 0$ ,

$$\phi'(m) = \int_0^2 \frac{y^{m+1} \log(y)}{1+y} dy, \quad \phi''(m) = \int_0^2 \frac{y^{m+1} (\log(y))^2}{1+y} dy.$$

Hence  $\phi$  is convex and furthermore, we show in Appendix C that  $\phi'(0) > 0$ . This implies that  $\phi$  and  $\phi'$  are increasing on  $\mathbb{R}^+$ . Hence, since

$$I^{(m)} = \frac{c_m}{2^{m+1}} (\phi(m) - \log(3)),$$

we get  $I^{(m)} > 0$  for  $m > 1$  and  $I^{(m)} < 0$  for  $m < 1$ . This can be interpreted as follows: from the point of view of  $A_1^{(m)}$ , for  $m > 1$ , the time spent by the Brownian motion in  $[0, 1]$  is dominant whereas for  $m < 1$ , the time spent in  $(-\infty, 0)$  is more important.

- Let  $(L_t^x, x \in \mathbb{R}, t \geq 0)$  denote the local time of the Brownian motion  $B$ . Within the proof of Theorem 3.1, we are led to show the following interesting result:

**Proposition 3.1.** *Let  $\mu > 0$ ,  $0 < b < 1$  and  $x \geq 0$ , we have*

$$\mathbb{E}[L_{T_1}^b \exp(-\mu^2 T_1/2)] = \frac{1}{\mu} \left( \exp(-\mu) - \exp(-\mu(3-2b)) \right)$$

and

$$\mathbb{E}[L_{T_1}^{-x} \exp(-\mu^2 T_1/2)] = \frac{1}{\mu} \left( \exp(-\mu(1+2x)) - \exp(-\mu(3+2x)) \right).$$

We also give another proof of Proposition 3.1, based on the Ray-Knight theorem, in Appendix D.

### 3.3 Uniform sampling up to hitting time

We now want to interpret Theorem 3.1 as a result about sampling independently and uniformly the properly rescaled Brownian motion up to its first hitting time  $T_1$ . More precisely, let us introduce  $(l_1^y, y \in \mathbb{R})$ , the local time at time 1 of the process

$$\left(\frac{B_{sT_1}}{\sqrt{T_1}}, s \leq 1\right).$$

Let  $f$  be a Borel non negative function and  $U$  a uniform random variable independent of any other random variable defined here. Using the occupation formula, we get

$$\mathbb{E}\left[f\left(\frac{B_{UT_1}}{\sqrt{T_1}}\right)\right] = \mathbb{E}\left[\int_0^1 f\left(\frac{B_{sT_1}}{\sqrt{T_1}}\right)ds\right] = \int_{-\infty}^{+\infty} f(y)\mathbb{E}[l_1^y]dy.$$

Hence  $h(y) = \mathbb{E}[l_1^y]$  is the density of  $\alpha$  at point  $y$ . The following result is easily deduced from Theorem 3.1, by injectivity of the Mellin transform.

**Theorem 3.2.** *The density  $h$  satisfies for  $y \geq 0$*

$$h(y) = \sqrt{\frac{2}{\pi}} \int_0^2 \frac{1}{1+w} \exp(-2y^2/w^2) dw$$

and for  $y \leq 0$

$$h(y) = \sqrt{\frac{2}{\pi}} \log(3) \exp(-2y^2).$$

Hence, conditional on  $\alpha > 0$ , the law of  $\alpha^+$  is a mixture of absolute Gaussian laws, whereas conditional on  $\alpha < 0$ ,  $\alpha^-$  is distributed as the absolute value of a Gaussian random variable.

Remark that for  $y \geq 0$ , we have the obvious inequality

$$h(y) \leq \sqrt{\frac{2}{\pi}} \log(3) \exp(-y^2/2).$$

Therefore, we have the following corollary:

**Corollary 3.1.** *For  $\varepsilon < 1/2$ , the random variable  $\alpha^+$  satisfies*

$$\mathbb{E}[\exp(\varepsilon(\alpha^+)^2)] < +\infty.$$

In fact, thanks to Proposition 3.1, we can even provide the density at point  $y$  of  $\alpha$  conditional on  $T_1 = t$ . We denote this density by  $h(y, t)$ . Obvious relations between  $(l_1^y)$  and  $(L_t^y)$  yield

$$h(y, t) = \mathbb{E}_{T_1=t}[l_1^y] = \frac{1}{\sqrt{t}} \mathbb{E}_{T_1=t}[L_t^{y\sqrt{t}}].$$

We easily obtain the following corollary from Proposition 3.1:

**Corollary 3.2.** *The conditional density  $h(y, t)$  satisfies for  $0 \leq y\sqrt{t} \leq 1$ ,*

$$h(y, t) \exp(-1/(2t)) t^{-1/2} = \exp(-1/(2t)) - \exp(-(3 - 2y\sqrt{t})^2/(2t))$$

and for  $x \geq 0$

$$h(-x, t) \exp(-1/(2t)) t^{-1/2} = \exp(-(1 + 2x\sqrt{t})^2/(2t)) - \exp(-(1 + 3x\sqrt{t})^2/(2t)).$$



### 3.4 Interpretation in terms of the Brownian meander

In the same spirit as in Corollary 2.2, we can give an interpretation of Theorem 3.1 in terms of the Brownian meander. Using Williams theorem in the same way as in the proof of Theorem 2.1, see Appendix A, together with Imhof's relation, see [3, 6], as already done for Corollary 2.2, we get that for any non negative measurable functions  $f$  and  $g$ ,

$$\mathbb{E}\left[\int_0^1 f\left(\frac{B_{sT_1}}{\sqrt{T_1}}\right) ds g\left(\frac{1}{\sqrt{T_1}}\right)\right] = \sqrt{\frac{2}{\pi}} \mathbb{E}\left[\int_0^1 f(m_1 - m_u) du \frac{g(m_1)}{m_1}\right],$$

where  $m$  denotes the Brownian meander. Let  $U$  be a uniform random variable, independent of all other quantities. The last relation is equivalent to

$$\mathbb{E}\left[f\left(\frac{B_{UT_1}}{\sqrt{T_1}}\right) g\left(\frac{1}{\sqrt{T_1}}\right)\right] = \sqrt{\frac{2}{\pi}} \mathbb{E}\left[f(m_1 - m_U) \frac{g(m_1)}{m_1}\right].$$

Thus, from Corollary 3.2, we are able to compute the density of  $m_U$  conditional on the value  $m_1$ . More precisely, we have the following theorem:

**Theorem 3.3.** *Let  $f$  be a Borel non negative function. We have*

$$\mathbb{E}_{m_1=y}[f(m_U)] = \int_0^y h\left(y-z, \frac{1}{y^2}\right) f(z) dz + \int_y^{+\infty} h\left(-(z-y), \frac{1}{y^2}\right) f(z) dz.$$

### 3.5 Future developments

In this work, we have studied some properties of random sampling through the random variable

$$\alpha = \frac{B_{UT_1}}{\sqrt{T_1}}.$$

Another interesting variable is the variable  $\beta$  defined by

$$\beta = \frac{B_{U\tau_1}}{\sqrt{\tau_1}},$$

with  $\tau_l = \inf\{t \geq 0, L_t > l\}$ . In fact the associated process

$$\left(\frac{B_{s\tau_1}}{\sqrt{\tau_1}}, s \leq 1\right)$$

is called pseudo Brownian bridge and has been considered more explicitly in the literature than

$$\left(\frac{B_{sT_1}}{\sqrt{T_1}}, s \leq 1\right).$$

In particular, it enjoys some absolute continuity property with respect to the standard Brownian bridge, see [3]. We intend to present results related to  $\beta$  in a forthcoming work, in a way which will help us to recover the interesting law of  $\alpha$ . For now, we only mention that  $\beta$  is distributed as  $N/2$ , where  $N$  is a standard Gaussian random variable.

### 3.6 Proof of Theorem 3.1

Let  $m \geq 0$ . We split the proof into several steps.

### Step 1: Introducing a natural measure

First, let us remark that

$$\begin{aligned} I_{\pm}^{(m)} &= \frac{1}{\Gamma(1+m/2)} \mathbb{E} \left[ \int_0^{+\infty} \lambda^{m/2} \exp(-\lambda T_1) d\lambda \int_0^{T_1} (B_s^{\pm})^m ds \right] \\ &= \frac{1}{2^{m/2} \Gamma(1+m/2)} \mathbb{E} \left[ \int_0^{+\infty} \mu^{1+m} \exp(-\mu^2 T_1/2) d\mu \int_0^{T_1} (B_s^{\pm})^m ds \right]. \end{aligned}$$

Hence, it is natural to introduce for  $\mu \geq 0$  the measure  $I_{\mu}$ , which to a positive function  $\psi$  associates

$$I_{\mu}(\psi) = \mathbb{E} \left[ \int_0^{T_1} \psi(B_s) \exp(-\mu^2 T_1/2) ds \right] = e^{-\mu} \mathbb{E} \left[ \int_0^{T_1} \psi(B_s) \exp(\mu - \mu^2 T_1/2) ds \right].$$

### Step 2: Computation of $I_{\mu}(\psi)$

Let  $(S_s) = (\sup_{u \leq s} B_u)$ . Using the martingale property of the process  $\exp(\mu B_s - \mu^2 s/2)$ , we get

$$I_{\mu}(\psi) = e^{-\mu} \mathbb{E} \left[ \int_0^{+\infty} \psi(B_s) 1_{\{S_s < 1\}} \exp(\mu B_s - \mu^2 s/2) ds \right].$$

We now use the following well known formula, see for example [9]: for  $s > 0$  and  $b \in \mathbb{R}$ ,

$$\mathbb{P}[S_s < 1 | B_s = b] = 1 - \exp\left(-\frac{2}{s}(1-b)^+\right).$$

It implies that  $I_{\mu}(\psi)$  is equal to

$$e^{-\mu} \int_0^{+\infty} \exp(-\mu^2 s/2) ds \int_{-\infty}^1 e^{\mu b} \psi(b) \frac{1}{\sqrt{2\pi s}} \exp(-b^2/(2s)) \left(1 - \exp\left(-\frac{2}{s}(1-b)\right)\right) db,$$

which can be rewritten

$$e^{-\mu} \int_{-\infty}^1 e^{\mu b} \psi(b) db \int_0^{+\infty} \exp(-\mu^2 s/2) \frac{1}{\sqrt{2\pi s}} \left( \exp(-b^2/(2s)) - \exp(-(2-b)^2/(2s)) \right) ds.$$

Then, using the density and the value of the first moment of an inverse Gaussian random variable, we get that for  $\mu > 0$  and  $y \in \mathbb{R}$ ,

$$\int_0^{+\infty} \frac{1}{\sqrt{2\pi s}} \exp(-y^2/(2s) - \mu^2 s/2) ds = \frac{1}{\mu} \exp(-\mu|y|).$$

From this, we deduce that when the support of  $\psi$  is included in  $[0, 1]$ ,

$$I_{\mu}(\psi) = \frac{1}{\mu} \int_0^1 \psi(b) \left( \exp(-\mu) - \exp(-\mu(3-2b)) \right) db, \quad (2)$$

and when the support of  $\psi$  is included in  $(-\infty, 0)$ ,

$$I_{\mu}(\psi) = \frac{1}{\mu} \int_0^{+\infty} \psi(-x) \left( \exp(-\mu(1+2x)) - \exp(-\mu(3+2x)) \right) dx. \quad (3)$$

Remark here that Proposition 3.1 immediately follows from Equation (2) and Equation (3).

**Step 3: End of the proof of Theorem 3.1**

We end the proof of Theorem 3.1 in this final step. We start with the following elementary lemma:

**Lemma 3.1.** For  $a > 0$ ,  $b > 0$  and  $m \geq 0$ , we define

$$L(a, b, m) = \int_0^{+\infty} y^m \left( \frac{1}{(a+y)^{m+1}} - \frac{1}{(b+y)^{m+1}} \right) dy.$$

The following equality holds:

$$L(a, b, m) = \log(b/a).$$

*Proof.* We have

$$\begin{aligned} L(a, b, m) &= \lim_{n \rightarrow +\infty} \int_0^n y^m \left( \frac{1}{(a+y)^{m+1}} - \frac{1}{(b+y)^{m+1}} \right) dy \\ &= \lim_{n \rightarrow +\infty} \left( \int_0^{n/a} \frac{y^m}{(1+y)^{m+1}} dy - \int_0^{n/b} \frac{y^m}{(1+y)^{m+1}} dy \right) \\ &= \lim_{n \rightarrow +\infty} \int_{1/b}^{1/a} \frac{n^{m+1} y^m}{(1+ny)^{m+1}} dy = \log(b/a). \end{aligned}$$

□

Now we take  $\psi(x) = (x^\pm)^m$  in Equation (2) and Equation (3). Integrating in  $\mu$ , we easily derive

$$I_+^{(m)} = \frac{\Gamma(1+m)}{2^{m/2}\Gamma(1+m/2)} \int_0^1 b^m \left( 1 - \frac{1}{(3-2b)^{m+1}} \right) db$$

and

$$I_-^{(m)} = \frac{\Gamma(1+m)}{2^{m/2}\Gamma(1+m/2)} \int_0^{+\infty} x^m \left( \frac{1}{(1+2x)^{m+1}} - \frac{1}{(3+2x)^{m+1}} \right) dx.$$

Applying Lemma 3.1, we obtain the result for  $I_-^{(m)}$ . For  $I_+^{(m)}$ , we write

$$I_+^{(m)} = \frac{\Gamma(1+m)}{2^{m/2}\Gamma(1+m/2)} \left( \int_0^1 b^m db - \int_0^1 \frac{1}{(3-2b)} \frac{b^m}{(3-2b)^m} db \right).$$

Then we use the change of variable  $y = b/(3-2b)$  in the second integral in order to retrieve the expression of  $I_+^{(m)}$  given in Theorem 3.1.

## Appendices

### A Proof of Theorem 2.1

Theorem 2.1 can be seen as a particular case of Theorem 3.1. Nevertheless, we give here a specific proof for this theorem which is interesting on its own. We split it in several steps.

#### Step 1: Time reversal

Let us recall Williams time reversal theorem, see for example [10]. We have the following equality:

$$(1 - B_{T_1-u}, u \leq T_1) \stackrel{\mathcal{L}}{=} (R_u, u \leq \gamma),$$

where  $R$  denotes a three dimensional Bessel process starting from 0 and  $\gamma$  is its last passage time at level 1:

$$\gamma = \sup\{t \geq 0, R_t = 1\}.$$

Consequently, since

$$A_1^{(1)} = \frac{1}{T_1^{3/2}} \int_0^{T_1} (1 - (1 - B_{T_1-s})) ds,$$

it has the same law as

$$\frac{1}{\gamma^{3/2}} \int_0^\gamma (1 - R_u) du = \frac{1}{\sqrt{\gamma}} - \int_0^1 \frac{R_{v\gamma}}{\sqrt{\gamma}} dv. \quad (4)$$

#### Step 2: Moments

We now show that  $A_1^{(1)}$  has moments of any order. First recall the following equalities:

$$\frac{1}{\sqrt{\gamma}} \stackrel{\mathcal{L}}{=} \frac{1}{\sqrt{T_1}} \stackrel{\mathcal{L}}{=} |B_1|.$$

Thus,  $\frac{1}{\sqrt{\gamma}}$  has moments of any order and therefore it is enough to prove the integrability of  $\xi^r$ , for any  $r > 0$ , with

$$\xi = \int_0^1 \frac{R_{v\gamma}}{\sqrt{\gamma}} dv.$$

Such integrability result will be deduced from the following absolute continuity relation that can be found in [3]:

**Lemma A.1.** *For any Borel functional  $F$  from  $(\mathbb{C}[0, 1], \mathbb{R}^+)$  into  $\mathbb{R}^+$ ,*

$$\mathbb{E}\left[F\left(\frac{R_{u\gamma}}{\sqrt{\gamma}}, u \leq 1\right)\right] = \mathbb{E}\left[F(R_u, u \leq 1) \frac{1}{R_1^2}\right].$$

Now take  $r > 0$ ,  $1 < p < 3/2$  and  $q$  such that  $1/p + 1/q = 1$ . From Lemma A.1 together with Hölder inequality, we obtain

$$\mathbb{E}[(\xi^r)] = \mathbb{E}\left[\left(\int_0^1 R_u du\right)^r \frac{1}{R_1^2}\right] \leq \left(\mathbb{E}\left[\left(\int_0^1 R_u du\right)^{rq}\right]\right)^{1/q} \left(\mathbb{E}\left[\frac{1}{R_1^{2p}}\right]\right)^{1/p}.$$

The first expectation on the right hand side of the last inequality is obviously finite. For the second one, recall that  $R_1^2$  has the distribution of  $2Z$ , with  $Z$  following a gamma law with parameter  $3/2$ . Therefore, the second expectation is also finite since  $p < 3/2$ .

### Step 3: Centering property

We end the proof of Theorem 2.1 in this step. We start with the following technical lemma.

**Lemma A.2.** *Let  $a > 0$ . We have*

$$\mathbb{E}[R_1 \exp(-R_1^2 a/2)] = \frac{\sqrt{2}}{\Gamma(3/2)(1+a)^2}.$$

*Proof.* Using again that  $R_1^2$  has the distribution of  $2Z$ , with  $Z$  following a gamma law with parameter  $3/2$ , we can write

$$\mathbb{E}[R_1 \exp(-R_1^2 a/2)] = \frac{\sqrt{2}}{\Gamma(3/2)} \int_0^{+\infty} x \exp(-x(1+a)) dx.$$

The result follows easily from this equality.  $\square$

We now prove that  $\mathbb{E}[A_1^{(1)}] = 0$ . From Equation (4) and Lemma A.1, using the fact that  $\mathbb{E}[1/\sqrt{\gamma}] = \mathbb{E}[|B_1|] = \sqrt{2/\pi}$ , this is equivalent to prove the following lemma:

**Lemma A.3.** *We have*

$$\mathbb{E}\left[\left(\int_0^1 R_u du\right) \frac{1}{R_1^2}\right] = \sqrt{\frac{2}{\pi}}.$$

*Proof.* First, using Markov property, we get

$$\mathbb{E}\left[\frac{R_u}{R_1^2}\right] = \mathbb{E}\left[R_u \mathbb{E}_{R_u}\left[\frac{1}{R_{1-u}^2}\right]\right],$$

where  $\mathbb{E}_r$  denotes the expectation of a three dimensional Bessel process starting from point  $r$ . From Proposition 2, page 99, in [11], we know that

$$\mathbb{E}_r\left[\frac{1}{R_t^2}\right] = \int_0^{1/(2t)} \exp(-r^2 v) (1-2tv)^{-1/2} dv.$$

Thus, using the last equality together with a change of variable and the scaling property of the Bessel process, we get

$$\mathbb{E}\left[\frac{R_u}{R_1^2}\right] = \sqrt{u} \mathbb{E}\left[R_1 \int_0^1 \frac{(1-w)^{-1/2}}{2(1-u)} \exp\left(-\frac{R_1^2 u w}{2(1-u)}\right) dw\right].$$

From Lemma A.2, we obtain

$$\mathbb{E}\left[\frac{R_u}{R_1^2}\right] = \frac{\sqrt{u}(1-u)}{\sqrt{2}\Gamma(3/2)} \int_0^1 \frac{1}{\sqrt{x}(1-ux)^2} dx = \frac{(1-u)}{\sqrt{2}\Gamma(3/2)} \int_0^u \frac{1}{\sqrt{y}(1-y)^2} dy.$$

Using Fubini's theorem when integrating in  $u$  from 0 to 1, and remarking that  $\Gamma(3/2) = \sqrt{\pi}/2$ , we easily conclude the proof of Lemma A.3 and so the proof of Theorem 2.1.  $\square$

## B Proof of Theorem 2.2

Let  $a > 0$ ,  $b > 0$  and  $\theta > 0$ . In this section, we consider  $\psi(a, b, \theta)$  defined by

$$\psi(a, b, \theta) = \mathbb{E}\left[\frac{1}{\tau^\theta} \int_0^\tau B_s ds\right],$$

where  $\tau$  is the exit time of the interval  $(-b, a)$  by the Brownian motion  $B$ .

## B.1 General result

We start with a general result. We give here a representation of  $\psi(a, b, \theta)$  in term of a Lebesgue integral. Let  $\delta > 0$ ,  $a > 0$ ,  $b > 0$  and  $p > -1$ . Recall that  $c_p$  denotes the  $p$ -th absolute moment of a standard Gaussian random variable and define  $\phi_\delta(a, b, p)$  by

$$\phi_\delta(a, b, p) = ab + b^2(p - 1 - (p - 2)\text{ch}(\delta(a + b))).$$

We have the following result.

**Theorem B.1.** *Let  $\theta > 0$ . We have*

$$\psi(a, b, \theta) = \frac{\sqrt{2}}{\sqrt{\pi}m_{2\theta-1}} \int_0^\infty \delta^{2\theta-1} E_\delta d\delta,$$

with

$$E_\delta = \frac{b\text{sh}(\delta a) - a\text{sh}(\delta b)}{2\delta^2\text{sh}(\delta(a + b))} + \frac{(a^2\text{sh}(\delta b) - b^2\text{sh}(\delta a))(\text{ch}(\delta(a + b)) - 1)}{2\delta\text{sh}(\delta(a + b))^2}.$$

For  $\theta \neq 1$ , another representation for  $\psi(a, b, \theta)$  is

$$\frac{\sqrt{2}}{\sqrt{\pi}m_{2\theta-1}} \int_0^\infty \frac{\delta^{2\theta-2}}{4(\theta-1)\text{sh}(\delta(a + b))^2} (\text{sh}(\delta a)\phi_\delta(a, b, 2\theta - 1) - \text{sh}(\delta b)\phi_\delta(b, a, 2\theta - 1)) d\delta.$$

*Proof.* Our proof is based on Feynman-Kac formula, see for example [5]. Note that in [8], the author used this formula in order to derive the joint Laplace transform of  $(\tau, \int_0^\tau B_s ds)$ . We propose here a specific method for our problem. We introduce the function

$$g : (x, \delta, \rho) \mapsto \mathbb{E}_x \left[ e^{-(\delta^2/2)\tau + \rho \int_0^\tau B_s ds} \right].$$

By Feynman-Kac formula,  $g$  solves on  $(-b, a)$

$$g_{xx}(x, \delta, \rho) - (\delta^2 - 2\rho x)g(x, \delta, \rho) = 0, \quad \text{with } g(a, \cdot) = g(-b, \cdot) = 1.$$

For  $\rho = 0$ , we denote  $g^0 : (x, \delta) \mapsto g(x, \delta, 0)$  which solves on  $(-b, a)$

$$g_{xx}^0(x, \delta, \rho) - \delta^2 g^0(x, \delta, \rho) = 0, \quad \text{with } g^0(a, \cdot) = g^0(-b, \cdot) = 1.$$

Thus,  $g^0$  is of the form

$$g^0(x, \delta) = A_\delta \text{ch}(\delta x) + B_\delta \text{sh}(\delta x).$$

Differentiating the dynamics of  $g$  with respect to  $\rho$  and introducing

$$f : (x, \delta) \mapsto g_\rho(x, \delta, 0),$$

we observe that  $f$  solves on  $(-b, a)$

$$f_{xx}(x, \delta) - \delta^2 f(x, \delta) + 2xg^0(x, \delta) = 0, \quad \text{with } f(a, \cdot) = f(-b, \cdot) = 0.$$

Furthermore, by definition of  $g$ ,  $f$  satisfies

$$f(x, \delta) = \mathbb{E}_x \left[ e^{-(\delta^2/2)\tau} \int_0^\tau B_s ds \right].$$

Due to its dynamics, we get that  $f$  is of the form

$$f(x, \delta) = E_\delta \text{ch}(\delta x) + F_\delta \text{sh}(\delta x) + f^0(x, \delta),$$

where  $f^0$  is a particular solution of the ODE of interest. Applying the variation of the constant method, we look for  $f^0$  of the form

$$f^0(x, \delta) = C_\delta(x) \text{ch}(\delta x) + D_\delta(x) \text{sh}(\delta x),$$

so that  $f$  rewrites

$$f(x, \delta) = (E_\delta + C_\delta(x)) \text{ch}(\delta x) + (F_\delta + D_\delta(x)) \text{sh}(\delta x).$$

This function  $f$  is of particular interest since for  $p > -1$ ,

$$\mathbb{E} \left[ \tau^{-(p+1)/2} \int_0^\tau B_s ds \right]$$

is equal to

$$\frac{\sqrt{2}}{\sqrt{\pi c_p}} \mathbb{E} \left[ \left( \int_0^\infty \delta^p e^{-(\delta^2/2)\tau} d\delta \right) \left( \int_0^\tau B_s ds \right) \right] = \frac{\sqrt{2}}{\sqrt{\pi c_p}} \int_0^\infty f(0, \delta) \delta^p d\delta.$$

Hence, denoting  $p = 2\theta - 1$ , the first part of Theorem B.1 boils down to the computation of

$$\int_0^\infty f(0, \delta) \delta^p d\delta = \int_0^\infty (E_\delta + C_\delta(0)) \delta^p d\delta, \quad (5)$$

so that we need to identify  $A_\delta, B_\delta, C_\delta(\cdot), D_\delta(\cdot), E_\delta$  and  $F_\delta$ .

Observe that from the boundary conditions  $g^0(a, \cdot) = g^0(-b, \cdot) = 1$ , we obtain that  $A_\delta$  and  $B_\delta$  satisfy

$$A_\delta \text{ch}(\delta a) + B_\delta \text{sh}(\delta a) = 1, \quad A_\delta \text{ch}(\delta b) - B_\delta \text{sh}(\delta b) = 1.$$

We recall now for later use the classical ch and sh formulas:

$$\begin{aligned} \text{ch}(x) \text{ch}(y) \pm \text{sh}(x) \text{sh}(y) &= \text{ch}(x \pm y), \\ \text{ch}(x) \text{sh}(y) \pm \text{sh}(x) \text{ch}(y) &= \text{sh}(x \pm y). \end{aligned}$$

We deduce that  $A_\delta$  and  $B_\delta$  are given by :

$$A_\delta = \frac{\text{sh}(\delta b) + \text{sh}(\delta a)}{\text{sh}(\delta(a+b))}, \quad B_\delta = \frac{\text{ch}(\delta b) - \text{ch}(\delta a)}{\text{sh}(\delta(a+b))}. \quad (6)$$

Similarly we can compute  $E_\delta$  and  $F_\delta$  in terms of  $C_\delta(\cdot)$  and  $D_\delta(\cdot)$ . Indeed, the boundary conditions of  $f$  imply

$$\begin{aligned} E_\delta \text{ch}(\delta a) + F_\delta \text{sh}(\delta a) &= -C_\delta(a) \text{ch}(\delta a) - D_\delta(a) \text{sh}(\delta a), \\ E_\delta \text{ch}(\delta b) - F_\delta \text{sh}(\delta b) &= -C_\delta(-b) \text{ch}(\delta b) + D_\delta(-b) \text{sh}(\delta b). \end{aligned}$$

Consequently, we get that  $E_\delta \text{sh}(\delta(a+b))$  is equal to

$$\begin{aligned} & -C_\delta(a) \text{ch}(\delta a) \text{sh}(\delta b) - D_\delta(a) \text{sh}(\delta a) \text{sh}(\delta b) \\ & -C_\delta(-b) \text{ch}(\delta b) \text{sh}(\delta a) + D_\delta(-b) \text{sh}(\delta b) \text{sh}(\delta a) \end{aligned}$$

and  $F_\delta \text{sh}(\delta(a+b))$  to

$$\begin{aligned} & -C_\delta(a) \text{ch}(\delta a) \text{ch}(\delta b) - D_\delta(a) \text{sh}(\delta a) \text{ch}(\delta b) \\ & +C_\delta(-b) \text{ch}(\delta b) \text{ch}(\delta a) - D_\delta(-b) \text{sh}(\delta b) \text{ch}(\delta a). \end{aligned}$$

It now remains to compute  $C_\delta(\cdot)$  and  $D_\delta(\cdot)$ , which are both defined up to a constant and satisfy

$$\begin{aligned} f^0(x, \delta) &= C_\delta(x) \text{ch}(\delta x) + D_\delta(x) \text{sh}(\delta x) \\ f_x^0(x, \delta) &= C_\delta(x) \delta \text{sh}(\delta x) + D_\delta(x) \delta \text{ch}(\delta x). \end{aligned}$$

Thus, since  $f_{xx}^0 - \delta^2 f^0 + 2xg^0(x) = 0$ ,  $C'_\delta$  and  $D'_\delta$  satisfy

$$\begin{aligned} C'_\delta(x) \text{ch}(\delta x) + D'_\delta(x) \text{sh}(\delta x) &= 0 \\ C'_\delta(x) \delta \text{sh}(\delta x) + D'_\delta(x) \delta \text{ch}(\delta x) &= -2xg^0(x). \end{aligned}$$

Therefore, we get

$$C'_\delta(x) = \frac{2xg^0(x)}{\delta} \text{sh}(\delta x), \quad D'_\delta(x) = -\frac{2xg^0(x)}{\delta} \text{ch}(\delta x).$$

We now compute  $C_\delta$ , which is given by

$$\begin{aligned} C_\delta(x) &= \frac{2}{\delta} \int_0^x t (A_\delta \text{ch}(\delta t) + B_\delta \text{sh}(\delta t)) \text{sh}(\delta t) dt \\ &= \frac{A_\delta}{\delta} \int_0^x t \text{sh}(2\delta t) dt + \frac{B_\delta}{\delta} \int_0^x t (\text{ch}(2\delta t) - 1) dt \\ &= \frac{A_\delta}{2\delta^2} \left( x \text{ch}(2\delta x) - \frac{\text{sh}(2\delta x)}{2\delta} \right) - \frac{B_\delta x^2}{2\delta} + \frac{B_\delta}{2\delta^2} \left( x \text{sh}(2\delta x) - \frac{\text{ch}(2\delta x)}{2\delta} + \frac{1}{2\delta} \right). \end{aligned}$$

In the same way,  $D_\delta$  is given by

$$\begin{aligned} D_\delta(x) &= -\frac{2}{\delta} \int_0^x t (A_\delta \text{ch}(\delta t) + B_\delta \text{sh}(\delta t)) \text{ch}(\delta t) dt \\ &= -\frac{B_\delta}{\delta} \int_0^x t \text{sh}(2\delta t) dt - \frac{A_\delta}{\delta} \int_0^x t (1 + \text{ch}(2\delta t)) dt \\ &= -\frac{B_\delta}{2\delta^2} \left( x \text{ch}(2\delta x) - \frac{\text{sh}(2\delta x)}{2\delta} \right) - \frac{A_\delta x^2}{2\delta} - \frac{A_\delta}{2\delta^2} \left( x \text{sh}(2\delta x) - \frac{\text{ch}(2\delta x)}{2\delta} + \frac{1}{2\delta} \right). \end{aligned}$$

Since  $C_\delta(0) = 0$ , observe that the quantity of interest (5) rewrites

$$\int_0^\infty \delta^p E_\delta d\delta,$$

where  $E_\delta$  is given above as a function of  $C_\delta(a)$ ,  $C_\delta(-b)$ ,  $D_\delta(a)$  and  $D_\delta(-b)$ . We now give an expression for

$$\text{sh}(\delta(a+b)) E_\delta.$$



First recall that it is equal to

$$-C_\delta(a)\text{ch}(\delta a)\text{sh}(\delta b) - D_\delta(a)\text{sh}(\delta a)\text{sh}(\delta b) \\ -C_\delta(-b)\text{ch}(\delta b)\text{sh}(\delta a) + D_\delta(-b)\text{sh}(\delta b)\text{sh}(\delta a).$$

Plugging the values for the coefficients, this can be rewritten

$$- \left[ \frac{A_\delta}{2\delta^2} \left( a\text{ch}(2\delta a) - \frac{\text{sh}(2\delta a)}{2\delta} \right) - \frac{B_\delta a^2}{2\delta} + \frac{B_\delta}{2\delta^2} \left( a\text{sh}(2\delta a) - \frac{\text{ch}(2\delta a)}{2\delta} + \frac{1}{2\delta} \right) \right] \text{ch}(\delta a)\text{sh}(\delta b) \\ - \left[ -\frac{B_\delta}{2\delta^2} \left( a\text{ch}(2\delta a) - \frac{\text{sh}(2\delta a)}{2\delta} \right) - \frac{A_\delta a^2}{2\delta} - \frac{A_\delta}{2\delta^2} \left( a\text{sh}(2\delta a) - \frac{\text{ch}(2\delta a)}{2\delta} + \frac{1}{2\delta} \right) \right] \text{sh}(\delta a)\text{sh}(\delta b) \\ - \left[ \frac{A_\delta}{2\delta^2} \left( -b\text{ch}(2\delta b) + \frac{\text{sh}(2\delta b)}{2\delta} \right) - \frac{B_\delta b^2}{2\delta} + \frac{B_\delta}{2\delta^2} \left( b\text{sh}(2\delta b) - \frac{\text{ch}(2\delta b)}{2\delta} + \frac{1}{2\delta} \right) \right] \text{ch}(\delta b)\text{sh}(\delta a) \\ + \left[ -\frac{B_\delta}{2\delta^2} \left( -b\text{ch}(2\delta b) + \frac{\text{sh}(2\delta b)}{2\delta} \right) - \frac{A_\delta b^2}{2\delta} - \frac{A_\delta}{2\delta^2} \left( b\text{sh}(2\delta b) - \frac{\text{ch}(2\delta b)}{2\delta} + \frac{1}{2\delta} \right) \right] \text{sh}(\delta b)\text{sh}(\delta a),$$

which leads to the expression:

$$\frac{1}{2\delta} [a^2\text{sh}(\delta b) (A_\delta\text{sh}(\delta a) + B_\delta\text{ch}(\delta a)) - b^2\text{sh}(\delta a) (A_\delta\text{sh}(\delta b) - B_\delta\text{ch}(\delta b))] \\ + \frac{a\text{sh}(\delta b)}{2\delta^2} [A_\delta (\text{sh}(2\delta a)\text{sh}(\delta a) - \text{ch}(2\delta a)\text{ch}(\delta a)) + B_\delta (\text{ch}(2\delta a)\text{sh}(\delta a) - \text{sh}(2\delta a)\text{ch}(\delta a))] \\ + \frac{b\text{sh}(\delta a)}{2\delta^2} [-A_\delta (\text{sh}(2\delta b)\text{sh}(\delta b) - \text{ch}(2\delta b)\text{ch}(\delta b)) + B_\delta (\text{ch}(2\delta b)\text{sh}(\delta b) - \text{sh}(2\delta b)\text{ch}(\delta b))] \\ + \frac{A_\delta}{4\delta^3} [\text{sh}(\delta b) (\text{sh}(2\delta a)\text{ch}(\delta a) - \text{ch}(2\delta a)\text{sh}(\delta a)) - \text{sh}(\delta a) (\text{sh}(2\delta b)\text{ch}(\delta b) - \text{ch}(2\delta b)\text{sh}(\delta b))] \\ + \frac{B_\delta}{4\delta^3} [\text{sh}(\delta b) (\text{ch}(2\delta a)\text{ch}(\delta a) - \text{sh}(2\delta a)\text{sh}(\delta a)) + \text{sh}(\delta a) (\text{ch}(2\delta b)\text{ch}(\delta b) - \text{sh}(2\delta b)\text{sh}(\delta b))] \\ - \frac{B_\delta}{4\delta^3} [\text{ch}(\delta a)\text{sh}(\delta b) + \text{ch}(\delta b)\text{sh}(\delta a)].$$

After obvious computations, we obtain that it is also equal to

$$\frac{1}{2\delta} [a^2\text{sh}(\delta b) (A_\delta\text{sh}(\delta a) + B_\delta\text{ch}(\delta a)) - b^2\text{sh}(\delta a) (A_\delta\text{sh}(\delta b) - B_\delta\text{ch}(\delta b))] \\ + \frac{a\text{sh}(\delta b)}{2\delta^2} [-A_\delta\text{ch}(\delta a) - B_\delta\text{sh}(\delta a)] + \frac{b\text{sh}(\delta a)}{2\delta^2} [A_\delta\text{ch}(\delta b) - B_\delta\text{sh}(\delta b)].$$

By definition,  $A_\delta\text{ch}(\delta a) + B_\delta\text{sh}(\delta a) = A_\delta\text{ch}(\delta b) - B_\delta\text{sh}(\delta b) = 1$ . Therefore, we get

$$\text{sh}(\delta(a+b))E_\delta = \frac{1}{2\delta} [a^2\text{sh}(\delta b) (A_\delta\text{sh}(\delta a) + B_\delta\text{ch}(\delta a)) - b^2\text{sh}(\delta a) (A_\delta\text{sh}(\delta b) - B_\delta\text{ch}(\delta b))] \\ - \frac{a\text{sh}(\delta b) - b\text{sh}(\delta a)}{2\delta^2}.$$

Recall also that  $A_\delta$  and  $B_\delta$  are explicitly given by (6) so that

$$A_\delta\text{sh}(\delta a) + B_\delta\text{ch}(\delta a) = \frac{\text{sh}(\delta b)\text{sh}(\delta a) + \text{sh}(\delta a)^2 + \text{ch}(\delta b)\text{ch}(\delta a) - \text{ch}(\delta a)^2}{\text{sh}(\delta(a+b))} \\ = \frac{\text{ch}(\delta(a+b)) - 1}{\text{sh}(\delta(a+b))}$$

and

$$\begin{aligned} A_\delta \text{sh}(\delta b) - B_\delta \text{ch}(\delta b) &= \frac{\text{sh}(\delta b)\text{sh}(\delta a) + \text{sh}(\delta b)^2 + \text{ch}(\delta b)\text{ch}(\delta a) - \text{ch}(\delta b)^2}{\text{sh}(\delta(a+b))} \\ &= \frac{\text{ch}(\delta(a+b)) - 1}{\text{sh}(\delta(a+b))}. \end{aligned}$$

Plugging these expressions in the previous one provides

$$E_\delta = \frac{b\text{sh}(\delta a) - a\text{sh}(\delta b)}{2\delta^2\text{sh}(\delta(a+b))} + \frac{(a^2\text{sh}(\delta b) - b^2\text{sh}(\delta a))(\text{ch}(\delta(a+b)) - 1)}{2\delta\text{sh}(\delta(a+b))^2}.$$

Recalling that  $p = 2\theta - 1$ , this ends the proof of the first part of Theorem B.1.

We now give the proof of the second part. By integration by parts we get that

$$\int_0^\infty \delta^p E_\delta d\delta$$

is equal to

$$\begin{aligned} &\int_0^\infty \frac{b\text{sh}(\delta a) - a\text{sh}(\delta b)}{2\text{sh}(\delta(a+b))} \delta^{p-2} d\delta + \int_0^\infty \frac{(a^2\text{sh}(\delta b) - b^2\text{sh}(\delta a))(\text{ch}(\delta(a+b)) - 1)}{2\text{sh}(\delta(a+b))^2} \delta^{p-1} d\delta \\ &= \left[ \frac{b\text{sh}(\delta a) - a\text{sh}(\delta b)}{2\text{sh}(\delta(a+b))} \frac{\delta^{p-1}}{p-1} \right]_0^\infty - \int_0^\infty \frac{b\text{ach}(\delta a) - b\text{ach}(\delta b)}{2\text{sh}(\delta(a+b))} \frac{\delta^{p-1}}{p-1} d\delta \\ &+ \int_0^\infty \frac{(b(a+b)\text{sh}(\delta a) - a(a+b)\text{sh}(\delta b))(\text{ch}(\delta(a+b)))}{2\text{sh}(\delta(a+b))^2} \frac{\delta^{p-1}}{p-1} d\delta \\ &+ \int_0^\infty \frac{(a^2\text{sh}(\delta b) - b^2\text{sh}(\delta a))(\text{ch}(\delta(a+b)) - 1)}{2\text{sh}(\delta(a+b))^2} \delta^{p-1} d\delta. \end{aligned}$$

Then, we easily obtain that the last expression is equal to

$$\begin{aligned} &- \int_0^\infty \frac{b\text{ach}(\delta a) - b\text{ach}(\delta b)}{2\text{sh}(\delta(a+b))} \frac{\delta^{p-1}}{p-1} d\delta + \int_0^\infty \frac{(b\text{ash}(\delta a) - a\text{bsh}(\delta b))(\text{ch}(\delta(a+b)))}{2\text{sh}(\delta(a+b))^2} \frac{\delta^{p-1}}{p-1} d\delta \\ &+ \int_0^\infty \delta^{p-1} \frac{(a^2\text{sh}(\delta b) - b^2\text{sh}(\delta a))}{2\text{sh}(\delta(a+b))^2} \text{ch}(\delta(a+b)) \frac{p-2}{p-1} d\delta - \int_0^\infty \frac{(a^2\text{sh}(\delta b) - b^2\text{sh}(\delta a))}{2\text{sh}(\delta(a+b))^2} \delta^{p-1} d\delta. \end{aligned}$$

After obvious simplifications, this can be rewritten

$$\begin{aligned} &ba \int_0^\infty \frac{-\text{sh}(\delta b) + \text{sh}(\delta a)}{2\text{sh}(\delta(a+b))^2} \frac{\delta^{p-1}}{p-1} d\delta - \int_0^\infty \frac{(a^2\text{sh}(\delta b) - b^2\text{sh}(\delta a))}{2\text{sh}(\delta(a+b))^2} \delta^{p-1} d\delta \\ &+ \int_0^\infty \delta^{p-1} \frac{(a^2\text{sh}(\delta b) - b^2\text{sh}(\delta a))}{2\text{sh}(\delta(a+b))^2} \text{ch}(\delta(a+b)) \frac{p-2}{p-1} d\delta. \end{aligned}$$

Thus, using the function  $\phi_\delta$  defined before Theorem B.1, we obtain

$$\int_0^\infty \delta^p E_\delta d\delta = \int_0^\infty \frac{\delta^{p-1}}{(p-1)2\text{sh}(\delta(a+b))^2} (\text{sh}(\delta a)\phi_\delta(a, b, p) - \text{sh}(\delta b)\phi_\delta(b, a, p)) d\delta.$$

□

## B.2 Proof of Theorem 2.2

We now give the proof of Theorem 2.2. From Theorem B.1, we get

$$\psi(a, b, 3/2) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\delta}{\text{sh}(\delta(a+b))^2} (\text{sh}(\delta a)\phi_\delta(a, b, 2) - \text{sh}(\delta b)\phi_\delta(b, a, 2)) d\delta.$$

Then we use that

$$\phi_\delta(a, b, 2) = ab + b^2, \quad \phi_\delta(b, a, 2) = ab + a^2$$

in order to obtain

$$\psi(a, b, 3/2) = \frac{1}{\sqrt{2\pi}} (a+b) \int_0^\infty \frac{\delta}{\text{sh}(\delta(a+b))^2} (b\text{sh}(\delta a) - a\text{sh}(\delta b)) d\delta.$$

Taking  $\lambda = b/a$ , we get

$$\psi(a, b, 3/2) = \frac{1}{\sqrt{2\pi}} (1+\lambda) \int_0^\infty \frac{\delta a}{\text{sh}(\delta a(1+\lambda))^2} (a\lambda\text{sh}(\delta a) - a\text{sh}(\delta a\lambda)) d\delta.$$

We finally obtain the result after the change of variable  $x = \delta a$ .

## C Some computations about the function $\phi$ defined in Theorem 3.1

Recall that the function  $\phi$  is defined for  $m > -2$  by

$$\phi(m) = \int_0^2 \frac{y^{m+1}}{1+y} dy.$$

We wish to compute

$$\phi'(0) = \int_0^2 \frac{y \log(y)}{1+y} dy.$$

We denote by  $\text{Li}_2$  the dilogarithm function defined for  $x$  such that  $|x| \leq 1$  by

$$\text{Li}_2(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n^2},$$

see [1] for more details. We start with the following general lemma:

**Lemma C.1.** *For  $C \geq 1$ , we define the function  $\Delta$  by*

$$\Delta(C) = \int_0^C \frac{y \log(y)}{1+y} dy.$$

*We have*

$$\Delta(C) = C \log(C) - C - (\log(C)) \log(C+1) + \frac{\pi^2}{6} + \frac{1}{2} (\log(C))^2 + \text{Li}_2\left(-\frac{1}{C}\right).$$

*Proof.* We get the equality of the two functions in Lemma C.1 by showing that they have the same derivatives and that they coincide for  $C = 1$ . To show the equality of the derivatives, after straightforward computations, we see that we need to prove that

$$-\log\left(\frac{1}{C} + 1\right) + \frac{1}{C}(\text{Li}_2)'\left(-\frac{1}{C}\right)$$

is equal to zero. Now we use the fact that for  $|x| \leq 1$ ,

$$(\text{Li}_2)'(x) = -\frac{\log(1-x)}{x},$$

see [1], in order to get the result.

We now show that the values of the two functions in Lemma C.1 coincide for  $C = 1$ . We have

$$\int_0^1 \frac{y \log(y)}{1+y} dy = \sum_{n=0}^{+\infty} (-1)^n \int_0^1 y^{1+n} \log(y) dy.$$

Using integration by parts arguments, we deduce

$$\int_0^1 \frac{y \log(y)}{1+y} dy = -\sum_{n=2}^{+\infty} \frac{(-1)^n}{n^2} = -(\text{Li}_2(-1) + 1).$$

We conclude using the fact that  $\text{Li}_2(-1) = -\pi^2/12$ , see again [1]. □

Recall that  $\phi'(0) = \Delta(2)$ . Using Lemma C.1 together with the facts that  $\text{Li}_2(-1/2) > -1/2$  and

$$2\log(2) - 2 - \log(2)\log(3) + \frac{\pi^2}{6} + \frac{1}{2}(\log(2))^2 > \frac{1}{2},$$

we get the following lemma:

**Lemma C.2.** *We have  $\phi'(0) > 0$  ( $\phi'(0) \approx 0.0615$ ). Therefore, the convex function  $\phi$  is increasing on  $\mathbb{R}^+$ .*

Eventually, we give the graphs of the functions  $\phi$ ,  $\phi'$  and  $\Delta$  in Figures 1, 2 and 3.

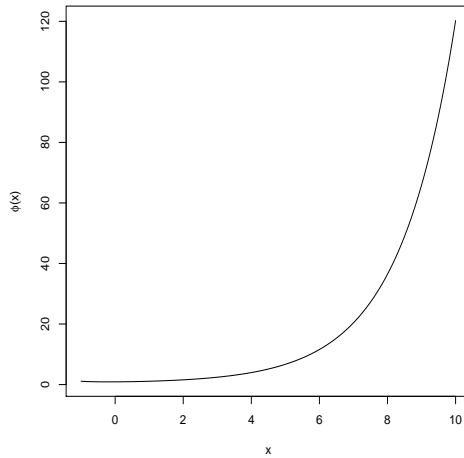


Figure 1: Function  $\phi$ , from  $-1$  to  $10$ .

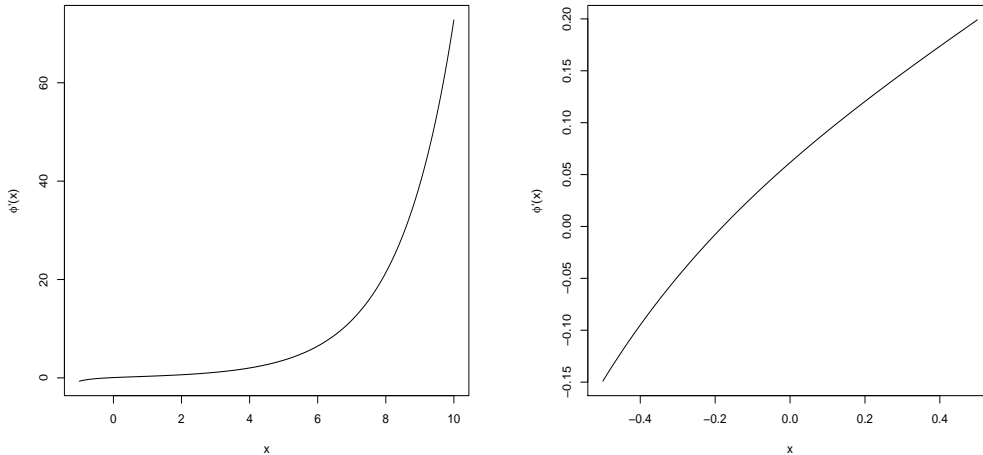


Figure 2: *Function  $\phi'$ , from  $-1$  to  $10$  (left) and from  $-0.5$  to  $0.5$  (right).*

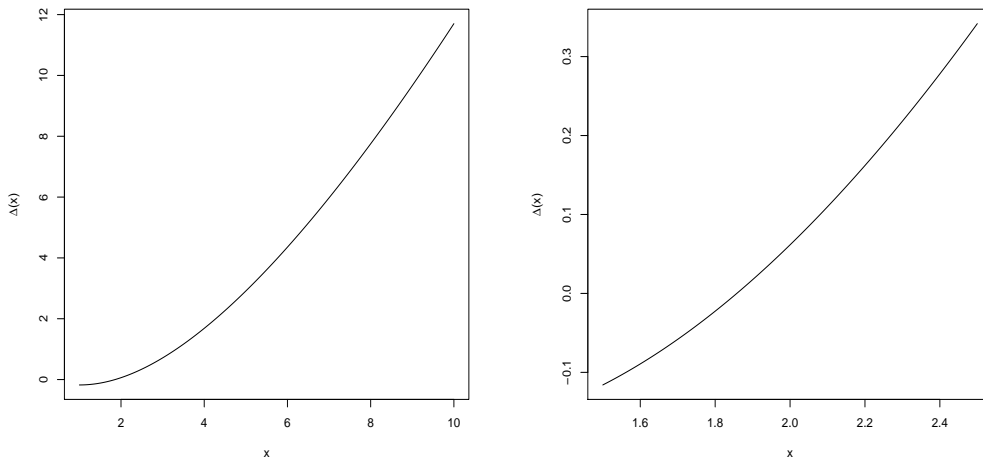


Figure 3: *Function  $\Delta$ , from  $1$  to  $10$  (left) and from  $1.5$  to  $2.5$  (right).*

## D Another proof of Proposition 3.1

In this section, we give a proof of Proposition 3.1 which is based on the Ray-Knight theorem. First note that multiplying both sides of the equalities in Proposition 3.1 by  $\exp(\mu)$  and using Girsanov's theorem, we see it is equivalent for  $0 < b < 1$  and  $x \geq 0$  to

$$\mathbb{E}[L_{T_1}^{1-b}(\mu)] = \frac{1}{\mu}(1 - \exp(-2\mu b))$$

and

$$\mathbb{E}[L_{T_1}^{-x}(\mu)] = \frac{1}{\mu} \left( \exp(-2\mu x) - \exp(-2\mu(1+2x)) \right),$$

where  $L_{T_1}^y(\mu)$  denotes the local time at level  $y$  of the Brownian motion with drift  $\mu$ ,  $B^\mu$ , considered up to its first hitting time of 1. Let us write  $X_b = L_{T_1}^{1-b}(\mu)$ . Ray-Knight's theorem

tells us that for  $0 < b < 1$ ,  $X_b$  is a (weak) solution of the following stochastic differential equation (SDE):

$$X_b = 2 \int_0^b \sqrt{X_s} d\beta_s - 2\mu \int_0^b X_s ds + 2b,$$

where  $\beta$  is a Brownian motion, see [5], pages 74-79. We now wish to compute  $u(b) = \mathbb{E}[X_b]$ . From the preceding SDE, we get

$$u(b) = -2\mu \int_0^b u(c) dc + 2b.$$

This ordinary differential equation can be easily solved using the variation of the constant method so that we get

$$u(b) = \frac{1}{\mu} (1 - \exp(-2\mu b)).$$

The proof for  $L_{T_1}^{-x}(\mu)$  goes similarly.

## Acknowledgments

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## References

- [1] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*. Encyclopedia Math. Appl, 1999.
- [2] J. Bertoin and J. Pitman. Path transformations connecting Brownian bridge, excursion and meander. *Bulletin des sciences mathématiques*, 118(2):147–166, 1994.
- [3] P. Biane, J.-F. Le Gall, and M. Yor. Un processus qui ressemble au pont brownien. In *Séminaire de Probabilités XXI*, pages 270–275. Springer, 1987.
- [4] P. Biane and M. Yor. Quelques précisions sur le méandre brownien. *Bulletin des sciences mathématiques*, 112(1):101–109, 1988.
- [5] A. N. Borodin and P. Salminen. *Handbook of Brownian motion: facts and formulae*. Springer, 2002.
- [6] J.-P. Imhof. Density factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications. *Journal of Applied Probability*, pages 500–510, 1984.
- [7] F. B. Knight. Inverse local times, positive sojourns, and maxima for Brownian motion. In *Colloque Paul Lévy sur les Processus Stochastiques*, pages 233–247, 1988.
- [8] A. Lachal. Sur le premier instant de passage de l'intégrale du mouvement brownien. In *Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques*, volume 27, pages 385–405. Gauthier-Villars, 1991.

- [9] J. Pitman. The distribution of local times of a Brownian bridge. In *Séminaire de probabilités XXXIII*, pages 388–394. Springer, 1999.
- [10] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293. Springer, 1999.
- [11] M. Yor. *On exponential functionals of Brownian motion and related processes*. Springer, 2001.