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The quenched limiting distributions of a one-dimensional random walk in random scenery

Nadine Guillotin-Plantard∗, Yueyun Hu†, and Bruno Schapira‡

Université Lyon 1, Université Paris 13, and Université Aix-Marseille

Summary. For a one-dimensional random walk in random scenery (RWRS) on \(\mathbb{Z}\), we determine its quenched weak limits by applying Strassen [13]'s functional law of the iterated logarithm. As a consequence, conditioned on the random scenery, the one-dimensional RWRS does not converge in law, in contrast with the multi-dimensional case.

Keywords: Random walk in random scenery; Weak limit theorem; Law of the iterated logarithm; Brownian motion in Brownian Scenery; Strong approximation.

AMS Subject Classification: 60F05, 60G52.

1. Introduction

Random walks in random sceneries were introduced independently by Kesten and Spitzer [9] and by Borodin [3, 4]. Let \(S = (S_n)_{n \geq 0}\) be a random walk in \(\mathbb{Z}^d\) starting at 0, i.e., \(S_0 = 0\) and \((S_n - S_{n-1})_{n \geq 1}\) is a sequence of i.i.d. \(\mathbb{Z}^d\)-valued random variables. Let \(\xi = (\xi_x)_{x \in \mathbb{Z}^d}\) be a field of i.i.d. real random variables independent of \(S\). The field \(\xi\) is called the random scenery. The random walk in random scenery (RWRS) \(K := (K_n)_{n \geq 0}\) is defined by setting \(K_0 := 0\) and, for \(n \in \mathbb{N}^*\),

\[
K_n := \sum_{i=1}^{n} \xi_{S_i}.
\]

We will denote by \(\mathbb{P}\) the joint law of \(S\) and \(\xi\). The law \(\mathbb{P}\) is called the annealed law, while the conditional law \(\mathbb{P}(\cdot|\xi)\) is called the quenched law.

Limit theorems for RWRS have a long history, we refer to [7] or [8] for a complete review. Distributional limit theorems for quenched sceneries (i.e. under the quenched law) are however quite recent. The first result in this direction that we are aware of was obtained by Ben Arous and Černý [1], in the case of a heavy-tailed scenery and planar random walk. In [7], quenched central limit theorems (with the usual \(\sqrt{n}\)-scaling and Gaussian law in the limit) were proved for a large class of transient random walks. More recently, in [8], the case of the planar random walk was studied, the authors proved a quenched version of the annealed central limit theorem obtained by Bolthausen in [2].

∗ Institut Camille Jordan, Université Lyon 1, 43, boulevard du 11 novembre 1918 69622 Villeurbanne. Research partially supported by ANR (MEMEMOII) 2010 BLAN 0125 Email: nadine.guillotin@univ-lyon1.fr
† Département de Mathématiques (LAGA CNRS-UMR 7530) Université Paris 13, 99 avenue J.B. Clément, 93430 Villetaneuse. Research partially supported by ANR (MEMEMOII) 2010 BLAN 0125 Email: yueyun@math.univ-paris13.fr
‡ Centre de Mathématiques et Informatique, Aix-Marseille Université, 39, rue F. Joliot Curie, 13453 Marseille Cedex 13. Research partially supported by ANR (MEMEMOII) 2010 BLAN 0125. Email: bruno.schapira@latp.univ-mrs.fr
In this note we consider the case of the simple symmetric random walk \((S_n)_{n \geq 0}\) on \(\mathbb{Z}\), the random scenery \((\xi_x)_{x \in \mathbb{Z}}\) is assumed to be centered with finite variance equal to one and there exists some \(\delta > 0\) such that \(\mathbb{E}[|\xi_0|^{2+\delta}] < \infty\). We prove that under these assumptions, there is no quenched distributional limit theorem for \(K\). In the sequel, for \(-\infty \leq a < b \leq \infty\), we will denote by \(\mathcal{AC}([a, b] \to \mathbb{R})\) the set of absolutely continuous functions defined on the interval \([a, b]\) with values in \(\mathbb{R}\). Recall that if \(f \in \mathcal{AC}([a, b] \to \mathbb{R})\), then the derivative of \(f\) (denoted by \(\dot{f}\)) exists almost everywhere and is Lebesgue integrable on \([a, b]\).

Define

\[
K^* := \left\{ f \in \mathcal{AC}(\mathbb{R} \to \mathbb{R}) : f(0) = 0, \int_{-\infty}^{\infty} (\dot{f}(x))^2 dx \leq 1 \right\}. \tag{2}
\]

**Theorem 1.** For \(\mathbb{P}\)-a.e. \(\xi\), under the quenched probability \(\mathbb{P}(\cdot \mid \xi)\), the process

\[
\tilde{K}_n := \frac{K_n}{(2n^{3/2} \log \log n)^{1/2}}, \quad n > e^e,
\]
does not converge in law. More precisely, for \(\mathbb{P}\)-a.e. \(\xi\), under the quenched probability \(\mathbb{P}(\cdot \mid \xi)\), the limit points of the law of \(\tilde{K}_n\), as \(n \to \infty\), under the topology of weak convergence of measures, are equal to the set of the laws of random variables in \(\Theta_B\), with

\[
\Theta_B := \left\{ \int_{-\infty}^{\infty} f(x) dL_1(x) : f \in K^* \right\}, \tag{3}
\]

where \((L_1(x), x \in \mathbb{R})\) denotes the family of local times at time 1 of a one-dimensional Brownian motion \(B\) starting from 0.

The set \(\Theta_B\) is closed for the topology of weak convergence of measures, and is a compact subset of \(L^2((B_t)_{t \in [0,1]})\).

Let us mention that the set \(K^*\) directly comes from Strassen [13]’s limiting set. The precise meaning of \(\int_{-\infty}^{\infty} f(x) dL_1(x)\) can be given by the integration by parts and the occupation times formula:

\[
\int_{-\infty}^{\infty} f(x) dL_1(x) = -\int_{-\infty}^{\infty} L_1(x) \dot{f}(x) dx = -\int_0^1 \dot{f}(B_s) ds, \tag{4}
\]

where as before, \(\dot{f}\) denotes the almost everywhere derivative of \(f\).

Instead of Theorem 1, we shall prove that there is no quenched limit theorem for the continuous analogue of \(K\) introduced by Kesten and Spitzer [9] and deduce Theorem 1 by using a strong approximation for the one-dimensional RWRS. Let us define this continuous analogue: Assume that \(B := (B(t))_{t \geq 0}\), \(W := (W(t))_{t \geq 0}\), \(\dot{W} := (\dot{W}(t))_{t \geq 0}\) are three real Brownian motions starting from 0, defined on the same probability space and independent of each other. For brevity, we shall write \(W(x) := W(t)\) if \(x \geq 0\) and \(\dot{W}(-x)\) if \(x < 0\) and say that \(W\) is a two-sided Brownian motion. We denote by \(\mathbb{P}_B\), \(\mathbb{P}_W\) the law of these processes. We will also denote by \((L_t(x))_{t \geq 0, x \in \mathbb{R}}\) a continuous version with compact support of the local time of the process \(B\). We define the continuous version of the RWRS, also called Brownian motion in Brownian scenery, as

\[
Z_t := \int_0^{+\infty} L_t(x) dW(x) + \int_0^{+\infty} L_t(-x) d\dot{W}(x) \equiv \int_{-\infty}^{+\infty} L_t(x) dW(x).
\]

In dimension one, under the annealed measure, Kesten and Spitzer [9] proved that the process \((n^{-3/4}K(|nt|))_{t \geq 0}\) weakly converges in the space of continuous functions to the continuous process \(Z = (Z(t))_{t \geq 0}\). Zhang [14] (see also [6, 10]) gave a stronger version of this result in the special case when the scenery has a finite moment of order \(2 + \delta\) for some \(\delta > 0\), more precisely, there is
a coupling of $\xi$, $S$, $B$ and $W$ such that $(\xi, W)$ is independent of $(S, B)$ and for any $\varepsilon > 0$, almost surely,

\[ \max_{0 \leq m \leq n} |K(m) - Z(m)| = o(n^{\frac{1}{2} + \frac{1}{2\lambda + \varepsilon}}), \quad n \to +\infty. \] (5)

Theorem 1 will follow from this strong approximation and the following result.

**Theorem 2.** $\mathbb{P}_W$-almost surely, under the quenched probability $\mathbb{P}(\cdot|W)$, the limit points of the law of

\[ \tilde{Z}_t := \frac{Z_t}{(2t^{3/2} \log \log t)^{1/2}}, \quad t \to \infty, \]

under the topology of weak convergence of measures, are equal to the set of the laws of random variables in $\Theta_B$ defined in Theorem 1. Consequently under $\mathbb{P}(\cdot|W)$, as $t \to \infty$, $\tilde{Z}_t$ does not converge in law.

To prove Theorem 2, we shall apply Strassen [13]'s functional law of the iterated logarithm applied to the two-sided Brownian motion $W$; we shall also need to estimate the stochastic integral $\int g(x)dW_t(x)$ for a Borel function $g$, see Section 2 for the details.

2. **Proofs**

For a two-sided one-dimensional Brownian motion $(W(t), t \in \mathbb{R})$ starting from 0, let us define for any $\lambda > e^e$,

\[ W_\lambda(t) := \frac{W(\lambda t)}{(2\lambda \log \log \lambda)^{1/2}}, \quad t \in \mathbb{R}. \]

**Lemma 3.** (i) Almost surely, for any $s < 0 < r$ rational numbers, $(W_\lambda(t), s \leq t \leq r)$ is relatively compact in the uniform topology and the set of its limit points is $K_{s,r}$, with

\[ K_{s,r} := \left\{ f \in AC([s, r] \to \mathbb{R}) : f(0) = 0, \int_s^r (f(x))^2 dx \leq 1 \right\}. \]

(ii) There exists some finite random variable $A_W$ only depending on $(W(x), x \in \mathbb{R})$ such that for all $\lambda \geq e^{36}$,

\[ \sup_{t \in \mathbb{R}, t \neq 0} \frac{|W_\lambda(t)|}{\sqrt{|t| \log \log(|t| + 1/|t| + 36)}} \leq A_W < \infty. \]

**Remark 4.** The statement (i) is a reformulation of Strassen’s theorem and holds in fact for all real numbers $s$ and $r$. Moreover, using the notation $K^*$ in (2), we remark that $K_{s,r}$ coincides with the restriction of $K^*$ on $[s, r]$: for any $s < 0 < r$,

\[ K_{s,r} = \left\{ f_{|[s,r]} : f \in K^* \right\}. \]

**Proof:** (i) For any fixed $s < 0 < r$, by applying Strassen’s theorem ([13]) to the two-dimensional rescaled Brownian motion: 

\[ \frac{W(\lambda u)}{\sqrt{2\lambda \log \log \lambda}}, \quad \frac{W(\lambda u)}{\sqrt{2\lambda |s| \log \log \lambda}} \] 

for $0 \leq u \leq 1$, we get that a.s., $(W_\lambda(t), s \leq t \leq r)$ is relatively compact in the uniform topology with $K_{s,r}$ as the set of limit points. By inverting a.s. and $s, r$, we obtain (i).

(ii) By the classical law of the iterated logarithm for the Brownian motion $W$ (both at 0 and at $\infty$), we get that

\[ \bar{A}_W := \sup_{x \in \mathbb{R}, x \neq 0} \frac{|W(x)|}{\sqrt{|x| \log \log(|x| + 1/|x| + 36)}} \]
is a finite variable. Observe that for any $t > 0$ and $\lambda > e^{36}$, $\log \log (\lambda t + \frac{1}{M} + 36) \leq \log \lambda + \log \log (t + \frac{1}{t} + 36)$. The Lemma follows if we take for e.g. $A_W := 2A_W$. □

Next, we recall some properties of Brownian local times: The process $x \mapsto L_1(x)$ is a (continuous) semimartingale (by Perkins [11]), moreover, the quadratic variation of $x \mapsto L_1(x)$ equals $4 \int_{0}^{x} L_1(z)dz$. By Revuz and Yor ([12], Exercise VI (1.28)), for any locally bounded Borel function $f$,

$$\frac{1}{2} \int_{-\infty}^{\infty} f(x) dL_1(x) = -\int_{0}^{B_1} f(u) du + \int_{0}^{1} f(B_u) dB_u. \quad (6)$$

Let us define for all $\lambda > e^\varepsilon$ and $n \geq 0$,

$$H_\lambda := \int_{-\infty}^{\infty} W_\lambda(x) dL_1(x), \quad H_\lambda^{(n)} := \int_{-n}^{n} W_\lambda(x) dL_1(x),$$

with $H_\lambda^{(0)} = 0$. Denote by $E_B$ the expectation with respect to the law of $B$.

**Lemma 5.** There exists some positive constant $c_1$ such that for any $\lambda > e^{36}$ and $n \geq 0$, we have

$$E_B |H_\lambda - H_\lambda^{(n)}| \leq c_1 e^{-\frac{n^2}{2}} A_W,$$  

$$E_B \left( \int_{-\infty}^{\infty} f(x) dL_1(x) \right)^2 \leq 16 s(f), \quad (8)$$

$$E_B \left| \int_{-\infty}^{\infty} f(x) dL_1(x) - \int_{-n}^{n} f(x) dL_1(x) \right| \leq 4 \sqrt{2s(f)} e^{-\frac{n^2}{4}}, \quad (9)$$

for any Borel function $f : \mathbb{R} \to \mathbb{R}$ such that $s(f) := \sup_{0 \leq u \leq 1} E_B \left[ f^2(B_u) \right] < \infty$.

Remark that if $f$ is bounded, then $s(f) \leq \sup_{x \in \mathbb{R}} f^2(x)$.

**Proof:** We first prove that there exists some positive constant $c_2$ such that for all $n \geq 0$ and $\lambda > e^{36}$,

$$E_B \left[ (H_\lambda - H_\lambda^{(n)})^2 \right] \leq c_2 A_W^2. \quad (10)$$

In fact, by applying (6) and using the Brownian isometry for $f(x) = W_\lambda(x)1_{(|x| > n)}$, we get that

$$E_B \left[ (H_\lambda - H_\lambda^{(n)})^2 \right] \leq 8E_B \left[ F_{n,\lambda}(B_1)^2 \right] + 8E_B \left[ \int_{0}^{1} (W_\lambda(B_u))^2 1_{(|B_u| > n)} du \right],$$

with $F_{n,\lambda}(x) := \int_{0}^{x} W_\lambda(y) 1_{(|y| > n)} dy$ for any $x \in \mathbb{R}$. By Lemma 3 (ii),

$$|F_{n,\lambda}(x)| \leq A_W \int_{0}^{x} \left( |y| \log |y| + \frac{1}{|y|} + 36 \right)^{1/2} dy \leq c_3 A_W (1 + x^2), \quad \forall x \in \mathbb{R},$$

with some constant $c_3 > 0$. Hence $E_B \left[ F_{n,\lambda}(B_1)^2 \right] \leq 6 c_3^2 A_W^2$. In the same way, $E_B \left[ (W_\lambda(B_u))^2 \right] \leq A_W^2 E_B \left[ |B_u| \log |B_u| + \frac{1}{|B_u|} + 36 \right]$ which is integrable for $u \in (0, 1]$. Then (10) follows.
To check (7), we remark that \( H_\lambda - H_\lambda^{(n)} = 0 \) if \( \sup_{0 \leq u \leq 1} |B_u| \leq n \). Then by Cauchy-Schwarz’ inequality and (10), we have that
\[
\mathbb{E}_B \left| H_\lambda - H_\lambda^{(n)} \right| = \mathbb{E}_B \left[ \left| H_\lambda - H_\lambda^{(n)} \right| 1_{\left( \sup_{0 \leq u \leq 1} |B_u| > n \right)} \right] \\
\leq \sqrt{\mathbb{E}_B \left[ (H_\lambda - H_\lambda^{(n)})^2 \right]} \sqrt{\mathbb{P}_B \left( \sup_{0 \leq u \leq 1} |B_u| > n \right)} \\
\leq \sqrt{c_2 A_W} \sqrt{2} e^{-\frac{c_2}{4}},
\]
by the standard Gaussian tail: \( \mathbb{P}_B \left( \sup_{0 \leq u \leq 1} |B_u| > x \right) \leq 2e^{-x^2/2} \) for any \( x > 0 \). Then we get (7).

To prove (8), we use again (6) and the Brownian isometry to arrive at
\[
\mathbb{E}_B \left( \int_{-\infty}^{\infty} f(x)dL_1(x) \right)^2 \leq 8\mathbb{E}_B \left[ G^2(B_1) \right] + 8 \int_0^1 \mathbb{E}_B \left[ f^2(B_u) \right] du \leq 8\mathbb{E}_B \left[ G^2(B_1) \right] + 8s(f),
\]
with \( G(x) := \int_0^x f(y)dy \) for any \( x \in \mathbb{R} \). By Cauchy-Schwarz’ inequality, \( (G(x))^2 \leq \left| x \int_0^x f^2(y)dy \right| \) for any \( x \in \mathbb{R} \), from which we use the integration by parts for the density of \( B_1 \) and deduce that \( \mathbb{E}_B \left[ G^2(B_1) \right] \leq \mathbb{E}_B \left[ f^2(B_1) \right] \). Then (8) follows.

Finally for (9), we use (8) to see that
\[
\mathbb{E}_B \left( \int_{-\infty}^{\infty} f(x)dL_1(x) - \int_{-n}^{n} f(x)dL_1(x) \right)^2 = \mathbb{E}_B \left( \int_{-\infty}^{\infty} f(x)1_{\{|x| > n\}}dL_1(x) \right)^2 \leq 16s(f),
\]
for any \( n \). Then (9) follows from the Cauchy-Schwarz inequality and the Gaussian tail, exactly in the same way as (7).

Recalling (3) for the definition of \( \Theta_B \). For any \( p > 0 \), it is easy to see that \( \Theta_B \subset L^p(B) \), since from Cauchy-Schwarz’ inequality, using the relation (4), we deduce that
\[
\left( \int_{-\infty}^{\infty} f(x)dL_1(x) \right)^2 \leq \left( \int_{-\infty}^{\infty} (L_1(x))^2 dx \right) \left( \int_{-\infty}^{\infty} (f(x))^2 dx \right) \leq \sup_{x} L_1(x) \in L^p(B),
\]
see Csáki [5], Lemma 1 for the tail of \( \sup_x L_1(x) \). Write \( d_{L^1(B)}(\xi, \eta) \) for the distance in \( L^1(B) \) for any \( \xi, \eta \in L^1(B) \).

**Lemma 6.** \( \mathbb{P}_W \)-almost surely,
\[
d_{L^1(B)}(H_\lambda, \Theta_B) \rightarrow 0, \quad \text{as } \lambda \rightarrow \infty,
\]
where \( \Theta_B \) is defined in (3). Moreover, \( \mathbb{P}_W \)-almost surely for any \( \xi \in \Theta_B \), \( \lim_{\lambda \rightarrow \infty} d_{L^1(B)}(H_\lambda, \xi) = 0 \).

**Proof:** Let \( \varepsilon > 0 \). Choose a large \( n = n(\varepsilon) \) such that \( c_1 e^{-n^2/4} \leq \varepsilon \). By Lemma 3 (i), for all large \( \lambda \geq \lambda_0(W, \varepsilon, n) \), there exists some function \( g = g_{\lambda, W, \varepsilon, n} \in K_{-n,n} \) such that \( \sup_{|x| \leq n} |W_\lambda(x) - g(x)| \leq \varepsilon \). Applying (8) to \( f(x) = (W_\lambda(x) - g(x))1_{\{|x| \leq n\}} \) which is a bounded by \( \varepsilon \), we get that
\[
\mathbb{E}_B \left| H_\lambda^{(n)} - \int_{-n}^{n} g(x)dL_1(x) \right| \leq 4\sqrt{s(f)} \leq 4\varepsilon.
\]
Proof of Theorem 2.

Firstly, we remark that by Brownian scaling, to find a sequence \( \xi \in \mathcal{K}^* \) for any \( W \) continuous semi-martingale, independent from the process \( P \) by parts, we get that

\[
\mathbb{E}_B \left[ H_\lambda - \int_{-\infty}^{\infty} g(x) dL_1(x) \right] \leq 4\varepsilon + \mathbb{E}_B [H_\lambda - H_\lambda^{(n)}] \leq (4 + c_1 A_W)\varepsilon.
\]

It follows that \( d_{L^1(B)}(H_\lambda, \Theta_B) \leq (4 + c_1 A_W)\varepsilon \). Hence \( \mathbb{P}_W \)-a.s., \( \limsup_{\lambda \to \infty} d_{L^1(B)}(H_\lambda, \Theta_B) \leq (4 + c_1 A_W)\varepsilon \), showing the first part in the Lemma.

For the other part of the Lemma, let \( h \in \mathcal{K}^* \) such that \( \xi = \int_{-\infty}^{\infty} h(x) dL_1(x) \). Observe that

\[
|h(x)| \leq \sqrt{\int |x| h(x)^2 dy} \leq \sqrt{\int |x|} \quad \text{for all } x \in \mathbb{R},
\]

\[
s(h) = \sup_{0 \leq s \leq 1} \mathbb{E}_B[h^2(B_n)] \leq \mathbb{E}_B[|B_1|],
\]

then for any \( \varepsilon > 0 \), we may use (9) and choose an integer \( n = n(\varepsilon) \) such that \( (c_1 + 4\sqrt{2})e^{-n^2/4} \leq \varepsilon \) and

\[
d_{L^1(B)}(H_\lambda^{(n)}, \xi_n) \leq \varepsilon,
\]

where \( \xi_n := \int_{-n}^{n} h(x) dL_1(x) \). Applying Lemma 3 (i) to the restriction of \( h \) on \([-n, n]\), we may find a sequence \( \lambda_j = \lambda_j(\varepsilon, W, n) \to \infty \) such that \( \sup_{x \leq n} |W_{\lambda_j}(x) - h(x)| \leq \varepsilon \). By applying (8) to \( f(x) = (W_{\lambda_j}(x) - h(x))1_{\{|x| \leq n\}} \), we have that

\[
d_{L^1(B)}(H_\lambda^{(n)}(x), \xi_n) \leq 4\varepsilon.
\]

By (7) and the choice of \( n \), \( d_{L^1(B)}(H_\lambda^{(n)}, H_{\lambda_j}) \leq \varepsilon A_W \) for all large \( \lambda_j \), it follows from the triangular inequality that

\[
d_{L^1(B)}(\xi, H_{\lambda_j}) \leq (5 + A_W)\varepsilon,
\]

implying that \( \mathbb{P}_W \)-a.s., \( \liminf_{\lambda \to \infty} d_{L^1(B)}(H_{\lambda}, \xi) \leq (5 + A_W)\varepsilon \to 0 \) as \( \varepsilon \to 0 \). \( \square \)

We now are ready to give the proof of Theorems 2 and 1.

**Proof of Theorem 2.** Firstly, we remark that by Brownian scaling, \( \mathbb{P}_W \)-a.s.,

\[
\frac{Z(t)}{t^{3/4}} \overset{(d)}{=} -\int_{m_1}^{M_1} \frac{1}{t^{1/4}} W(\sqrt{t}y) dL_1(y).
\] (11)

In fact, by the change of variables \( x = y\sqrt{t} \), we get

\[
\int_{-\infty}^{+\infty} L_t(x) dW(x) = \sqrt{t} \int_{-\infty}^{+\infty} \left( \frac{L_t(y\sqrt{t})}{\sqrt{t}} \right) dW(y\sqrt{t})
\]

which has the same distribution as

\[
\sqrt{t} \int_{-\infty}^{+\infty} L_1(y) dW(y\sqrt{t})
\]

from the scaling property of the local time of the Brownian motion. Since \( \{L_1(x)\}_{x \in \mathbb{R}} \) is a continuous semi-martingale, independent from the process \( W \), from the formula of integration by parts, we get that \( \mathbb{P}_W \)-a.s.,

\[
\sqrt{t} \int_{-\infty}^{+\infty} L_1(y) dW(y\sqrt{t}) = -t^{3/4} \int_{m_1}^{M_1} \left( \frac{W(\sqrt{y}t)}{t^{1/4}} \right) dL_1(y),
\]

yielding (11). The first part of Theorem 2 follows from Lemma 6.

Let \( \{\zeta_n\}_n \) be a sequence of random variables in \( \Theta_B \), each \( \zeta_n \) being associated to a function \( f_n \in \mathcal{K}^* \). The sequence of the (almost everywhere) derivatives of \( f_n \) is then a bounded sequence in the Hilbert space \( L^2(\mathbb{R}) \), so we can extract a subsequence which weakly converges. Using the definition of the weak convergence and the relation (4), \( \{\zeta_n\}_n \) converges almost surely and
the closure of $\Theta_B$ follows. Since the sequence $(\zeta_n)_n$ is bounded in $L^p(B)$ for any $p \geq 1$, the convergence also holds in $L^2(B)$. Therefore $\Theta_B$ is a compact set of $L^2(B)$ as closed and bounded subset. □

**Proof of Theorem 1.** We use the strong approximation of Zhang [14]: there exists on a suitably enlarged probability space, a coupling of $\xi, S, B$ and $W$ such that $(\xi, W)$ is independent of $(S, B)$ and for any $\varepsilon > 0$, almost surely,

$$\max_{0 \leq m \leq n} |K(m) - Z(m)| = o(n^{\frac{1}{2} + \frac{1}{2(2+\delta) + \varepsilon}}), \quad n \to +\infty.$$  

From the independence of $(\xi, W)$ and $(S, B)$, we deduce that for $\mathbb{P}$-a.e. $(\xi, W)$, under the quenched probability $\mathbb{P}(.|\xi, W)$, the limit points of the laws of $\tilde{K}_n$ and $\tilde{Z}_n$ are the same ones. Now, by adapting the proof of Theorem 2, we have that for $\mathbb{P}$-a.e. $(\xi, W)$, under the quenched probability $\mathbb{P}(.|\xi, W)$, the limit points of the laws of $\tilde{Z}_n$, as $n \to \infty$, under the topology of weak convergence of measures, are equal to the set of the laws of random variables in $\Theta_B$. It gives that for $\mathbb{P}$-a.e. $(\xi, W)$, under the quenched probability $\mathbb{P}(.|\xi, W)$, the limit points of the laws of $\tilde{K}_n$, as $n \to \infty$, under the topology of weak convergence of measures, are equal to the set of the laws of random variables in $\Theta_B$ and Theorem 1 follows. □

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**References**