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To cite this version:

HAL Id: hal-00861418
https://hal.archives-ouvertes.fr/hal-00861418
Submitted on 12 Sep 2013

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Sym-Bobenko formula for minimal surfaces in Heisenberg space

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September 12, 2013

Abstract

We give an immersion formula, the Sym-Bobenko formula, for minimal surfaces in the 3-dimensional Heisenberg space. Such a formula can be used to give a generalized Weierstrass type representation and construct explicit examples of minimal surfaces.

Mathematics Subject Classification: Primary 53A10, Secondary 53C42.

1 Introduction

A Sym-Bobenko formula is the expression of an immersion in terms of a one-parameter family of moving frames, called the extended frame. This idea was first used by A. Sym [9] in the case of surfaces with negative constant (Gauss) curvature in euclidean space. A. I. Bobenko applied the method to numerous cases [1] [2] [3], including constant mean curvature (CMC for short) surfaces in space forms — euclidean 3-space, 3-sphere and hyperbolic 3-space — and T. Taniguchi applied it to CMC spacelike surfaces in Minkowski 3-space [10].

In the mean time, the work of J. Dorfmeister, F. Pedit and H. Wu [8] and D. Brander, W. Rossman and N. Schmitt [4] show that Sym-Bobenko formulæ can be seen as generalized Weierstrass type representations for CMC surfaces, extended frames coming from holomorphic data.

In Heisenberg 3-space, the classical method does not apply, since the isometry group is of dimension only 4 — contrary to the ones of space forms that are 6-dimensional — and does not act transitively on orthonormal frames; there are “not enough” isometries to define a moving frame. We show that nevertheless, for minimal immersions a Sym-Bobenko formula can be established using an ad-hoc matrix-valued map.

In [7], J. F. Dorfmeister, J. Inoguchi and S. Kobyashi link this formula with pairs of meromorphic and anti-meromorphic 1-forms, which they call pairs of normalized potentials, in a way to get a generalized Weierstrass type representation for minimal surfaces.

*This work is part of the author’s Ph. D. thesis [5].
2 Surfaces in Heisenberg space

We see the 3-dimensional Heisenberg space $\text{Nil}_3$ as $\mathbb{R}^3$, with generic coordinates $(x_1, x_2, x_3)$, endowed with the following riemannian metric:

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 + \left(\frac{1}{2}(x_2dx_1 - x_1dx_2) + dx_3\right)^2.$$ We call canonical frame the orthonormal frame $(E_1, E_2, E_3)$ defined by:

$$E_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3}, \quad E_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} \quad \text{and} \quad E_3 = \frac{\partial}{\partial x_3},$$ and the Levi-Civita connection $\nabla$ writes:

$$\nabla_{E_1} E_1 = 0 \quad \nabla_{E_2} E_1 = -\frac{1}{2} E_3 \quad \nabla_{E_3} E_1 = -\frac{1}{2} E_2$$
$$\nabla_{E_1} E_2 = \frac{1}{2} E_3 \quad \nabla_{E_2} E_2 = 0 \quad \nabla_{E_3} E_2 = \frac{1}{2} E_1$$
$$\nabla_{E_1} E_3 = -\frac{1}{2} E_2 \quad \nabla_{E_2} E_3 = \frac{1}{2} E_1 \quad \nabla_{E_3} E_3 = 0.$$

Note that the vector field $E_3$ is a Killing field and that the projection $\pi : (x_1, x_2, x_3) \in \text{Nil}_3 \mapsto (x_1, x_2) \in \mathbb{R}^2$ on the first two coordinates is a Riemannian submersion. From now on, we identify $\mathbb{R}^2$ with $\mathbb{C}$.

We may also write $\text{Nil}_3$ as a subset of $\mathcal{M}_2(\mathbb{C})$. Consider the matrices:

$$\sigma_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ The identification is the following:

$$(x_1, x_2, x_3) \in \text{Nil}_3 \longleftrightarrow x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3 = \begin{pmatrix} x_3 \\ x_1 - ix_2 \\ x_3 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}). \quad (1)$$

Note that this identification is purely formal and does not involve any manifold related structure.

Let $\Sigma$ be a simply connected Riemann surface and $z$ be a conformal parameter on $\Sigma$. A conformal immersion is denoted $f : \Sigma \rightarrow \text{Nil}_3$ with unit normal $N$ and conformal factor $\rho : \Sigma \rightarrow (0, +\infty)$ meaning:

$$\langle f_z, f_z \rangle = \langle f_\bar{z}, f_\bar{z} \rangle = 0, \quad \langle f_z, f_\bar{z} \rangle = \frac{\rho}{2},$$
$$\langle f_z, N \rangle = \langle f_\bar{z}, N \rangle = 0 \quad \text{and} \quad \langle N, N \rangle = 1.$$ Consider also $\varphi = \langle N, E_3 \rangle : \Sigma \rightarrow (-1, 1)$ denote the angle function of $N$, $A = \langle f_z, E_3 \rangle : \Sigma \rightarrow \mathbb{C}$ the vertical part of $f_z$ and $p dz^2 = \langle \nabla f_z f_z, N \rangle dz^2$ the Hopf differential of $f$. 2
The Abresch-Rosenberg differential expresses $Q dz^2 = (ip + A^2) dz^2$, and a necessary and sufficient condition for $f$ to be minimal is $\nabla f_z f_{\bar{z}} = 0$.

We also decompose $f$ into $f = (F, h)$ with $F = \pi \circ f : \Sigma \to \mathbb{C}$ the horizontal projection of $f$ and $h : \Sigma \to \mathbb{R}$ its height function. We can express $A$ in terms of $F$ and $h$:

$$A = h_z - i\frac{1}{4} \left( F F_z - F_{\bar{z}} \right).$$

(2)

In the matrix model (1) of $\text{Nil}_3$, the map $F$ is given by the non-diagonal coefficients — precisely the $(1,2)$-coefficient — and $h$ by the diagonal ones.

The intuitive idea behind Sym-Bobenko formulæ in space forms is that, up to ambient isometries, the unit normal — or Gauss map — would locally determine the immersion up to ambient isometries. In $\text{Nil}_3$ such a map is defined as follows; see [6] for details. Since $\text{Nil}_3$ is a Lie group, the map $f^{-1} N$ takes values in the unit sphere $S^2$ of the Lie algebra. Moreover, for a local study, we can suppose $\varphi > 0$ so that the values of $f^{-1} N$ are actually in the northern hemisphere of $S^2$. If $s$ denotes the stereographic projection centered at the South Pole, we call Gauss map of an immersion $f$ the map $g = s \circ (f^{-1} N)$ with values in the unit disk. Actually, endowing the unit disk with the Poincaré metric, we see the Gauss map $g$ as a map with values into the hyperbolic disk $\mathbb{H}^2$.

We use the following criterion to show that a conformal immersion in Heisenberg space is minimal:

**Proposition 2.1** (Daniel [5]). A conformal immersion $f = (F, h) : \Sigma \to \text{Nil}_3$ is minimal if and only if:

$$F_{zz} = \frac{i}{2} \left( \overline{AF_z} + A_{\bar{z}} \right) \quad \text{and} \quad A_{\bar{z}} + \overline{A_z} = 0.$$

Furthermore, when $f$ is minimal its Gauss map $g : \Sigma \to \mathbb{H}^2$ is harmonic.

3 The Sym-Bobenko formula

Consider the family $(\Psi_t)_{t \in \mathbb{R}}$ of matrix fields over $\Sigma$ which are solutions of the system:

$$\Psi_t^{-1} d\Psi_t = \frac{1}{4} \left( \begin{array}{cc} \frac{\log \rho_0}{\sqrt{\rho_0}} & i \sqrt{\rho_0} e^{2it} \\ -i \sqrt{\rho_0} e^{-2it} & -\frac{\log \rho_0}{\sqrt{\rho_0}} \end{array} \right) dz$$

$$+ \frac{1}{4} \left( \begin{array}{cc} -\frac{\log \rho_0}{\sqrt{\rho_0}} e^{2it} \\ i \sqrt{\rho_0} e^{-2it} \end{array} \right) d\bar{z},$$

$$\Psi_t(z = 0) = \sigma_3$$
where $\rho_0: \Sigma \to (0, +\infty)$ and $Q_0: \Sigma \to \mathbb{C}$ are smooth. Such a family $(\Psi_t)$ exists if and only if:

$$\frac{\log \rho_0}{8} - \frac{2|Q_0|^2}{\rho_0} \quad \text{and} \quad (Q_0)_{\bar{z}} = 0.$$ 

**Theorem 3.1** (Sym-Bobenko formula). Using the matrix model (1), define the map $f_t: \Sigma \to \text{Nil}_3$ for any $t \in \mathbb{R}$ as:

$$f_t = -\frac{1}{2} \left( \sigma_0 \frac{\partial \hat{f}_t}{\partial t} \right)^d + \left( \hat{f}_t \right)^{\text{nd}} \quad \text{with} \quad \hat{f}_t = -2 \frac{\partial \Psi_t}{\partial t} \Psi_t^{-1} + 2 \Psi_t \sigma_0 \Psi_t^{-1}, \quad (3)$$

where the superscripts $^d$ and $^{\text{nd}}$ denote respectively the diagonal and non-diagonal terms. Then $f_t$ is a conformal minimal immersion in Heisenberg space and the family $(f_t)$ is the so-called associated family.

**Proof.** Fix $t \in \mathbb{R}$. From Equation (3), we get:

$$\left( f_t \right)^{\text{nd}} = (\hat{f}_t)^{\text{nd}}, \quad ((f_t)_z)^d = \frac{1}{2} \left( \sigma_0 \left[ (\hat{f}_t)_z, \hat{f}_t \right] \right)^d + 2 \left( \sigma_0 (\hat{f}_t)_z \right)^d \quad \text{and} \quad (\hat{f}_t)_{\bar{z}z} = \frac{i}{4} \left( (\hat{f}_t)_z, (\hat{f}_t)_{\bar{z}} \right), \quad (4)$$

where $[\cdot, \cdot]$ denotes the commutator. From the first equation in (4), we have that matrices $f_t$ and $\hat{f}_t$ write:

$$f_t = \begin{pmatrix} h & F \\ \overline{F} & h \end{pmatrix} \quad \text{and} \quad \hat{f}_t = \begin{pmatrix} i\hbar & F \\ -i\overline{F} & -i\hbar \end{pmatrix},$$

with $F: \Sigma \to \mathbb{C}$ and $h, \hat{h}: \Sigma \to \mathbb{R}$ smooth. We show that $F$ and $h$ verify the conditions of Proposition 2.1. Using Equation (2) and the second identity in (4), we deduce $A = i\hbar z$ and since $\hat{h}$ is real-valued, we obtain $A_{\bar{z}} + \overline{A}_{\bar{z}} = 0$. Finally, the $(1, 2)$-coefficient of the third equation in (4) verify:

$$F_{\bar{z}\bar{z}} = \frac{1}{2} \left( \hbar F_{\bar{z}} - \hbar F_{\bar{z}} \right) = \frac{i}{2} \left( \overline{A} F_{\bar{z}} + A F_{\bar{z}} \right),$$

which concludes the proof. 

**References**


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