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Sym-Bobenko formula for minimal surfaces in Heisenberg space*

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Abstract

We give an immersion formula, the Sym-Bobenko formula, for minimal surfaces in the 3-dimensional Heisenberg space. Such a formula can be used to give a generalized Weierstrass type representation and construct explicit examples of minimal surfaces.

Mathematics Subject Classification: Primary 53A10, Secondary 53C42.

1 Introduction

A Sym-Bobenko formula is the expression of an immersion in terms of a one-parameter family of moving frames, called the extended frame. This idea was first used by A. Sym [9] in the case of surfaces with negative constant (Gauss) curvature in euclidean space. A. I. Bobenko applied the method to numerous cases [1] [2] [3], including constant mean curvature (CMC for short) surfaces in space forms — euclidean 3-space, 3-sphere and hyperbolic 3-space — and T. Taniguchi applied it to CMC spacelike surfaces in Minkowski 3-space [10]. In the mean time, the work of J. Dorfmeister, F. Pedit and H. Wu [8] and D. Brander, W. Rossman and N. Schmitt [4] show that Sym-Bobenko formulæ can be seen as generalized Weierstrass type representations for CMC surfaces, extended frames coming from holomorphic data.

In Heisenberg 3-space, the classical method does not apply, since the isometry group is of dimension only 4 — contrary to the ones of space forms that are 6-dimensional — and does not act transitively on orthonormal frames; there are “not enough” isometries to define a moving frame. We show that nevertheless, for minimal immersions a Sym-Bobenko formula can be established using an ad-hoc matrix-valued map.

In [7], J. F. Dorfmeister, J. Inoguchi and S. Kobyashi link this formula with pairs of meromorphic and anti-meromorphic 1-forms, which they call pairs of normalized potentials, in a way to get a generalized Weierstrass type representation for minimal surfaces.

*This work is part of the author’s Ph. D. thesis [5].
2 Surfaces in Heisenberg space

We see the 3-dimensional Heisenberg space $\text{Nil}_3$ as $\mathbb{R}^3$, with generic coordinates $(x_1, x_2, x_3)$, endowed with the following riemannian metric:

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 + \left( \frac{1}{2} (x_2 dx_1 - x_1 dx_2) + dx_3 \right)^2.$$ 

We call canonical frame the orthonormal frame $(E_1, E_2, E_3)$ defined by:

$$E_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3}, \quad E_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} \quad \text{and} \quad E_3 = \frac{\partial}{\partial x_3},$$

and the Levi-Civita connection $\nabla$ writes:

$$\nabla_{E_1} E_1 = 0, \quad \nabla_{E_2} E_1 = -\frac{1}{2} E_3, \quad \nabla_{E_3} E_1 = -\frac{1}{2} E_2,$$

$$\nabla_{E_1} E_2 = \frac{1}{2} E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_3} E_2 = \frac{1}{2} E_1,$$

$$\nabla_{E_1} E_3 = -\frac{1}{2} E_2, \quad \nabla_{E_2} E_3 = \frac{1}{2} E_1, \quad \nabla_{E_3} E_3 = 0.$$

Note that the vector field $E_3$ is a Killing field and that the projection $\pi : (x_1, x_2, x_3) \in \text{Nil}_3 \mapsto (x_1, x_2) \in \mathbb{R}^2$ on the first two coordinates is a Riemannian submersion. From now on, we identify $\mathbb{R}^2$ with $\mathbb{C}$.

We may also write $\text{Nil}_3$ as a subset of $\mathcal{M}_2(\mathbb{C})$. Consider the matrices:

$$\sigma_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

The identification is the following:

$$(x_1, x_2, x_3) \in \text{Nil}_3 \longleftrightarrow x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3$$

$$= \begin{pmatrix} x_3 \\ x_1 - i x_2 \\ x_3 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}). \quad (1)$$

Note that this identification is purely formal and does not involve any manifold related structure.

Let $\Sigma$ be a simply connected Riemann surface and $z$ be a conformal parameter on $\Sigma$. A conformal immersion is denoted $f : \Sigma \to \text{Nil}_3$ with unit normal $N$ and conformal factor $\rho : \Sigma \to (0, +\infty)$ meaning:

$$\langle f_z, f_z \rangle = (f_\bar{z}, f_\bar{z}) = 0, \quad \langle f_z, f_\bar{z} \rangle = \frac{\rho}{2},$$

$$\langle f_z, N \rangle = (f_\bar{z}, N) = 0 \quad \text{and} \quad \langle N, N \rangle = 1.$$ 

Consider also $\varphi = \langle N, E_3 \rangle : \Sigma \to (-1, 1)$ denote the angle function of $N$, $A = \langle f_z, E_3 \rangle : \Sigma \to \mathbb{C}$ the vertical part of $f_z$ and $pdz^2 = \langle \nabla f_z f_\bar{z}, N \rangle dz^2$ the Hopf differential of $f$. 

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The Abresch-Rosenberg differential expresses \( Qdz^2 = (ip + A^2)dz^2 \), and a necessary and sufficient condition for \( f \) to be minimal is \( \nabla f_z f_{\bar{z}} = 0 \).

We also decompose \( f \) into \( f = (F, h) \) with \( F = \pi \circ f : \Sigma \to \mathbb{C} \) the horizontal projection of \( f \) and \( h : \Sigma \to \mathbb{R} \) its height function. We can express \( A \) in terms of \( F \) and \( h \):

\[
A = h_z - \frac{i}{4} \left( F F_z - \overline{F} F_{\bar{z}} \right). \tag{2}
\]

In the matrix model \((\Pi)\) of \( \text{Nil}_3 \), the map \( F \) is given by the non-diagonal coefficients — precisely the \((1, 2)\)-coefficient — and \( h \) by the diagonal ones.

The intuitive idea behind Sym-Bobenko formulæ in space forms is that, up to ambient isometries, the unit normal — or Gauss map — would locally determine the immersion up to ambient isometries. In \( \text{Nil}_3 \) such a map is defined as follows; see [6] for details. Since \( \text{Nil}_3 \) is a Lie group, the map \( f^{-1}N \) takes values in the unit sphere \( S^2 \) of the Lie algebra. Moreover, for a local study, we can suppose \( \varphi > 0 \) so that the values of \( f^{-1}N \) are actually in the northern hemisphere of \( S^2 \). If \( s \) denotes the stereographic projection centered at the South Pole, we call Gauss map of an immersion \( f \) the map \( g = s \circ (f^{-1}N) \) with values in the unit disk. Actually, endowing the unit disk with the Poincaré metric, we see the Gauss map \( g \) as a map with values into the hyperbolic disk \( \mathbb{H}^2 \).

We use the following criterion to show that a conformal immersion in Heisenberg space is minimal:

**Proposition 2.1** (Daniel [6]). A conformal immersion \( f = (F, h) : \Sigma \to \text{Nil}_3 \) is minimal if and only if:

\[
F_{z\bar{z}} = \frac{i}{2} \left( \overline{AF_z} + AF_{\bar{z}} \right) \quad \text{and} \quad A_z + \overline{A}_{\bar{z}} = 0.
\]

Furthermore, when \( f \) is minimal its Gauss map \( g : \Sigma \to \mathbb{H}^2 \) is harmonic.

### 3 The Sym-Bobenko formula

Consider the family \( (\Psi_t)_{t \in \mathbb{R}} \) of matrix fields over \( \Sigma \) which are solutions of the system:

\[
\begin{aligned}
\Psi_t^{-1} d\Psi_t &= \frac{1}{4} \begin{pmatrix}
\frac{\log \rho_0}_z & i\sqrt{\rho_0} \\
\frac{4iQ_0}{\sqrt{\rho_0}} e^{2it} & -(\log \rho_0)_z
\end{pmatrix} d\bar{z} \\
&\quad + \frac{1}{4} \begin{pmatrix}
-(\log \rho_0)_{\bar{z}} & \frac{4iQ_0}{\sqrt{\rho_0}} e^{-2it} \\
-i\sqrt{\rho_0} & (\log \rho_0)_{\bar{z}}
\end{pmatrix} d\bar{z}, \\
\Psi_t(z = 0) &= \sigma_3
\end{aligned}
\]

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where \( \rho_0 : \Sigma \to (0, +\infty) \) and \( Q_0 : \Sigma \to \mathbb{C} \) are smooth. Such a family \((\Psi_t)\) exists if and only if:

\[
(\log \rho_0)_{\bar{z}z} = \frac{\rho_0}{8} - \frac{2|Q_0|^2}{\rho_0} \quad \text{and} \quad (Q_0)_{\bar{z}} = 0.
\]

**Theorem 3.1** (Sym-Bobenko formula). Using the matrix model (1), define the map \( f_t : \Sigma \to \text{Nil}_3 \) for any \( t \in \mathbb{R} \) as:

\[
f_t = -\frac{1}{2} \left( \sigma_0 \frac{\partial \hat{f}_t}{\partial t} \right)^d + \left( \hat{f}_t \right)^{nd} \quad \text{with} \quad \hat{f}_t = -2 \frac{\partial \Psi_t}{\partial t} \Psi_t^{-1} + 2 \Psi_t \sigma_0 \Psi_t^{-1}, \quad (3)
\]

where the superscripts \(^d\) and \(^{nd}\) denote respectively the diagonal and non-diagonal terms. Then \( f_t \) is a conformal minimal immersion in Heisenberg space and the family \((f_t)\) is the so-called associated family.

**Proof.** Fix \( t \in \mathbb{R} \). From Equation (3), we get:

\[
(f_t)^{nd} = (\hat{f}_t)^{nd}, \quad ((f_t)_{z})^d = \frac{1}{2} \left( \sigma_0 \left[ (\hat{f}_t)_z, \hat{f}_t \right] \right)^d + 2 \left( \sigma_0 (\hat{f}_t)_z \right)^d
\]

and

\[
(\hat{f}_t)_{\bar{z}z} = i \left[ (\hat{f}_t)_z, (\hat{f}_t)_{\bar{z}} \right], \quad (4)
\]

where \([\cdot, \cdot]\) denotes the commutator. From the first equation in (1), we have that matrices \( f_t \) and \( \hat{f}_t \) write:

\[
f_t = \begin{pmatrix} h & F \\ F & h \end{pmatrix} \quad \text{and} \quad \hat{f}_t = \begin{pmatrix} i\hat{h} & F \\ F & -i\hat{h} \end{pmatrix},
\]

with \( F : \Sigma \to \mathbb{C} \) and \( h, \hat{h} : \Sigma \to \mathbb{R} \) smooth. We show that \( F \) and \( h \) verify the conditions of Proposition 2.1. Using Equation (2) and the second identity in (4), we deduce \( A = i\hat{h}z \) and since \( \hat{h} \) is real-valued, we obtain \( A_{\bar{z}} + \bar{A}_z = 0 \). Finally, the \((1, 2)\)-coefficient of the third equation in (4) verify:

\[
F_{\bar{z}z} = \frac{1}{2} \left( \bar{h}_z F_z - \bar{h}_z F_{\bar{z}} \right) = \frac{i}{2} \left( \bar{A} F_z + A F_{\bar{z}} \right),
\]

which concludes the proof. \( \square \)

**References**


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