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STRATIFIED CRITICAL POINTS ON THE REAL MILNOR FIBRE AND INTEGRAL-GEOMETRIC FORMULAS

NICOLAS DUTERTRE

Dedicated to professor David Trotman on his 60th birthday

Abstract. Let \((X, 0) \subset (\mathbb{R}^n, 0)\) be the germ of a closed subanalytic set and let \(f\) and \(g : (X, 0) \to (\mathbb{R}, 0)\) be two subanalytic functions. Under some conditions, we relate the critical points of \(g\) on the real Milnor fibre \(X \cap f^{-1}(\delta) \cap B_\epsilon, 0 < |\delta| \ll \epsilon \ll 1\), to the topology of this fibre and other related subanalytic sets. As an application, when \(g\) is a generic linear function, we obtain an “asymptotic” Gauss-Bonnet formula for the real Milnor fibre of \(f\). From this Gauss-Bonnet formula, we deduce “infinitesimal” linear kinematic formulas.

1. Introduction

Let \(F = (f_1, \ldots, f_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0), 2 \leq k \leq n\), be a complete intersection with isolated singularity. The Lê-Greuel formula [21, 22] states that
\[
\mu(F') + \mu(F) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} I,
\]
where \(F' : (\mathbb{C}^n, 0) \to (\mathbb{C}^{k-1}, 0)\) is the map with components \(f_1, \ldots, f_{k-1}, I\) is the ideal generated by \(f_1, \ldots, f_{k-1}\) and the \((k \times k)\)-minors \(\frac{\partial (f_1, \ldots, f_k)}{\partial (x_1, \ldots, x_k)}\) and \(\mu(F)\) (resp. \(\mu(F')\)) is the Milnor number of \(F\) (resp. \(F'\)). Hence the Lê-Greuel formula gives an algebraic characterization of a topological data, namely the sum of two Milnor numbers. However, since the right-hand side of the above equality is equal to the number of critical points of \(f_k\), counted with multiplicity, on the Milnor fibre of \(F'\), the Lê-Greuel formula can be also viewed as a topological characterization of this number of critical points.

Many works have been devoted to the search of a real version of the Lê-Greuel formula. Let us recall them briefly. We consider an analytic map-germ \(F = (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0), 2 \leq k \leq n\), and we denote by \(F'\) the map-germ \((f_1, \ldots, f_{k-1}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^{k-1}, 0)\). Some authors investigated the following difference:
\[
D_{\delta, \delta'} = \chi(F'^{-1}(\delta) \cap \{f_k \geq \delta'\} \cap B_\epsilon) - \chi(F'^{-1}(\delta) \cap \{f_k \leq \delta'\} \cap B_\epsilon),
\]
where \((\delta, \delta')\) is a regular value of \(F\) such that \(0 \leq |\delta'| \ll |\delta| \ll \epsilon\).

In [12], we proved that
\[
D_{\delta, \delta'} \equiv \dim_{\mathbb{R}} \mathcal{O}_{\mathbb{R}^n, 0} I \mod 2,
\]
where $O_{\mathbb{R}^n,0}$ is the ring of analytic function-germs at the origin and $I$ is the ideal generated by $f_1, \ldots, f_{k-1}$ and all the $k \times k$ minors $\frac{\partial(f_{i_1}, f_{i_2}, \ldots, f_{i_k})}{\partial(x_{i_1}, \ldots, x_{i_k})}$. This is only a mod 2 relation and we may ask if it is possible to get a more precise relation.

When $k = n$ and $f_k = x_1^2 + \cdots + x_n^2$, according to Aoki et al. ([1], [3]), $D_{\delta,0} = \chi(F^{r-1}(\delta) \cap B_{\epsilon}) = 2\deg_0 H$ and $2\deg_0 H$ is the number of semi-branches of $F^{r-1}(0)$, where

$$H = \left( \frac{\partial(f_n, f_1, \ldots, f_{n-1})}{\partial(x_1, \ldots, x_n)}, f_1, \ldots, f_{n-1} \right).$$

They proved a similar formula in the case $f_k = x_n$ in [2] and Szafraniec generalized all these results to any $f_k$ in [23].

When $k = 2$ and $f_2 = x_1$, Fukui [18] stated that

$$D_{\delta,0} = -\text{sign}(-\delta)^n \deg_0 H,$$

where $H = (f_1, \frac{\partial f_2}{\partial x_2}, \ldots, \frac{\partial f_2}{\partial x_n})$. Several generalizations of Fukui’s formula are given in [19], [11], [20] and [13].

In all these papers, the general idea is to count algebraically the critical points of a Morse perturbation of $f_k$ on $F^{r-1}(\delta) \cap B_{\epsilon}$ and to express this sum in two ways: as a difference of Euler characteristics and as a topological degree. Using the Eisenbud-Levine formula [16], this latter degree can be expressed as a signature of a quadratic form and so, we obtain an algebraic expression for $D_{\delta,0}$.

In this paper, we give a real and stratified version of the Lé-Greuel formula. We restrict ourselves to the topological aspect and relate a sum of indices of critical points on a real Milnor fibre to some Euler characteristics (this is also the point of view adopted in [7]). More precisely, we consider a germ of a closed subanalytic set $(X, 0) \subset (\mathbb{R}^n, 0)$ and a subanalytic function $f : (X, 0) \rightarrow (\mathbb{R}, 0)$. We assume that $X$ is contained in an open set $U$ of $\mathbb{R}^n$ and that $f$ is the restriction to $X$ of a $C^2$-subanalytic function $F : U \rightarrow \mathbb{R}$. We denote by $X^f$ the set $X \cap f^{-1}(0)$ and we equip $X$ with a Thom stratification adapted to $X^f$. If $0 < |\delta| \ll \epsilon \ll 1$ then the real Milnor fibre of $f$ is defined by

$$M^k_{\delta, \epsilon} = f^{-1}(\delta) \cap X \cap B_{\epsilon}.$$  

We consider another subanalytic function $g : (X, 0) \rightarrow (\mathbb{R}, 0)$ and we assume that it is the restriction to $X$ of a $C^2$-subanalytic function $G : U \rightarrow \mathbb{R}$. We denote by $X^g$ the intersection $X \cap g^{-1}(0)$. Under two conditions on $g$, we study the topological behaviour of $g|_{M^k_{\delta, \epsilon}}$.

We recall that if $Z \subset \mathbb{R}^n$ is a closed subanalytic set, equipped with a Whitney stratification and $p \in Z$ is an isolated critical point of a subanalytic function $\phi : Z \rightarrow \mathbb{R}$, restriction to $Z$ of a $C^2$-subanalytic function $\Phi$, then the index of $\phi$ at $p$ is defined as follows:

$$\text{ind}(\phi, Z, p) = 1 - \chi(Z \cap \{\phi = \phi(p) - \eta\} \cap B_{\epsilon}(p)).$$
where $0 < \eta \ll \epsilon \ll 1$ and $B_{\epsilon}(p)$ is the closed ball of radius $\epsilon$ centered at $p$. Let $p_1^{\delta, \epsilon}, \ldots, p_r^{\delta, \epsilon}$ be the critical points of $g$ on $X \cap f^{-1}(\delta) \cap B_{\epsilon}$, where $B_{\epsilon}$ denotes the open ball of radius $\epsilon$. We set
\[
I(\delta, \epsilon, g) = \sum_{i=1}^{r} \text{ind}(g, X \cap f^{-1}(\delta), p_i^{\delta, \epsilon}),
\]
\[
I(\delta, \epsilon, -g) = \sum_{i=1}^{r} \text{ind}(-g, X \cap f^{-1}(\delta), p_i^{\delta, \epsilon}).
\]

Our main theorem (Theorem 3.10) is the following:
\[
I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2 \chi(M^{\delta, \epsilon}_{f}) - \chi(X \cap f^{-1}(\delta) \cap S_{\epsilon}) - \chi(X \cap f^{-1}(\delta) \cap S_{\epsilon}).
\]

As a corollary (Corollary 3.11), when $f : (X, 0) \to (\mathbb{R}, 0)$ has an isolated stratified critical point at $0$, we obtain that
\[
I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2 \chi(M^{\delta, \epsilon}_{f}) - \chi(\text{Lk}(X^{f})) - \chi(\text{Lk}(X^{f} \cap X^{g})),
\]
where $\text{Lk}(-)$ denotes the link at the origin.

Then we apply these results when $g$ is a generic linear form to get an asymptotic Gauss-Bonnet formula for $M^{\delta, \epsilon}_{f}$ (Theorem 4.5). In the last section, we use this asymptotic Gauss-Bonnet formula to prove infinitesimal linear kinematic formulas for closed subanalytic germs (Theorem 5.5), that generalize the Cauchy-Crofton formula for the density due to Comte [8].

The paper is organized as follows. In Section 2, we prove several lemmas about critical points on the link of a subanalytic set. Section 3 contains real stratified versions of the Lê-Greuel formula. In Section 4, we establish the asymptotic Gauss-Bonnet formula and in Section 5, the infinitesimal linear kinematic formulas.

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2. LEMMAS ON CRITICAL POINTS ON THE LINK OF A STRATUM

In this section, we study the behaviour of the critical points of a $C^2$-subanalytic function on the link of stratum that contains $0$ in its closure, for a generic choice of the distance function to the origin.

Let $Y \subset \mathbb{R}^n$ be a $C^2$-subanalytic set such that $0$ belongs to its closure $\overline{Y}$. Let $\theta : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$-subanalytic function such that $\theta(0) = 0$. We will first study the behaviour of the critical points of $\theta|_Y : Y \to \mathbb{R}$ in
the neighborhood of 0, and then the behaviour of the critical points of the restriction of \( \theta \) to the link of 0 in \( Y \).

**Lemma 2.1.** The critical points of \( \theta|_Y \) lie in \( \{ \theta = 0 \} \) in a neighborhood of 0.

*Proof.* By the Curve Selection Lemma, we can assume that there is a \( C^1 \)-subanalytic curve \( \gamma : [0, \nu[ \to \mathbb{Y} \) such that \( \gamma(0) = 0 \) and \( \gamma(t) \) is a critical point of \( \theta|_Y \) for \( t \in ]0, \nu[ \). Therefore, we have

\[
(\theta \circ \gamma)'(t) = \langle \nabla \theta|_Y(\gamma(t)), \gamma'(t) \rangle = 0,
\]

since \( \gamma'(t) \) is tangent to \( Y \) at \( \gamma(t) \). This implies that \( \theta \circ \gamma(t) = \theta \circ \gamma(0) = 0 \). \( \square \)

Let \( \rho : \mathbb{R}^n \to \mathbb{R} \) be another \( C^2 \)-subanalytic function such that \( \rho^{-1}(a) \) intersects \( Y \) transversally. Then the set \( Y \cap \{ \rho \leq a \} \) is a manifold with boundary. Let \( p \) be a critical point of \( \theta|_Y \cap \{ \rho \leq a \} \) which lies in \( Y \cap \{ \rho = a \} \) and which is not a critical point of \( \theta|_Y \). This implies that

\[
\nabla \theta|_Y(p) = \lambda(p) \nabla \rho|_Y(p),
\]

with \( \lambda(p) \neq 0 \).

**Definition 2.2.** We say that \( p \in Y \cap \{ \rho = a \} \) is an outwards-pointing (resp. inwards-pointing) critical point of \( \theta|_{Y \cap \{ \rho \leq a \}} \) if \( \lambda(p) > 0 \) (resp. \( \lambda(p) < 0 \)).

Now let us assume that \( \rho : \mathbb{R}^n \to \mathbb{R} \) is a distance function to the origin which means that \( \rho \geq 0 \) and \( \rho^{-1}(0) = \{ 0 \} \) in a neighborhood of 0. By Lemma 2.1, we know that for \( \epsilon > 0 \) small enough, the level \( \rho^{-1}(\epsilon) \) intersects \( Y \) transversally. Let \( p^\epsilon \) be a critical point of \( \theta|_{Y \cap \rho^{-1}(\epsilon)} \) such that \( \theta(p^\epsilon) \neq 0 \). This means that there exists \( \lambda(p^\epsilon) \) such that

\[
\nabla \theta|_Y(p^\epsilon) = \lambda(p^\epsilon) \nabla \rho|_Y(p^\epsilon).
\]

Note that \( \lambda(p^\epsilon) \neq 0 \) because \( \nabla \theta|_Y(p^\epsilon) \neq 0 \) for \( \theta(p^\epsilon) \neq 0 \).

**Lemma 2.3.** The point \( p^\epsilon \) is an outwards-pointing (resp. inwards-pointing) for \( \theta|_{Y \cap \{ \rho \leq a \}} \) if and only if \( \theta(p^\epsilon) > 0 \) (resp. \( \theta(p^\epsilon) < 0 \)).

*Proof.* Let us assume that \( \lambda(p^\epsilon) > 0 \). By the Curve Selection Lemma, there exists a \( C^1 \)-subanalytic curve \( \gamma : [0, \nu[ \to \mathbb{Y} \) passing through \( p^\epsilon \) such that \( \gamma(0) = 0 \) and for \( t \neq 0 \), \( \gamma(t) \) is a critical point of \( \theta|_{Y \cap \{ \rho = \rho(\gamma(t)) \}} \) with \( \lambda(\gamma(t)) > 0 \). Therefore we have

\[
(\theta \circ \gamma)'(t) = \langle \nabla \theta|_Y(\gamma(t)), \gamma'(t) \rangle = \lambda(\gamma(t)) \langle \nabla \rho|_Y(\gamma(t)), \gamma'(t) \rangle.
\]

But \( \langle \rho \circ \gamma \rangle' > 0 \) for otherwise \( \rho \circ \gamma \) would be decreasing. Since \( \rho(\gamma(t)) \) tends to 0 as \( t \) tends to 0, this would imply that \( \rho \circ \gamma(t) \leq 0 \), which is impossible. We can conclude that \( (\theta \circ \gamma)' > 0 \) and that \( \theta \circ \gamma \) is strictly increasing. Since \( \theta \circ \gamma(t) \) tends to 0 as \( t \) tends to 0, we see that \( \theta \circ \gamma(t) > 0 \) for \( t > 0 \). Similarly if \( \lambda(p^\epsilon) < 0 \) then \( \theta(p^\epsilon) < 0 \). \( \square \)
Now we will study these critical points for a generic choice of the distance function. We denote by $\text{Sym}(\mathbb{R}^n)$ the set of symmetric $n \times n$-matrices with real entries, by $\text{Sym}^*(\mathbb{R}^n)$ the open dense subset of such matrices with non-zero determinant and by $\text{Sym}^+\text{+}(\mathbb{R}^n)$ the open subset of these invertible matrices that are positive definite or negative definite. Note that these sets are semi-algebraic. For each $A \in \text{Sym}^+\text{+}(\mathbb{R}^n)$, we denote by $\rho_A$ the following quadratic form:

$$\rho_A(x) = \langle Ax, x \rangle.$$  

We denote by $\Gamma_{\theta,A}^Y$ the following subanalytic polar set:

$$\Gamma_{\theta,A}^Y = \left\{ x \in Y \mid \text{rank} \left[ \nabla \theta|_Y(x), \nabla \rho_A|_Y(x) \right] < 2 \right\},$$  

and by $\Sigma_\theta^Y$ the set of critical points of $\theta|_Y$. Note that $\Sigma_\theta^Y \subset \{ \theta = 0 \}$ by Lemma 2.1.

**Lemma 2.4.** For almost all $A \in \text{Sym}^+\text{+}(\mathbb{R}^n)$, $\Gamma_{\theta,A}^Y \setminus (\Sigma_\theta^Y \cup \{ 0 \})$ is a $C^1$-subanalytic curve (possible empty) in a neighborhood of 0.

**Proof.** We can assume that $\dim Y > 1$. Let

$$Z = \left\{ (x, A) \in \mathbb{R}^n \times \text{Sym}^+\text{+}(\mathbb{R}^n) \mid x \in Y \setminus (\Sigma_\theta^Y \cup \{ 0 \}) \text{ and } \text{rank} \left[ \nabla \theta|_Y(x), \nabla \rho_A|_Y(x) \right] < 2 \right\}.$$  

Let $(y, B)$ be a point in $Z$. We can suppose that around $y$, $Y$ is defined by the vanishing of $k$ subanalytic functions $f_1, \ldots, f_k$ of class $C^2$. Hence in a neighborhood of $(y, B)$, $Z$ is defined be the vanishing of $f_1, \ldots, f_k$ and the minors

$$\frac{\partial (f_1, \ldots, f_k, \theta, \rho_A)}{\partial (x_{i1}, \ldots, x_{ik+2})}.$$  

Furthermore, since $y$ does not belong to $\Sigma_\theta^Y$, we can assume that

$$\frac{\partial (f_1, \ldots, f_k, \theta)}{\partial (x_1, \ldots, x_k, x_{k+1})} \neq 0,$$  

in a neighborhood of $y$. Therefore $Z$ is locally defined by $f_1 = \cdots = f_k = 0$ and

$$\frac{\partial (f_1, \ldots, f_k, \theta, \rho_A)}{\partial (x_1, \ldots, x_{k+1}, x_{k+2})} = \cdots = \frac{\partial (f_1, \ldots, f_k, \theta, \rho_A)}{\partial (x_1, \ldots, x_{k+1}, x_n)} = 0.$$  

Let us write $M = \frac{\partial (f_1, \ldots, f_k, \theta)}{\partial (x_1, \ldots, x_k, x_{k+1})}$ and for $i \in \{k+2, \ldots, n\}$, $m_i = \frac{\partial (f_1, \ldots, f_k, \theta, \rho_A)}{\partial (x_1, \ldots, x_{k+1}, x_i)}$.

If $A = [a_{ij}]$ then

$$\rho_A(x) = \sum_{i=1}^{n} a_{ii}x_i^2 + 2 \sum_{i \neq j} a_{ij}x_ix_j,$$  

and so $\frac{\partial \rho_A}{\partial x_i}(x) = 2\sum_{j=1}^{n} a_{ij}x_j$. For $i \in \{k+1, \ldots, n\}$ and $j \in \{1, \ldots, n\}$, we have

$$\frac{\partial m_i}{\partial a_{ij}} = 2x_jM.$$

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Since $y \neq 0$, one of the $x_j$’s does not vanish in the neighborhood of $y$ and we can conclude that the rank of
\[ [\nabla f_1(x), \ldots, \nabla f_k(x), \nabla m_{k+2}(x, A), \ldots, \nabla m_n(x, A)] \]
is $n - 1$ and that $Z$ is a $C^1$-subanalytic manifold of dimension $\frac{n(n+1)}{2} + 1$.

Now let us consider the projection $\pi_2 : Z \to \Sym^{+,*}(\mathbb{R}^n)$, $(x, A) \mapsto A$. Bertini-Sard’s theorem implies that the set $D_{\pi_2}$ of critical values of $\pi_2$ is a subanalytic set of dimension strictly less than $\frac{n(n+1)}{2}$. Hence, for all $A \notin D_{\pi_2}$, $\pi_2^{-1}(A)$ is a $C^1$-subanalytic curve (possibly empty). But this set is exactly $\Gamma_{Y,A} \setminus (\Sigma_Y \cup \{0\})$. \( \square \)

Let $R \subset Y$ be a subanalytic set of dimension strictly less than $\dim Y$. We will need the following lemma.

**Lemma 2.5.** For almost all $A$ in $\Sym^{+,*}(\mathbb{R}^n)$, $\Gamma_{Y,A} \setminus (\Sigma_Y \cup \{0\}) \cap R$ is a subanalytic set of dimension at most $0$ in a neighborhood of $0$.

**Proof.** Let us put $l = \dim Y$. Since $R$ admits a locally finite subanalytic stratification, we can assume that $R$ is a $C^2$-subanalytic manifold of dimension $d$ with $d < l$. Let $W$ be the following subanalytic set:
\[
W = \left\{ (x, A) \in \mathbb{R}^n \times \Sym^{+,*}(\mathbb{R}^n) \mid x \in R \setminus (\Sigma_Y \cup \{0\}) \right\}
\]
and $\operatorname{rank} \left[ \nabla \theta_{Y}(x), \nabla \rho_{A|Y}(x) \right] < 2$.

Using the same method as in the previous lemma, we can prove that $W$ is a $C^1$-subanalytic manifold of dimension $\frac{n(n+1)}{2} + 1 + d - l$ and conclude, remarking that $d - l \leq -1$. \( \square \)

Now we introduce a new $C^2$-subanalytic function $\beta : \mathbb{R}^n \to \mathbb{R}$ such that $\beta(0) = 0$. We denote by $\Gamma_{Y,\beta,A}$ the following subanalytic polar set:
\[
\Gamma_{Y,\beta,A} = \left\{ x \in Y \mid \operatorname{rank} \left[ \nabla \theta_{Y}(x), \nabla \beta_{Y}(x), \nabla \rho_{A|Y}(x) \right] < 3 \right\},
\]
and by $\Gamma_{Y,\beta}$ the following subanalytic polar set:
\[
\Gamma_{Y,\beta} = \left\{ x \in Y \mid \operatorname{rank} \left[ \nabla \theta_{Y}(x), \nabla \beta_{Y}(x) \right] < 2 \right\}.
\]

**Lemma 2.6.** For almost all $A$ in $\Sym^{+,*}(\mathbb{R}^n)$, $\Gamma_{Y,\beta,A} \setminus (\Gamma_{Y,\beta} \cup \{0\})$ is a $C^1$-subanalytic set of dimension at most $2$ (possibly empty) in a neighborhood of $0$.

**Proof.** We can assume that $\dim Y > 2$. Let
\[
Z = \left\{ (x, A) \in \mathbb{R}^n \times \Sym^{+,*}(\mathbb{R}^n) \mid x \in Y, \operatorname{rank} \left[ \nabla \theta_{Y}(x), \nabla \beta_{Y}(x) \right] = 2 \right\}
\]
and $\operatorname{rank} \left[ \nabla \theta_{Y}(x), \nabla \beta_{Y}(x), \nabla \rho_{A|Y}(x) \right] < 3$.

Let $(y, B)$ be a point in $Z$. We can suppose that around $y$, $Y$ is defined by the vanishing of $k$ subanalytic functions $f_1, \ldots, f_k$ of class $C^2$. Hence in a
neighborhood of \((y, B)\), \(Z\) is defined by the vanishing of \(f_1, \ldots, f_k\) and the minors
\[
\frac{\partial(f_1, \ldots, f_k, \theta, \beta, \rho_A)}{\partial(x_{i_1}, \ldots, x_{i_{k+3}})}.
\]
Since \(y\) does not belong to \(\Gamma_{\theta, \beta}^Y\), we can assume that
\[
\frac{\partial(f_1, \ldots, f_k, \theta)}{\partial(x_1, \ldots, x_k, x_{k+1}, x_{k+2})} \neq 0,
\]
in a neighborhood of \(y\). Therefore \(Z\) is locally defined by \(f_1, \ldots, f_k = 0\) and
\[
\frac{\partial(f_1, \ldots, f_k, \theta, \beta)}{\partial(x_1, \ldots, x_{k+2}, x_{k+3})} = \cdots = \frac{\partial(f_1, \ldots, f_k, \theta, \beta, \rho_A)}{\partial(x_1, \ldots, x_{k+2}, x_n)} = 0.
\]
It is clear that we can apply the same method as Lemma 2.4 to get the result. \(\square\)

3. LÊ-GREUEL TYPE FORMULA

In this section, we prove the LÊ-Greuel type formula announced in the introduction.

Let \((X, 0) \subset (\mathbb{R}^n, 0)\) be the germ of a closed subanalytic set and let \(f : (X, 0) \to (\mathbb{R}, 0)\) be a subanalytic function. We assume that \(X\) is contained in an open set \(U\) of \(\mathbb{R}^n\) and that \(f\) is the restriction to \(X\) of a \(C^2\)-subanalytic function \(F : U \to \mathbb{R}\). We denote by \(X^f\) the set \(X \cap f^{-1}(0)\) and by [4], we can equip \(X\) with a Thom stratification \(\mathcal{V} = \{V_\alpha\}_{\alpha \in A}\) adapted to \(X^f\). This means that \(\{V_\alpha \in \mathcal{V} \mid V_\alpha \nsubseteq X^f\}\) is a Whitney stratification of \(X \setminus X^f\) and that for any pair of strata \((V_\alpha, V_\beta)\) with \(V_\alpha \nsubseteq X^f\) and \(V_\beta \subset X^f\), the Thom condition is satisfied.

Let us denote by \(\Sigma_{\mathcal{V}} f\) the critical locus of \(f\). It is the union of the critical loci of \(f\) restricted to each stratum, i.e. \(\Sigma_{\mathcal{V}} f = \bigcup_\alpha \Sigma(f|_{V_\alpha})\), where \(\Sigma(f|_{V_\alpha})\) is the critical set of \(f|_{V_\alpha} : V_\alpha \to \mathbb{R}\). Since \(\Sigma_{\mathcal{V}} f \subset f^{-1}(0)\) (see Lemma 2.1), the fibre \(f^{-1}(\delta)\) intersects the strata \(V_{\alpha}\)'s, \(V_{\alpha} \nsubseteq X^f\), transversally if \(\delta\) is sufficiently small. Hence it is Whitney stratified with the induced stratification \(\{f^{-1}(\delta) \cap V_\alpha \mid V_\alpha \nsubseteq X^f\}\).

By Lemma 2.1, we know that if \(\epsilon > 0\) is sufficiently small then the sphere \(S_\epsilon\) intersects \(X^f\) transversally. By the Thom condition, this implies that there exists \(\delta(\epsilon) > 0\) such that for each \(\delta\) with \(0 < |\delta| \leq \delta(\epsilon)\), the sphere \(S_\epsilon\) intersects the fibre \(f^{-1}(\delta)\) transversally as well. Hence the set \(f^{-1}(\delta) \cap B_\epsilon\) is a Whitney stratified set equipped with the following stratification:
\[
\{f^{-1}(\delta) \cap V_\alpha \cap B_\epsilon, f^{-1}(\delta) \cap V_\alpha \cap S_\epsilon \mid V_\alpha \nsubseteq X^f\}.
\]

**Definition 3.1.** We call the set \(f^{-1}(\delta) \cap X \cap B_\epsilon\), where \(0 < |\delta| \ll \epsilon \ll 1\), a real Milnor fibre of \(f\).

We will use the following notation: \(M^\delta \epsilon = f^{-1}(\delta) \cap X \cap B_\epsilon\).

Now we consider another subanalytic function \(g : (X, 0) \to (\mathbb{R}, 0)\) and we assume that it is the restriction to \(X\) of a \(C^2\)-subanalytic function \(G : U \to \mathbb{R}\).
We denote by $X^g$ the intersection $X \cap g^{-1}(0)$. Under some restrictions on $g$, we will study the topological behaviour of $g|_{M_{f,\epsilon}^\delta}$.

First we assume that $g$ satisfies the following Condition (A):

- **Condition (A):** $g : (X, 0) \rightarrow (\mathbb{R}, 0)$ has an isolated stratified critical point at 0.

This means that for each strata $V_\alpha$ of $\mathcal{V}$, $g : V_\alpha \setminus \{0\} \rightarrow \mathbb{R}$ is a submersion in a neighborhood of the origin.

In order to give the second assumption on $g$, we need to introduce some polar sets. Let $V_\alpha$ be a stratum of $\mathcal{V}$ not contained in $X^f$. Let $V_\alpha^\epsilon$ be

\[ V_\alpha^\epsilon = \{ x \in V_\alpha \mid \text{rank} [\nabla f|_{V_\alpha}(x), \nabla g|_{V_\alpha}(x)] < 2 \} \]

and let $\Gamma_{f,g}$ be the union $\bigcup V_\alpha^\epsilon$, where $V_\alpha \not\subset X^f$. We call $\Gamma_{f,g}$ the relative polar set of $f$ and $g$ with respect to the stratification $\mathcal{V}$. We will assume that $g$ satisfies the following Condition (B):

- **Condition (B):** the relative polar set $\Gamma_{f,g}$ is a 1-dimensional $C^1$-subanalytic set (possibly empty) in a neighborhood of the origin.

Note that Condition (B) implies that $\Gamma_{f,g} \cap X^f \subset \{0\}$ in a neighborhood of the origin because the frontiers of the $V_\alpha^\epsilon$’s are 0-dimensional.

From Condition (A) and Condition (B), we can deduce the following result.

**Lemma 3.2.** We have $\Gamma_{f,g} \cap X^g \subset \{0\}$ in a neighborhood of the origin.

**Proof.** If it is not the case then there is a $C^1$-subanalytic curve $\gamma : [0, \nu[ \rightarrow \Gamma_{f,g} \cap X^g$ such that $\gamma(0) = 0$ and $\gamma([0, \nu]) \subset X^g \setminus \{0\}$. We can also assume that $\gamma([0, \nu])$ is contained in a stratum $V$. For $t \in [0, \nu[$, we have

\[ 0 = (g \circ \gamma)'(t) = \langle \nabla g|_{V}(\gamma(t)), \gamma'(t) \rangle. \]

Since $\gamma(t)$ belongs to $\Gamma_{f,g}$ and $\nabla g|_{V}(\gamma(t))$ does not vanish for $g : (X, 0) \rightarrow (\mathbb{R}, 0)$ has an isolated stratified critical point at 0, we can conclude that $\langle \nabla f|_{V}(\gamma(t)), \gamma'(t) \rangle = 0$ and that $(f \circ \gamma)'(t) = 0$ for all $t \in [0, \nu[$. Therefore $f \circ \gamma \equiv 0$ because $f(0) = 0$ and $\gamma([0, \nu])$ is included in $X^f$. This is impossible by the above remark. \qed

Let $B_1, \ldots, B_l$ be the connected components of $\Gamma_{f,g}$, i.e. $\Gamma_{f,g} = \bigcup_{i=1}^l B_i$. Each $B_i$ is a $C^1$-subanalytic curve along which $f$ is strictly increasing or decreasing and the intersection points of the $B_i$’s with the fibre $M_{f,\epsilon}^\delta$ are exactly the critical points (in the stratified sense) of $g$ on $X \cap f^{-1}(\delta) \cap \bar{B}_\epsilon$.

Let us write

\[ M_{f,\epsilon}^\delta \cap \bigcup_{i=1}^l B_i = \{ p_1^{\delta,\epsilon}, \ldots, p_r^{\delta,\epsilon} \}. \]

Note that $r \leq l$.

Let us recall now the definition of the index of an isolated stratified critical point.
Definition 3.3. Let $Z \subset \mathbb{R}^n$ be a closed subanalytic set, equipped with a Whitney stratification. Let $p \in Z$ be an isolated critical point of a subanalytic function $\phi : Z \to \mathbb{R}$, which is the restriction to $Z$ of a $C^2$-subanalytic function $\Phi$. We define the index of $\phi$ at $p$ as follows:

$$\text{ind}(\phi, Z, p) = 1 - \chi(Z \cap \{\phi = \phi(p) - \eta \} \cap B_\epsilon(p)),$$

where $0 < \eta \ll \epsilon < 1$ and $B_\epsilon(p)$ is the closed ball of radius $\epsilon$ centered at $p$.

Our aim is to give a topological interpretation to the following sum:

$$\sum_{i=1}^{r} \text{ind}(g, X \cap f^{-1}(\delta), p_{i}^{\delta, \epsilon}) + \text{ind}(-g, X \cap f^{-1}(\delta), p_{i}^{\delta, \epsilon}).$$

For this, we will apply stratified Morse theory to $g|_{M_f^{\delta, \epsilon}}$. Note that the points $p_i$’s are not the only critical points of $g|_{M_f^{\delta, \epsilon}}$ and other critical points can occur on the “boundary” $M_f^{\delta, \epsilon} \cap S_\epsilon$.

The next step is to study the behaviour of these “boundary” critical points for a generic choice of the distance function to the origin. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$-subanalytic function which is a distance function to the origin. We denote by $\tilde{S}_\epsilon$ the level $\rho^{-1}(\epsilon)$ and by $\tilde{B}_\epsilon$ the set $\{\rho \leq \epsilon\}$, with $0 < |\delta| < \epsilon < 1$.

For each stratum $V$ of $X_f$, let

$$\Gamma_{V}^{g, \rho} = \{x \in V \mid \text{rank}[\nabla g_{|V}(x), \nabla \rho_{|V}(x)] < 2\},$$

and let $\Gamma_{g, \rho}^{X_f} = \bigcup_{V \subset X_f} \Gamma_{g, \rho}^{V}$. By Lemma 2.4 and the fact that $g : (X_f, 0) \to (\mathbb{R}, 0)$ has an isolated stratified critical point at 0, we can assume that $\Gamma_{g, \rho}^{X_f}$ is a $C^1$-subanalytic curve in a neighborhood of the origin.

Lemma 3.4. We have $\Gamma_{g, \rho}^{X_f} \cap X^g \subset \{0\}$ in a neighborhood of the origin.

Proof. Same proof as Lemma 3.2. \qed

Therefore if $\epsilon > 0$ is small enough, $g|_{\tilde{S}_\epsilon \cap X_f}$ has a finite number of critical points. They do not lie in the level $\{g = 0\}$ so by Lemma 2.3, they are outwards-pointing for $g|_{X_f \cap \tilde{B}_\epsilon}$ if they lie in $\{g > 0\}$ and inwards-pointing if they lie in $\{g < 0\}$.

Let us study now the critical points of $g|_{X \cap f^{-1}(\delta) \cap \tilde{S}_\epsilon}$. We will need the following lemma.

Lemma 3.5. For every $\epsilon > 0$ sufficiently small, there exists $\delta(\epsilon) > 0$ such that for $0 < |\delta| \leq \delta(\epsilon)$, the points $p_i^{\delta, \epsilon}$ lie in $\tilde{B}_{\epsilon/4}$.

Proof. Let

$$W = \{(x, r, y) \in U \times \mathbb{R} \times \mathbb{R} \mid \rho(x) = r, y = f(x) \text{ and } x \in \Gamma_{f, g}\}.$$
Then $W$ is a subanalytic set of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ and since it is a graph over $\Gamma_{f,g}$, its dimension is less or equal to 1. Let

$$
\pi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \\
(x, r, y) \mapsto (r, y),
$$

be the projection on the last two factors. Then $\pi|_W : W \to \pi(W)$ is proper and $\pi(W)$ is a closed subanalytic set in a neighborhood of the origin.

Let us write $Y_1 = \mathbb{R} \times \{0\}$ and let $Y_2$ be the closure of $\pi(W) \setminus Y_1$. Since $Y_2$ is a curve for $W$ is a curve, 0 is isolated in $Y_1 \cap Y_2$. By Lojasiewicz’s inequality, there exists a constant $C > 0$ and an integer $N > 0$ such that $|y| \geq C r^N$ for $(r, y)$ in $Y_2$ sufficiently close to the origin. So if $x \in \Gamma_{f,g}$ then $|f(x)| \geq C r(x)^N$ if $r(x)$ is small enough.

Let us fix $\epsilon > 0$ small. If $0 < |\delta| \leq \frac{1}{4}(\frac{1}{\epsilon})^N$ and $x \in f^{-1}(\delta) \cap \Gamma_{f,g}$ then $\rho(x) \leq \frac{\epsilon}{4}$.

For each stratum $V \not\subseteq X^f$, let

$$
\Gamma_{f,g,\rho}^V = \{ x \in V \mid \rank[\nabla f|_V(x), \nabla g|_V(x), \nabla \rho|_V(x)] < 3 \},
$$

and let $\Gamma_{f,g,\rho} = \bigcup_{V \not\subseteq X^f} \Gamma_{f,g,\rho}^V$. By Lemma 2.6, we can assume that $\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}$ is a $C^1$-subanalytic manifold of dimension 2. Let us choose $\epsilon > 0$ small enough so that $\tilde{S}_e$ intersects $\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}$ transversally. Therefore $(\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}) \cap \tilde{S}_e$ is subanalytic curve. By Lemma 3.4, we can find $\delta(\epsilon) > 0$ such that $f^{-1}([-\delta(\epsilon), \delta(\epsilon)]) \cap \tilde{S}_e \cap \Gamma_{f,g}$ is empty and so

$$
f^{-1}([-\delta(\epsilon), \delta(\epsilon)]) \cap (\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}) \cap \tilde{S}_e = f^{-1}([-\delta(\epsilon), \delta(\epsilon)]) \cap \Gamma_{f,g,\rho} \cap \tilde{S}_e.
$$

Let $C_1, \ldots, C_t$ be the connected components of $f^{-1}([-\delta(\epsilon), \delta(\epsilon)]) \cap \Gamma_{f,g,\rho} \cap \tilde{S}_e$ whose closure intersects $X^f \cap \tilde{S}_e$. Note that by Thom’s $(a_f)$-condition, for each $i \in \{1, \ldots, t\}$, $\overline{C_i} \cap X^f$ is subset of $\Gamma_{g,\rho}^X$. Let $z_i$ be a point in $\overline{C_i} \cap X^f$. Since $C_i \cap X^f = \emptyset$, there exists $0 < \delta_i'(\epsilon) \leq \delta(\epsilon)$ such that the fibre $f^{-1}(\delta)$, $0 < |\delta| \leq \delta_i'(\epsilon)$, intersects $C_i$ transversally in a neighborhood of $z_i$.

Let us choose $\delta$ such that $0 < |\delta| \leq \min\{\delta_i'(\epsilon) \mid i = 1, \ldots, t\}$. Then the fibre $f^{-1}(\delta)$ intersect the $C_i$’s transversally and $f^{-1}(\delta) \cap (\cup_i C_i)$ is exactly the set of critical points of $g|_{f^{-1}(\delta) \cap X \cap \tilde{S}_e}$. We have proved:

**Lemma 3.6.** For $0 < |\delta| \ll \epsilon \ll 1$, $g|_{f^{-1}(\delta) \cap X \cap \tilde{S}_e}$ has a finite number of critical points, which are exactly the points in $\Gamma_{f,g,\rho} \cap \tilde{S}_e \cap f^{-1}(\delta)$.

Let $\{s_{1,\epsilon}^\delta, \ldots, s_{u,\epsilon}^\delta\}$ be the set of critical points of $g|_{f^{-1}(\delta) \cap X \cap \tilde{S}_e}$.

**Lemma 3.7.** For $i \in \{1, \ldots, u\}$, $g(s_{i,\epsilon}^\delta) \neq 0$ and $s_{i,\epsilon}^\delta$ is outwards-pointing (resp. inwards-pointing) if and only if $g(s_{i,\epsilon}^\delta) > 0$ (resp. $g(s_{i,\epsilon}^\delta) < 0$).

**Proof.** Note that $s_{i,\epsilon}^\delta$ is necessarily outwards-pointing or inwards-pointing because $s_{i,\epsilon}^\delta \notin \Gamma_{f,g}$. 

Assume that for each $\delta > 0$ small enough, there exists a point $s_i^{\delta, \epsilon}$ such that $g(s_i^{\delta, \epsilon}) = 0$. Then we can construct a sequence of points $(s_n)_{n \in \mathbb{N}}$ such that $g(s_n) = 0$ and $s_n$ is a critical point of $g|_{f^{-1}(\varepsilon) \cap X \cap \tilde{S}_n}$. We can also assume that the points $\sigma_n$’s belong to the same stratum $S$ and that they tend to $\sigma \in V$ where $V \subseteq X'$ and $V \subset \partial \tilde{S}$. Therefore we have a decomposition:

$$\nabla g|_S(\sigma_n) = \lambda_n \nabla f|_S(\sigma_n) + \mu_n \nabla \rho|_S(\sigma_n).$$

Now by Whitney’s condition (a), $T_{\sigma_n}S$ tends to a linear space $T$ such that $T_{\sigma}V \subset T$. So $\nabla g|_S(\sigma_n)$ tends to a vector in $T$ whose orthogonal projection on $T_{\sigma}V$ is exactly $\nabla g|_V(\sigma)$. Similarly $\nabla \rho|_S(\sigma_n)$ tends to a vector in $T$ whose orthogonal projection on $T_{\sigma}V$ is exactly $\nabla \rho|_V(\sigma)$. By Thom’s condition, $\nabla f|_S(\sigma_n)$ tends to a vector in $T$ which is orthogonal to $T_{\sigma}V$, so we see that $\nabla g|_V(\sigma)$ and $\nabla \rho|_V(\sigma)$ are colinear which means that $\sigma$ is a critical point of $g|_{X' \cap \tilde{S}_n}$. But since $g(\sigma_n) = 0$, we find that $g(\sigma) = 0$, which is impossible by Lemma 3.4. This proves the first assertion.

To prove the second one, we use the same method. Assume that for each $\delta > 0$ small enough, there exists a point $s_i^{\delta, \epsilon}$ such that $g(s_i^{\delta, \epsilon}) > 0$ and $s_i^{\delta, \epsilon}$ is an inwards-pointing critical point for $g|_{X' \cap f^{-1}(\delta) \cap \tilde{S}_n}$. Then we can construct a sequence of points $(\tau_n)_{n \in \mathbb{N}}$ such that $g(\tau_n) > 0$ and $\tau_n$ is an inwards-pointing critical point for $g|_{f^{-1}(\delta) \cap X \cap \tilde{S}_n}$. We can also assume that the points $\tau_n$’s belong to the same stratum $S$ and that they tend to $\tau \in V$ where $V \subseteq X'$ and $V \subset \partial \tilde{S}$. Therefore, we have a decomposition:

$$\nabla g|_S(\tau_n) = \lambda_n \nabla f|_S(\tau_n) + \mu_n \nabla \rho|_S(\tau_n),$$

with $\mu_n < 0$. Using the same arguments as above, we find that $\nabla g|_V(\tau) = \mu \nabla \rho|_V(\tau)$ with $\mu \leq 0$ and $g(\tau) \geq 0$. This contradicts the remark after Lemma 3.4. Of course, this proof works for $\delta < 0$. \hfill $\square$

Let $\Gamma_{g, \rho}$ be the following polar set:

$$\Gamma_{g, \rho} = \{ x \in U \mid \text{rank}[\nabla g(x), \nabla \rho(x)] < 2 \}.$$  

By Lemma 2.5 and Lemma 2.1, we can assume that $\Gamma_{g, \rho} \setminus \{ g = 0 \}$ does not intersect $X' \setminus \{ 0 \}$ in a neighborhood of $0$ and so $\Gamma_{g, \rho} \setminus \{ g = 0 \}$ does not intersect $X' \cap \tilde{S}_\epsilon$ for $\epsilon > 0$ sufficiently small. Since the critical points of $g|_{X' \cap \tilde{S}_n}$ lie outside $\{ g = 0 \}$, they do not belong to $\Gamma_{g, \rho} \cap \tilde{S}_\epsilon$ and so the critical points of $g|_{f^{-1}(\delta) \cap X \cap \tilde{S}_n}$ do not neither if $\delta$ is sufficiently small. Hence at each critical point of $g|_{f^{-1}(\delta) \cap X \cap \tilde{S}_n}$, $g|_{\tilde{S}_n}$ is a submersion. We are in position to apply Theorem 3.1 and Lemma 2.1 in [15]. For $0 < |\delta| \ll \epsilon \ll 1$, we set

$$I(\delta, \epsilon, g) = \sum_{i=1}^{r} \text{ind}(g, X \cap f^{-1}(\delta), p_i^{\delta, \epsilon}),$$

$$I(\delta, \epsilon, -g) = \sum_{i=1}^{r} \text{ind}(-g, X \cap f^{-1}(\delta), p_i^{\delta, \epsilon}).$$
Theorem 3.8. We have
\[
I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2\chi(X \cap f^{-1}(\delta) \cap \tilde{B}_e)
- \chi(X \cap f^{-1}(\delta) \cap \tilde{S}_e)\]
\[
- \chi(X^g \cap f^{-1}(\delta) \cap \tilde{S}_e).
\]

Proof. Let us denote by \(\{a_j^+\}_{j=1}^{\alpha^+}\) (resp. \(\{a_j^-\}_{j=1}^{\alpha^-}\)) the outwards-pointing (resp. inwards-pointing) critical points of \(g : X \cap f^{-1}(\delta) \cap \tilde{S}_e \to \mathbb{R}\). Applying Morse theory type theorem ([15], Theorem 3.1) and using Lemma 2.1 in [15], we can write
\[
I(\delta, \epsilon, g) + \sum_{j=1}^{\alpha^-} \text{ind}(g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^-) = \chi(X \cap f^{-1}(\delta) \cap \tilde{B}_e) \quad (1),
\]
\[
I(\delta, \epsilon, -g) + \sum_{j=1}^{\alpha^+} \text{ind}(-g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, -a_j^+) = \chi(X \cap f^{-1}(\delta) \cap \tilde{B}_e) \quad (2).
\]
Let us evaluate
\[
\sum_{j=1}^{\alpha^-} \text{ind}(g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^-) + \sum_{j=1}^{\alpha^+} \text{ind}(-g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^+).
\]
Since the outwards-pointing critical points of \(g|_{X \cap f^{-1}(\delta) \cap \tilde{S}_e}\) lie in \(\{g > 0\}\) and the inwards-pointing critical points of \(g|_{X \cap f^{-1}(\delta) \cap \tilde{S}_e}\) lie in \(\{g < 0\}\), we have
\[
\chi(X \cap f^{-1}(\delta) \cap \tilde{S}_e \cap \{g \geq 0\}) - \chi(X \cap f^{-1}(\delta) \cap \tilde{S}_e \cap \{g = 0\}) =
\sum_{j=1}^{\alpha^+} \text{ind}(g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^+) \quad (3),
\]
and
\[
\chi(X \cap f^{-1}(\delta) \cap \tilde{S}_e \cap \{g \leq 0\}) - \chi(X \cap f^{-1}(\delta) \cap \tilde{S}_e \cap \{g = 0\}) =
\sum_{j=1}^{\alpha^-} \text{ind}(-g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^-) \quad (4).
\]
Therefore making (3) + (4) and using the Mayer-Vietoris sequence, we find
\[
\chi(X \cap f^{-1}(\delta) \cap \tilde{S}_e) - \chi(X \cap f^{-1}(\delta) \cap \tilde{S}_e \cap \{g = 0\}) =
\sum_{j=1}^{\alpha^+} \text{ind}(g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^+) + \sum_{j=1}^{\alpha^-} \text{ind}(-g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^-) \quad (5).
\]
Moreover we have
\[
\chi(X \cap f^{-1}(\delta) \cap \tilde{S}_e) = \sum_{j=1}^{\alpha^+} \text{ind}(g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^+)
\]
\begin{align*}
\chi(X \cap f^{-1}(\delta) \cap \tilde{S}_e) &= \sum_{j=1}^{\alpha^-} \text{ind}(-g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^-) \\
&\quad + \sum_{j=1}^{\alpha^+} \text{ind}(-g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^+) \\
&\quad + \sum_{j=1}^{\alpha^-} \text{ind}(g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^-) \quad (6),
\end{align*}

\begin{align*}
\chi(X \cap f^{-1}(\delta) \cap \tilde{S}_e) &= \sum_{j=1}^{\alpha^+} \text{ind}(-g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^+) \\
&\quad + \sum_{j=1}^{\alpha^-} \text{ind}(-g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^-) \quad (7).
\end{align*}

The combination $-(5) + (6) + (7)$ leads to

\begin{align*}
\chi(X \cap f^{-1}(\delta) \cap \tilde{S}_e) &= \chi(X \cap f^{-1}(\delta) \cap \tilde{S}_e \cap \{g = 0\}) \\
&= \sum_{j=1}^{\alpha^+} \text{ind}(-g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^+) + \sum_{j=1}^{\alpha^-} \text{ind}(g, X \cap f^{-1}(\delta) \cap \tilde{S}_e, a_j^-).
\end{align*}

Let us assume now that $(X, 0)$ is equipped with a Whitney stratification

$W = \cup_{\alpha \in A} W_{\alpha}$ and $f : (X, 0) \to (\mathbb{R}, 0)$ has an isolated critical point at 0. In this situation, our results apply taking for $\mathcal{V}$ the following stratification:

$\{W_{\alpha} \setminus f^{-1}(0), W_{\alpha} \cap f^{-1}(0) \setminus \{0\}, \{0\} \mid W_{\alpha} \in W\}.$

**Corollary 3.9.** If $f : (X, 0) \to (\mathbb{R}, 0)$ has an isolated stratified critical point at 0, then

$\begin{align*}
I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) &= 2\chi(X \cap f^{-1}(\delta) \cap \tilde{S}_e) \\
&\quad - \chi(X^f \cap \tilde{S}_e) - \chi(X^g \cap \tilde{S}_e).
\end{align*}$

**Proof.** For each stratum $W$ of $X$, let

$\Gamma^W_{f, \rho} = \{x \in W \mid \text{rank}[\nabla f|_W(x), \nabla \rho|_W(x)] < 2\},$

and let $\Gamma_{f, \rho} = \cup_W \Gamma^W_{f, \rho}$. By Lemma 3.4 applied to $X$ and $f$ instead of $X^f$ and $g$, $\Gamma_{f, \rho} \cap \{f = 0\} \subset \{0\}$ in a neighborhood of the origin and so 0 is a regular value of $f : X \cap \tilde{S}_e \to \mathbb{R}$ for $\epsilon$ sufficiently small. By Thom-Mather’s second isotopy lemma, $X \cap f^{-1}(0) \cap \tilde{S}_e$ is homeomorphic to $X \cap f^{-1}(\delta) \cap \tilde{S}_e$ for $\delta$ sufficiently small.

Now let $p$ be a stratified critical point of $f : X^g \to \mathbb{R}$. By Lemma 2.1, we know that $p$ belongs to $f^{-1}(0) \cap X^g$ and so $p$ is also a critical point of $g : X^f \to \mathbb{R}$. Hence $p = 0$ by Condition (A), and $f : X^g \to \mathbb{R}$ has an isolated stratified critical point at 0. As above, we conclude that $X^f \cap X^g \cap \tilde{S}_e$ is homeomorphic to $X^g \cap f^{-1}(\delta) \cap \tilde{S}_e$. \qed

Let $\omega(x) = \sqrt{x_1^2 + \cdots + x_n^2}$ be the euclidian distance to the origin. As explained by Durfee in [10], Lemma 1.8 and Lemma 3.6, there is a neighborhood $\Omega$ of 0 in $\mathbb{R}^n$ such that for every stratum $V$ of $X^f$, $\nabla \omega|_V$ and $\nabla \rho|_V$ are non-zero and do not point in opposite direction in $\Omega \setminus \{0\}$. Applying
Durfee’s argument ([10], Proposition 1.7 and Proposition 3.5), we see that $X^f \cap \hat{S}_\epsilon$ is homeomorphic to $X^f \cap S_{\epsilon'}$ for $\epsilon, \epsilon' > 0$ sufficiently small. Similarly $X^f \cap X^g \cap \hat{S}_\epsilon$ and $X^f \cap X^g \cap S_{\epsilon'}$ are homomorphic. Now let us compare $X \cap f^{-1}(\delta) \cap \hat{B}_\epsilon$ and $X \cap f^{-1}(\delta) \cap B_{\epsilon'}$. Let us choose $\epsilon'$ and $\epsilon$ such that $X \cap f^{-1}(\delta) \cap B_{\epsilon'} \subset X \cap f^{-1}(\delta) \cap \hat{B}_\epsilon \subset \Omega$.

If $\delta$ is sufficiently small then, for every stratum $V \not\subset X^f$, $\nabla \omega|_{V \cap f^{-1}(\delta)}$ and $\nabla \rho|_{V \cap f^{-1}(\delta)}$ are non-zero and do not point in opposite direction in $\hat{B}_\epsilon \setminus B_{\epsilon'}$. Otherwise, by Thom’s $(a_f)$-condition, we would find a point $p$ in $X^f \cap (\hat{B}_\epsilon \setminus B_{\epsilon'})$ such that either $\nabla \omega|_S(p)$ or $\nabla \rho|_S(p)$ vanish or $\nabla \omega|_S(p)$ and $\nabla \rho|_S(p)$ point in opposite direction, where $S$ is the stratum of $X^f$ that contains $p$. This is impossible if we are sufficiently close to the origin. Now, applying the same arguments as Durfee [10], Proposition 1.7 and Proposition 3.5, we see that $X \cap f^{-1}(\delta) \cap \hat{B}_\epsilon$ is homeomorphic to $X \cap f^{-1}(\delta) \cap B_{\epsilon'}$ and that $X \cap f^{-1}(\delta) \cap \hat{S}_\epsilon$ is homeomorphic to $X \cap f^{-1}(\delta) \cap S_{\epsilon'}$.

**Theorem 3.10.** We have

$I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2 \chi(M^\delta_\epsilon) - \chi(X \cap f^{-1}(\delta) \cap S_\epsilon) - \chi(X^g \cap f^{-1}(\delta) \cap S_\epsilon)$.

□

**Corollary 3.11.** If $f : (X, 0) \to (\mathbb{R}, 0)$ has an isolated stratified critical point at 0, then

$I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2 \chi(M^\delta_\epsilon) - \chi(\text{Lk}(X^f)) - \chi(\text{Lk}(X^f \cap X^g))$.

□

Let us remark if dim $X = 2$ then in Theorem 3.10 and in Corollary 3.11, the last term of the right-hand side of the equality vanishes. If dim $X = 1$ then in Theorem 3.10 and in Corollary 3.11, the last two terms of the right-hand side of the equality vanish.

4. **An infinitesimal Gauss-Bonnet formula**

In this section, we apply the results of the previous section to the case of linear forms and we establish a Gauss-Bonnet type formula for the real Milnor fibre.

We will first show that generic linear forms satisfy Condition (A) and Condition (B). For $v \in S^{n-1}$, let us denote by $v^* \colon X \to \mathbb{R}$ the function $x \mapsto \langle v, x \rangle$.

**Lemma 4.1.** There exists a subanalytic set $\Sigma_1 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma_1$, $\{v^* = 0\}$ intersects $X \setminus \{0\}$ transversally (in the stratified sense) in a neighborhood of the origin.

**Proof.** It is a particular case of Lemma 3.8 in [14]. □

**Corollary 4.2.** If $v \notin \Sigma_1$ then $v^*_X : (X, 0) \to (\mathbb{R}, 0)$ has an isolated stratified point at 0.
Proof. By Lemma 2.1, we know that the stratified critical points of \( v^*_X \) lie in \( \{ v^* = 0 \} \). But since \( \{ v^* = 0 \} \) intersects \( X \setminus \{ 0 \} \) transversally, the only possible critical point of \( v^*_X : (X, 0) \to (\mathbb{R}, 0) \) is the origin. \( \square \)

Lemma 4.3. There exists a subanalytic set \( \Sigma_2 \subset S^{n-1} \) of positive codimension such that if \( v \notin \Sigma_2 \), then \( \Gamma_{f, v^*} \) is a \( C^1 \)-subanalytic curve (possibly empty) in a neighborhood of 0.

Proof. Let \( V \) be stratum of dimension \( e \) such that \( V \not\subset X^f \). We can assume that \( e \geq 2 \). Let \( M_V = \{(x, y) \in V \times \mathbb{R}^n \mid \text{rank} [\nabla f|_V(x), \nabla y^*_V(x)] < 2 \} \).

It is a subanalytic manifold of class \( C^1 \) and of dimension \( n + 1 \). To see this, let us pick a point \((x, y)\) in \( M_V \). In a neighborhood of \( x \), \( V \) is defined by the vanishing of \( k = n - e \) \( C^2 \)-subanalytic functions \( f_1, \ldots, f_k \). Since \( V \) is not included in \( X^f \), \( f : V \to \mathbb{R} \) is a submersion and we can assume that in a neighborhood of \( x \), the following \((k + 1) \times (k + 1)\)-minor:

\[
\frac{\partial (f_1, \ldots, f_k, f)}{\partial (x_1, \ldots, x_k, x_{k+1})},
\]

does not vanish. Therefore, in a neighborhood of \((x, y)\), \( M_V \) is defined by the vanishing of the following \((k + 2) \times (k + 2)\)-minors:

\[
\frac{\partial (f_1, \ldots, f_k, f, y^*)}{\partial (x_1, \ldots, x_k, x_{k+1}, x_{k+2})}, \ldots, \frac{\partial (f_1, \ldots, f_k, f, y^*)}{\partial (x_1, \ldots, x_k, x_{k+1}, x_n)}.
\]

A simple computation of determinants shows that the gradient vectors of these minors are linearly independent. As in previous lemmas, we show that \( \Sigma_{f, v^*} \) is one-dimensional considering the projection

\[
\pi_2 : M_V \to \mathbb{R}^n \quad (x, y) \mapsto y.
\]

Since \( \Gamma_{f, v^*} = \cup_{V \not\subset X^f} \Gamma_{f, v^*}^V \), we get the result. \( \square \)

Let \( \Sigma = \Sigma_1 \cup \Sigma_2 \), it is a subanalytic subset of \( S^{n-1} \) of positive codimension and if \( v \notin \Sigma \) then \( v^* \) satisfies Conditions (A) and (B). In particular, \( v^*_{f^{-1}(\delta) \cap X \cap B_\epsilon} \) has a finite number of critical points \( p_1^{\delta, \epsilon}, \ldots, p_{r_v}^{\delta, \epsilon} \). We recall that

\[
I(\delta, \epsilon, v^*) = \sum_{i=1}^{r_v} \text{ind}(v^*, X \cap f^{-1}(\delta), p_i^{\delta, \epsilon}),
\]

\[
I(\delta, \epsilon, -v^*) = \sum_{i=1}^{r_v} \text{ind}(-v^*, X \cap f^{-1}(\delta), p_i^{\delta, \epsilon}).
\]

In this situation, Theorem 3.10 and Corollary 3.11 become
Corollary 4.4. If \( v \notin \Sigma \) then
\[
I(\delta, \epsilon, v^*) + I(\delta, \epsilon, -v^*) = 2\chi(M_f^{\delta, \epsilon}) - \chi(X \cap f^{-1}(\delta) \cap S) - \chi(X^v \cap f^{-1}(\delta) \cap S).
\]
Furthermore, if \( f : (X, 0) \to (\mathbb{R}, 0) \) has an isolated stratified critical point at 0, then
\[
I(\delta, \epsilon, v^*) + I(\delta, \epsilon, -v^*) = 2\chi(M_f^{\delta, \epsilon}) - \chi(Lk(X^f)) - \chi(Lk(X^f \cap X^v)).
\]

As an application, we give a Gauss-Bonnet formula for the Milnor fibre \( M_f^{\delta, \epsilon} \). Let \( \Lambda_0(X \cap f^{-1}(\delta), -) \) be the Gauss-Bonnet measure on \( X \cap f^{-1}(\delta) \) defined by
\[
\Lambda_0(X \cap f^{-1}(\delta), U') = \frac{1}{s_{n-1}} \int_{S_{n-1}} \sum_{x \in U} \text{ind}(v^*, X \cap f^{-1}(\delta), x) dx,
\]
where \( U' \) is a Borel set of \( X \cap f^{-1}(\delta) \) (see [6], page 299) and \( s_{n-1} \) is the volume of the unit sphere \( S^{n-1} \). Note that if \( x \) is not a critical point of \( v^*_|X \cap f^{-1}(\delta) \) then \( \text{ind}(v^*, X \cap f^{-1}(\delta), x) = 0 \). We are going to evaluate
\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(X \cap f^{-1}(\delta), M_f^{\delta, \epsilon}).
\]

Theorem 4.5. We have
\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(X \cap f^{-1}(\delta), M_f^{\delta, \epsilon}) = \chi(M_f^{\delta, \epsilon}) - \frac{1}{2} \chi(X \cap f^{-1}(\delta) \cap S).
\]
Furthermore, if \( f : (X, 0) \to (\mathbb{R}, 0) \) has an isolated stratified critical point at 0, then
\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(X \cap f^{-1}(\delta), M_f^{\delta, \epsilon}) = \chi(M_f^{\delta, \epsilon}) - \frac{1}{2} \chi(Lk(X^f))
\]
\[
- \frac{1}{2s_{n-1}} \int_{S^{n-1}} \chi(Lk(X^f \cap X^v)) dv.
\]

Proof. By definition, we have
\[
\Lambda_0(X \cap f^{-1}(\delta), M_f^{\delta, \epsilon}) = \frac{1}{s_{n-1}} \int_{S_{n-1}} \sum_{x \in M_f^{\delta, \epsilon}} \text{ind}(v^*, X \cap f^{-1}(\delta), x) dv.
\]
It is not difficult to see that
\[
\Lambda_0(X \cap f^{-1}(\delta), M_f^{\delta, \epsilon}) =
\]
\[
\frac{1}{2s_{n-1}} \int_{S_{n-1}} \left[ \sum_{x \in M_f^{\delta, \epsilon}} \text{ind}(v^*, X \cap f^{-1}(\delta), x) + \text{ind}(-v^*, X \cap f^{-1}(\delta), x) \right] dv.
\]
Note that if $v \notin \Sigma$ then
\[ \sum_{x \in \mathcal{M}_{f}^{\delta, \epsilon}} \text{ind}(v^{*}, X \cap f^{-1}(\delta), x) + \text{ind}(-v^{*}, X \cap f^{-1}(\delta), x) \]
is equal to $I(\delta, \epsilon, v^{*}) + I(\delta, \epsilon, -v^{*})$ and is uniformly bounded by Hardt’s theorem. By Lebesgue’s theorem, we obtain
\[ \lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_{0}(X \cap f^{-1}(\delta), M^{\delta, \epsilon}_{f}) = \frac{1}{2s_{n-1}} \int_{S^{n-1}} \lim_{\epsilon \to 0} \lim_{\delta \to 0} [I(\delta, \epsilon, v^{*}) + I(\delta, \epsilon, -v^{*})]dv. \]
We just have to apply the previous corollary to conclude. \hfill \square

5. Infinitesimal linear kinematic formulas

In this section, we apply the results of the previous section to the case of a linear function in order to obtain “infinitesimal” linear kinematic formulas for closed subanalytic germs.

We start recalling known facts on the geometry of subanalytic sets. We need some notations:

- for $k \in \{0, \ldots, n\}$, $G_{n}^{k}$ is the Grassmann manifold of $k$-dimensional linear subspaces in $\mathbb{R}^{n}$ and $g_{n}^{k}$ is its volume,
- for $k \in \mathbb{N}$, $b_{k}$ is the volume of the $k$-dimensional unit ball and $s_{k}$ is the volume of the $k$-dimensional unit sphere.

In [17], Fu developed integral geometry for compact subanalytic sets. Using the technology of the normal cycle, he associated with every compact subanalytic set $X \subset \mathbb{R}^{n}$ a sequence of curvature measures $\Lambda_{0}(X, -), \ldots, \Lambda_{n}(X, -)$, called the Lipschitz-Killing measures. He proved several integral geometry formulas, among them a Gauss-Bonnet formula and a kinematic formula. Later another description of the measures using stratified Morse theory was given by Broecker and Kuppe [6] (see also [5]). The reader can refer to [14], Section 2, for a rather complete presentation of these two approaches and for the definition of the Lipschitz-Killing measures.

Let us give some comments on these Lipschitz-Killing curvatures. If $\dim X = d$ then
\[ \Lambda_{d+1}(X, U') = \cdots = \Lambda_{d}(X, U') = 0, \]
for any Borel set $U'$ of $X$ and $\Lambda_{d}(X, U') = \mathcal{L}_{d}(U')$, where $\mathcal{L}_{d}$ is the $d$-dimensional Lebesgue measure in $\mathbb{R}^{n}$. Furthermore if $X$ is smooth then for any Borel set $U'$ of $X$ and for $k \in \{0, \ldots, d\}$, $\Lambda_{k}(X, U')$ is related to the classical Lipschitz-Killing-Weil curvature $K_{d-k}$ through the following equality:
\[ \Lambda_{k}(X, U') = \frac{1}{s_{n-d-k-1}} \int_{U'} K_{d-k}(x)dx. \]
In [14], Section 5, we studied the asymptotic behaviour of the Lipschitz-Killing measures in the neighborhood of a point of $X$. Namely we proved the following theorem ([14], Theorem 5.1).

**Theorem 5.1.** Let $X \subset \mathbb{R}^n$ be a closed subanalytic set such that $0 \in X$. We have:

$$
\lim_{\epsilon \to 0} \Lambda_0(X, X \cap B_\epsilon) = 1 - \frac{1}{2} \chi(Lk(X)) - \frac{1}{2g_n^{n-1}} \int_{G_n^{n-1}} \chi(Lk(X \cap H))dH.
$$

Furthermore for $k \in \{1, \ldots, n-2\}$, we have:

$$
\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k\epsilon^k} = -\frac{1}{2g_n^{n-k-1}} \int_{G_n^{n-k-1}} \chi(Lk(X \cap H))dH
$$

and:

$$
\lim_{\epsilon \to 0} \frac{\Lambda_{n-1}(X, X \cap B_\epsilon)}{b_{n-1}\epsilon^{n-1}} = \frac{1}{2g_n^2} \int_{G_n^2} \chi(Lk(X \cap H))dH,
$$

$$
\lim_{\epsilon \to 0} \frac{\Lambda_n(X, X \cap B_\epsilon)}{b_n\epsilon^n} = \frac{1}{2g_n^n} \int_{G_n^n} \chi(Lk(X \cap H))dH,
$$

In the sequel, we will use these equalities and Theorem 4.5 to establish linear kinematic types formulas for the quantities $\lim_{\epsilon \to 0} \Lambda_k(X, X \cap B_\epsilon)$, $k = 1, \ldots, n$. Let us start with some lemmas. We work with a closed subanalytic set $X$ such that $0 \in X$, equipped with a Whitney stratification $\{W_\alpha\}_{\alpha \in A}$.

**Lemma 5.2.** Let $f$ be a $C^2$-subanalytic function such that $f|_X : X \to \mathbb{R}$ has an isolated stratified critical point at $0$. Then for $0 < \delta \ll \epsilon \ll 1$, we have

$$
\chi(M^\delta_f) + \chi(M^{-\delta}_f) = \chi(Lk(X)) + \chi(Lk(X^f)).
$$

**Proof.** With the same technics and arguments as the ones we used in order to establish Corollary 3.11, we can prove that

$$
\text{ind}(f, X, 0) + \text{ind}(-f, X, 0) = 2\chi(X \cap B_\epsilon) - \chi(Lk(X)) - \chi(Lk(X^f)).
$$

We conclude thanks to the following equalities

$$
\text{ind}(f, X, 0) = 1 - \chi(M^{-\delta}_f), \quad \text{ind}(-f, X, 0) = 1 - \chi(M^\delta_f),
$$

and

$$
\chi(X \cap B_\epsilon) = 1.
$$

□

**Corollary 5.3.** There exist a subanalytic set $\Sigma_1 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma$ then for $0 < \delta \ll \epsilon \ll 1$,

$$
\chi(M^\delta_v) + \chi(M^{-\delta}_v) = \chi(Lk(X)) + \chi(Lk(X \cap \{v^* = 0\})).
$$

**Proof.** Apply Corollary 4.2 and Lemma 5.2.

□
Lemma 5.4. Let $S \subset \mathbb{R}^n$ be $C^2$-subanalytic manifold. Let $H \in G_{n-k}^n$, $k \in \{1, \ldots, n\}$ and let $G_{H^+}^1$ be the Grassmann manifold of lines in the orthogonal complement $H^\perp$ of $H$. There exists a subanalytic set $\Sigma_H \subset G_{H^+}^1$ of positive codimension such that if $\nu \notin \Sigma_H$ then $H \oplus \nu$ intersects $S \setminus \{0\}$ transversally.

Proof. Assume that $S$ has dimension $e$ and that $H$ is given by the equations $x_1 = \ldots = x_k = 0$ so that $H^\perp = \mathbb{R}^k$ with coordinate system $(x_1, \ldots, x_k)$. Let $W$ be defined by

$$W = \left\{ (x, v_1, \ldots, v_{k-1}) \in \mathbb{R}^n \times (\mathbb{R}^k)^{k-1} \mid x \in S \setminus \{0\} \right\},$$

where $v_i \in \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$. Let us show that $W$ is a $C^2$-subanalytic manifold of dimension $e + (k - 1)^2$. Let $(y, w)$ be a point in $W$. We can assume that around $y$, $S$ is defined by the vanishing of $n - e$ $C^2$-subanalytic functions $f_1, \ldots, f_{n-e}$. Hence in a neighborhood of $(y, w)$, $W$ is defined by the equations:

$$f_1(x) = \ldots = f_{n-e}(x) = 0 \text{ and } \langle x, v_1 \rangle = \ldots = \langle x, v_{k-1} \rangle = 0.$$

Because $y \neq 0$, we see that the gradient vectors of this $n - e + k - 1$ functions are linearly independent at $(y, w)$. This enables us to conclude that $W$ is a $C^2$-subanalytic manifold of dimension $e + (k - 1)^2$. Let $\pi_2$ be the following projection:

$$\pi_2 : W \to (\mathbb{R}^n)^{n-k}, (x, v_1, \ldots, v_{n-k}) \mapsto (v_1, \ldots, v_{n-k}).$$

Bertini-Sard’s theorem implies that the set of critical values of $\pi_2$ is a subanalytic set of positive codimension. If $(v_1, \ldots, v_{k-1})$ lies outside this subanalytic set then the $(n - k + 1)$-plane $\{x \in \mathbb{R}^n \mid \langle x, v_1 \rangle = \ldots = \langle x, v_{k-1} \rangle = 0\}$ contains $H$ and intersects $S \setminus \{0\}$ transversally. \hfill $\Box$

Now we can present our infinitesimal linear kinematic formulas. Let $H \in G_{n-k}^n$, $k \in \{1, \ldots, n\}$, and let $S^{k-1}_{H^+}$ be the unit sphere of the orthogonal complement of $H$. Let $\nu$ be an element in $S^{k-1}_{H^+}$. For $\delta > 0$, we denote by $H_{\nu, \delta}$ the $(n - k)$-dimensional affine space $H + \delta \nu$ and we set

$$\beta_0(H, \nu) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(H_{\nu, \delta} \cap X, H_{\nu, \delta} \cap X \cap B_\epsilon).$$

Then we set

$$\beta_0(H) = \frac{1}{s_{k-1}} \int_{S^{k-1}_{H^+}} \beta_0(H, \nu) d\nu.$$

Theorem 5.5. For $k \in \{1, \ldots, n\}$, we have

$$\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = \frac{1}{g_{n-k}} \int_{G_{n-k}^n} \beta_0(H) dH.$$
Proof. We treat first the case \( k \in \{1, \ldots, n-2\} \). By Theorem 5.1, we know that
\[
\lim_{\epsilon \to 0} \frac{\Delta_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = -\frac{1}{2g_n^{n-k-1}} \int_{G_n^{n-k-1}} \chi(\text{Lk}(X \cap H)) dH + \frac{1}{2g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap L)) dL.
\]
By Lemma 3.8 in [14], we know that generically \( H \) intersects \( X \setminus \{0\} \) transversally in a neighborhood of the origin. Let us fix \( H \) that satisfies this generic property. For any \( v \in S^{n-1}_{H} \), let \( \nu \) be the line generated by \( v \) and let \( L_\nu \) be the \((n-k+1)\)-plane defined by \( L_\nu = H \oplus \nu \). By Lemma 5.4, we know that for \( v \) generic in \( S^{n-1}_{H} \), \( L_\nu \) intersects \( X \setminus \{0\} \) transversally in a neighborhood of the origin. Therefore, \( v|_{X \cap L_\nu} \) has an isolated singular point at 0 and we can apply Theorem 4.5. We have
\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \Delta_0(X \cap L_\nu \cap \{v^* = \delta\}, X \cap L_\nu \cap \{v^* = \delta\} \cap B_\epsilon) = \chi(X \cap L_\nu \cap \{v^* = \delta\} \cap B_\epsilon) - \frac{1}{2} \chi(\text{Lk}(X \cap L_\nu \cap \{v^* = 0\})) - \frac{1}{2s_{n-k}} \int_{S_{L_\nu}^{n-k}} \chi(\text{Lk}(X \cap L_\nu \cap \{v^* = 0\} \cap \{w^* = 0\})) dw,
\]
where \( S^{n-k}_{L_\nu} \) is the unit sphere of \( L_\nu \). Let us remark that \( L_\nu \cap \{v^* = \delta\} \) is exactly \( H_r,\delta \) and that \( L_\nu \cap \{v^* = 0\} \) is \( H \). We can also apply Lemma 5.2 to \( v|_{X \cap L_\nu} \) to obtain the following relation:
\[
\beta_0(H, v) + \beta_0(H, -v) = \chi(\text{Lk}(X \cap L_\nu)) - \frac{1}{s_{n-k}} \int_{S_{L_\nu}^{n-k}} \chi(\text{Lk}(X \cap H \cap \{w^* = 0\})) dw.
\]
Since \( \beta(H) \) is equal to
\[
\frac{1}{2s_{k-1}} \int_{S^{k-1}_H} \chi(\text{Lk}(X \cap L_\nu)) dv,
\]
we find that
\[
\beta(H) = \frac{1}{2s_{k-1}} \int_{S^{k-1}_H} \chi(\text{Lk}(X \cap L_\nu)) dv - \frac{1}{2s_{k-1}s_{n-k}} \int_{S^{k-1}_H} \int_{S_{L_\nu}^{n-k}} \chi(\text{Lk}(X \cap H \cap \{w^* = 0\})) dwdv.
\]
Replacing spheres with Grassman manifolds in this equality, we obtain
\[
\beta(H) = \frac{1}{2g_k^{1}} \int_{G^1_H} \chi(\text{Lk}(X \cap H \oplus \nu)) d\nu - \frac{1}{2g_k g^{n-k}_{n-k+1}} \int_{G^1_H} \int_{G^{n-k}_{H\oplus \nu}} \chi(\text{Lk}(X \cap H \cap K)) dK d\nu.
\]
Therefore, we have
\[
\frac{1}{g_n^{n-k}} \int_{c_n^{n-k}} \beta(H)dH = \frac{1}{2g_1^kg_n^{n-k}} \int_{G_n^{n-k}} \int_{G_{H^\perp}} \chi(\text{Lk}(X \cap H + \nu))d\nu dH - \frac{1}{2g_n^{n-k}} g_k g_{n-k+1} \int_{G_n^{n-k}} \int_{G_{H^\perp}} \int_{G_{H^\oplus}} \chi(\text{Lk}(X \cap H \cap K))dK d\nu dH.
\]

Let us compute
\[
I = \frac{1}{2g_n^{n-k}} g_k \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap L))dL = \frac{1}{2g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap L))dL.
\]

Let \( \mathcal{H} \) be the flag variety of pairs \((L, H), L \in G_n^{n-k+1} \text{ and } H \in G_n^{n-k}\). This variety is a bundle over \( G_n^{n-k} \), each fibre being a \( G_k \). Hence we have
\[
\int_{G_n^{n-k}} \int_{G_{H^\perp}} \chi(\text{Lk}(X \cap H + \nu))d\nu dH = \int_{G_n^{n-k+1}} \int_{G_n^{n-k}} \chi(\text{Lk}(X \cap L))dH dL = g_n^{n-k} g_{n-k+1} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap L))dL.
\]

Finally, we get that
\[
I = \frac{g_n^{n-k+1}}{2g_n^{n-k}} g_k \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap L))dL = \frac{1}{2g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap L))dL.
\]

Let us compute now
\[
J = \frac{1}{2g_n^{n-k}} g_k \int_{G_n^{n-k+1}} \int_{G_n^{n-k}} \int_{G_{H^\perp}} \int_{G_{H^\oplus}} \chi(\text{Lk}(X \cap H \cap K))dK d\nu dH dL.
\]

First, as we have just done above, we can write
\[
J = \frac{1}{2g_n^{n-k+1}} g_k \int_{G_n^{n-k+1}} \int_{G_n^{n-k}} \int_{G_n^{n-k}} \int_{G_n^{n-k}} \chi(\text{Lk}(X \cap H \cap K))dK dH dL.
\]

Then we remark (see [14], Corollary 3.11 for a similar argument) that
\[
\frac{1}{g_n^{n-k+1}} \int_{G_n^{n-k}} \chi(\text{Lk}(X \cap H \cap K))dK = \frac{1}{g_n^{n-k-1}} \int_{c_n^{n-k-1}} \chi(\text{Lk}(X \cap J))dJ,
\]

and so
\[
J = \frac{1}{2g_n^{n-k}} g_k \frac{1}{g_n^{n-k-1}} \int_{G_n^{n-k+1}} \int_{G_n^{n-k}} \int_{G_n^{n-k}} \int_{G_n^{n-k-1}} \chi(\text{Lk}(X \cap J))dJ dH dL.
\]

Considering the flag variety of pairs \((H, J), H \in G_{n-k} \text{ and } J \in G_{n-k-1}^n\), and proceeding as above, we find
\[
\int_{G_n^{n-k}} \int_{G_n^{n-k-1}} \chi(\text{Lk}(X \cap J))dJ dH = g_n^1 \int_{G_n^{n-k-1}} \chi(\text{Lk}(X \cap J))dJ,
\]

and
so
\[ J = \frac{g_2}{2g_n n^k} \int_{G_{n-k+1}} \int_{G_{n-k-1}} \chi(\text{Lk}(X \cap J)) dJ. \]

To finish the computation, we consider the flag variety of pairs \((L, J), \ L \in G_{n-k+1}^n \text{ and } J \in G_{L}^{n-k-1}. \quad \text{It is a bundle over } G_{n-k+1}^n, \text{ each fibre being a } G_{k+1}^2. \quad \text{Hence we have } \\
J = \frac{g_2^2}{2g_n n^k} \int_{G_{n-k+1}} \int_{G_{n-k-1}} \chi(\text{Lk}(X \cap J)) dJ dM, \\
J = \frac{g_2^2}{2g_n n^k} \int_{G_{n-k+1}} \int_{G_{n-k-1}} \chi(\text{Lk}(X \cap J)) dJ = \\
\frac{1}{2g_n n^k} \int_{G_{n-k+1}} \chi(\text{Lk}(X \cap J)) dJ.

This ends the proof for the case \( k \in \{1, \ldots, n-2 \}. \) For \( k = n-1 \text{ or } n, \) the proof is the same. We just have to remark that in these cases 
\( \beta_0(H, v) + \beta_0(H, -v) = \chi(\text{Lk}(X \cap L_v)), \) 
and if \( k = n-1, \dim L_v = 2 \) and if \( k = n, \dim L_v = 1. \quad \square \)

Let us end with some remarks on the limits \( \lim_{\epsilon \to 0} \frac{\Lambda_k(X, Y \cap B_\epsilon)}{b_k \epsilon^k}. \) We already know that if \( \dim X = d \) then \( \lim_{\epsilon \to 0} \frac{\Lambda_k(X, Y \cap B_\epsilon)}{b_k \epsilon^k} = 0 \) for \( k \geq d+1. \) This is also the case if \( l < d_0, \) where \( d_0 \) is the dimension of the stratum that contains \( 0. \) To see this let us first relate the limits \( \lim_{\epsilon \to 0} \frac{\Lambda_k(X, Y \cap B_\epsilon)}{b_k \epsilon^k} \) to the polar invariants defined by Comte and Merle in [9]. They can be defined as follows. Let \( H \in G_{n-k}^n, k \in \{1, \ldots, n\}, \) and let \( v \) be an element in \( S_{k-1} H_{H^\perp}. \)

For \( \delta > 0, \) we set
\[ \lambda_0(H, v) = \lim_{\epsilon \to 0} \frac{\Lambda_k(X, Y \cap B_\epsilon)}{b_k \epsilon^k}, \]
and then
\[ \sigma_k(X, 0) = \frac{1}{s_{k-1}} \int_{S_{k-1} H_{H^\perp}} \lambda_0(H, v) dv. \]

Moreover, we put \( \sigma_0(X, 0) = 1. \)

**Theorem 5.6.** For \( k \in \{0, \ldots, n-1\}, \) we have
\[ \lim_{\epsilon \to 0} \frac{\Lambda_k(X, Y \cap B_\epsilon)}{b_k \epsilon^k} = \sigma_k(X, 0) - \sigma_{k+1}(X, 0). \]
Furthermore, we have
\[ \lim_{\epsilon \to 0} \frac{\Lambda_n(X, Y \cap B_\epsilon)}{b_n \epsilon^n} = \sigma_n(X, 0). \]
Proof. It is the same proof as Theorem 5.5. For example if \( k \in \{0, \ldots, n-1\} \), we just have to remark that
\[
\lambda_0(H,v) + \lambda_0(H,-v) = \chi(\text{Lk}(X \cap L_v)) + \chi(\text{Lk}(X \cap H)),
\]
by Lemma 5.2, which implies that
\[
\sigma_k(X,0) = \frac{1}{2g^{n-k+1}_n} \int_{G^{n-k+1}} \chi(\text{Lk}(X \cap L))dL + \frac{1}{2g^{n-k}_n} \int_{G^{n-k}} \chi(\text{Lk}(X \cap H))dH.
\]
□

It is explained in [9] that \( \sigma_k(X,0) = 1 \) if \( 0 \leq k \leq d_0 \), so if \( k < d_0 \) then
\[
\lim_{\varepsilon \to 0} \Lambda_k(X,X \cap B_\varepsilon) = 0.
\]

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Aix-Marseille Université, LATP, 39 rue F. Joliot-Curie, 13453 Marseille Cedex 13, France.

E-mail address: nicolas.dutertre@univ-amu.fr