Identification and state realization of non-rational convolutive models by means of diffusive representation

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Abstract

We introduce a new identification method for general causal convolution models of the form $u \mapsto h * u = H(\partial_t)u$, where h is the impulse response of the system, to be identified from measurement data. This method is based on a suitable parameterization of operator $H(\partial_t)$ deduced from the so-called *diffusive representation*, devoted to state representations of such integral operators. Following this approach, the complex dynamic features of $H(\partial_t)$ can be summarized by a few numerical parameters on which the identification method will focus. The class of concerned convolution operators includes rational as well as non rational ones, even of complex nature. For illustration, we implement this method on a numerical example.

1 Introduction

Identification of linear input-output convolution systems of the form:

$$u \mapsto H(\partial_t)\, u, \ \ [H(\partial_t)\, u](t) = \int_0^t h(t-s)\, u(s)ds \ (1)$$

is all the more difficult as the operator $H(\partial_t) = h*(.)$ is of complex nature, in general non rational. Indeed, in cases where the associated transfer function $H(p) = (\mathcal{L}h)(p)$ is rational, the identification problem is classically tackled by means of well known adapted methods, such as ARMAX for example. On the other hand, systems with non rational transfer functions $H(p)$ are associated, in the time domain, to infinite-dimensional state representations, which makes the identification problem tricky.

As the operator $H(\partial_t)$ is linear, it can be interesting to work in the frequency domain; the unknown object to be identified is then the symbol $H(\mathrm{i}\omega)$ of the operator, that is the Fourier transform of its impulse response h, which can be identified by means of Fourier techniques. However, frequency identification methods present some well-known shortcomings. In particular, the so-identified symbol $H(\mathrm{i}\omega)$ is in general ill adapted to the construction of efficient time-realizations of the associated identified operator. Frequency methods are also incompatible with real-time identification (and so with pursuit when the symbol has the ability to evolve slowly). But above all, the number of unknown numerical parameters is in general excessive, which makes the problem sensitive to measurement noises.

Time domain techniques, which do not present such drawbacks, have also been developed for the identification of such input-output linear models. Among them, we can mention the approaches based on the approximation of the non rational transfer function by a rational one (by means for example of the well-known Padé approximations). Nevertheless such approximations in general do not enable to correctly represent from a small number of identified numerical parameters some of the complex dynamic phenomena present in many physical systems.

The proposed identification method is based on a suitable parameterization of operator $H(\partial_t)$ deduced from the so-called *diffusive representation* [1, 2]. According to this approach, a wide class of integral

causal operators (including both rational and non rational ones) can be studied and realized by means of suitable state representations. In these state formulations, a new mathematical object associated with the operator is introduced: the so-called γ-symbol, in general denoted by μ. When it exists, the γ-symbol allows to realize the operator under consideration as an output of a universal state representation of diffusive nature, the state of which is namely the diffusive representation of the input u, denoted ψ_u. As ψ_u is a function of a continuous parameter $\xi \in \mathbb{R}$, the state representation is of infinite dimension. However, cheap and precise finite dimension approximations can be easily built, thanks to the diffusive nature of the state equation, in such a way that only a small number of unknown numerical parameters in general suffices to get a good accuracy. As a consequence, and because the output is expressed linearly with respect to the γ-symbol μ, an approximate γ-symbol can be identified from measurement data by means of a simple least squares method.

Identifying the γ-symbol of a convolution operator presents numerous advantages. First the so-identified model can be expressed under the form of a stable input-output state equation. Then, some recursive identification algorithms, which are compatible with real-time identification or even pursuit, can be easily deduced. Finally, as for purely frequency methods, the identification process can be used for rational as well as non rational models without any distinction.

The paper is organized as follows. In section 2, we briefly present a simplified version of the diffusive representation approach. In sections 3, 4, we describe the identification method in a general framework and we give some indications for numerical implementations. In section 5, we finally test the method on a simple example of non rational model and we comment the obtained results, which allows us to highlight the relevance of the approach.

2 Diffusive formulation of causal convolution operators

A complete statement of diffusive representation can be found in [1]; a shortened one is presented in [2]. For various applications and questions relating to this approach, we can refer for example to [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

2.1 Principle

We consider a causal convolution operator defined, on any continuous function $u : \mathbb{R}^+ \to \mathbb{R}$, by

$$u \mapsto \int_0^t h(t-s)\, u(s)\, ds. \qquad (2)$$

We denote H the Laplace transform of h and $H(\partial_t)$ the convolution operator defined by (2).

Let $u^t(s) = \mathbf{1}_{]-\infty,t]}(s)\, u(s)$ be the restriction of u to its past and $u_t(s) = u^t(t-s)$ the so-called "history" of u. From causality of $H(\partial_t)$, we deduce:

$$[H(\partial_t)(u - u^t)](t) = 0 \text{ for all } t; \qquad (3)$$

then, we have for any continuous function u:

$$[H(\partial_t)\, u](t) = \left[\mathcal{L}^{-1}\left(H\,\mathcal{L}u\right)\right](t) = \left[\mathcal{L}^{-1}\left(H\,\mathcal{L}u^t\right)\right](t). \qquad (4)$$

We define:

$$\Psi_u(t,p) := \mathrm{e}^{p\,t}\left(\mathcal{L}u^t\right)(p) = (\mathcal{L}u_t)(-p); \qquad (5)$$

by computation of $\partial_t \mathcal{L}u_t$, Laplace inversion and use of (4), we then have:

Lemma 1 *1. The function Ψ_u is solution of the differential equation:*

$$\partial_t \Psi(t,p) = p\,\Psi(t,p) + u, \quad t > 0, \quad \Psi(0,p) = 0. \qquad (6)$$

2. For any $b \geqslant 0$,

$$[H(\partial_t)\, u]\,(t) = \frac{1}{2i\pi}\int_{b-i\infty}^{b+i\infty} H(p)\,\Psi_u(t,p)\, dp. \qquad (7)$$

Let γ be a closed[1] simple arc in $\mathbb{C}^-$; we denote Ω_γ^+ the exterior domain defined by γ, and Ω_γ^- the complementary of $\overline{\Omega_\gamma^+}$. By use of standard techniques (Cauchy theorem, Jordan lemma), it can be shown:

Lemma 2 *For γ such that H is holomorphic in Ω_γ^+, if $H(p) \to 0$ when $|p| \to \infty$ in Ω_γ^+, then:*

$$[H(\partial_t)\, u]\,(t) = \frac{1}{2i\pi} \int_{\tilde{\gamma}} H(p)\, \Psi_u(t,p)\, dp, \qquad (8)$$

where $\tilde{\gamma}$ is any closed simple arc in Ω_γ^+ such that $\gamma \subset \ \Omega_{\tilde{\gamma}}^-$.

We now suppose that γ, $\tilde{\gamma}$ are defined by functions of the Sobolev space[2]). $W_{\text{loc}}^{1,\infty}\,(\mathbb{R};\mathbb{C})$, also denoted γ, $\tilde{\gamma}$ and such that, for simplicity:

$$\gamma(0) = 0. \qquad (9)$$

We also suppose that there exists $\alpha_\gamma \in]\frac{\pi}{2}, \pi[$ and $a \in \mathbb{R}$ such that:

$$e^{i[-\alpha_\gamma,\, \alpha_\gamma]}\mathbb{R}_+ + a \ \subset \ \Omega_\gamma^+. \qquad (10)$$

By use of the convenient notation[3] $\langle \mu, \psi \rangle = \int \mu\,\psi\,d\xi$, and under hypotheses of lemma 2, we have [1]:

Theorem 1 *If the possible singularities of H on γ are simple poles or branching points such that $|H \circ \gamma|$ is locally integrable in their neighbourhood, then:*
1. with $\tilde{\mu} = \frac{\tilde{\gamma}'}{2i\pi} H \circ \tilde{\gamma}$ and $\tilde{\psi}(t,.) = \Psi_u(t,.) \circ \tilde{\gamma}$:

$$[H(\partial_t)\, u]\,(t) = \langle \tilde{\mu}, \tilde{\psi}(t,.)\rangle; \qquad (11)$$

2. with[4] $\tilde{\gamma}_n \to \gamma$ in $W_{\text{loc}}^{1,\infty}$ and $\mu = \frac{\tilde{\gamma}'}{2i\pi} \lim K \circ \tilde{\gamma}_n$ in the sense of measures:

$$[H(\partial_t)u]\,(t) = \langle \mu, \psi(t,.)\rangle, \qquad (12)$$

where $\psi(t,\xi)$ is solution of the following evolution problem on $(t,\xi) \in \mathbb{R}^{*+} \times \mathbb{R}$ (of diffusive type):

$$\partial_t \psi(t,\xi) = \gamma(\xi)\,\psi(t,\xi) + u(t),\ \psi(0,\xi) = 0. \qquad (13)$$

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[1]Possibly at infinity

[2]$W_{\text{loc}}^{1,\infty}\,(\mathbb{R};\mathbb{C})$ is the topological space of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that $f, f' \in L_{\text{loc}}^\infty$ (that is f and f' are locally essentially bounded

[3]Note that in particular, when μ is atomic, that is $\mu = \sum_k a_k\, \delta_{\xi_k}$, we have: $\langle \mu, \psi \rangle = \sum_k a_k\, \psi(\xi_k)$.

[4]This convergence mode means that on any bounded set P, $\tilde{\gamma}_{n|P} - \gamma_{|P} \to 0$ and $\tilde{\gamma}'_{n|P} - \gamma'_{|P} \to 0$ uniformly.

Definition 1 *The measure μ defined in theorem 1 is called the γ-symbol of operator $H(\partial_t)$. The function ψ solution of (14) is called the γ-representation of u.*

Example 1 *Note in particular that thanks to (15), the Dirac measure δ is clearly a γ-symbol of the operator $u \mapsto \int_0^t u(s)\,ds$, denoted ∂_t^{-1}. We indeed have $(\partial_t^{-1}u)(t) = \langle \delta, \psi(t,.)\rangle = \psi(t,0)$, with $\partial_t \psi(t,0) = u$, $\psi(0,0) = 0$.*

Beyond the measure framework, the general space of γ-symbols is a quotient space of distributions, denoted Δ'_γ; it is the topological dual of the topological vector space $\Delta_\gamma \ni \psi(t,.)$ [1].

Thanks to the sector condition (10) verified by γ, the state representation is of diffusive type; this property allows to easily build cheap and precise numerical approximations of (14,15) as explained in section 2.4.

2.2 Summary

Given γ as defined above, the diffusive representation of an operator $H(\partial_t)$ is the following state-representation:

$$
\begin{aligned}
\partial_t \psi(t,\xi) &= \gamma(\xi)\psi(t,\xi) + u(t),\ \psi(0,.) = 0 & (14)\\
(H(\partial_t)u)(t) &= <\mu, \psi(t,.)>_{\Delta'_\gamma, \Delta_\gamma}, & (15)
\end{aligned}
$$

where $\mu \in \Delta'_\gamma$ is the γ-symbol of $H(\partial_t)$; the main conditions the operator has to satisfy to admit such a representation are:

- H holomorphic in Ω_γ^+, $\qquad\qquad$ (16)
- $H(p) \to 0$ when $|p| \to +\infty$ in Ω_γ^+. $\quad$ (17)

2.3 Extension to higher order operators

Formulation (14,15) can be extended to operators of the form $H(\partial_t) = K(\partial_t) \circ \partial_t^n$ where $K(\partial_t)$ admits a γ-symbol ν in Δ'_γ. We have (formally):

$$[K(\partial_t) \circ \partial_t^n u](t) = \langle \nu, \partial_t^n \psi(t,.)\rangle, \qquad (18)$$

with $\psi(t,\xi)$ solution of (14). In the particular case where $n = 1$, (18) becomes:

$$[K(\partial_t) \circ \partial_t\, u](t) = \langle \nu, \gamma\,\psi(t,.) + u(t)\rangle. \qquad (19)$$

2.4 About numerical approximations

The state equation (14) is infinite-dimensional. To get numerical approximations, we consider a sequence $\mathcal{M}_L$ of L-dimensional spaces of atomic measures on suitable meshes $\{\xi_l^L\}_{l=1:L}$ on the variable ξ; L-dimensional approximations μ^L of the γ-symbol $\mu \in \Delta'_\gamma$ are then defined in the sense of atomic measures, that is:

$$\mu^L = \sum_{l=1}^{L} \mu_l^L \, \delta_{\xi_l^L}, \quad \mu_l^L \in \mathbb{C}. \qquad (20)$$

If $\cup_L \mathcal{M}_L$ is dense in the topological space Δ'_γ (that is, concretely, if $\cup_L \{\xi_l^L\}$ is dense in $\mathbb{R}$), then we can have [1]:

$$\langle \mu^L, \psi \rangle \underset{L \to +\infty}{\longrightarrow} \langle \mu, \psi \rangle \;\; \forall \psi \in \Delta_\gamma; \qquad (21)$$

we then deduce the following L-dimensional approximate state formulation of $H(\partial_t)$ (with γ-symbol μ):

$$\begin{cases} \partial_t \psi(t, \xi_l^L) = \gamma(\xi_l^L) \, \psi(t, \xi_l^L) + u(t), \, l = 1 : L, \\ \psi(0, \xi_l^L) = 0, \\ [H(\partial_t) \, u](t) \simeq \sum_{l=1}^{L} \mu_l^L \, \psi(t, \xi_l^L). \end{cases} \qquad (22)$$

Note that in the particular case where $H(\partial_t) = K(\partial_t) \circ \partial_t$ with $K(\partial_t)$ an operator which admits a γ-symbol ν in Δ'_γ, an approximate state formulation of operator $H(\partial_t)$ is given, from (19), under the form:

$$H(\partial_t)u \simeq \sum_l \gamma(\xi_l^L) \, \nu_l^L \, \psi(., \xi_l^L) \; + \sum_l \nu_l^L \, u. \quad (23)$$

One of the properties of the approach presented above is that most of non rational operators encountered in practice can be closely approximate with small L (see for example [14]). In the context of identification of convolution models, this is a great advantage because only a few numerical parameters μ_l^L have to be identified from experimental data, while the property (21) ensures the well-posedness and the robustness of the problem as soon as the operator to be identified admits a γ-symbol in Δ'_γ.

3 Identification of an operator by means of its γ-symbol

We consider in the sequel the problem of identification of the model:

$$H(\partial_t)u = x, \qquad (24)$$

where $H(\partial_t)$ is an integral operator. The goal is to build an estimation (if possible optimal) of the γ-symbol of $H(\partial_t)$ from (possibly noisy) measurements u_m and x_m of the input u and the associated output x.

3.1 Principle

Let $H(\partial_t)$ be a linear integral operator. We suppose there exists $n \in \mathbb{N}$ and $K(\partial_t)$ an operator admitting a γ-symbol ν in Δ'_γ, such that $H(\partial_t) = K(\partial_t) \circ \partial_t^n$. We then have (see section 2):

$$x = H(\partial_t)u = \langle \nu, \partial_t^n \psi_u \rangle = \langle \nu, \psi_{\partial_t^n u} \rangle. \qquad (25)$$

By denoting $\mathcal{A}_{\partial_t^n u}$ the operator defined by:

$$\mathcal{A}_{\partial_t^n u} : \nu \longmapsto \langle \nu, \psi_{\partial_t^n u} \rangle, \qquad (26)$$

we get a new formulation, linear with respect to the γ-symbol ν:

$$x = \mathcal{A}_{\partial_t^n u} \nu, \qquad (27)$$

from which ν can be identified by classical least squares methods. Indeed, from (possibly noisy) measurements x_m and u_m of the trajectories x and u, let us consider the minimisation problem:

$$\min_{\nu \in \mathcal{E}} \left\| \mathcal{A}_{\partial_t^n u_\mathrm{m}} \nu - x_\mathrm{m} \right\|_{\mathcal{F}}^2, \qquad (28)$$

where $\mathcal{E}$ is a Hilbert subspace of Δ'_γ and $\mathcal{F}$ another Hilbert space chosen *a priori*. The solution of this problem then gives an (optimal) estimation ν^* of the unknown γ-symbol ν; it is expressed:

$$\nu^* = \mathcal{A}_{\partial_t^n u_\mathrm{m}}^\dagger x_\mathrm{m}, \qquad (29)$$

where $\mathcal{A}_{\partial_t^n u_\mathrm{m}}^\dagger$ designates the pseudo-inverse[15] of operator $\mathcal{A}_{\partial_t^n u_\mathrm{m}}$.

Remark 1 *For example, we can consider the space* $\mathcal{F} := L^2(0;T)$ *with* $T > 0$, *and the associated norm:*

$$\|f\|_{\mathcal{F}} = \left(\int_0^T |f(t)|^2 \, dt \right)^{\frac{1}{2}}. \qquad (30)$$

By denoting $K^*(\partial_t)$ the operator with γ-symbol ν^* and $H^*(\partial_t) = K^*(\partial_t) \circ \partial_t^n$, the identified model is then written:

$$x = H^*(\partial_t)u, \qquad (31)$$

or, under diffusive representation:

$$\begin{cases} \partial_t \psi = \gamma \, \psi + \partial_t^n u, \ \psi(0,.) = 0 \\ x = \langle \nu^*, \psi_{\partial_t^n u} \rangle. \end{cases} \qquad (32)$$

Remark 2 *Recursive formulations of (29) can be established under the form (see [7]):*

$$\nu_t^* = \nu_{t-\Delta t}^* + K_{t-\Delta t}(x_\mathrm{m} - \mathcal{A}_{\partial_t^n u_\mathrm{m}} \nu_{t-\Delta t}^*)_{|[0,t]}; \quad (33)$$

such formulations allow real-time identification (or even the pursuit of ν in case of slowly varying operators $H(t,\partial_t)$).

3.2 On the bias of the estimator ν^*

The exact (but unknown) value of ν is denoted ν_0; it verifies:

$$x = \mathcal{A}_{\partial_t^n u} \, \nu_0. \qquad (34)$$

We suppose in the sequel that $\nu_0 \in \mathcal{E}$. In the sense of the hilbertian norm of $\mathcal{F}$, the estimator ν^* of ν_0 is optimal. Let suppose $x_\mathrm{m} = x + \eta_x$ and $u_\mathrm{m} = u + \eta_u$ with η_x and η_u two Gaussian noises such that $E[\eta_x] = E[\eta_u] = 0$. If $\eta_u \neq 0$, the estimator ν^* is *biased* because $\mathcal{A}_{\partial_t^n u_\mathrm{m}}^{\dagger}$ depends on the measurement noise. To mitigate this problem, it could be interesting to consider some classical bias reduction methods as the ones used in time-continuous system identification.

Note that in the case where $\eta_x = 0$, we can get an unbiased estimation by identifying the operator $H(\partial_t)^{-1}$ with input x and output u instead of $H(\partial_t)$. If we suppose that there exists $n \in \mathbb{N}$ and $K(\partial_t)$ an operator admitting a γ-symbol $\nu \in \Delta'_\gamma$ such that $H(\partial_t)^{-1} = K(\partial_t) \circ \partial_t^n$, we indeed have:

$$u = \mathcal{H}(\partial_t)^{-1}x = \mathcal{A}_{\partial_t^n x}\nu, \qquad (35)$$

and so we get, after identification:

$$\nu^* = \mathcal{A}_{\partial_t^n x}^{\dagger} u_\mathrm{m}. \qquad (36)$$

The identified model can then be written:

$$x = K^*(\partial_t)^{-1} \circ \partial_t^{-n} u, \qquad (37)$$

that is, under diffusive representation:

$$\begin{cases} \partial_t \psi = \gamma \, \psi + u, \ \psi(0,.) = 0 \\ x = \langle (\nu^*)^{-1} \# \delta^n, \psi_u \rangle, \end{cases} \qquad (38)$$

where $(\nu^*)^{-1}$, δ^n and $(\nu^*)^{-1} \# \delta^n$ are the respective γ-symbols of $K^*(\partial_t)^{-1}$, ∂_t^{-n} and $K^*(\partial_t)^{-1} \circ \partial_t^{-n}$. The computation of $(\nu^*)^{-1} \# \delta^n$ can be numerically performed from ν^* as shown in [16].

3.3 Prefiltering with an invertible convolution operator

The identification model (27) can be equivalently transformed by composition with any invertible causal convolution operator $Q(\partial_t)$. Indeed we have:

$$Q(\partial_t)x = Q(\partial_t) \circ H(\partial_t)u = H(\partial_t) \circ Q(\partial_t)u; \quad (39)$$

by denoting $\tilde{x} = Q(\partial_t)x$ and $\tilde{u} = Q(\partial_t)u$, the model is then rewritten:

$$\tilde{x} = H(\partial_t)\tilde{u}. \qquad (40)$$

When applying the identification method to model (40), the estimator of ν_0 is written:

$$\nu^* = \mathcal{A}_{\partial_t^n \tilde{u}_\mathrm{m}}^{\dagger} \, \tilde{x}_\mathrm{m}, \qquad (41)$$

with $\tilde{u}_\mathrm{m} = Q(\partial_t)u_\mathrm{m}$ and $\tilde{x}_\mathrm{m} = Q(\partial_t)x_\mathrm{m}$.

When $n > 1$, such a transformation is necessary; otherwise, due to unbounded high frequency amplification resulting from operator ∂_t^n, the high frequency noise present in the term $\partial_t^n u_\mathrm{m}$ would make the identification quite impossible. When $n = 1$, we use the

formulation (19) instead of (18); thus, the term $\partial_t u_{\rm m}$ is not involved and the noise is not amplified.

So to avoid high frequency noise amplification, the operator $Q(\partial_t)$ has to be chosen in such a way that high frequencies are sufficiently attenuated, without amplifying low and middle ones, that is:

$$|Q(\mathrm{i}\omega)| \underset{H.F}{\sim} \frac{1}{\omega^n}, \ |Q(\mathrm{i}\omega)| \underset{L.F}{\sim} 1; \qquad (42)$$

basically, it behaves like a nth order low-pass filter. We simply consider in the sequel the transfer function:

$$Q(p) = \frac{\sigma^n}{(p+\sigma)^n}, \qquad (43)$$

where $\sigma > 0$ (the cut-off frequency) will be chosen in such a way that $\left\| \mathcal{A}_{\partial_t^n \tilde{u}_{\rm m}} \nu - \tilde{x}_{\rm m} \right\|_{\mathcal{F}}^2$ is "as small as possible".

Note that the transfer function $Q(p)$ could also be optimized in order to minimize the estimation error.

3.4 Case of multiple measured trajectories

Consider a set of input trajectories u^j, $j = 1 : J$ and the associated output trajectories $x^j = H(\partial_t)u^j$. Let $u_{\rm m}^j$ and $x_{\rm m}^j$ be some measurements of u^j and x^j. Then, without any change of notations, model (27) can be extended to the general case of multiple trajectories simply by defining:

$$u = (u^1, ..., u^J)^T, \ x = (x^1, ..., x^J)^T, \qquad (44)$$

$$u_{\rm m} = (u_{\rm m}^1, ..., u_{\rm m}^J)^T, \ x_{\rm m} = (x_{\rm m}^1, ..., x_{\rm m}^J)^T, \qquad (45)$$

$$\text{and } \mathcal{A}_{\partial_t^n u} : \nu \longmapsto \begin{bmatrix} < \nu, \psi_{\partial_t^n u^1} > \\ \vdots \\ < \nu, \psi_{\partial_t^n u^J} > \end{bmatrix}. \qquad (46)$$

4 Numerical formulation

In this section, we show how to numerically solve the problem (28). For that, we first have to choose the contour γ on which the problem is based. Then, the discretization of the variable ξ leads to a time continuous approximate problem of finite dimension. Finally, we discretize the problem in the time variable t and get the numerical solution by classical pseudo inversion of a matrix.

4.1 Discrete measurement data

We consider J solutions (u^j, x^j), $j = 1 : J$ of (24) and a discretization $\{t_k\}_{k=0:K}$ of the variable t defined by:

$$t_0 = 0; \ t_k = t_{k-1} + \Delta t_k, \ k = 1 : K. \qquad (47)$$

We will denote in the sequel $T = t_K$.

Let $\{u_{\rm m}^{j,k}, x_{\rm m}^{j,k}\}_{k=0:K, j=1:J}$ be some sets of discrete data with $u_{\rm m}^{j,k}$ and $x_{\rm m}^{j,k}$ the respective measurements of u^j and x^j at time t_k. We also denote $u_{\rm m}^j$ and $x_{\rm m}^j$ the time continuous measurement trajectories such that $u_{\rm m}^j(t_k) = u_{\rm m}^{j,k}$ and $x_{\rm m}^j(t_k) = x_{\rm m}^{j,k}$.

4.2 Choice of contour γ

The operator $K(\partial_t) = H(\partial_t) \circ \partial_t^{-n}$ is supposed to admit a γ-symbol ν in Δ_γ', which implies, from lemma 2, that K is analytic in Ω_γ^+. So, all the singularities of K have to be inside the domain $\overline{\Omega_\gamma^-}$ delimited by γ. However, as the operator $H(\partial_t)$ is unknown, so is the position of the singularities of K. As a consequence, the contour γ will be chosen in such a way that the domain $\overline{\Omega_\gamma^-}$ is sufficiently big to contain all the singularities of K. In practice, we often take a contour of sector type (see figure 1):

$$\gamma(\xi) = |\xi| \, e^{\mathrm{i}\,\mathrm{sign}(\xi)(\frac{\pi}{2}+\alpha)}, \qquad (48)$$

with $\alpha \in]0, \frac{\pi}{2}]$ small enough.

If the identification results are good with a small angle α, then we can iterate the process with a greater α. Note that in practice, available informations about the operator $H(\partial_t)$ to be identify can help us in the choice of γ.

Remark 3 *Note however that the more γ is close to the axis $\mathrm{i}\mathbb{R}$ (stability limit axis), the finer (and so the more expensive numerically) the discretization in ξ has to be in order to get a good approximation of the model.*
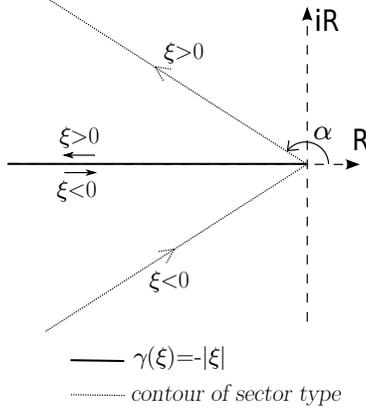
6

Figure 1: Examples of contours γ

4.3 Approximate problem of finite dimension

We consider a discretization $\{\xi_l\}_{l=1:L}$ of the variable ξ, and the approximation of $H(\partial_t)u^j = \mathcal{A}_{\partial_t^n u^j} \nu$ given by (see paragraph 2.4):

$$\left[\mathcal{A}_{\partial_t^n u^j} \nu\right](t) \simeq \sum_{l=1}^{L} \nu_l^L \psi_{\partial_t^n u^j}(t, \xi_l), \text{ with } \nu_l^L \in \mathbb{C}. \tag{49}$$

The time-continuous approximate model can then be written:

$$\sum_{l=1}^{L} \nu_l^L \psi_{\partial_t^n u^j}(t, \xi_l) = x^j(t), \tag{50}$$

that is, under matrix form:

$$\Psi_{\partial_t^n u} \nu^L = x, \tag{51}$$

where u and x are defined by (44), $\nu^L = (\nu_1^L, ..., \nu_L^L)^T$, and $\Psi_{\partial_t^n u} : t \in [0, T] \longrightarrow \Psi_{\partial_t^n u}(t)$ with $\Psi_{\partial_t^n u}(t)$ the J-by-L matrix defined by:

$$\left[\Psi_{\partial_t^n u}(t)\right]_{j,l} = \psi_{\partial_t^n u^j}(t, \xi_l). \tag{52}$$

This leads to the minimization problem:

$$\min_{\nu \in \mathbb{C}^L} \left\|\Psi_{\partial_t^n u} \nu - x\right\|_{\mathcal{F}}^2, \tag{53}$$

with $\mathcal{F} := (L^2(0; T))^J$ and:

$$\|f\|_{\mathcal{F}}^2 = \sum_{j=1}^{J} \int_0^T \left|f^j(t)\right|^2 dt. \tag{54}$$

Note that the available data $\{u_{\mathrm{m}}^{j,k}, x_{\mathrm{m}}^{j,k}\}$ allow to correctly approximate the frequency response $H(\mathrm{i}\omega)$ only in the frequency band $\left[\frac{2\pi}{t_K}, \frac{2\pi}{2 \max\{\Delta t_k\}}\right]$. Consequently, the band $[\xi_1, \xi_L]$ covered by the ξ−discretization will be chosen in such a way that $\left[\frac{2\pi}{t_K}, \frac{2\pi}{2 \max\{\Delta t_k\}}\right] \subset |\gamma([\xi_1, \xi_L])|$. In the particular case of sector contours of the form (48), we have $|\gamma(\xi)| = |\xi|$; so, by noting that $|\gamma(\xi)|$ can be considered as a cut-off frequency (associated with the input-output realization $\partial_t \psi = \gamma(\xi)\psi + u$), we can state the practical condition:

$$\left[\frac{2\pi}{t_K}, \frac{2\pi}{2 \max\{\Delta t_k\}}\right] \subset \left[\min|\gamma(\xi_l)|, \max|\gamma(\xi_l)|\right]. \tag{55}$$

Some further indications about the choice of the discretization points ξ_l can be found in [1].

4.4 Time discretization

By means of suitable quadrature of the integral, problem (53) is approximated by:

$$\min_{\nu \in \mathbb{C}^L} \|A_{u_{\mathrm{m}}} \nu - \mathrm{x}_{\mathrm{m}}\|_{\mathbb{C}^{J \times K}}^2, \tag{56}$$

with:

- $\|z\|_{\mathbb{C}^{J \times K}}^2 = \sum_{j=1}^{J} \sum_{k=1}^{K} |c_{j,k}|^2 \Delta t_k$,

- $\mathrm{x}_{\mathrm{m}} = (x_{\mathrm{m}}^{1,1}, ..., x_{\mathrm{m}}^{1,K}, x_{\mathrm{m}}^{2,1}, ..., x_{\mathrm{m}}^{J,1}, ..., x_{\mathrm{m}}^{J,K})^T$,

- and $A_{u_{\mathrm{m}}}$ the matrix defined by:

$$A_{u_{\mathrm{m}}} = \begin{bmatrix} \Psi^1 \\ \hline \vdots \\ \hline \Psi^J \end{bmatrix}$$

where $\Psi_{k,l}^j = \psi_{\partial_t^n u_{\mathrm{m}}^j}^{l,k}$ is an approximation of the diffusive representation of $\partial_t^n u_{\mathrm{m}}^j$ at $\xi = \xi_l$ and

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$t = t_k$, deduced from numerical integration of the state equation of $\psi_{\partial_t^n u_{\mathrm{m}}^j}$, namely:

$$\begin{cases} \Psi_{k,l}^j = e^{\gamma(\xi_l)\Delta t_k}\,\Psi_{k-1,l}^j + \dfrac{e^{\gamma(\xi_l)\Delta t_k} - 1}{\gamma(\xi_l)}\,\partial_t^n u_{\mathrm{m}}^{j,k-1} \\[2mm] \Psi_{0,l}^j = 0 \end{cases}$$

$$(57)$$

The solution of (56) is then given by:

$$\nu^* = A_{u_{\mathrm{m}}}^\dagger \mathrm{x}_{\mathrm{m}}, \qquad (58)$$

with:

$$\begin{aligned} A_{u_{\mathrm{m}}}^\dagger &= (A_{u_{\mathrm{m}}}^* D\, A_{u_{\mathrm{m}}})^{-1} A_{u_{\mathrm{m}}}^* D, & (59) \\ D &= \mathrm{diag}(\Delta t_1, ..., \Delta t_K). & (60) \end{aligned}$$

Remark 4 *The above minimization problem can sometimes be ill conditioned: as usual, a penalization term can judiciously be added, so that (59) becomes:*

$$A_{u_{\mathrm{m}}}^\dagger = (A_{u_{\mathrm{m}}}^* D\, A_{u_{\mathrm{m}}} + \varepsilon J)^{-1} A_{u_{\mathrm{m}}}^* D, \qquad (61)$$

with ε a small positive parameters and J a suitable matrix (for example the identity one).

5 Validation on a simple numerical example

In this section, we implement the method presented above on a simple example in order to validate it and highlight its efficiency.

5.1 Model under consideration

We consider the operator $H(\partial_t)$ with Laplace-symbol:

$$H(p) = \frac{\sqrt{p}}{p+1}. \qquad (62)$$

We have:

- H is holomorphic in $\mathbb{C} \setminus \mathbb{R}^-$,

- $H(p) \to 0$ when $|p| \to +\infty$ in $\mathbb{C} \setminus \mathbb{R}^-$.

So, with the contour γ defined by (see figure 1):

$$\gamma(\xi) = -|\xi|, \qquad (63)$$

operator $H(\partial_t)$ admits a γ-symbol (denoted μ) and therefore a diffusive realization.

5.2 Data

The data $\{u_{\mathrm{m}}^{j,k}, x_{\mathrm{m}}^{j,k}\}$ used for identification are obtained by numerical simulation of the diffusive realization of $H(\partial_t)u = x$, with $L = 120$ discretization points ξ_l geometrically spaced between $\xi_1 = 10^{-1}$ and $\xi_L = 10^3$, and γ defined by (63). We take $J = 1$ (that is only one measurement set is available).

In figure 2, the Bode diagram of the frequency response of the so-approximate operator is compared to the one of the theoretical operator with symbol $H(p)$. Note that the approximation is quasi exact in the frequency band $[10^{-1}, 10^3]$ covered by the mesh $\{\xi_l\}$.
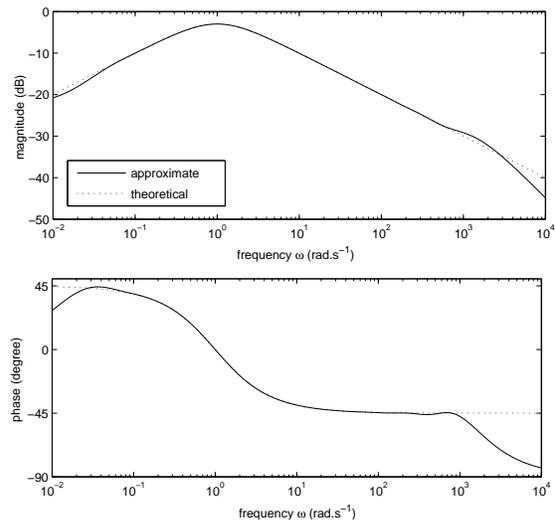

Figure 2: Bode diagrams of the theoretical operator $H(\partial_t) = \partial_t^{\frac{1}{2}} \circ (\partial_t + 1)^{-1}$ and of the approximate one used for the simulation of data.

The time discretization $\{t_k\}_{k=0:K}$ is defined by:

$$t_k = k\,\Delta t, \qquad (64)$$

with $K = 150\,000$ and $\Delta t = 10^{-3}$. The input is a Dirac impulse at time $t = 0$, that is $u = \delta_{t=0}$, associated with the discrete input $\{u^k\}_{k=0:K}$ defined by:

$$u^k = \begin{cases} 1 \text{ if } k = 0 \\ 0 \text{ otherwise.} \end{cases} \qquad (65)$$

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The associated discrete output is denoted $\{x^k\}_{k=0:K}$. The input trajectory is supposed to be known (without noise) and the (discrete) output measured trajectory under consideration is obtained from $\{x^k\}$ by addition of numerical white noise η_x^k with standard deviation σ_x, that is:

$$u_{\mathrm{m}}^k = u^k, \; x_{\mathrm{m}}^k = x^k + \eta_x^k. \qquad (66)$$

5.3 Quantitative evaluation of identification errors

In order to estimate the accuracy of the identification of $H(i\omega)$, we consider the following quantities E and $\mathrm{E}_{\mathrm{sup}}$:

$$
\begin{aligned}
\mathrm{E} \quad &= \frac{\sum_i \left| \sum_{l=0}^{L} \frac{\mu_l^*}{i\omega_j - \gamma(\xi_l)} - H(i\omega_j) \right| \frac{\Delta\omega_j}{\omega_j}}{\sum_j |H(i\omega_j)| \frac{\Delta\omega_j}{\omega_j}} \\
&\simeq \frac{\int_{\omega_1}^{\omega_2} \frac{1}{\omega} \left| \sum_{l=0}^{L} \frac{\mu_l^*}{i\omega - \gamma(\xi_l)} - H(i\omega) \right| d\omega}{\int_{\omega_1}^{\omega_2} \frac{1}{\omega} |H(i\omega)| \, d\omega}
\end{aligned} \qquad (67)
$$

and

$$
\begin{aligned}
\mathrm{E}_{\mathrm{sup}} \quad &= \sup_{\omega_j} \frac{\left| \sum_{l=0}^{L} \frac{\mu_l^*}{i\omega_j - \gamma(\xi_l)} - H(i\omega_j) \right|}{|H(i\omega_j)|} \\
&\simeq \sup_{\omega} \frac{\left| \sum_{l=0}^{L} \frac{\mu_l^*}{i\omega - \gamma(\xi_l)} - H(i\omega) \right|}{|H(i\omega)|}.
\end{aligned} \qquad (68)
$$

5.4 Case where the exact μ_0 is in the solution space $\mathcal{E}$

In order to validate the method, we identify the operator $H(\partial_t)$ with the same contour γ and the same discretization points $\{\xi_l\}$ than the ones used for the simulation of data. In such a case, there exists (a unique) $\mu_0 \in \mathcal{E}$ such that $\mathcal{A}_u \mu_0 = x$ and so, the identified solution μ^* should be equal to μ_0, up to some negligible numerical errors.

The results obtained on the one hand without any measurement noise ($\eta_x^k = 0$), and one the other hand

with standard deviation of the measurement noise $\sigma_x = 10^{-4}$ are given in figure 3. In the two cases, the frequency response is well identified in the frequency band accessible by the data and the ξ_l, that is $[\frac{2\pi}{K\Delta t}, \frac{2\pi}{2\Delta t}] = [4.19 \times 10^{-2}, 3.14 \times 10^3]$. In the noise free case, the Bode diagrams of the identified operator and of the one used for the simulation of the data are even superimposed.
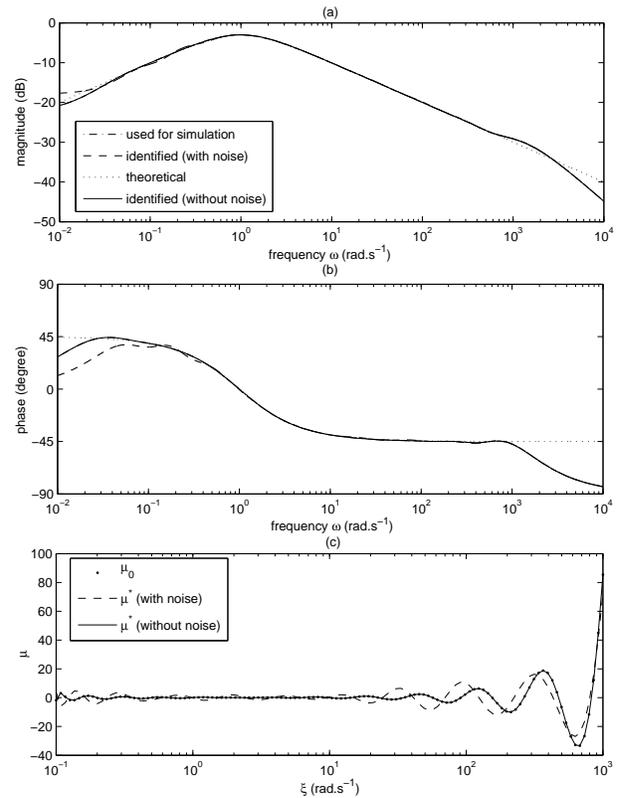

Figure 3: **Validation of the identification method in the ideal case** : (a) and (b): Magnitude (dB) and phase (degree) of the frequency responses of $H(\partial_t) = \partial_t^{\frac{1}{2}} \circ (\partial_t + 1)^{-1}$, of its approximation under diffusive representation and of the identified operator in the case without and with measurement noise ($\sigma_x = 10^{-4}$) - (c): Graphs of μ_0 and μ^*.

To quantify the identification error on μ, we intro-

σ_x	E	E_{sup}	E^{μ}	ε
0	7.3121×10^{-15}	1.3064×10^{-6}	0.0065	10^{-16}
10^{-7}	6.6201×10^{-13}	9.0663×10^{-5}	0.0142	10^{-14}
10^{-6}	6.3046×10^{-12}	7.7146×10^{-4}	0.0895	10^{-13}
10^{-5}	5.3474×10^{-11}	0.0107	0.1218	10^{-12}
10^{-4}	6.7330×10^{-10}	0.0673	0.6314	10^{-11}
10^{-3}	3.2136×10^{-9}	0.3872	1.1134	10^{-10}

Table 1: Comparison of the values of E, E_{sup} and E^{μ} for different values of the standard deviation σ_x of the measurement noise.

duce the following quantity:

$$E^{\mu} = \frac{\|\mu^* - \mu_0\|^2}{\|\mu_0\|^2} = \frac{\sum_l \left|\mu_l^{*,L} - \mu_{0,l}^L\right|}{\sum_l \left|\mu_{0,l}^L\right|}. \qquad (69)$$

The values of E, E_{sup} and E^{μ} are given in table 1 for different values of σ_x. The value of the used conditioning parameter ε (see (61)) is also given. When $\sigma_x \to 0^+$, we clearly have $H^*(i\omega) \to H(i\omega)$ (in the accessible frequency band $[\frac{2\pi}{K\Delta t}, \frac{2\pi}{2\Delta t}]$) and $\mu^* \to \mu_0$ (when there is no measurement noise, the graphs of the identified μ^* and of μ_0 are perfectly superimposed). Note also that, as expected, the larger the measurement noise is, the greater the value of the conditioning parameter ε has to be.

5.5 General case

In general, due to approximation errors, there does not exist[5] any μ_0 in $\mathcal{E}$ such that $\mathcal{A}_u \mu_0 = x$.

So the operator $H(\partial_t)$ is now identified with a contour γ different from the one used for the simulation of the data, namely:

$$\gamma(\xi) = |\xi| \, e^{\frac{3i\pi}{4} \text{sign}(\xi)}. \qquad (70)$$

We take $L = 140$ points of ξ-discretization ξ_l geometrically spaced between $\xi_1 = 10^{-1}$ and $\xi_L = 10^3$. The output measurement noise has standard deviation $\sigma_x = 10^{-4}$ and the penalisation term is $\varepsilon = 10^{-6}$.

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[5] Of course, thanks to approximation properties, there exists some μ such that $\mathcal{A}_u \mu \simeq x$.

The Bode diagram of the identified operator and the graph of μ^* are given in figure 4. As expected, the identification is less accurate than in the previous case. The results however remain of good quality, with $H^*(i\omega)$ very close to $H(i\omega)$ (in the accessible frequency band). We have finally:

$$E = 6.0210 \times 10^{-10}, \quad E_{\text{sup}} = 0.1499. \qquad (71)$$

6 Conclusion

As a first attempt to identify non rational convolution models by use of the method presented in section 3, the numerical results obtained in section 5 can be considered as convincing, from both points of view of implementation simplicity and accuracy. So, it can be envisaged to implement this method in aim of identifying some unknown complex convolution operators involved in various physical problems. Some of these complex operators are studied in ideal cases for example in [13, 17].

Several questions must be studied in order to improve the results presented in this paper. For example, among the most significant ones, the involved hilbertian norms should be judiciously chosen and adapted to the specific properties of the class of models under consideration. Indeed, this choice is crucial in terms of sensitivity with respect to perturbations of any nature. It can also be shown that the measurement noise induces some estimation bias, which should be significantly reduced by appropriate treatments. These problems will be tackled in a further work.

References

[1] Montseny, G.: 'Représentation diffusive' (Hermès-science, 2005, 1st edn)

[2] Casenave, C., and Montseny, G.: 'Introduction to diffusive representation'. 4th IFAC Symposium on System, Structure and Control, Ancona, Italy, September 2010
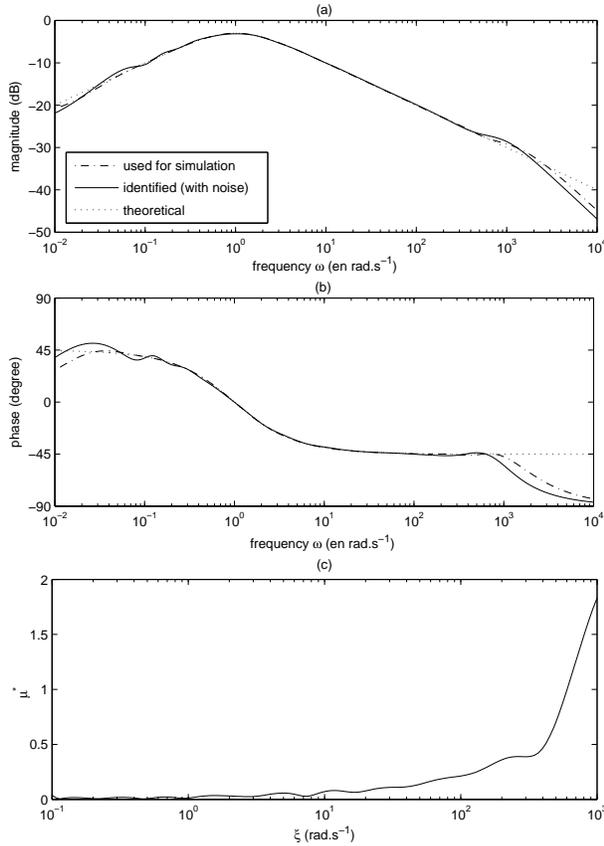
Figure 4: **Identification with measurement noise** $(\sigma_x = 10^{-4})$ **and a contour γ of sector type**: (a) and (b): Magnitude (dB) and phase (degree) of the frequency responses of $H(\partial_t) = \partial_t^{\frac{1}{2}} \circ (\partial_t + 1)^{-1}$, of its approximation under diffusive representation and of the identified operator - (c): Graph of $|\mu^*|$.

[3] Audounet, J., and Roquejoffre, J.M.: 'An asymptotic fractional differential model of spherical flame', European Series on Applied and Industrial Mathematics (ESAIM), 1998, 5, (4), pp. 15-28

[4] Carmona, Ph., and Coutin, L.: 'Fractional Brownian Motion and the Markov Property', Electronic Communications in Probability, 1998, 3, pp. 95-107

[5] Casenave, C., and Montseny, E.: 'Time-local dissipative formulation and stable numerical schemes for a class of integrodifferential wave equations', SIAM Journal on Applied Mathematics, 2008, 341, pp. 1763-1783

[6] Degerli, Y., and Lavernhe, F., and Magnan, P., and Farre, J.: 'Bandlimited $1/f^\alpha$ noise source', Electronics Letters, 1999, 35, (7), pp. 521-522

[7] Garcia, G., and Bernussou, J.: 'Identification of the dynamics of a lead acid battery by a diffusive model', European Series on Applied and Industrial Mathematics (ESAIM), 1998, 5, pp. 87-98

[8] Lenczner, M., and Montseny, G.: 'Diffusive realization of operator solutions of certain operational partial differential equations', Compte Rendus de l'Académie des Sciences - Mathématiques, 2005, 341, pp. 737-740

[9] Levadoux, D., and Montseny, G.: 'Diffusive formulation of the impedance operator on circular boundary for 2D wave equation'. The Sixth International Conference on Mathematical and Numerical Aspects of Wave Propagation, Jyväskylä, Finland, july 2003

[10] Montseny, G.: 'Diffusive representation for operators involving delays', in Loiseau, J.-J. and Chiasson, J. (Eds.): 'Lecture Notes in Control and Information Science (LNCIS): Applications of Time-delay Systems' (Springer Verlag, 2007, 1st edn), pp. 217-232

[11] Mouyon, Ph., and Imbert, N.: 'Identification of a 2D turbulent wind spectrum'. Aerospace Science and Technology, Kansas City, Missouri USA, October 2002, 6, (8) pp. 3599-605

[12] Rumeau, A., and Bidan, P., and Lebey, T., and Marchin, L., and Barbier, B., and Guillemet, S.: 'Behavior modeling of a $CaCu_3Ti_4O_{12}$ ceramic for capacitor applications'. IEEE Conference on Electrical Insulation and Dielectric Phenomena, Kansas City, Missouri USA, October 2006

[13] Casenave, C., and Montseny, G.: 'Optimal Identification of Delay-Diffusive Operators and Application to the Acoustic Impedance of Absorbent Materials', in Loiseau, J.-J., and Michiels, W. and Niculescu, S. and Sipahi, R. (Eds.): 'Lecture Notes in Control and Information Science (LNCIS): Topics in Time Delay Systems: Analysis, Algorithms and Control' (Springer Verlag, 2009, 1st edn), pp. 315

[14] Montseny, G.: 'Simple approach to approximation and dynamical realization of pseudodifferential time-operators such as fractional ones', IEEE Transactions on Circuits and System II, 51, (11), 2004, pp. 613-618

[15] Ben-Israel, A., and Greville, T.N.E.: 'Generalized inverses: Theory and applications' (Springer Verlag, 2003)

[16] Casenave, C.: 'Représentation diffusive et inversion opératorielle pour l'analyse et la résolution de problèmes dynamiques non locaux'. PhD thesis, University of Toulouse, 2009

[17] Chen, Z., and Tan, X., and Will, A., and Ziel, C.: 'A dynamic model for ionic polymer-metal composite sensors', Smart Materials and Structures, 2007, 16, pp. 1477-1488