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Transversals of Longest Paths and Cycles

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Abstract

Let G be a graph of order n . Let $\text{lpt}(G)$ be the minimum cardinality of a set X of vertices of G such that X intersects every longest path of G and define $\text{lct}(G)$ analogously for cycles instead of paths. We prove that

- $\text{lpt}(G) \leq \left\lceil \frac{n}{4} - \frac{n^{2/3}}{90} \right\rceil$, if G is connected,
- $\text{lct}(G) \leq \left\lceil \frac{n}{3} - \frac{n^{2/3}}{36} \right\rceil$, if G is 2-connected, and
- $\text{lpt}(G) \leq 3$, if G is a connected circular arc graph.

Our bound on $\text{lct}(G)$ improves an earlier result of Thomassen and our bound for circular arc graphs relates to an earlier statement of Balister *et al.* the argument of which contains a gap. Furthermore, we prove upper bounds on $\text{lpt}(G)$ for planar graphs and graphs of bounded tree-width.

Keywords: Longest path, longest cycle, transversal.

MSC2010: 05C38, 05C70

1 Introduction

It is well known that every two longest paths in a connected graph as well as every two longest cycles in a 2-connected graph intersect. While these observations are easy exercises, it is an open problem, originating from a question posed by Gallai [2], to determine the largest value of k such that for every connected graph and every k longest paths in that graph, there is a vertex that belongs to all of these k paths. The above remark along with examples constructed by Skupień [5] ensure that $2 \leq k \leq 6$.

We consider only simple, finite, and undirected graphs and use standard terminology. For a graph G , we define $\mathcal{P}(G)$ to be the collection of all longest paths of G and a *longest path transversal* of G to be a set of vertices that intersects every longest path of G . Let $\text{lpt}(G)$ be the minimum cardinality of a longest path transversal of G . We define $\mathcal{C}(G)$, a *longest cycle transversal*, and the parameter $\text{lct}(G)$ analogously for cycles instead of paths.

The intersections of longest paths and cycles have been studied in detail and Zamfirescu [8] gave a short survey. In the present paper we prove upper bounds on $\text{lpt}(G)$ and $\text{lct}(G)$. Our bound on $\text{lct}(G)$ for a 2-connected graph G improves an earlier result of Thomassen [6]. Balister *et al.* [1] showed that for every connected interval graph, there is a vertex that belongs to every longest path. Furthermore, their work [1] contains the statement that for every connected circular arc graph, there is a vertex that belongs to every longest path. Unfortunately, we believe that the argument they provide has a gap. We shall explain the approach of Balister *et al.*, the problem with their argument, and give a proof of a weaker result, specifically that every connected circular arc graph contains a longest path transversal of order at most 3.

2 Results

We start by proving a lemma that allows us to exploit the structure of some particular matchings to find long paths and cycles.

Lemma 1. *If $G = (P \cup Q) + M$ where $P : u_1 \dots u_\tau$ and $Q : v_1 \dots v_\tau$ are paths and M is a matching of edges between $V(P)$ and $V(Q)$ that has a partition $M = M_1 \cup \dots \cup M_q$ such that*

(a) $|M_i|$ is either 1 or even for $i \in [q]$ and

(b) if $u_{i_1}v_{i_2} \in M_i$ and $u_{j_1}v_{j_2} \in M_j$ for $i, j \in [q]$, then

$$(j_1 - i_1)(j_2 - i_2) \begin{cases} < 0, & \text{if } i = j \text{ and} \\ > 0, & \text{if } i \neq j, \end{cases}$$

that is, the edges in one of the sets M_i are pairwise “crossing” and the edges in distinct sets M_i are pairwise “parallel”,

then G contains a path between a vertex in $\{u_1, v_1\}$ and a vertex in $\{u_\tau, v_\tau\}$ of order at least $\tau + |M|$.

Proof. If $i_0 = 1$, $i_{|M|+1} = \tau$, and $u_{i_1}, \dots, u_{i_{|M|}}$ with $1 \leq i_1 < \dots < i_{|M|} \leq \tau$ are the vertices of P that are incident with edges in M , then a subpath of P of the form $u_{i_j} \dots u_{i_{j+1}}$ with odd/even j is called an *odd/even segment* of P , respectively. Odd/even segments of Q are defined analogously.

The odd segments of P , M , and the even segments of Q define a path P' . Similarly, the even segments of P , M , and the odd segments of Q define a path Q' . See Figure 1 for an illustration. Since $E(P') \cap E(Q') = M$ and $E(P') \cup E(Q') = E(P) \cup E(Q) \cup M$, the longer of the two paths satisfies the desired properties. \square

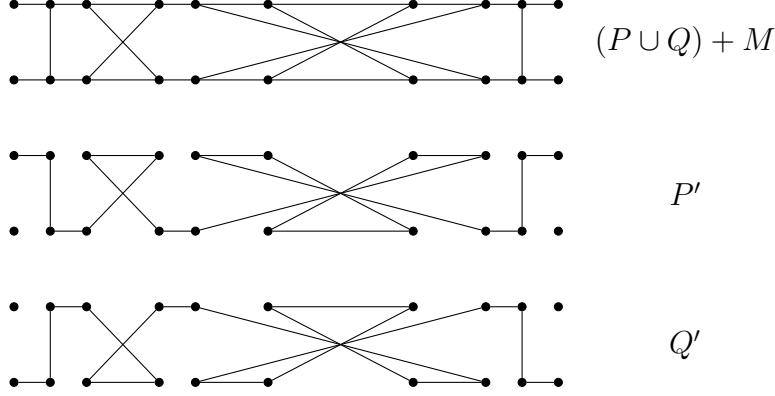


Figure 1: The two paths P' and Q' for a set M with $|M_1| = |M_4| = 1$, $|M_2| = 2$, and $|M_3| = 4$.

We proceed to our first main result. Note that in the proof of Theorem 2, as well as of Theorem 3 below, we did not try to minimize the factor of $n^{\frac{2}{3}}$. The point of these two results is that $\text{lpt}(G)$ is strictly less than $n/4$ and $\text{lct}(G)$ is strictly less than $n/3$, respectively.

Theorem 2. *If G is a connected graph of order n , then $\text{lpt}(G) \leq \left\lceil \frac{n}{4} - \frac{n^{2/3}}{90} \right\rceil$.*

Proof. Let G be a connected graph of order n . Let $\epsilon = \frac{1}{90}n^{-\frac{1}{3}}$ and $\tau = \left\lceil \left(\frac{1}{4} - \epsilon\right)n \right\rceil$. For a contradiction, we assume that $\text{lpt}(G) > \tau$. Let $P : u_1 \dots u_\ell$ be a longest path of G . Since $V(P)$ as well as every set of $n - \ell + 1$ vertices of G are longest path transversals, we obtain

$$\left(\frac{1}{4} - \epsilon\right)n \leq \tau < \ell < n - \tau + 1 \leq \left(\frac{3}{4} + \epsilon\right)n + 1. \quad (1)$$

Let $p = \left\lceil \frac{\ell - \tau}{2} \right\rceil$. Since the set $T = \{u_i : p + 1 \leq i \leq p + \tau\}$ is too small to be a longest path transversal of G , there is a path $P' : v_1 \dots v_\ell$ in $G - T$. Since G is connected, the paths P and P' intersect.

If $V(P) \cap V(P') \subseteq \{u_1, \dots, u_p\}$, then let $v_x = u_r \in V(P) \cap V(P')$ be such that r is maximum. We may assume that $x \geq \frac{\ell+1}{2}$. Now $v_1 \dots v_x u_{r+1} \dots u_\ell$ is a path of order at least $x + \ell - p \geq \frac{\ell+1}{2} + \ell - \frac{\ell-\tau+1}{2} = \ell + \frac{\tau}{2} > \ell$, which is a contradiction. Hence P and P' intersect in a vertex in $\{u_1, \dots, u_p\}$ as well as a vertex in $\{u_{p+\tau+1}, \dots, u_\ell\}$. Let $v_x = u_r$ be in $V(P') \cap \{u_1, \dots, u_p\}$ such that r is maximum and $v_y = u_s$ be in $V(P') \cap \{u_{p+\tau+1}, \dots, u_\ell\}$ such that s is minimum. We may assume that $x < y$.

Since $v_1 \dots v_x u_{r+1} \dots u_{s-1} v_y \dots v_\ell$ is a path of order at least $\ell - (y - x - 1) + \tau$, we obtain $y - x - 1 \geq \tau$. Since $u_{s-1} \dots u_{r+1} v_x \dots v_\ell$ is a path of order at least $\tau + \ell - (x - 1)$, we obtain $x - 1 \geq \tau$. Since $u_{r+1} \dots u_{s-1} v_y \dots v_1$ is a path of order at least $\tau + y$, we obtain $\ell - y \geq \tau$.

Choosing four vertex-disjoint paths $A : a_1 \dots a_\tau$, $B : b_1 \dots b_\tau$, $C : c_1 \dots c_\tau$, and $D : d_1 \dots d_\tau$ as subpaths of the four paths $P'[\{v_1, \dots, v_{x-1}\}]$, $P'[\{u_{r+1}, \dots, u_{s-1}\}]$, $P'[\{v_{x+1}, \dots, v_{y-1}\}]$, and $P'[\{v_{y+1}, \dots, v_\ell\}]$, respectively, we obtain the existence of two vertex-disjoint sets X and Y in

$V(G) \setminus (V(A) \cup V(B) \cup V(C) \cup V(D))$ with $|X \cup Y| \leq n - 4\tau \leq 4\epsilon n$ such that X contains a path between some neighbors of any two of the vertices a_1, b_1 , and c_1 , and Y contains a path between some neighbors of any two of the vertices b_τ, c_τ , and d_1 . See Figure 2 for an illustration.

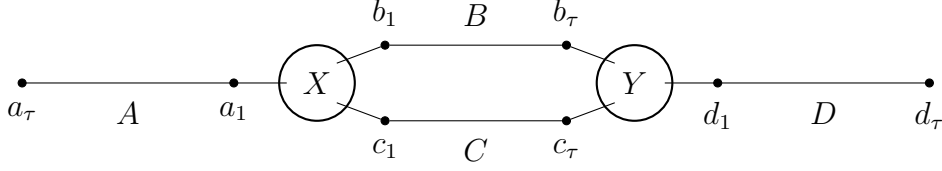


Figure 2: The paths A, B, C , and D and the two sets X and Y .

If $a_i b_j$ is an edge of G with $j \geq i$, then the path $a_\tau \dots a_i b_j \dots b_1$, a path in X between neighbors of b_1 and c_1 , the path C , a path in Y between neighbors of c_τ and d_1 , and the path D form a path of order at least $3\tau + (j - i + 1) + 2$. By (1), this implies that

$$j - i \leq \lceil 4\epsilon n \rceil - 3. \quad (2)$$

Our goal is now to prove the existence of a vertex cover $T_{A,B}$ of small order for the bipartite graph $G_{A,B}$ with bipartition $V(A)$ and $V(B)$ formed by the edges between these two sets. First, note that if $4\epsilon n \leq 2$, then (2) implies that this bipartite graph is edgeless, so it is enough to set $T_{A,B} = \emptyset$.

Assume now that $4\epsilon n > 2$. Let N be a maximum matching of $G_{A,B}$. Let $I = \left\lceil \left\lceil \frac{\tau}{2(\lceil 4\epsilon n \rceil - 3)} \right\rceil \right\rceil$. For $i \in I$, let N_i be the set of edges in N that are incident with a vertex in

$$\{a_j : 2(\lceil 4\epsilon n \rceil - 3)(i - 1) + 1 \leq j \leq \min\{\tau, 2(\lceil 4\epsilon n \rceil - 3)i\}\}.$$

By (2), if $a_{i_1} b_{i_2} \in N_i$ and $a_{j_1} b_{j_2} \in N_j$ with $j - i \geq 2$, then $(j_1 - i_1)(j_2 - i_2) > 0$, that is, the two edges are parallel in the sense of Lemma 1. Without loss of generality, we may assume that $\bigcup_{i \in I: i \text{ odd}} N_i$ contains at least half the edges of N . Since N is a matching, $|N_i| \leq 2(\lceil 4\epsilon n \rceil - 3)$ for every $i \in I$. Since permutation graphs are perfect [3, Chapter 7], each N_i contains a set of at least $\sqrt{|N_i|} \geq \frac{|N_i|}{\sqrt{2(\lceil 4\epsilon n \rceil - 3)}}$ edges that are either all pairwise parallel or all pairwise crossing in the sense of Lemma 1. This implies that N contains a subset M_0 that satisfies condition (b) from Lemma 1 with $|M_0| \geq \sum_{i \in I: i \text{ odd}} \frac{|N_i|}{\sqrt{2(\lceil 4\epsilon n \rceil - 3)}} \geq \frac{1}{2\sqrt{2(\lceil 4\epsilon n \rceil - 3)}} |N|$. By removing a set of at most $|M_0|/3$ edges from M_0 , we obtain a matching M that satisfies both conditions from Lemma 1 with $|M| \geq \frac{2}{3} |M_0| \geq \frac{1}{3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}} |N|$.

If N has order at least $(4\epsilon n - 1)3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}$, then N contains a matching M as in Lemma 1 of order at least $4\epsilon n - 1$. Thus by Lemma 1, the graph $(A \cup B) + M$ contains a path Q between $\{a_1, b_1\}$ and $\{a_\tau, b_\tau\}$ of order at least $\tau + |M|$. Now the path Q , a path in X between neighbors of a vertex in $\{a_1, b_1\}$ and c_1 , the path C , a path in Y between neighbors of c_τ and d_1 , and the path D form a path of order at least $3\tau + |M| + 2 \geq \left(\frac{3}{4} + \epsilon\right)n + 1$, which contradicts (1). Hence the bipartite graph $G_{A,B}$ has no matching of order at least $(4\epsilon n - 1)3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}$. Now König's theorem [4] implies that $G_{A,B}$ has a vertex cover $T_{A,B}$ of order less than $(4\epsilon n - 1)3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}$.

Similar arguments yield that for every two distinct paths $Q, R \in \{A, B, C, D\}$, the bipartite graph $G_{Q,R}$ with bipartition $V(Q)$ and $V(R)$ formed by the edges between these two sets has a vertex cover $T_{Q,R}$ of order less than $(4\epsilon n - 1)3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}$ if $4\epsilon n > 2$ and of order 0 otherwise. (If, for instance, $a_i d_j$ is an edge of G with $j \geq i$, then the path $a_\tau \dots a_i d_j \dots d_1$, a path in Y between neighbors of d_1 and b_τ , the path B , a path in X between neighbors of b_1 and c_1 , and the path C again form a path of order at least $3\tau + (j - i + 1) + 2$ and we can argue as above.)

Let $T' = X \cup Y \cup \bigcup_{\{Q,R\} \in \binom{\{A,B,C,D\}}{2}} T_{Q,R}$. Since every component of $G - T'$ has order at most τ , the set T' is a longest path transversal of G of order less than $4\epsilon n + 6(4\epsilon n - 1)3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}$ if $4\epsilon n > 2$ and of order at most $4\epsilon n$ otherwise. For $\epsilon = \frac{1}{90}n^{-\frac{1}{3}}$, it follows that $|T'| \leq \left(\frac{1}{4} - \epsilon\right)n$, which yields the final contradiction. \square

For the fractional version of the longest path transversal problem, a much stronger result is possible. In fact, for every connected graph G , there is a function $t: V(G) \rightarrow [0, 1]$ such that

$$\begin{aligned} \sum_{u \in V(G)} t(u) &\leq \sqrt{n} \quad \text{and} \\ \sum_{u \in V(P)} t(u) &\geq 1 \quad \text{for every } P \in \mathcal{P}(G). \end{aligned}$$

Indeed, if the largest order of the paths in $\mathcal{P}(G)$ is at most \sqrt{n} , then let t be the characteristic function of $V(P)$ for some $P \in \mathcal{P}(G)$, otherwise let t be the constant function of value $\frac{1}{\sqrt{n}}$.

Confirming a conjecture by Zamfirescu [7], Thomassen [6] proved that $\text{lct}(G) \leq \left\lceil \frac{|V(G)|}{3} \right\rceil$ for every graph G , which is best possible for the class of connected graphs in view of a disjoint union of cycles of length 3 to which bridges are added. For 2-connected graphs, though, this bound can be improved as follows.

Theorem 3. *If G is a 2-connected graph of order n , then $\text{lct}(G) \leq \left\lceil \frac{n}{3} - \frac{n^{2/3}}{36} \right\rceil$.*

Proof. Let G be a 2-connected graph of order n . Let $\epsilon = \frac{1}{36}n^{-\frac{1}{3}}$ and $\tau = \left\lceil \left(\frac{1}{3} - \epsilon\right)n \right\rceil$. For a contradiction, we assume that $\text{lct}(G) > \tau$. Let $C: u_0 \dots u_{\ell-1} u_0$ be a longest cycle of G . Since $V(C)$ as well as every set of $n - \ell + 1$ vertices of G are longest cycle transversals, we obtain

$$\left(\frac{1}{3} - \epsilon\right)n \leq \tau < \ell < n - \tau + 1 \leq \left(\frac{2}{3} + \epsilon\right)n + 1. \quad (3)$$

Since the set $T = \{u_0, \dots, u_{\tau-1}\}$ is too small to be a longest cycle transversal of G , there is a cycle $C': v_0 \dots v_{\ell-1} v_0$ in $G - T$. Since G is 2-connected, the cycles C and C' intersect in at least two vertices. We may assume that $v_0 = u_r$ is the first and $v_k = u_s$ is the last common vertex of C and C' following the path $C - T$ from u_τ to $u_{\ell-1}$, that is $r < s$.

Since $v_0 \dots v_k u_{s+1} \dots u_{\ell-1} u_0 \dots u_{r-1}$ is a cycle of length at least $k+1+\tau$, we obtain $\ell - k - 1 \geq \tau$. Since $v_{k+1} \dots v_{\ell-1} u_r \dots u_0 u_{\ell-1} \dots u_s$ is a cycle of length at least $\ell - (k-1) + \tau$, we obtain $k-1 \geq \tau$.

Choosing three vertex-disjoint paths $P: x_1 \dots x_\tau$, $Q: y_1 \dots y_\tau$, and $R: z_1 \dots z_\tau$ as subpaths of the three paths $C[T]$, $C'[\{v_1, \dots, v_{k-1}\}]$, and $C'[\{v_{k+1}, \dots, v_{\ell-1}\}]$, respectively, we obtain the existence of two vertex-disjoint sets X and Y in $V(G) \setminus (V(P) \cup V(Q) \cup V(R))$ with $|X \cup Y| \leq n - 3\tau \leq 3\epsilon n$ such that X contains a path between some neighbors of every two of

the vertices $x_1, y_1,$ and $z_1,$ and Y contains a path between some neighbors of every two of the vertices $x_\tau, y_\tau,$ and $z_\tau.$ See Figure 3 for an illustration.

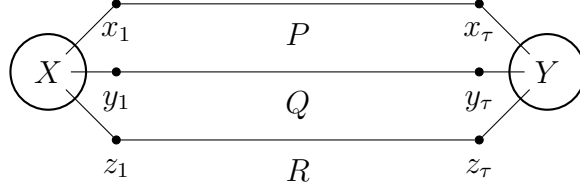


Figure 3: The paths $P, Q,$ and R and the two sets X and $Y.$

If $x_i y_j$ is an edge of G with $j \geq i,$ then the path $y_1 \dots y_j x_i \dots x_\tau,$ a path in Y between neighbors of x_τ and z_τ the path $R,$ and a path in X between neighbors of y_1 and z_1 form a cycle of length at least $2\tau + (j - i + 1) + 2.$ By (3), this implies that $j - i \leq \lceil 3\epsilon n \rceil - 3.$ Using Lemma 1 as in the proof of Theorem 2, we infer that for every two distinct $A, B \in \{P, Q, R\},$ the bipartite graph $G_{A,B}$ with bipartition $V(A)$ and $V(B)$ formed by the edges between these two sets has a vertex cover $T_{A,B}$ of order less than $(3\epsilon n - 1)3\sqrt{2(\lceil 3\epsilon n \rceil - 3)}$ if $3\epsilon n > 2$ and of order 0 otherwise. Since every component of $G - (X \cup Y \cup T_{P,Q} \cup T_{P,R} \cup T_{Q,R})$ has order at most $\tau,$ the set $T' = X \cup Y \cup T_{P,Q} \cup T_{P,R} \cup T_{Q,R}$ is a longest cycle transversal of G of order less than $3\epsilon n + 3(3\epsilon n - 1)3\sqrt{2(\lceil 3\epsilon n \rceil - 3)}$ if $3\epsilon n > 2$ and of order at most $3\epsilon n$ otherwise. For $\epsilon = \frac{1}{36}n^{-\frac{1}{3}},$ it follows that $|T'| \leq (\frac{1}{3} - \epsilon)n,$ which yields the final contradiction. \square

Since every two longest paths of a connected graph G intersect, it follows that $\text{lpt}(G) \leq \lceil \frac{|\mathcal{P}(G)|}{2} \rceil.$ Similarly, if every k longest paths of a connected graph G would intersect for some $k \geq 3,$ then it would follow that $\text{lpt}(G) \leq \lceil \frac{|\mathcal{P}(G)|}{k} \rceil.$ The next result shows how to decrease the multiplicative constant $1/2$ in the former bound at the cost of adding a square-root proportion of the total number of vertices in the graph.

Proposition 4. *If G is a connected graph and $\alpha \geq 2,$ then*

$$\text{lpt}(G) \leq \frac{|\mathcal{P}(G)|}{\alpha} + \sqrt{\alpha |V(G)|}.$$

Proof. We proceed by induction on the order n of $G,$ the statement being true if $n = 1.$ Let $n \geq 2$ and assume that the statement holds for all connected graphs of order less than $n.$ Let G be a connected graph of order n and let ℓ be the order of the longest paths in $G.$

We may assume that $|\mathcal{P}(G)| > \sqrt{\alpha n}$ since otherwise we obtain a longest path transversal of the desired size by picking one vertex in each longest path of $G.$ Next, since the vertex set of a longest path in G is a longest path transversal, we may also assume that $\ell > \sqrt{\alpha n}.$

For a vertex $v \in V(G),$ let p_v be the number of paths in $\mathcal{P}(G)$ that contain $v.$ We may assume that $p_v < \alpha$ for every vertex $v \in V(G).$ Indeed, suppose that v is a vertex such that $p_v \geq \alpha.$ In particular, $p_v \geq 1.$ If the set $\{v\}$ is a longest path transversal of $G,$ then G satisfies the desired property. Otherwise let $G' = G - v$ and note that G' contains a path of order $\ell.$ Furthermore, $|\mathcal{P}(G')| = |\mathcal{P}(G)| - p_v \leq |\mathcal{P}(G)| - \alpha.$ Note that all paths of order ℓ in G' must belong to the same component of $G',$ since every two longest paths intersect. Let C be this component; thus $\mathcal{P}(C) = \mathcal{P}(G').$ The induction hypothesis applied to C yields that $\text{lpt}(G') \leq \frac{|\mathcal{P}(G')|}{\alpha} - 1 + \sqrt{\alpha(n-1)}.$ As $\text{lpt}(G) \leq \text{lpt}(G') + 1,$ we deduce that $\text{lpt}(G) \leq \frac{|\mathcal{P}(G)|}{\alpha} + \sqrt{\alpha n}.$

We now consider the number N of pairs (v, P) such that $P \in \mathcal{P}(G)$ and $v \in V(P)$. On the one hand, since $N = \sum_{v \in V(G)} p_v$, we deduce from the previous observations that $N < \alpha n$. On the other hand, since $N = \sum_{P \in \mathcal{P}(G)} |V(P)| = \ell |\mathcal{P}(G)|$, the previous observations also imply that $N > \alpha n$. This contradiction concludes the proof. \square

The minimum sizes of transversals of longest paths can be bounded in classes of graphs with small separators, such as planar graphs and graphs of bounded tree-width. As before, no effort is made to minimize the constant multiplicative factors appearing in the next two results.

Proposition 5. *If G is a connected planar graph of order at least 2, then*

$$\text{lpt}(G) \leq 9\sqrt{|V(G)|} \log |V(G)|.$$

Proof. We proceed by induction on the order n of G , the result being true if $n = 2$. Let $n \geq 3$ and assume that the statement holds for all connected planar graphs of order at least 2 and less than n . Let G be a connected planar graph of order n and let ℓ be the order of the longest paths in G . In particular, $\ell \geq 2$. Since G is planar, the separator theorem of Lipton and Tarjan ensures that G contains a set X of order at most $2\sqrt{2}\sqrt{n}$ such that every component of $G - X$ has order at most $2n/3$.

If X is a longest path transversal of G , then G satisfies the desired property. Otherwise, $G - X$ contains a path of order ℓ . Note that, since every two longest paths of G intersect, all paths of order ℓ in $G - X$ must be contained in the same component of $G - X$, which we call C . Moreover, the order of C is at most $2n/3$ and at least ℓ , so the induction hypothesis implies that C has a longest path transversal X' of order at most $9\sqrt{2n/3} \log(2n/3)$. Therefore, $X \cup X'$ is a longest path transversal of G of order at most

$$\begin{aligned} 2\sqrt{2}\sqrt{n} + 9\sqrt{2n/3} \log(2n/3) &= 9\sqrt{2n/3} \log n + \sqrt{n} \cdot \left(2\sqrt{2} - 9\sqrt{2/3} \log(3/2)\right) \\ &\leq 9\sqrt{n} \log n \end{aligned}$$

since $9\sqrt{2/3} \log(3/2) > 2\sqrt{2}$. This concludes the proof. \square

An analogous statement is true for graphs of bounded tree-width. Indeed, if G is a graph with tree-width at most k , then there is a set X of vertices of G of order at most $k + 1$ such that every component of $G - X$ has order at most $|V(G)|/2$. Consequently, an inductive reasoning similar to that made in the proof of Proposition 5 yields the following statement.

Proposition 6. *If G is a connected graph of tree-width at most k and order at least 2, then*

$$\text{lpt}(G) \leq 3k \log |V(G)|.$$

We proceed to circular arc graphs. We explain the approach of Balister *et al.* [1], the problem with their argument, and prove the following weaker result.

Theorem 7. *Let G be a circular-arc graph.*

If G is connected, then $\text{lpt}(G) \leq 3$, and if G is 2-connected, then $\text{lct}(G) \leq 3$.

Let G be a connected circular arc graph. Let C be a circle and let \mathcal{F} be a collection of open arcs of C such that G is the intersection graph of \mathcal{F} . In view of the result for interval graphs mentioned in the introduction, we may assume that $C \subseteq \bigcup_{A \in \mathcal{F}} A$. Furthermore, we may assume that all endpoints of arcs in \mathcal{F} are distinct.

Balister *et al.* [1] consider a collection $\mathcal{K} = \{K_0, \dots, K_{n-1}\}$ of arcs in \mathcal{F} such that

- (0) $C \subseteq \bigcup_{A \in \mathcal{K}} A$,
- (1) n is minimal, and
- (2) each K_i is maximal, that is, no arc in \mathcal{F} properly contains an arc in \mathcal{K} .

They may assume that $n \geq 2$, because otherwise, G has a universal vertex that belongs to every longest path or cycle. We consider the indices of the arcs in \mathcal{K} as elements of \mathbf{Z}_n , that is, modulo n .

A chain of order ℓ in \mathcal{F} is a sequence $\mathcal{P} : A_1 \dots A_\ell$ of distinct arcs in \mathcal{F} such that $A_i \cap A_{i+1} \neq \emptyset$ for $i \in [\ell - 1]$. The chain \mathcal{P} is *closed*, if $A_\ell \cap A_1 \neq \emptyset$. Thus chains and closed chains in \mathcal{F} correspond to paths and cycles in G . For a chain $\mathcal{P} : A_1 \dots A_\ell$ in \mathcal{F} , let $\mathcal{K}(\mathcal{P}) = \{A_1, \dots, A_\ell\} \cap \mathcal{K}$.

If $\mathcal{P} : A_1 \dots A_\ell$ is a chain in \mathcal{F} of largest order, then Balister *et al.* [1, Lemma 3.1] proved that $\mathcal{K}(\mathcal{P})$ is of the form $\{K_i : i \in I\}$ for some contiguous and non-empty subset I of \mathbf{Z}_n . Their argument actually yields the same statement for closed chains, that is, if \mathcal{C} is a closed chain in \mathcal{F} of largest order, then $\mathcal{K}(\mathcal{C})$ is of the form $\{K_i : i \in J\}$ for some contiguous and non-empty subset J of \mathbf{Z}_n .

In the proof of their main result [1, Theorem 3.3] — stating that $\text{lpt}(G) = 1$ — Balister *et al.* choose a chain \mathcal{P} in \mathcal{F} of largest order such that $\mathcal{K}(\mathcal{P})$ has minimum order. They let $\mathcal{K}(\mathcal{P})$ be $\{K_{a+1}, \dots, K_{b-1}\}$ and assert that K_{b-1} belongs to $\mathcal{K}(\mathcal{Q})$ for every chain \mathcal{Q} in \mathcal{F} of largest order, that is, the vertex of G corresponding to the arc K_{b-1} would belong to every longest path of G .

For a contradiction, they consider a chain \mathcal{Q} in \mathcal{F} of largest order such that $K_{b-1} \notin \mathcal{K}(\mathcal{Q})$. They set $\mathcal{K}(\mathcal{Q}) = \{K_{\ell+1}, \dots, K_{m-1}\}$. They deduce from the choice of \mathcal{P} that $K_{\ell+1} \in \mathcal{K}(\mathcal{Q}) \setminus \mathcal{K}(\mathcal{P})$ since $K_{b-1} \in \mathcal{K}(\mathcal{P}) \setminus \mathcal{K}(\mathcal{Q})$. Using their Lemma 3.2 [1], they reorder the arcs in the chains \mathcal{P} and \mathcal{Q} and obtain chains \mathcal{P}^* and \mathcal{Q}^* containing the same arcs as \mathcal{P} and \mathcal{Q} in a possibly different order, respectively. They split these chains at K_{b-1} and $K_{\ell+1}$ writing them as $\mathcal{P}^* : \mathcal{P}_1 K_{b-1} \mathcal{P}_2$ and $\mathcal{Q}^* : \mathcal{Q}_1 K_{\ell+1} \mathcal{P}_2$, respectively.

Their core statement is that $\mathcal{C}_1 : \mathcal{P}_1 K_{b-1} \mathcal{R} K_{\ell+1} \mathcal{Q}_1^r$ and $\mathcal{C}_2 : \mathcal{P}_2^r K_{b-1} \mathcal{R} K_{\ell+1} \mathcal{Q}_2$ are chains that satisfy the inequality $|\mathcal{C}_1| + |\mathcal{C}_2| \geq 2 + |\mathcal{P}| + |\mathcal{Q}|$, where \mathcal{R} is the possibly empty chain $K_b \dots K_\ell$ and the exponent “ r ” means reversal. In order to prove this statement, they have to show that no arc appears twice in these sequences. They give details only for \mathcal{C}_1 . Their argument that \mathcal{C}_1 is a chain heavily relies on the properties of the reordered chains \mathcal{P}^* and \mathcal{Q}^* guaranteed by their Lemma 3.2. In the proof of Lemma 3.2 these properties are established by iteratively shifting within \mathcal{P} the arc K_{b-1} to the beginning of \mathcal{P} and, similarly, by iteratively shifting within \mathcal{Q} the arc $K_{\ell+1}$ to the beginning of \mathcal{Q} . After proving that \mathcal{C}_1 is indeed a chain, they say that the same type of argument shows that \mathcal{C}_2 is a chain as well.

This is the gap in their argument.

In order to use the same type of argument for \mathcal{C}_2 , they would need reversed versions of the properties guaranteed by Lemma 3.2, that is, in order to establish these properties they would have to iteratively shift within \mathcal{P} the arc K_{b-1} to the end of \mathcal{P} and, similarly, to iteratively shift within \mathcal{Q} the arc $K_{\ell+1}$ to the end of \mathcal{Q} . This may easily result in reorderings that are distinct

from \mathcal{P}^* and \mathcal{Q}^* . In view of this asymmetry, the suitably adapted chain \mathcal{C}_2 , which would use the different reorderings of \mathcal{P} and \mathcal{Q} , need not satisfy the crucial inequality $|\mathcal{C}_1| + |\mathcal{C}_2| \geq 2 + |\mathcal{P}| + |\mathcal{Q}|$ and the argument breaks down.

We proceed to the proof of our Theorem 7.

Proof of Theorem 7. Let G be a connected circular arc graph. We choose C , \mathcal{F} , and \mathcal{K} exactly as above and we start by proving the following statement.

Assertion. *If \mathcal{P} and \mathcal{Q} are chains of largest order in \mathcal{F} such that*

$$\begin{aligned}\mathcal{K}(\mathcal{P}) &= \{K_{a+1}, \dots, K_{b-1}\} = \{K_i : i \in I(\mathcal{P})\} \quad \text{and} \\ \mathcal{K}(\mathcal{Q}) &= \{K_{\ell+1}, \dots, K_{m-1}\} = \{K_i : i \in I(\mathcal{Q})\}\end{aligned}$$

are disjoint, then $a = m - 1$ or $b = \ell + 1$, that is, the subsets $I(\mathcal{P})$ and $I(\mathcal{Q})$ of \mathcal{Z}_n are contiguous.

To establish this assertion, assume on the contrary that $a \neq m - 1$ and $b \neq \ell + 1$. Select a set $S(\mathcal{P})$ of points of C such that $S(\mathcal{P})$ contains a point in the intersection of every two consecutive arcs of \mathcal{P} . Define $S(\mathcal{Q})$ analogously. If K_a or K_b would intersect $S(\mathcal{P})$ or $S(\mathcal{Q})$, then K_a or K_b could be inserted into \mathcal{P} or \mathcal{Q} , respectively, contradicting the assumption that these chains are of largest order. If $S(\mathcal{P})$ or $S(\mathcal{Q})$ would intersect both arcs of $C \setminus (K_a \cup K_b)$, then some arc of \mathcal{P} or \mathcal{Q} would properly contain K_a or K_b , which yields a contradiction to the condition (2) in the choice of \mathcal{K} . Since $\mathcal{K}(\mathcal{P})$ and $\mathcal{K}(\mathcal{Q})$ are disjoint, the sets $S(\mathcal{P})$ and $S(\mathcal{Q})$ are contained in different of the two arcs of $C \setminus (K_a \cup K_b)$. Since G is connected, \mathcal{P} and \mathcal{Q} have a common arc A . This arc A intersects $S(\mathcal{P})$ as well as $S(\mathcal{Q})$, that is, it intersects both arcs of $C \setminus (K_a \cup K_b)$. Hence either K_a or K_b is properly contained in A , which again yields a contradiction to the condition (2) in the choice of \mathcal{K} . This concludes the proof of the assertion.

Again let \mathcal{P} be a chain in \mathcal{F} of largest order and, subject to this, such that $\mathcal{K}(\mathcal{P})$ has minimum order. Let $\mathcal{K}(\mathcal{P}) = \{K_{a+1}, \dots, K_{b-1}\}$.

In view of the desired statement, we may assume that \mathcal{F} contains a chain \mathcal{Q} of largest order such that $K_{a+1}, K_{b-1} \notin \mathcal{K}(\mathcal{Q})$. Among all such chains, we assume that \mathcal{Q} is chosen such that $\mathcal{K}(\mathcal{Q})$ has minimum order. Let $\mathcal{K}(\mathcal{Q}) = \{K_{\ell+1}, \dots, K_{m-1}\}$. By the choice of \mathcal{P} , the sets $\mathcal{K}(\mathcal{P})$ and $\mathcal{K}(\mathcal{Q})$ are disjoint. By the assertion, we may assume that $b = \ell + 1$.

In view of the desired statement, we may assume that \mathcal{F} contains a chain \mathcal{R} of largest order such that $K_{a+1}, K_{b-1}, K_{m-1} \notin \mathcal{K}(\mathcal{R})$. Among all such chains, we assume that \mathcal{R} is chosen such that $\mathcal{K}(\mathcal{R})$ has minimum order. Let $\mathcal{K}(\mathcal{R}) = \{K_{p+1}, \dots, K_{q-1}\}$. By the choice of \mathcal{P} and \mathcal{Q} , the sets $\mathcal{K}(\mathcal{P}) \cup \mathcal{K}(\mathcal{Q})$ and $\mathcal{K}(\mathcal{R})$ are disjoint. Applying the assertion to \mathcal{P} and \mathcal{R} as well as to \mathcal{Q} and \mathcal{R} , we obtain $p = m - 1$ and $q = a + 1$, that is, $\mathcal{K}(\mathcal{P}) \cup \mathcal{K}(\mathcal{Q}) \cup \mathcal{K}(\mathcal{R})$ is a partition of \mathcal{K} .

In view of the desired statement, we may assume that \mathcal{F} contains a chain \mathcal{S} of largest order such that $K_{a+1}, K_{\ell+1}, K_{p+1} \notin \mathcal{K}(\mathcal{S})$. We deduce from the choice of \mathcal{P} that $\mathcal{K}(\mathcal{S})$ has at least as many elements as $\mathcal{K}(\mathcal{P})$. This implies that $\mathcal{K}(\mathcal{S})$ is disjoint from $\mathcal{K}(\mathcal{P})$. Now, by the choice of \mathcal{Q} , this implies that $\mathcal{K}(\mathcal{S})$ has at least as many elements as $\mathcal{K}(\mathcal{Q})$. This in turn implies that the set $\mathcal{K}(\mathcal{S})$ is disjoint from $\mathcal{K}(\mathcal{P}) \cup \mathcal{K}(\mathcal{Q})$. Finally, by the choice of \mathcal{R} , this implies that $\mathcal{K}(\mathcal{S})$ has at least as many elements as $\mathcal{K}(\mathcal{R})$. This in turn implies that the set $\mathcal{K}(\mathcal{S})$ equals $\mathcal{K}(\mathcal{R})$, that is, $\mathcal{K}(\mathcal{S})$ contains K_{p+1} , which is a contradiction.

This completes the proof that $\text{lpt}(G)$ is at most 3.

From now on we assume that G is a 2-connected circular arc graph, that is, every two longest cycles in G — closed chains of largest order in \mathcal{F} — intersect. It is straightforward to see that the assertion also applies to closed chains instead of chains. Arguing exactly as above for closed chains in \mathcal{F} instead of chains in \mathcal{F} implies that $\text{lct}(G) \leq 3$. \square

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