ANOTHER LOOK AT SOBOLEV SPACES

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Dedicated to Alain Bensoussan with esteem and affection

1. Introduction

Our initial concern was to study the limiting behavior of the norms of fractional Sobolev spaces $W^{s,p}, 0 < s < 1$, $1 < p < \infty$ as $s \to 1$. Recall that a commonly used (semi) norm on $W^{s,p}$ is given by

$$\|f\|^p_{W^{s,p}} = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+sp}} dxdy$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ (see e.g. Adams[1]). A well-known "defect" of this scale of norms is that $\|f\|_{W^{s,p}}$ does not converge, as $s \nearrow 1$, to $\|f\|_{W^{1,p}}$, given by the (semi) norm

$$\|f\|^p_{W^{1,p}} = \int_\Omega |\nabla f|^p dx,$$

where $|\ \ |$ denotes the euclidean norm.

In fact, it is clear that if f is any smooth nonconstant function, then $\|f\|_{W^{s,p}} \to \infty$ as $s \nearrow 1$. The factor$(1-s)^{1/p}$ in front of $\|f\|W^{s,p}$ "rectifies" the situation (see Corollary 2 and Remark 5). This analysis has led us to a new characterization of the Sobolev space $W^{1,p}, 1 < p < \infty$.

The first observation is

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Theorem 1. *Assume $f \in W^{1,p}(\Omega)$, $1 \le p < \infty$ and let $\rho \in L^1(\mathbb{R}^n)$, $\rho \ge 0$. Then*

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho(x-y)\,dx\,dy \le C \|f\|^p_{W^{1,p}} \|\rho\|_{L^1}$$

where C depends only on p and Ω.

Next, we take a sequence (ρ_n) of radial mollifiers, i.e.

$$\rho_n(x) = \rho_n(|x|)$$

$$\rho_n \ge 0, \quad \int \rho_n(x)\,dx = 1$$

$$\lim_{n\to\infty} \int_\delta^\infty \rho_n(r) r^{n-1}\,dr = 0 \quad \text{for every } \delta > 0.$$

Theorem 2. *Assume $f \in L^p(\Omega), 1 < p < \infty$. Then*

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_n(x-y)\,dx\,dy = K_{p,N} \|f\|^p_{W^{1,p}},$$

with the convention that $\|f\|_{W^{1,p}} = \infty$ if $f \notin W^{1,p}$. Here $K_{p,N}$ depends only on p and N.

When $p = 1$ we have the following variants

Theorem 3. *Assume $f \in W^{1,1}$. Then*

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|} \rho_n(x-y)\,dx\,dy = K_{1,N} \|f\|_{W^{1,1}}$$

where $K_{1,N}$ depends only on N.

Theorem 3'. *Assume $f \in L^1(\Omega)$. Then $f \in BV(\Omega)$ if and only if*

$$\liminf_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|} \rho_n(x-y)\,dx\,dy < \infty,$$

and then

$$C_1\|f\|_{BV} \le \liminf_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy$$

$$\le \limsup_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy \le C_2\|f\|_{BV}. \tag{1}$$

Here C_1 and C_2 depend only on Ω, and

$$\|f\|_{BV} = \int_\Omega |\nabla f| = Sup\left\{\int_\Omega f div\varphi \quad \varphi \in C_0^\infty(\Omega;\mathbb{R}^N), |\varphi(x)| \le 1 \text{ on } \Omega\right\}.$$

Remark 1. In dimension $N = 1$ we can prove that for every $f \in BV$

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy = K_{1,1}\int_\Omega |\nabla f|,$$

where $K_{1,1}$ is the same constant as in Theorem 3. We do not know whether the same conclusion holds when $N \ge 2$ (even for a special sequence of mollifiers).

Here are some simple consequences of the above results (and their proofs), where K denotes various constants depending only on p and N.

Corollary 1. *Assume $f \in W^{1,p}(\Omega)$ with $1 \le p < \infty$. Then*

$$\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)\, dy \longrightarrow K_{p,N} \; |\nabla f(x)|^p \text{ in } L^1(\Omega).$$

Corollary 2. *Assume $f \in L^p, 1 < p < \infty$. Then*

$$\lim_{\varepsilon\to 0} \; \varepsilon \, \|f\|^p_{W^{1-\varepsilon,p}} = K \; \|f\|^p_{W^{1,p}}.$$

Corollary 3. *Assume $f \in L^p, 1 < p < \infty$. Then*

$$\lim_{\varepsilon\to 0} \; \varepsilon^N \iint_{|x-y|<\varepsilon} \frac{|f(x)-f(y)|^p}{|x-y|^p} \; dxdy = K\|f\|^p_{W^{1,p}}.$$

Corollary 4. *Assume $f \in L^p, 1 < p < \infty$. Then*

$$\lim_{\varepsilon\to 0} \frac{1}{|log\varepsilon|} \iint\limits_{|x-y|>\varepsilon} \frac{|f(x)-f(y)|^p}{|x-y|^{N+p}} dxdy = K\|f\|^p_{W^{1,p}}.$$

Remark 2. P. Mironescu and I. Shafrir [] have studied related limits, e.g., when $N = 1$ and $f \in BV(0,1)$

$$\lim_{\varepsilon\to 0} \frac{1}{|log\varepsilon|} \iint\limits_{|x-y|>\varepsilon} \frac{|f(x)-f(y)|^2}{|x-y|^2} dxdy.$$

The case where $f \in BV(\Omega)$ is not fully satisfactory; we have only partial results, for example

Corollary 5. *Assume $f \in L^1$. Then*

$$\begin{aligned} C_1\|f\|_{BV} &\le \liminf_{\varepsilon\to 0} \varepsilon \int\limits_\Omega \int\limits_\Omega \frac{|f(x)-f(y)|}{|x-y|^{N+1-\varepsilon}} dxdy \\ &\le \limsup_{\varepsilon\to 0} \varepsilon \int\limits_\Omega \int\limits_\Omega \frac{|f(x)-f(y)|}{|x-y|^{N+1-\varepsilon}} dxdy \le C_2\|f\|_{BV}. \end{aligned}$$

Remark 3. In particular when $f = \chi_A$ is the characteristic function of a measurable set A having finite perimeter, then

$$\|\chi_A\|_{BV} \le C \liminf_{\varepsilon\to 0} \varepsilon \int\limits_{\Omega\setminus A} \int\limits_A \frac{dxdy}{|x-y|^{N+1-\varepsilon}}$$

and in view of the isoperimetric inequality

$$\big(|A||\Omega \setminus A|\big)^{(N-1)/2N} \le C \liminf_{\varepsilon\to 0} \varepsilon \int\limits_{\Omega\setminus A} \int\limits_A \frac{dxdy}{|x-y|^{N+1-\varepsilon}}$$

If A is a measurable subset of $\Omega \subset \mathbb{R}^N, N \ge 1$, such that

$$\int\limits_{\Omega\setminus A} \int\limits_A \frac{dxdy}{|x-y|^{N+1}} < \infty,$$

then either $|A| = 0$ or $|\Omega \setminus A| = 0$. This fact was already established in Bourgain, Brezis and Mironescu [1] (Appendix B) with a different proof (see also Bourgain, Brezis and Mironescu [2] and Brezis []).

2. Proofs

Proof of Theorem 1. By standard extension we may always assume that $f \in W^{1,p}(\mathbb{R}^N)$ and then there is some constant $C = C(p, N)$ such that

$$(2) \qquad \Big(\int\limits_{\mathbb{R}^N} |f(x+h) - f(x)|^p dx \Big)^{1/p} \le |h| \|f\|_{W^{1,p}(\mathbb{R}^N)} \le C|h| \|f\|_{W^{1,p}(\Omega)},$$

for all $f \in W^{1.p}$ and $h \in \mathbb{R}^N$ (see, e.g., Brezis [], Proposition IX.3). By (2), we obtain

$$\int\limits_{\Omega}\int\limits_{\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho(x-y) dx dy \le \int\limits_{\mathbb{R}^N} \frac{1}{|h|^p} \int\limits_{\mathbb{R}^N} |f(x+h) - f(x)| dx dh^p$$

$$\le C^p \|f\|^p_{W^{1,p}} \int\limits_{\mathbb{R}^n} \rho(h) dh = C^p \|f\|^p_{W^{1,p}} \|\rho\|^1_L.$$

Proof of Theorem 2. For $f \in L^p$, let

$$F_n(x,y) \;=\; \frac{|f(x) - f(y)|}{|x-y|} \rho_n^{1/p}(x-y).$$

Assuming first that $f \in W^{1,p}$, we have to prove that

$$(3) \qquad \lim_{n\to\infty} \;\; \|F_n\|^p_{L^p} \;=\; K\|f\|^p_{W^{1,p}},$$

for some $K = K_{p,N}$.

By Theorem 1, we have, for any n and $f, g \in W^{1,p}$,

$$(4) \qquad \|F_n - G_n\|_{L^p} \le C\|f - g\|_{W^{1,p}},$$

for some constant C independent of n, f and g. Therefore it suffices to establish (3) for f in some dense subset of $W^{1,p}$, e.g., for $f \in C^2(\bar{\Omega})$.

Fix some $f \in C^2(\bar{\Omega})$. Then

$$\frac{|f(x) - f(y)|}{|x-y|} = |(\nabla f)(x) \cdot \frac{x-y}{|x-y|}| + 0(|x-y|).$$

For each fixed $x \in \Omega$,

$$\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dy =$$

(5)
$$\int_{|x-y|<dist(x,\partial\Omega)} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dy + \int_{|x-y|\geq dist(x,\partial\Omega)} \frac{|f(x)-(y)|^p}{|x-y|^p}\rho_n(x-y)dy.$$

Clearly, the last integral in (5) tends to 0 as $n \to \infty$. On the other hand, with $R = dist(x, \partial\Omega)$, we have

$$\int_{|y-x|<R} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dy$$
$$= \int_0^R \rho_n(r) \int_{|y-x|=r} \Big(\Big|(\nabla f)(x).\frac{x-y}{|x-y|}\Big|^p + (|x-y|)^p\Big)ds_y dr$$
$$= \int_0^R \rho_n(r) \int_{|\omega|=r} (|(\nabla f)(x)\cdot \frac{\omega}{|\omega|}|^p + 0(r^p))ds_w dr$$
$$= \int_0^R |S^{N-1}|K|\nabla f(x)|^p r^{N-1}\rho_n(r)dr + 0(\int_0^R r^{n+p-1}\rho_n(r)dr),$$

where $K = K_{p,n} =$

$$\int_{\omega\in S^{n-1}} |\omega_N|^p ds_\omega / |S^{n-1}| = \!\!\!\!\!\!-\!\!\!\!\!\! \int_{\omega \in S^{N-1}} |\omega_N|^p ds_\omega$$

Therefore,

(6)
$$\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dy \longrightarrow K|\nabla f(x)|^p, \quad \forall x \in \Omega.$$

If L is such that $|f(x)-f(y)| \leq L|x-y|, \ \forall x, y \in \Omega$, then

(7)
$$\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dy \leq L^p, \quad \forall x \in \Omega.$$

Hence (3) for $f \in C^2(\bar{\Omega})$ follows by dominated convergence from (6) and (7).

In order to complete the proof of Theorem 2, to suffices to prove that, if $f \in L^p$ and

$$\liminf_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dxdy < \infty,$$

then $f \in W^{1,p}$.

Recall that, for some $C_1 > 0$,

$$\|f\|_{W^{1,p}} \le C_1 sup\{\int_\Omega f\partial_i\varphi \, ; \varphi \in C_0^\infty(\Omega), \|\varphi\|_{L^{p1}} \le 1, i = 1, ..., N\}.$$

Fix some $\varphi \in C_0^\infty(\Omega)$ and some $i \in \{1, ..., n\}$ and consider the functions

$$x \overset{H_n}{\longmapsto} \int_{(y-x)\cdot e_i \ge 0} \frac{f(x)-f(y)}{|x-y|}\rho_n(x-y)\varphi(y)dy.$$

On the one hand, we have

$$\int_\Omega |H_n(x)|dx \le$$

$$\Big(\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dxdy\Big)^{1/p} \quad \Big(\int_\Omega \int_\Omega \rho_n(x-y)|\varphi(x-y)|^{p'}dxdy\Big)^{1/p}$$

$$= \|\varphi\|_{L^{p'}}\Big(\int\Omega \int \Omega \frac{f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-)dxdy\Big)^{1/p}. \tag{8}$$

As above,

$$\begin{aligned} H_n(x) \overset{n\to\infty}{\longmapsto} f(x) ⨍(\nabla\varphi)(x)\cdot\omega ds_\omega &= \{\omega \in S^{N-1}, \cdot\omega_i \ge 0\} f(x)(\nabla\varphi)(x)\cdot ⨍ \omega ds_\omega \\ &= C_2 f(x)(\nabla\varphi)\cdot e_i \\ &= C_2 f(x)\partial_i\varphi(x), \{w \in \S^{N-1}; w_i \ge 0\} \end{aligned}$$

for some $C_2 > 0$ depending only on N.

Therefore, by combining (8) and(9) we find

$$|\int_\Omega f(x)\partial_i\varphi(x)dx| \le C_3 \liminf_{n\to\infty}\Big(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x-y)dxdy\Big)^{1/p}\|\varphi\|_{L^p},$$

for any $\varphi \in C_0^\infty(\Omega)$.

The proof of Theorem 2 is complete.

Proof of Corollary 1. The conclusion is clear when $f \in C^2(\bar{\Omega})$. For a general $\in W^{1,p}$, the statement follows by density using (4).

The proof of Theorem 3 is the same as the first part of the proof of Theorem 2, since smooth functions are dense in $W^{1,1}$.

Proof of Theorem 3. The last inequality in (1) is proved as in Theorem 1. The first inequality in (1) is proved as in the second part of the proof of Theorem 2 (using duality).

Finally, we return to Remark 1 and prove that , for $f \in L^1((0,1))$,

$$\lim_{n\to\infty} \int_0^1\int_0^1 \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy = \|f\|_{BV} \tag{10}$$

Clearly, if $f \in BV((0,1))$, then

$$\int |f(x+h)-f(x)|dx \leq |h|\|f\|_{BV},$$

and therefore, as in the proof of Theorem 1,

$$\limsup_{n\to\infty} \int_0^1\int_0^1 \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy \leq \|f\|_{BV}. \tag{11}$$

Assume now that $f \in L^1$ is such that

$$\liminf_{n\to\infty} \int_0^1\int_0^1 \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy < \infty$$

Then, for any fixed $\varphi \in C_0^\infty((0,1))$ with $|\varphi| \leq 1$, we have

$$\int_0^1\int_0^1 \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy \leq \int_0^1\int_0^1 \frac{f(x)-f(y)}{y-x}\rho_n(x-y)\varphi(y)dy =$$

$$\int_0^1\int_0^1 f(x)\rho_n(x-y)\frac{\varphi(y)-\varphi(x)}{y-x}dydx. \tag{12}$$

As in the proof of Theorem 3, we find that

$$\int_0^1 f(x)\varphi'(x)dx \leq \liminf_{n\to\infty} \int_0^1\int_0^1 \frac{|f(x)-f(y)|}{|x-y|}\rho_n(x-y)dxdy, \quad \forall\varphi \in C_0^\infty(\Omega) \text{ with} |\varphi| \leq 1. \tag{13}$$

Equality (10) follows from (11) and (13).

3. The case of a sequence (f_n)

In the previous sections, f was a fixed function. Throughout this section, we assume that (f_n) is a sequence of L^p functions satisying the uniform estimate

$$\int_\Omega\int_\Omega \frac{|f_n(x)-f_n(y)|^p}{|x-y|^p}\rho_n(x-y)dxdy \leq C_0, \tag{14}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N, 1 \leq p < \infty$, and (ρ_n) is a sequence of radial mollifiers. Without loss of generality, we may also assume the normalization condition

$$\int_\Omega f_n(x)dx = 0, \quad \forall n. \tag{15}$$

Theorem 4. *Assume (14), (15) and*

$$\text{For each } n, \text{the function } t \in (0,\infty) \mapsto \rho_n(t) \text{is non-increasing.} \tag{16}$$

Then the sequence (f_n) is relatively compact in L^p and (up to a subsequence) we may assume that $f_n \to f$ in L^p. Moreover,

a) if $1 < p < \infty$, then $f \in W^{1,p}$ and $\|f\|^p_{W^{1,p}} \leq C(p,\Omega)C_0$;

b) if $p = 1$, then $f \in BV$ and $\|f\|_{BV} \leq C(\Omega)C_0$.

Remark 4. In view of Theorems 2 and 3, the additional assumption (16) may seem artificial. Actually, it is possible to slightly weaken (16); for example we may assume

$$\rho_n(t) \geq C_1 \rho_n(s), \quad \forall n, \forall t \leq s, \tag{17}$$

Form some C_1 independent of n, t, s.

However, the conclusions of Theorem 4 fail for bf general $\rho_n' s$. We shall give below a counterexample where the sequence (f_n) need not be relatively compact in L^p (Counterexample 2).

Here are two examples of interest

Corollary 6. *For $1 \leq p < \infty$, let (f_ε) be a family of L^p functions such that*

$$\iint\limits_{|x-y|<\varepsilon} \frac{|f_\varepsilon(x) - f_\varepsilon(y)|^p}{|x-y|^p} dxdy \leq C_0 \varepsilon^N.$$

Then, up to a subsequence, $(f\varepsilon)$ converges in L^p to some $f \in W^{1,p}($ for$1 < p < \infty)$ or $f \in BV($ for$p = 1)$.

Corollary 7. *For$1 < p < \infty$, let $f_\varepsilon \in W^{1-\varepsilon,p}$. Assume that*

$$\varepsilon \| f\varepsilon \|^p_{W^{1-\varepsilon,p}} \leq C_0.$$

Then, up to a subsequence, $(f\varepsilon)$ converges in L^p (and, in fact, in $W^{1-\delta,p}$, for all $\delta > 0$) to some $f \in W^{1,p}$.

Proof of Theorem 4

The heart of the proof consists of showing that (f_n) is relatively compact in L^p. The rest is done as in the second part of the proof of Theorem 2.

Without loss of generality, we may assume that $\Omega = \mathbb{R}^N$ and that supp $f_n \subset B$, a ball in $\mathbb{R}^N$ of diameter 1. This can be achieved by extendeing each function f_n by reflection across the boundary in a neighborhood of $\partial\Omega$. Using the monotonicity assumption (16), we see that assumption (14) still holds.

In order to prove compactness in L^p, we rely on the Riesz-Fréchet-Kolmogorov theorem(see, e.g., Brezis [], Théorème IV.25) or rather its proof: let, for $\delta > 0, \varPhi_\delta$ be the mollifier

$$\varPhi_\delta = \frac{1}{|B_\delta(0)|} \chi_{B_\delta(0)}.$$

Then (f_n) is relatively compact in $L^p(\Omega)$ if and only if

$$\|f_n\|_{L^p} \le C \tag{18}$$

and

$$\lim_{\delta\to 0}\,(\limsup_{n\to\infty}\,\|f_n - f_n * \Phi_\delta\|_{L^p}) = 0. \tag{19}$$

For each n and $t > 0$, let

$$F_n(t) = \int\limits_{\omega\in S^{N-1}}\ \int\limits_{\mathbb{R}^N} |f_n(x+t\omega) - f_n(x)|^p dx d\sigma =$$
$$\frac{1}{t^{N-1}} \int\limits_{|h|=t}\ \int\limits_{\mathbb{R}^N} |f_n(x+h) - f_n(x)|^p dx d\sigma.$$

Unsing the triangle inequality, we obtain

$$F_n(2t) \le 2^p F_n(t). \tag{20}$$

In terms of F_n, assumption (14) can be expressed as

$$\int_0^1 t^{N-1}\frac{F_n(t)}{t^p}\,\rho_n(t)dt \le C_0. \tag{21}$$

We claim that

$$\int |f_n(x)|^p dx \ \le\ C\int_0^1 t^{N-1}F_n(t)dt \tag{22}$$

and

$$\int |f_n(x) - (f_n * \Phi_\delta)(x)|^p dx \le C\delta^{-N}\int_0^\delta t^{N-1}F_n(t)dt, \tag{23}$$

for some C independent of n and δ.

We prove for example (23):

$$\int |f_n(x) - (f_n * \Phi_\delta)(x)|^p dx = \int |f_n(x) - \frac{1}{|S^{N-1}|\delta^n} \int\limits_{|y-x|<\delta} f_n(y)dy|dx^p$$

$$= \frac{1}{(|S^{N-1}|\delta^N)^p} \int | \int\limits_{|y-x|<\delta} (f_n(x) - f_n(y))dy|^p dx$$

$$\leq \frac{1}{|S^{N-1}|}\delta^{-N} \iint\limits_{|y-x|<\delta} |f_n(x) - f_n(y)|^p dxdy$$

$$= \frac{1}{|S^{N-1}|}\delta^{-N} \int\limits_{|h|<\delta} (\int |f_n(x+h) - f_n(x)|^p dx)dh$$

$$= C\delta^{-N}\int_0^\delta t^{N-1}F_n(t)dt.$$

The proof of (22) is similar, since

$$f_n(x) = f_n(x) - \fint_B f_r(y)dy.$$

We are going to establish below the key inequality

$$\delta^{-N}\int_0^\delta t^{N-1}\frac{F_n(t)}{t^p}dt \leq C(\int_0^\delta t^{N-1}\frac{F_n(t)}{t^p}\rho_n(t)dt)/(\int\limits_{|x|<\delta} \rho_n(x)dx). \tag{24}$$

Assume (24) has been moved, then we proceed as follows since.

$$\lim_{N\to\infty} \int\limits_{|x|<\delta} \rho_n(x)dx = 1,$$

by combining (14) with (24) we find

$$\delta^{-N}\int_0^\delta ddt^{N-1}\frac{F_n(t)}{t^p}dt \leq C \text{ for } n \geq n_\delta. \tag{25}$$

In particular, we have

$$\delta^{-N}\int_0^\delta t^{N-1}F_n(t)dt \le C\delta^p \text{ for } n\ge n_\delta \tag{26}$$

Inequalities (18), (19)– and thus the conclusion of Theorem 4– follow from (22), (23) and (26).

It remains to establish inquality (24). Note that it is a particular case ($g(t)=\frac{F_n(t)}{t^p}, h(t)=\rho_n(t)$) of the following varian of an inequality due to Chebyshev:

Lemma. *Let $g,h:(0,\delta)\to\mathbb{R}_+$. Assume that $g(t)\le g(t/2), t\in(0,\delta)$, and that h is non-increasing.*

Then, for some $C=C(N)>0$,

$$\int_0^\delta t^{N-1}g(t)h(t)dt \ge C\delta^{-N}\int_0^\delta t^{N-1}g(t)dt\int_0^\delta t^{N-1}h(t)dt.$$

Proof of the lemma: It suffices to consider the case $\delta=1$;

the general case follows by scaling. We have

$$\int_0^1 t^{N-1}g(t)h(t)dt = \sum_{j\ge0}\int_{1/2^{j+1}}^{1/2^j} t^{N-1}g(t)h(t)dt$$

$$= \sum_{j\ge0}\frac{1}{2^{Nj}}\int_{1/2}^1 S^{N-1}g(\frac{s}{2^j})h(\frac{s}{2^j})ds$$

$$= \int_{1/2}^1\sum_{j\ge0}\frac{1}{2^{Nj}}g(\frac{s}{2^j})h(\frac{s}{2^j})ds, \tag{27}$$

and a similar equality holds for $\int_0^1 t^{N-1}g(t)dt$. We recall the classical Chebyshev inequality: if

$G,H:X\to\mathbb{R}, u$ a positive measure on X and $(G(x)-G(y))(H(x)-H(y))\ge 0, \quad \forall x,y\in X$,then

$$\int_x GH d\mu \geq \frac{1}{|x|} \quad \int_{(} x)G d\mu \int_{(} x)H d\mu.$$

In particular, if $\alpha_j \geq 0$ and the sequences $(a_j), (b_j)$ have the same nonotonicity, then

$$\sum \alpha_j a_j b_j \geq \frac{1}{\sum \alpha_j} \sum \alpha_j a_j \sum \alpha_j b_j. \tag{28}$$

Since for each $s \in (1/2, 1)$, the sequence $(g(\frac{s}{2^j}))$ and $(h(\frac{s}{2^j})$ are non-decreasing, (28) with $\alpha_j = \frac{1}{2^{Nj}}$ yields

$$\sum_{j\geq 0} \frac{1}{2^{Nj}} g(\frac{s}{2^j}) h(\frac{s}{2^j}) \ \geq \ C \sum_{j\geq 0} \frac{1}{2^{Nj}} g(\frac{s}{2^j}) \sum_{j\geq 0} \frac{1}{2^N j} h(\frac{s}{2_j}). \tag{29}$$

Now clearly, for each $s \in (1/2, 1)$ and each $j \geq 1$,

$$\frac{1}{2^{Nj}} h(\frac{s}{2^j}) \geq \frac{1}{2^{Nj}} h(\frac{s}{2^j}) \geq C \int^{1/2^{j-1}} t^{N-1} h(t) dt,$$

for some C independent of j, so that $1/2^j$

$$\sum_{j\geq 0} \frac{1}{2^{Nj}} h(\frac{s}{2^j}) \geq C \int_0^1 t^{N-1} h(t) dt. \tag{30}$$

It follows from $(2g)$ and (30) that

$$\sum_{j\geq 0} \frac{1}{2^{Nj}} g(\frac{s}{2^j}) h(\frac{s}{2^j}) \geq C \int_0^1 t^{N-1} h(t) dt \sum_{j\geq 0} \frac{1}{2^{Nj}} g(\frac{s}{2^j}). \tag{31}$$

Inserting (31) into (27), we find

$$\begin{aligned}\int_0^1 t^{N-1} g(t) j(t) dt &\geq C \int_0^1 t^{N-1} j(t) dt \int_{1/2}^1 \sum_{j\geq 0} \frac{1}{2^{Nj}} g(\frac{s}{2^j}) ds \\ &= C \int_0^1 t^{N-1} h(t) dt \int_0^1 g(t) dt.\end{aligned}$$

the proof of Theorem4 is complete.

Returning to Corollary7, we still have to prove that, for any $\delta > 0$, we have $\varepsilon > 0$

$$\|f\varepsilon\|_{W^{1-s,p}} \leq C$$

Considering the same functions $f\varepsilon(t)$, as above (relative to the parameter ε instead of n we have to prove that

$$\int_0^1 \frac{F\varepsilon(t)}{t^{(1-\delta)p+1}}dt \leq C, \text{ for small}\varepsilon > 0, \tag{32}$$

under the assumption

$$\varepsilon\int_0^1 \frac{F\varepsilon(t)}{t^{(1-\varepsilon)p+1}}dt \ \leq C. \tag{33}$$

the proof is similar to that of the Lemma, so we just sketch it. We start by rewritting (32) and (33) as

$$\int\limits_0^1 \frac{1}{t^{1-sp}} \quad \frac{F\varepsilon(t)}{t^p}dt \leq C \tag{34}$$

and

$$\int\limits_0^1 \frac{1}{t^{1-sp}} \quad \frac{F\varepsilon(t)}{t^p} \quad \frac{\varepsilon}{t^{(s-\varepsilon)p}}dt \leq C. \tag{35}$$

We continue as in the proof of the lemma, with

$$g(t) = \frac{F\varepsilon(t)}{t^p} \text{ and } h(t) = \frac{\varepsilon}{t^{(s-\varepsilon)p}}, \text{ and take } 0 < \varepsilon < \delta.$$

We finally find

$$\int\limits_0^1 \frac{1}{t^{1-sp}} \quad \frac{F\varepsilon(t)}{t^p} \ \frac{\varepsilon}{t^(s-\varepsilon)p}dt \geq C\int\limits_0^1 \frac{1}{t^{1-sp}}\frac{F\varepsilon(t)}{t^p}dt \tag{36}$$

for some C depending possibly on S, but not on ε.

Remark 5. If we renorm the $W^{s,p}(\Omega)$ spaces by

$$|f|^p_{W^{s,p}} \quad = \ \{\begin{cases} (1-s)\|f\|^p_{W^{s,p}} & , 0 < s < 1 \\ \|f\|^p_{W^{1,p}} & , s = 1, \end{cases}$$

the above computation yields

$$|f|^p_{W^{\sigma,p}} \ \leq \ C|f|^p_{W^{s,p}} \ .0 < \sigma < s \leq 1 \text{ for}$$

some constant C **independent of** s **and** σ.

Counterexample 1: a sequence (f_n) unbounded in L^p and a sequence of radial mollifiers(ρ_n) such that

$$\int_\Omega \int_\Omega \frac{|f_n(x) - f_n(y)|^p}{|x-y|^p} \rho_n(x-y) dx dy \le C. \tag{37}$$

We take $\Omega = (0,1)$.

Fix some function $f \in L^p_{loc}(\mathbb{R})$, non-constant, periodical of period 1, such that

$$\int_0^1 f(x)dx = 0(\text{ e.g. } , f(x) = \sin(2\pi x)).$$

Define $g_n(x) = f(nx)$,so that $\|g_n\|^p_{L^p} = \int_0^1 |f(x)|^p dx = C$.

Clearly, $\int_0^1 |g_n(x = \frac{1}{n}) - g_n(x)|^p dx = 0$. Since that translations are continous in l^p, we may find some $0 < \delta_n < \frac{1}{2_n}$

such that $\int_0^1 |g_n(x+h) - g_n(x)|^p dx \le \frac{1}{n2p}$ for $|h \pm \frac{1}{n}| < \delta_n$.

Let $\rho - N = \frac{1}{4\delta_n}(\chi_{\frac{1}{n}-\delta_n, \frac{1}{n}+\delta_n)} \ + \ \chi_{(-\frac{1}{n}-\delta_n, -\frac{1}{n}=\delta_n)})$.

Then clearly

$$\int_\Omega \int_\Omega \frac{|g_n(x) - g_n(y)|}{|x-y|^p} \rho_n(x-y) dx dy \ \le \ \frac{C}{n^p}.$$

Finally, the functions $f_n = n g_n$ satisfy the desired inequality (37) and $\|f-N\| - L^p \sim n$.

Counterexample 2: the sequence (g_n) constructed above is bounded in l^p, is not relatively compact in L^p, and yet it satisfies

$$\int_\Omega \int_\Omega \frac{|g_n(x) - g_n(y)|^p}{|x-y|^p} \rho_n(x-y) dx dy \le C.$$

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