Refined support and entropic uncertainty inequalities

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Abstract

Generalized versions of the entropic (Hirschman-Beckner) and support (Elad-Bruckstein) uncertainty principle are presented for frames representations. Moreover, a sharpened version of the support inequality has been obtained by introducing a generalization of the coherence. In the finite dimensional case and under certain conditions, minimizers of this inequalities are given as constant functions on their support. In addition, ℓ^p-norms inequalities are introduced as byproducts of the entropic inequalities.

Index Terms

Uncertainty principles, support inequalities, Shannon entropy, Renyi entropy, lp norms, frames, mutual coherence.

I. Introduction

The uncertainty principle is originally a quantum physics principle stating that some families of observable quantities cannot be measured simultaneously with infinite precision. The uncertainty principle can be turned into quantitative statements thanks to uncertainty inequalities, which provide bounds on precision of simultaneous measurements of such quantities.

The prototype of uncertainty inequality is the celebrated Heisenberg inequality, first formulated in [12], which uses a variance measure as criterion for the measurement

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precision. Namely, for all $x \in L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} t^2 |x(t)|^2 \, dt \, . \int_{-\infty}^{\infty} \nu^2 |\hat{x}(\nu)|^2 \, d\nu \geq \frac{1}{16\pi^2} \ ,$$

where the Fourier transform is normalized in such a way that the Fourier transform of $e^{-\pi t^2}$ equals $e^{-\pi \nu^2}$. Originally stated for position and momentum, the Heisenberg inequality has been extended to more general observable pairs, under the name of Robertson inequality [21], [22]. Particular cases have been analyzed by various authors, see e.g. [5], [4], [10], [15] and references therein. The Robertson variance inequality has been criticized in the physics literature, mainly because the bound in the inequality sometimes depends explicitely on the left hand side, which has motivated to seek alternative formulations. Besides, Robertson-type inequalities do not generalize well to all situations: for example, the notion of variance is not necessarily easy to define in some contexts, such as for periodic sequences or functions, functions defined on compact manifolds or graphs, more generally in situations where the notion of spreading away from a reference point is not straightforward. Among the generalizations, entropic inequalities, that use entropy measures to quantize measurement precision have enjoyed renewed interest recently. In the particular case of the position-momentum situation, the corresponding entropic uncertainty inequality, called the Hirschman-Beckner inequality [13], is intimately related to the sharp form of the Hausdorff-Young inequality, the so-called Babenko-Beckner inequality [1]. In signal processing terms, this uncertainty principle limits the simultaneous concentration or sparsity of a function and its Fourier transform. The inequality provides a lower bound on the differential entropies of their respective square moduli.

Uncertainty inequalities have received a renewed interest in the context of sparse approximation and signal processing applications. Often in a finite-dimensional setting, ℓ^p norms (with $p < 2$) are used to measure dispersion of signals. This provides some quantities in order to compare the sharpness of different representations (ℓ^2 vectors) of a signal x or probe the concentration of information inside them. For example, the signal itself and its Fourier transform are two representations of the same mathematical object. More generally, any projection of x on a basis of the Hilbert space gives a representation of the signal. In this context, uncertainty bounds involving ℓ^0 quasi-norm and ℓ^1 norm have been derived. A prototype of such bounds is the Elad-Bruckstein ℓ^0 inequality: given two orthonormal bases in a finite-dimensional Hilbert space and any vector x in that space with set of coefficients a and b with respect to the two bases,

$$\|a\|_0 \, \|b\|_0 \geq \frac{1}{\boldsymbol{\mu}^2} \, ,$$

where $\boldsymbol{\mu}$ is a constant called mutual coherence, that depends on the two bases (and not on x). Such results have important implications for practical problems, as shown in the pioneering work of Donoho and Stark [8]. For instance, ℓ^0 bounds have been used to prove the equivalence of ℓ^0 and ℓ^1-based sparse recovery algorithms, under suitable sparsity assumptions [7], [9]. Results of similar nature have also been obtained in the context of the Fourier transform on abelian groups (see for example [24], [14], [18]). As is well known in information theory, and remarked also in [20], Shannon entropy and ℓ^p norms are closely connected, through Rényi entropies. Inequalities involving Rényi entropies [16], [6] actually imply both Shannon entropy inequalities and ℓ^p inequalities.

This study has been divided into two parts (Sec. II and Sec. III). In the first part, we analyze such support (ℓ^0) inequalities in the context of frame decompositions, as follows. Given two frames $\mathcal{U}$ and $\mathcal{V}$ in a Hilbert space $\mathcal{H}$, denote by U and V the corresponding analysis operator. Ux and Vx are the two representations of x with respect to the frames. For any $x \in \mathcal{H}$, we prove for example a bound of the form

$$\|Ux\|_0 \|Vx\|_0 \geq \frac{1}{\boldsymbol{\mu}_*^2} \, ,$$

where $\boldsymbol{\mu}_*$ is a constant that only depends on the two frames. In the case of orthonormal bases, these inequalities yield refined forms for support inequalities ($\boldsymbol{\mu}_* \leq \boldsymbol{\mu}$), for which we can analyze conditions for equality. The refined inequalities involve cumulated coherence measures, instead of the standard coherence measures used classically. In the case of frame decompositions, the inequalities we obtain concern analysis coefficients, while most recent contributions (in the domain of sparse decompositions and approximation) focus on inequalities involving synthesis coefficients. Therefore, exact recovery results such as those derived in [9], [11] do not apply directly to the new results. Though, given the renewed interest on analysis-based sparse decompositions and co-sparsity (see e.g. [19]), we believe that these new inequalities are of interest, as they can yield bounds for the performances of cosparse signal recovery methods. Consider for instance the following signal separation problem: given two frames $\mathcal{U}$ and $\mathcal{V}$, and some observed signal $u \in \mathcal{H}$, we want to split u as a sum of two components whose respective analysis coefficients with respect to frames $\mathcal{U}$ and $\mathcal{V}$ are sparse. In other words, we want to solve

$$\min_{x,y\in\mathcal{H}} [\|Ux\|_0 + \|Vy\|_0] \quad \text{under constraint} \quad u = x + y \ .$$

where U and V are the analysis operators of two frames under consideration. Given two such decompositions $u = x + y = x' + y'$, the above support inequality directly leads to

$$\begin{aligned} \|Ux\|_0 + \|Vy\|_0 + &\|Ux'\|_0 + \|Vy'\|_0 \\ &\geq \|U(x - x')\|_0 + \|V(y' - y)\|_0 \geq \frac{2}{\boldsymbol{\mu}_*} \ . \end{aligned}$$

Therefore, if one is given a splitting of the form $u = x + y$ such that $\|Ux\|_0 + \|Vy\|_0 < 1/\boldsymbol{\mu}_*$, this splitting is automatically the solution of the above optimization problem.

Besides support size estimates, we also obtain entropic inequalities for analysis coefficients with respect to frames, that explicitely involve the frame bounds. This is developed in the second part of the study. As a particular case, Shannon entropy bounds are derived, and it is shown that the latter are only informative for tight frames. In the latter case, the entropy inequalities take a fairly simple form; for example, denoting by $S(a)$ the Shannon entropy of a vector a, we show that given two tight frames $\mathcal{U}$ and $\mathcal{V}$, with respective analysis operators U and V, then

$$S(Ux) + S(Vx) \geq -2\ln(\boldsymbol{\mu}_*) \ .$$

Such inequality also turns out to yield the above mentioned support inequalities as a by-product. Finally, we also derive new ℓ^p inequalities as consequences of Rényi entropic inequalities.

II. Refined Elad-Bruckstein ℓ^0 inequalities

A. Notations

We first introduce the general setting we shall be working with. Throughout this paper, we shall denote by $\mathcal{U} = \{u_k, k \in \Lambda\}$ and $\mathcal{V} = \{v_\ell, \ell \in \Lambda\}$ two countable frames for the Hilbert space $\mathcal{H}$ (we refer to [2] for a self contained account of frame theory). Here, the index set Λ will be finite when $\mathcal{H}$ is finite-dimensional, and infinite otherwise. $\|x\|$ also written $\|x\|_2$ is the norm of x in $\mathcal{H}$. We denote by $A_\mathcal{U}, B_\mathcal{U}$ and $A_\mathcal{V}, B_\mathcal{V}$ the corresponding frame bounds, i.e. we have for all $x \in \mathcal{H}$

$$A_\mathcal{U}\|x\|^2 \leq \sum_k |\langle x, u_k\rangle|^2 \leq B_\mathcal{U}\|x\|^2 \ , \tag{1}$$

$$A_\mathcal{V}\|x\|^2 \leq \sum_\ell |\langle x, v_\ell\rangle|^2 \leq B_\mathcal{V}\|x\|^2 \ . \tag{2}$$

Let $U : \mathcal{H} \to \ell^2(\Lambda)$ and $V : \mathcal{H} \to \ell^2(\Lambda)$ be the corresponding analysis operators, i.e.

$$a_k \stackrel{\Delta}{=} (Ux)_k = \langle x, u_k\rangle \ , \quad b_\ell \stackrel{\Delta}{=} (Vx)_\ell = \langle x, v_\ell\rangle \ , \qquad x \in \mathcal{H} \ , \tag{3}$$

and denote by $T = VU^\dagger : a \to b$ the change of frame operator (with $U^\dagger$ the Moore-Penrose pseudo-inverse of U). We shall also denote by $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ corresponding (generic) dual frames, among which the canonical dual frames will be denoted by $\widetilde{\mathcal{U}}^\circ = (U^*U)^{-1}\mathcal{U}$ and $\widetilde{\mathcal{V}}^\circ = (V^*V)^{-1}\mathcal{V}$. As is well known (see [2]), the corresponding frame bounds are respectively $A_{\widetilde{\mathcal{U}}^\circ} = 1/B_\mathcal{U}$, $B_{\widetilde{\mathcal{U}}^\circ} = 1/A_\mathcal{U}$, and similarly for $\widetilde{\mathcal{V}}$. We recall that in the particular case where $\mathcal{U}$ and/or $\mathcal{V}$ are (Riesz) bases, $\widetilde{\mathcal{U}}$ and/or $\widetilde{\mathcal{V}}$ are the corresponding bi-orthogonal bases.

In the following, we shall make use of the following quantity:

Definition 1: Let $r \in [1,2]$, let r' be conjugate to r, i.e. such that $1/r + 1/r' = 1$. The mutual coherence of order r of two frames $\mathcal{U}$ and $\mathcal{V}$ is defined by

$$\boldsymbol{\mu}_r(\mathcal{U},\mathcal{V}) \stackrel{\Delta}{=} \sup_\ell \left(\sum_k |\langle u_k, v_\ell\rangle|^{r'}\right)^{r/r'} \ , \tag{4}$$

In the case $r = 1$, this corresponds to the standard mutual coherence, simply denoted by $\boldsymbol{\mu}(\mathcal{U},\mathcal{V})$.

This quantity is clearly well-defined in finite-dimensional settings. Notice also that in infinite-dimensional situations (i.e. when Λ is an infinite index set), this quantity is well-defined for all $r \in [1,2]$. Indeed, $\boldsymbol{\mu}_2(\mathcal{U},\mathcal{V}) \leq B_\mathcal{U} \sup_\ell \|v_\ell\|^2$, and $\boldsymbol{\mu}_r^{r'/r}(\mathcal{U},\mathcal{V}) \leq \boldsymbol{\mu}_2(\mathcal{U},\mathcal{V}) \sup_{k,\ell} |\langle u_k, v_\ell\rangle|^{r'-2}$, which is finite since $r' \geq 2$.

In finite-dimensional situations, the notion of mutually unbiased bases has been introduced in the physics literature by Schwinger [23] (see [25] for a review).

Definition 2: Two orthonormal bases $\mathcal{U}$ and $\mathcal{V}$ in an N-dimensional Hilbert space $\mathcal{H}$ are mutually unbiased (MUB) if

$$|\langle u_k, v_\ell\rangle| = \frac{1}{\sqrt{N}}\ , \qquad \forall k,\ell = 0,\dots N-1\ .$$

$\mathcal{U}$ and $\mathcal{V}$ are blockwise mutually unbiased bases (BMUB) of $\mathcal{H}$ if $\mathcal{U} = \{\mathcal{U}^{(1)},\dots\mathcal{U}^{(K)}\}$, $\mathcal{V} = \{\mathcal{V}^{(1)},\dots\mathcal{V}^{(K)}\}$, where for all $k = 1,\dots K$, $\mathcal{U}^{(k)}$ and $\mathcal{V}^{(k)}$ span the same subspace $\mathcal{H}^{(k)}$, of dimension N_k, and are MUBs for $\mathcal{H}^{(k)}$, with $\bigoplus_k \mathcal{H}^{(k)} = \mathcal{H}$.
Notice that the coherence of a MUB in an N-dimensional Hilbert space equals $\boldsymbol{\mu}(\mathcal{U},\mathcal{V}) = 1/\sqrt{N}$, the corresponding r-coherence equals $\boldsymbol{\mu}_r(\mathcal{U},\mathcal{V}) = N^{r/2-1}$, and the r-coherence of a BMUB equals $\boldsymbol{\mu}_r(\mathcal{U},\mathcal{V}) = \max_k N_k^{r/2-1}$.

B. Refined Elad-Bruckstein inequality

The classical Elad-Bruckstein ℓ^0 inequality [9] (a strong form of which has been given in [11]) gives a lower bound for the product of support sizes of two orthonormal basis representations of a single vector. The inequality can be extended to the frame case and generalized as follows.

Theorem 1: Let $\mathcal{U}$ and $\mathcal{V}$ be two frames of the Hilbert space $\mathcal{H}$. For any $x \in \mathcal{H}$, $x \neq 0$, denote by $a = Ux$ and $b = Vx$ the analysis coefficients of x with respect to these two frames.

1) For all $r \in [1,2]$, coefficients a and b satisfy the uncertainty inequality

$$\|a\|_0.\|b\|_0 \geq \frac{1}{\boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})\boldsymbol{\mu}_r(\widetilde{\mathcal{V}},\mathcal{U})}\ . \tag{5}$$

Therefore, $\|a\|_0.\|b\|_0 \geq 1/\boldsymbol{\mu}_*(\mathcal{U},\widetilde{\mathcal{U}},\mathcal{V},\widetilde{\mathcal{V}})^2$, where

$$\boldsymbol{\mu}_*(\mathcal{U},\widetilde{\mathcal{U}},\mathcal{V},\widetilde{\mathcal{V}}) \stackrel{\Delta}{=} \inf_{r\in[1,2]} \sqrt{\boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})\boldsymbol{\mu}_r(\widetilde{\mathcal{V}},\mathcal{U})}\ . \tag{6}$$

2) For all $r \in [1,2]$, the inequality can only be sharp if the following three properties hold true:

   i. the sequences $|a|$ and $|b|$ are constant on their support,
   ii. for all $k \in \mathrm{supp}(a)$ (resp. $\ell \in \mathrm{supp}(b)$) the sequence $\ell \to |\langle \tilde{u}_k, v_\ell\rangle|$ (resp. $k \to |\langle \tilde{v}_\ell, u_k\rangle|$) is constant on $\mathrm{supp}(b)$ (resp. $\mathrm{supp}(a)$).
   iii. for all $k \in \mathrm{supp}(a), \ell \in \mathrm{supp}(b)$, $\arg(\langle \tilde{u}_k, v_\ell\rangle) = \arg(b_\ell) - \arg(a_k) = -\arg(\langle \tilde{v}_\ell, u_k\rangle)$.

Proof:

1) Let $r \in [1,2]$, let $x \in \mathcal{H}$, $x \neq 0$. First remark that

$$\begin{aligned} \|b\|_\infty &= \sup_\ell |\langle x, v_\ell\rangle| \\ &= \sup_\ell \left|\left\langle \sum_k a_k \tilde{u}_k, v_\ell \right\rangle\right| \\ &\leq \sup_\ell \sum_k |a_k|\, |\langle \tilde{u}_k, v_\ell\rangle| \ , \end{aligned}$$

and Hölder's inequality yields

$$\|b\|_\infty \leq \|a\|_r\ \boldsymbol{\mu}_r(\widetilde{\mathcal{U}}, \mathcal{V})^{1/r} \tag{7}$$

Similarly,

$$\|a\|_\infty \leq \|b\|_r\ \boldsymbol{\mu}_r(\widetilde{\mathcal{V}}, \mathcal{U})^{1/r}\ . \tag{8}$$

Then, notice that

$$\|a\|_r^r \leq \|a\|_0\, \|a\|_\infty^r \leq \|a\|_0\, \|b\|_r^r\ \boldsymbol{\mu}_r(\widetilde{\mathcal{V}}, \mathcal{U})\ .$$

The same estimate on $\|b\|_r$ proves the first part of the theorem.

2) Assume first $r \neq 1$. As for the sharpness of the bound, notice first that the inequality $\|a\|_r^r \leq \|a\|_0\, \|a\|_\infty^r$ is sharp if and only if $|a|$ is constant on its support (similarly, $|b|$ has to be constant on its support). Now in the first inequality, Hölder's inequality is an equality if and only if the sequence $k \to |\langle \tilde{u}_k, v_\ell\rangle|^{r'}$ is proportional to $|a|^r$, meaning that the sequence $k \to |\langle \tilde{u}_k, v_\ell\rangle|$ is constant on its support, which coincides with the support of a. A similar reasoning is done for b and the sequence $\ell \to |\langle \tilde{v}_\ell, u_k\rangle|$. The last inequality to be investigated is $|b_\ell| \leq \sum_k |a_k|\, |\langle \tilde{u}_k, v_\ell\rangle|$. The latter becomes an equality if and only if the sum only involves positive numbers, i.e. iff $\arg(\langle \tilde{u}_k, v_\ell\rangle) = \arg(b_\ell) - \arg(a_k)$. A similar reasoning yields the condition $\arg(\langle \tilde{v}_\ell, u_k\rangle) = \arg(a_k) - \arg(b_\ell)$.

   Finally, consider the case $r = 1$. The above argument can be reproduced exactly, except for the tightness argument for Hölder's inequality. The latter can now be an equality only if the sequence $k \to |\langle \tilde{u}_k, v_\ell\rangle|$ is equal to a constant (namely, $\boldsymbol{\mu}(\widetilde{\mathcal{U}}, \mathcal{V})$) on $\mathrm{supp}(a)$, and smaller outside the support. This does not change the conclusion.

This concludes the proof. ♠

Remark 1: 1) Clearly, by the arithmetic-geometric inequality, we also obtain the bound

$$\|a\|_0 + \|b\|_0 \geq \frac{2}{\boldsymbol{\mu}_*(\mathcal{U},\widetilde{\mathcal{U}},\mathcal{V},\widetilde{\mathcal{V}})} \tag{9}$$

2) Using exactly the same techniques, the uncertainty inequality can be extended to K frames. Given K frames $\mathcal{U}^{(k)},\ k = 1,\dots K$ and denoting by $a^{(k)}$ the corresponding sequences of analysis coefficients of any $x \in \mathcal{H}$, we readily obtain the bound

$$\|a^{(1)}\|_0.\|a^{(2)}\|_0 \dots \|a^{(K)}\|_0 \geq \left(\prod_{k=1}^{K} \mu_*^{(k)}\right)^{-1} \tag{10}$$

where

$$\mu_*^{(k)} = \mu_*(\mathcal{U}^{(k)},\widetilde{\mathcal{U}}^{(k)},\mathcal{U}^{(k+1 \bmod K)},\widetilde{\mathcal{U}}^{(k+1 \bmod K)}),$$

and again by the arithmetic-geometric inequality,

$$\|a^{(1)}\|_0 + \|a^{(2)}\|_0 + \dots + \|a^{(K)}\|_0 \geq K \left(\prod_{k=1}^{K} \mu_*^{(k)}\right)^{-1/K} . \tag{11}$$

Remark 2: We notice that when $\mathcal{U}$ and $\mathcal{V}$ are orthonormal bases the result generalizes the Elad-Bruckstein inequality. When $\mathcal{U}$ and $\mathcal{V}$ are non-orthonormal bases $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ are their respective biorthogonal bases and we obtain a straightforward generalization. In the case of frames, let us point out that the generalization we obtain concerns analysis coefficients rather than synthesis coefficients.

Remark 3: Notice finally that these bounds involve arbitrary dual frames $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ of $\mathcal{U}$ and $\mathcal{V}$, not necessarily the canonical ones. Therefore the bound can be make more general, in the form

$$\|a\|_0.\|b\|_0 \geq 1/\boldsymbol{\mu}_{**}(\mathcal{U},\mathcal{V})^2 \ , \tag{12}$$

where

$$\boldsymbol{\mu}_{**}(\mathcal{U},\mathcal{V}) \stackrel{\Delta}{=} \inf_{\widetilde{\mathcal{U}},\widetilde{\mathcal{V}}} \inf_{r\in[1,2]} \sqrt{\boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})\boldsymbol{\mu}_r(\widetilde{\mathcal{V}},\mathcal{U})} \ , \tag{13}$$

the infimum running over the family of dual frames of $\mathcal{U}$ and $\mathcal{V}$. A characterization of such families can be found in [2], Theorem 5.6.5.

C. Examples and comments: the case of orthonormal bases

Consider first the case where $\mathcal{U}$ and $\mathcal{V}$ are two orthonormal bases in finite dimensional Hilbert spaces. First notice that the case $r = 1$ provides an elementary proof of the Elad-Bruckstein inequality (which involves $1/\boldsymbol{\mu}_1(\mathcal{U},\mathcal{V})^2$ as a lower bound), together with explicit conditions for sharpness. In the particular case of mutually unbiased bases, i.e. orthonormal bases such that $|\langle u_k, v_\ell\rangle|$ is constant, $\boldsymbol{\mu}_r(\mathcal{U},\mathcal{V}) = N^{r/2-1}$ is monotone and minimal for $r = 1$, which yields the usual coherence $\boldsymbol{\mu} = \boldsymbol{\mu}_1 = 1/\sqrt{N}$, N being the dimension of the considered Hilbert space. An example of mutually unbiased bases is provided by the Kronecker and Fourier bases in $\mathbb{R}^N$, in which case the Elad-Bruckstein inequality coincides with the inequality derived before by Donoho and Huo [7]. For blockwise mutually unbiased bases, we also obtain a monotone function of r for the r-coherence $\boldsymbol{\mu}_r(\mathcal{U},\mathcal{V}) = \max_k N_k^{r/2-1}$, which means that again the optimal bound is provided by $\boldsymbol{\mu}_1$.

In the case of orthonormal bases, the smallest possible value for the coherence is provided by the Welch bound: $\boldsymbol{\mu} \geq 1/\sqrt{N}$. Therefore, we obtain

Corollary 1: Assume $\mathcal{U}$ and $\mathcal{V}$ are orthonormal bases. The optimal bound for the refined Elad-Bruckstein uncertainty inequality is attained in the case of mutually unbiased bases, for $r = 1$.

Consider now the case where the inequality is an equality, in the case $r \neq 1$. By Theorem 1, the analysis coefficients a and b of the corresponding optimizer are such that $|a|$ and $|b|$ are constant on their support. The proof of part 2. of the theorem also implies that for $k \in \text{supp}(a)$, the sequence $\ell \to |\langle u_k, v_\ell\rangle|$ vanishes outside $\text{supp}(b)$ and equals a constant on $\text{supp}(b)$; The latter constant equals necessarily $\|b\|_0^{-1/2}$, and $\boldsymbol{\mu}_r(\mathcal{U},\mathcal{V}) = \|b\|_0^{r/2-1}$. Similarly, $\boldsymbol{\mu}_r(\mathcal{V},\mathcal{U}) = \|a\|_0^{r/2-1}$. Assume finally that the inequality be an equality, the latter thus reads

$$\|a\|_0.\|b\|_0 = \|a\|_0^{1-r/2}.\|b\|_0^{1-r/2} \ ,$$

which implies (for nonzero signals x) $\|a\|_0.\|b\|_0 = 1$, i.e. the two bases have at least one common element, and the signal is a multiple of one of these common elements.

Corollary 2: Assume $\mathcal{U}$ and $\mathcal{V}$ are orthonormal bases. For $r \neq 1$, the corresponding refined Elad-Bruckstein inequality cannot be an equality, unless the two bases have a common element.

Notice however that the case $r \in [1,2]$ constitutes a true generalization; indeed, for general pairs of orthonormal bases, it turns out that $\sup_r[1/\boldsymbol{\mu}_r(\mathcal{U},\mathcal{V})\boldsymbol{\mu}_r(\mathcal{V},\mathcal{U})] > 1/\boldsymbol{\mu}_1(\mathcal{U},\mathcal{V})^2$. This is examplified in Figure 1, where are displayed the functions $\boldsymbol{\mu}_r(\mathcal{U},\mathcal{V})$, $\boldsymbol{\mu}_r(\mathcal{V},\mathcal{U})$ and $\sqrt{\boldsymbol{\mu}_r(\mathcal{U},\mathcal{V})\boldsymbol{\mu}_r(\mathcal{V},\mathcal{U})}$ as a function of r, in a generic situation: the two bases $\mathcal{U}$ and $\mathcal{V}$ are random bases, obtained by diagonalization of random (Gaussian) symmetric matrices. As can be seen in this picture, the minimum of these three functions is not attained for $r=1$ but for a larger value. For the sake of comparison, the case of mutually unbiased bases is also represented and exhibit a power law behavior as a function of r (represented as a straight line in the logarithmic plot). This shows that the coherence based bounds are not the best possible ones in general. Elementary infinitesimal calculus yields the following expression for the behavior of the r-coherence near $r=2$:

$$\begin{aligned}\boldsymbol{\mu}_r(\mathcal{U},\mathcal{V}) =& 1-(2-r)\max_\ell\left(-\sum_k |\langle u_k,v_\ell\rangle|^2\ln\left(|\langle u_k,v_\ell\rangle|^2\right)\right)\\ &+O((2-r)^2)\ ,\end{aligned}$$

i.e. the slope of the tangent at $r=2$ is given by the entropy-like expression

$$\text{slope} = -\sum_k |\langle u_k,v_\ell\rangle|^2\ln\left(|\langle u_k,v_\ell\rangle|^2\right)$$

(see section below), which is known to be minimal (in finite dimensional situations, see [3] for more details) when the $|\langle u_k,v_\ell\rangle|^2$ are all equal.

More generally, we have the following result on r-coherences.

Proposition 1: Let $\{u_k\}_k$ and $\{v_l\}_l$ be two frames in the Hilbert space $\mathcal{H}$ of dimension N.Fix l, let $s_l = \max_k |\langle u_k,v_l\rangle|$ and denote by $n_l = \sharp(|\langle u_k,v_l\rangle| = s_l)$ the multiplicity of this maximal value. If $\max_l n_l s_l < 1$, then there exists $r>1$ such that $\mu_r < \mu_1$.

Proof: It is enough to show that the derivative of μ_r is negative at $r=1$ under the stated conditions. Let us introduce the notation

$$S_{kl} = |\langle u_k,v_l\rangle| \quad \text{and} \quad L_l(r) = \ln\left(\sum_k S_{kl}^{r'}\right)^{r/r'}\ ,$$

so that $\mu_r = \sup_l L_l(r)$. If for all l the derivative of L_l is negative, so is the derivative of μ_r. Since $\{u_k\}_k$ and $\{v_l\}_l$ are frames, $\sum_k S_{kl} > 0$ for all l and L_l is well-defined as well

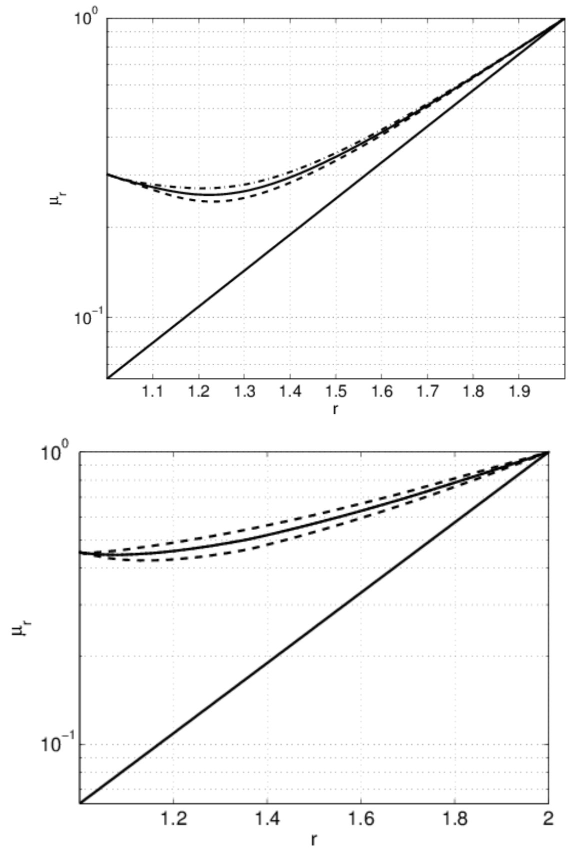

Fig. 1. Logarithm of r-coherence functions as a function of r. $\boldsymbol{\mu}_r(\mathcal{U},\mathcal{V})$ (dashed), $\boldsymbol{\mu}_r(\mathcal{V},\mathcal{U})$ (dash-dotted) and $\sqrt{\boldsymbol{\mu}_r(\mathcal{U},\mathcal{V})\boldsymbol{\mu}_r(\mathcal{V},\mathcal{U})}$ (full); straight line: mutually unbiased bases. Top: random bases $\mathcal{U}$ and $\mathcal{V}$. Bottom: MDCT bases with different window sizes.

as its derivative near $r = 1$, $r \geq 1$. This latter reads:

$$L_l'(r) = \ln n_l s_l + \sum_{k\in\Lambda} \alpha_{kl}^{\frac{r}{r-1}} + \frac{\ln s_l}{r-1} \sum_{k\in\Lambda} \alpha_{kl}^{\frac{r}{r-1}} + \mathcal{O}\left(\frac{1}{r-1}\left(\sum_{k\in\Lambda} \alpha_{kl}^{\frac{r}{r-1}}\right)^2\right),$$

where $\alpha_{kl} = |\langle u_k, v_l\rangle|/n_l s_l$ and Λ is the set of k such that $|\langle u_k, v_l\rangle| \neq s_l$. For r close to one, the dominant term is $\ln n_l s_l$. In this case, if $\max_l n_l s_l < 1$, the derivative of μ_r is negative. ♠

Remark 4: If two orthonormal bases have a high mutual coherence (μ_1), this implies that a single term is dominant and since in this case $s_l \leq 1$, Proposition 1 holds. However for frames with high coherence s_l is large for some l, and it is highly probable that its multiplicity be large as well. In this case, the conditions of the proposition are not satisfied and μ_r may not be smaller than μ_1. If there is too little coherence (like in MUB), n_l may be large and again μ_r may not be smaller than μ_1. Finally, for MUB, $n_l s_l = \sqrt{N}$, this is the slope of the curve plotted on Fig. 1.

III. Entropic inequalities

The support inequalities described above can also be obtained as particular limits of entropic inequalities, which have been derived during the last 20 years in the mathematical physics and information theory communities.

A. Entropies

In information theory, the notion of entropy is often used to measure disorder, or information content of a random source; entropy measures are basically related to measures of dispersion of the probability density function of the random variables under consideration.

In the context of sparse analysis, the coefficients of the decomposition of any finite-norm vector with respect to a frame can be turned into a probability distribution, after suitable normalization. With the same notations as before, denote by a the sequence of analysis coefficients of $x \in \mathcal{H}$ $(x \neq 0)$ with respect to the frame $\mathcal{U}$, and we set $\tilde{a} = a/\|a\|_2$. Given $\alpha \in [0,\infty]$ we introduce the corresponding Rényi entropy

$$R_\alpha(a) \stackrel{\Delta}{=} \frac{1}{1-\alpha} \ln\Big(\|\tilde{a}\|_{2\alpha}^{2\alpha} \Big) \ . \tag{14}$$

Rényi entropies fulfill a number of simple properties, among which we will use the following two: monotonicity and limit to Shannon's entropy. More precisely, for a given coefficient sequence a,

$$\alpha \leq \beta \quad \Longrightarrow \quad R_\alpha(a) \geq R_\beta(a) \ , \tag{15}$$

and

$$\lim_{\alpha\to 1} R_\alpha(a) = -\sum_n |\tilde{a}_n|^2 \ln\left(|\tilde{a}_n|^2\right) \stackrel{\Delta}{=} S(a) \ . \tag{16}$$

$S(a)$ is the Shannon entropy of the coefficient sequence. Also, notice that $R_0(a) = \ln \|a\|_0$. This will lead to support inequalities as consequences of Rényi entropy inequalities.

Uncertainty inequalities involving entropy measures have been derived in several different contexts (see [1], [6], [17] for example). We derive below similar inequalities in a more general setting.

B. Entropic uncertainty inequalities for frame expansions

As above, let us consider two frames $\mathcal{U}$ and $\mathcal{V}$. We use the same notations as in the previous section, and introduce the following additional constants: the geometric mean of the upper frame bounds $\boldsymbol{\rho}$, the geometric mean of frame bounds ratios $\boldsymbol{\sigma}$, and the normalized r-coherence $\boldsymbol{\nu}_r$, written as

$$\boldsymbol{\rho}(\mathcal{U},\mathcal{V}) \triangleq \sqrt{\frac{B_{\mathcal{V}}}{A_{\mathcal{U}}}}\ , \qquad \boldsymbol{\sigma}(\mathcal{U},\mathcal{V}) \triangleq \sqrt{\frac{B_{\mathcal{U}}B_{\mathcal{V}}}{A_{\mathcal{U}}A_{\mathcal{V}}}} \geq 1\ ,$$

$$\boldsymbol{\nu}_r(\mathcal{U},\widetilde{\mathcal{U}},\mathcal{V}) = \frac{\boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})}{\boldsymbol{\rho}(\mathcal{U},\mathcal{V})^r}\ . \tag{17}$$

For the sake of simplicity, we shall drop the r index in the case $r = 1$, and set $\boldsymbol{\mu} = \boldsymbol{\mu}_1$ and $\boldsymbol{\nu} = \boldsymbol{\nu}_1$. We then have the following theorem, which can be seen as a frame generalization of the Maassen-Uffink uncertainty inequality [17], [6]:

Theorem 2: Let $\mathcal{H}$ be a separable Hilbert space, let $\mathcal{U}$ and $\mathcal{V}$ be two frames of $\mathcal{H}$, and let $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ denote corresponding dual frames. Let $r \in [1,2)$. For all $\alpha \in [r/2,1]$, let $\beta = \alpha(r-2)/(r-2\alpha) \in [1,\infty]$.

For $x \in \mathcal{H}$, denote by a and b the sequences of analysis coefficient of x with respect to $\mathcal{U}$ and $\mathcal{V}$. Then the Rényi entropies satisfy the following bound:

$$\begin{aligned}(2-r)R_\alpha(a) + rR_\beta(b) \ \geq\ & -2\ln(\boldsymbol{\nu}_r(\mathcal{U},\tilde{\mathcal{U}},\mathcal{V})) \\ & -\frac{2r\beta}{\beta-1}\ln(\boldsymbol{\sigma}(\mathcal{U},\mathcal{V}))\end{aligned}$$

Proof: the proof is both a refinement and a frame generalization of the proof in [17], [6]. Let $T : a \to b$ denote the linear operator of change of coordinate. From the frame bounds, we obviously have the inequalities

$$\|b\|_2 \leq \sqrt{\frac{B_{\mathcal{V}}}{A_{\mathcal{U}}}}\|a\|_2\ , \qquad \|a\|_2 \leq \sqrt{\frac{B_{\mathcal{U}}}{A_{\mathcal{V}}}}\|b\|_2\ ,$$

so that we have the estimate

$$\|T\|_{2\to 2} \leq \sqrt{\frac{B_{\mathcal{V}}}{A_{\mathcal{U}}}} = \boldsymbol{\rho}(\mathcal{U},\mathcal{V})\ . \tag{18}$$

A second bound is obtained as in (7), and yields

$$\|T\|_{r\to\infty} = \boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})^{1/r} \ . \tag{19}$$

Let $p_0 = q_0 = 2$, $p_1 = r$, $q_1 = \infty$, and set for $\theta \in [0,1]$

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{r}\ , \qquad \frac{1}{q} = \frac{1-\theta}{2}\ .$$

Clearly, $1-\theta = 2/q$ and $\theta = 1 - 2/q = r(1/p - 1/q)$, and the Riesz-Thorin lemma yields the following bound.

$$\|T\|_{p\to q} \le \boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})^{(1-2/q)/r}\boldsymbol{\rho}(\mathcal{U},\mathcal{V})^{2/q}\ , \tag{20}$$

Using the definition of $\tilde{a}$ and $\tilde{b}$ and the frame bounds, we deduce

$$\begin{aligned}\|\tilde{b}\|_q &\le \boldsymbol{\rho}(\mathcal{U},\mathcal{V})^{2/q}\boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})^{1/p-1/q}\sqrt{\frac{B_{\mathcal{U}}}{A_{\mathcal{V}}}}\|\tilde{a}\|_p \\ &\le \boldsymbol{\sigma}(\mathcal{U},\mathcal{V})\boldsymbol{\nu}_r(\mathcal{U},\widetilde{\mathcal{U}},\mathcal{V})^{1/p-1/q}\|\tilde{a}\|_p\ ,\end{aligned} \tag{21}$$

where we have used the bound $\|a\|_2/\|b\|_2 \le \boldsymbol{\rho}(\mathcal{V},\mathcal{U})$ and the definition of $\boldsymbol{\nu}_r$ and $\boldsymbol{\sigma}$ in (17).

Set now $p = 2\alpha$ and $q = 2\beta$; taking logarithms, we get

$$\begin{aligned}\frac{1-\alpha}{2\alpha}R_\alpha(a) - \frac{1-\beta}{2\beta}R_\beta(b) \ &\ge\ -\left(\frac{1}{2\alpha} - \frac{1}{2\beta}\right)\ln(\boldsymbol{\nu}_r(\mathcal{U},\widetilde{\mathcal{U}},\mathcal{V})) \\ &\qquad -\ln(\boldsymbol{\sigma}(\mathcal{U},\mathcal{V}))\ ,\end{aligned}$$

Since $(\beta-1)/\beta = 1 - 2/q = r(1/2\alpha - 1/2\beta)$, this implies

$$\begin{aligned}\frac{\beta(1-\alpha)}{\alpha(\beta-1)}R_\alpha(a) + R_\beta(b) \ &\ge\ -\frac{2}{r}\ln\boldsymbol{\nu}_r(\mathcal{U},\widetilde{\mathcal{U}},\mathcal{V}) \\ &\qquad -2\frac{\beta-1}{\beta}\ln(\boldsymbol{\sigma}(\mathcal{U},\mathcal{V}))\ .\end{aligned}$$

Finally, explicit calculations give $\alpha = \beta r/(r + 2(\beta-1))$, so that

$$\frac{\beta(1-\alpha)}{\alpha(\beta-1)} = \frac{2-r}{r} \in [0,1]\ ,$$

which yields the desired result. ♠

Notice that since $(2-r)/r \in [0,1]$, this implies the (generally non sharp) inequality

$$R_\alpha(a) + R_\beta(b) \ge -\frac{2}{r}\ln(\boldsymbol{\nu}_r(\mathcal{U},\widetilde{\mathcal{U}},\mathcal{V})) - \frac{2\beta}{\beta-1}\ln(\boldsymbol{\sigma}(\mathcal{U},\mathcal{V}))$$

It is also worth noticing that in general, the limit $\alpha \to 1$ (which yields the sum of the Shannon entropies as left hand side) is non-informative, since the right hand side tends to $-\infty$, unless $\sigma = 1$, i.e. $\mathcal{U}$ and $\mathcal{V}$ are tight. In that case the following simplified inequalities hold true:

Corollary 3: Assume $\mathcal{U}$ and $\mathcal{V}$ are tight frames, and let $r \in [1,2)$:

1) For all $\alpha \in [r/2, 1]$, with $\beta = \alpha(r-2)/(2\alpha - r) \in [1,\infty]$

$$(2-r)R_\alpha(a) + rR_\beta(b) \geq -2\ln(\boldsymbol{\nu}_r(\mathcal{U},\widetilde{\mathcal{U}},\mathcal{V}))\ . \tag{22}$$

2) the following inequalities between Shannon entropies hold true:

$$S(a) + S(b) \geq -2\ln\left(\boldsymbol{\mu}_*(\mathcal{U},\widetilde{\mathcal{U}},\mathcal{V},\widetilde{\mathcal{V}})\right)\ , \tag{23}$$

where $\boldsymbol{\mu}_*$ is defined in (6).

<u>Proof:</u> the first item is a direct consequence of the previous theorem in the case of tight frames. For the second item, remark that from the monotonicity of the Rényi entropy, we obtain $(2-r)S(a) + rS(b) \geq (2-r)R_\alpha(a) + rR_\beta(b)$. Remark also that for tight frames,

$$\begin{aligned}\boldsymbol{\nu}_r(\mathcal{U},\widetilde{\mathcal{U}},\mathcal{V})\boldsymbol{\nu}_r(\mathcal{V},\widetilde{\mathcal{V}},\mathcal{U}) &= \frac{\boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})\boldsymbol{\mu}_r(\widetilde{\mathcal{V}},\mathcal{U})}{\sigma(\mathcal{U},\mathcal{V})^r} \\ &= \boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})\boldsymbol{\mu}_r(\widetilde{\mathcal{V}},\mathcal{U})\ .\end{aligned}$$

Symmetrizing the bound on Shannon entropies yield the desired result. ♠

Notice that owing to the monotonicity property of Rényi entropies, $R_0(a) = \ln(\|a\|_0) \geq S(a)$, and we recover the generalized Elad Bruckstein inequality

$$\|a\|_0.\|b\|_0 \geq \frac{1}{\boldsymbol{\mu}_*(\mathcal{U},\widetilde{\mathcal{U}},\mathcal{V},\widetilde{\mathcal{V}})^2}\ .$$

Similar results in the general case are discussed below.

C. Consequence: ℓ^p inequalities for analysis frame coefficients

Let us start again from the modified entropic inequality in Theorem 2, and symmetrize it with respect to a and b. We obtain

$$\begin{aligned}&(2-r)(R_\alpha(a)+R_\alpha(b)) + r(R_\beta(a)+R_\beta(b)) \geq \\ &\qquad -2\ln\left(\frac{\boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})\boldsymbol{\mu}_r(\widetilde{\mathcal{V}},\mathcal{U})}{\sigma(\mathcal{U},\mathcal{V})^r}\right) - \frac{4r\beta}{\beta-1}\ln(\sigma(\mathcal{U},\mathcal{V}))\ .\end{aligned}$$

Using the monotonicity of Rényi entropies, i.e. $R_\alpha \geq R_\beta$, we then get for all $\alpha \in [r/2, 1]$

$$\begin{aligned} R_\alpha(a) + R_\alpha(b) &\geq -\ln\left(\boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})\boldsymbol{\mu}_r(\widetilde{\mathcal{V}},\mathcal{U})\right) \\ &\quad -r\frac{\beta+1}{\beta-1}\ln(\sigma(\mathcal{U},\mathcal{V}))\ , \end{aligned}$$

thus

$$\begin{aligned} \ln\left(\|\tilde{a}\|_{2\alpha}.\|\tilde{b}\|_{2\alpha}\right) &\geq -\frac{1-\alpha}{2\alpha}\ln\left(\boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})\boldsymbol{\mu}_r(\widetilde{\mathcal{V}},\mathcal{U})\right) \\ &\quad -r\frac{(\beta+1)(1-\alpha)}{2\alpha(\beta-1)}\ln(\sigma(\mathcal{U},\mathcal{V})) \\ &\geq \left(\frac{1}{2}-\frac{1}{2\alpha}\right)\ln\left(\boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})\boldsymbol{\mu}_r(\widetilde{\mathcal{V}},\mathcal{U})\right) \\ &\quad -\left(1-\frac{r}{2}\right)\left(1+\frac{r-2\alpha}{\alpha r-2\alpha}\right)\ln(\sigma(\mathcal{U},\mathcal{V})) \end{aligned}$$

finally yields the bound, for $p \in [r, 2]$

$$\begin{aligned} \|a\|_p.\|b\|_p \geq &\left(\boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})\boldsymbol{\mu}_r(\widetilde{\mathcal{V}},\mathcal{U})\right)^{\frac{1}{2}-\frac{1}{p}} \\ &\sigma(\mathcal{U},\mathcal{V})^{-(1-\frac{r}{2})\left(1+\frac{r-p}{p}(1-\frac{r}{2})\right)}\|a\|_2.\|b\|_2\ . \end{aligned} \tag{24}$$

Also, using the fact that $R_0(a) = \ln(\|a\|_0) \geq R_\alpha(a)$ for all $\alpha \in [r/2, 1]$, and specifying to the sharpest bound $\alpha = r/2$, we also obtain

$$\ln\left(\|a\|_0.\|b\|_0\right) \geq -\ln\left(\boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})\boldsymbol{\mu}_r(\widetilde{\mathcal{V}},\mathcal{U})\right) - r\ln(\sigma(\mathcal{U},\mathcal{V}))\ ,$$

which yields

$$\|a\|_0.\|b\|_0 \geq \sigma(\mathcal{U},\mathcal{V})^{-r}\frac{1}{\boldsymbol{\mu}_r(\widetilde{\mathcal{U}},\mathcal{V})\boldsymbol{\mu}_r(\widetilde{\mathcal{V}},\mathcal{U})}\ . \tag{25}$$

It is worth noticing that this bound is similar to the support inequalities obtained previously, except for the factor $\sigma(\mathcal{U},\mathcal{V})^{-r}$, which makes it weaker. Thus the bound is equivalent to the previous one if and only if the frames are tight. Notice also that sharper bounds are obtained, as before, by optimizing with respect to r and the dual frames $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ of $\mathcal{U}$ and $\mathcal{V}$.

D. Remark: necessary conditions for equality in the tight case

We now examine conditions for the entropic inequalities be saturated. Our aim is to make the connection with the *constant on support* property we already met in Theorem 1 and its proof. Since the entropic bounds we could prove are not sharp in generic situations, we limit the present discussion to the particular case of tight frames. Let $\mathcal{U}$ and $\mathcal{V}$ be two tight frames, denote by $A_{\mathcal{U}} = B_{\mathcal{U}}$ and $A_{\mathcal{V}} = B_{\mathcal{V}}$ the corresponding frame constants, and set

$$g_{k\ell} = \langle u_\ell, v_k \rangle \ . \tag{26}$$

Straightforward calculations give

$$\frac{\partial}{\partial \overline{a}_\ell} \ln(\|a\|_{2\alpha}^{2\alpha}) = \frac{\alpha}{\overline{a}_\ell} \frac{|a_\ell|^{2\alpha}}{\|a\|_{2\alpha}^{2\alpha}} \ ,$$

and

$$\frac{\partial}{\partial \overline{a}_\ell} \ln(\|b\|_{2\beta}^{2\beta}) = \sum_{k=0}^{N-1} \overline{g}_{k\ell} \frac{\beta}{\overline{b}_k} \frac{|b_k|^{2\beta}}{\|b\|_{2\beta}^{2\beta}}$$

and therefore the variational equations associated with the optimization of $(2/r - 1)R_\alpha(a) + R_\beta(b)$ under constraint $\|x\| = 1 = \|a\|/\sqrt{A}$ read

$$\frac{2-r}{r} \frac{\alpha}{1-\alpha} \frac{1}{\overline{a}_\ell} \frac{|a_\ell|^{2\alpha}}{\|a\|_{2\alpha}^{2\alpha}} + \frac{\beta}{1-\beta} \sum_{k=0}^{N-1} \overline{g}_{k\ell} \frac{1}{\overline{b}_k} \frac{|b_k|^{2\beta}}{\|b\|_{2\beta}^{2\beta}} = \lambda \frac{a_\ell}{A_{\mathcal{U}}} \ ,$$

where λ is a Lagrange multiplier. Now remark that $\beta/(1-\beta) = -\alpha(2-r)/r(1-\alpha)$; multiplying both sides with $\overline{a}_k$ and summing over k, the constraint $\|x\| = \|a\|/\sqrt{A_{\mathcal{U}}} = 1$ gives $\lambda = 0$, so that the variational equations take the form, for $\alpha \neq 1$

$$\frac{|a_\ell|^{2(\alpha-1)}}{\|a\|_{2\alpha}^{2\alpha}} a_\ell = \frac{1}{A_{\mathcal{U}}} \sum_{k=0}^{N-1} \overline{g}_{k\ell} \frac{|b_k|^{2(\beta-1)}}{\|b\|_{2\beta}^{2\beta}} b_k \ . \tag{27}$$

Remark 5: From the above expression, we can observe that $|a|$ is constant on its support if and only if $|b|$ is, since $\sum_k \overline{g}_{k\ell} b_k = a_l$. In this situation, we have $|a_k| = \sqrt{A_{\mathcal{U}}/\|a\|_0}$ for all $k \in \text{supp}(a)$ and $|b_k| = \sqrt{A_{\mathcal{V}}/\|b\|_0}$ for all $k \in \text{supp}(b)$, so that

$$R_\alpha(a) + R_\beta(b) = \ln\left(\|a\|_0 . \|b\|_0\right) \ ,$$

which therefore saturates the inequalities.
Similar calculations on the Shannon entropy yield a comparable result.

IV. Conclusions

We have examined in this paper entropic and ℓ^p uncertainty principles in the framework of frame expansions. Our main results are extensions of support and entropic uncertainty principles to the case of frames, which turn out to generalize some known results when specializing to orthonormal bases. We showed in particular that in general situations, bounds involving the classical mutual coherence of the frames or bases under considerations are outperformed by the new bounds involving generalized coherences.

While ℓ^p uncertainty principles have been mainly exploited in the framework of sparse expansion problems, i.e. synthesis based approaches, our results fit better into the so-called *analysis frameworks* (see e.g. [19]), as shortly explained in the introduction. Practical consequences for co-sparse signal approximation and decomposition approaches are still to be investigated further. This is ongoing work by the authors of the present paper.

Let us mention that the finite dimensional case is by now fairly well understood, and the existence of optimizers for the uncertainty inequalities is closely connected to coefficient sequences that are constant on their support, as already remarked by [20]. In the infinite-dimensional case, such *constant on support* properties do not make much sense in general situations, and the optimization problem is still to be investigated much further.

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