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Decentralized Laplacian Eigenvalues Estimation and Collaborative Network Topology Identification

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Abstract: In this paper we first study observability conditions on networks. Based on spectral properties of graphs, we state new sufficient or necessary conditions for observability. These conditions are based on properties of the Khatri-Rao product of matrices. Then we consider the problem of estimating the eigenvalues of the Laplacian matrix associated with the graph modeling the interconnections between the nodes of a given network. Eventually, we extend the study to the identification of the network topology by estimating both eigenvalues and eigenvectors of the network matrix. In addition, we show how computing, in finite-time, some linear functionals of the state initial condition, including average consensus. Specifically, based on properties of the observability matrix, we show that Laplacian eigenvalues can be recovered by solving a local eigenvalue decomposition on an appropriately constructed matrix of observed data. Unlike FFT based methods recently proposed in the literature, in the approach considered herein, we are also able to estimate the multiplicities of the eigenvalues. Then, for identifying the network topology, the eigenvectors are estimated by means of a consensus-based least squares method.

Keywords: Observability, Laplacian eigenvalues, Consensus, Network topology identification.

1. INTRODUCTION

Identification of large-scale graphs or networks of systems is an important task for solving many problems arising in different science and engineering domains. Indeed, complex networks widely exist in both natural and man-made systems. Several current research works are focused on studying the impact of the network topology on the network dynamic behavior. In particular, several recent contributions are related to stability of networks, convergence study in consensus networks, among others. The reverse side of the interaction between the network structure and the network dynamics allows inferring the network topology from the observations of the network dynamics.

Many network dynamics can be modeled as linear dynamic systems where the interaction matrix, the state matrix in a state-space representation, is related to the network topology. Therefore, from a system theory point of view, the question of network topology identification is strongly related to that of state matrix identification with unknown inputs and initial condition. Such a problem is also called a blind identification problem. Sometimes, instead of identifying the overall network matrix, important informations on the properties of the network topology, or equivalently of the underlying graph, can be extracted from the eigenvalues of the associated adjacency, Laplacian, or any other type of matrix associated with the graph, Godsil and Royle (2001). Indeed, from the graph spectrum, one can infer the algebraic connectivity that influences the performance and robustness of network controlled systems, the graph diameter, and the connectedness of the graph among others.

The eigenvalues of the adjacency matrix and that of the Laplacian matrix were particularly investigated in the recent past (see for example Chung (1997) and Merris (1994)). Most of the algorithms for estimating the Laplacian eigenvalues are based on centralized methods requiring the knowledge of the overall network matrix. Some contributions for decentralizing such an estimation have been recently reported in the literature. In Yang et al. (2008), a power iteration based method is proposed. Due to approximations for decentralizing the power iteration method, only approximate solution can be guaranteed. More recently, FFT based methods have been proposed in Franceschelli et al. (2009) and in Sahai et al. (2010). The main idea is: based on a specific interaction protocol, the time series observed at each node exhibit a harmonic behavior related to the Laplacian spectrum. As a consequence, by performing a local FFT, the locations of the peaks on the computed spectrogram are related to the Laplacian eigenvalues whereas their magnitudes can serve to retrieve some components of the eigenvectors (in Sahai et al. (2010)). With such approaches, the estimation of the Laplacian spectrum is incomplete since the multiplicities of the eigenvalues cannot be inferred. Moreover, the resolution of the estimated eigenvalues is strongly dependent on that of the FFT method.

In contrast to these approaches, in this paper, we resort to an algebraic method using observability properties of the network. The observability of networks, and consensus networks in particular, is the topic of several recent contributions. In Ji and Egerstedt (2007), necessary conditions for observability are derived based on equitable partitions of a graph and the interlacing theory. More recently, in Parlangeli and Notarstefano (2012), necessary and sufficient conditions have been provided for path and cycle graphs. Based on some of these observability properties, we show that the eigenvalues of the network matrix can be recovered by solving a local eigenvalue decomposition on an appropriately constructed matrix of observed data. Obviously, the proposed method is well indicated for networks having nodes with sufficient storage and computation capabilities.

The paper is organized as follows. We first study the case where the unknown network is a Laplacian consensus network. Then, we extend the study to any arbitrary network. As in Sundaram and Hadjicostis (2008), we show how computing eigenvalues and some linear functionals of the initial condition with a finite number of observations. While the above mentioned quantities can be computed strictly locally, we show that estimating the network eigenvectors necessitates collaboration among nodes.

Notations: Vectors are written as boldface lower-case letters ($\mathbf{a},\mathbf{b},\cdots$) and matrices as boldface capitals ($\mathbf{A},\mathbf{B},\cdots$). $\mathbf{A}_{i.}$ and $\mathbf{A}_{.j}$ denote respectively the ith row and the jth column of the $I\times J$ matrix $\mathbf{A}$. $diag(.)$ is the operator that forms a diagonal matrix from its vector argument whereas $vec(.)$ forms a vector by stacking the columns of its matrix argument. $vecd(.)$ represents the vector formed with the diagonal entries of the matrix in argument. $\mathbf{e}_{i,N}$ stands for the ith vector of the canonical basis of $\Re^N$.

Definition 1. For $\mathbf{X}\in\mathbb{C}^{I\times R}$, and $\mathbf{Y}\in\mathbb{C}^{J\times R}$, the Khatri-Rao product, denoted by $\odot$, is defined as follows:

$$\mathbf{X}\odot\mathbf{Y}=\begin{pmatrix}\mathbf{Y}diag(\mathbf{X}_{1.})\\ \mathbf{Y}diag(\mathbf{X}_{2.})\\ \vdots\\ \mathbf{Y}diag(\mathbf{X}_{I.})\end{pmatrix}\in\mathbb{C}^{IJ\times R}. \quad (1)$$

It can be viewed as a columnwise Kronecker product:

$$\mathbf{X}\odot\mathbf{Y}=\left(\,\mathbf{X}_{.1}\otimes\mathbf{Y}_{.1}\ \cdots\ \mathbf{X}_{.R}\otimes\mathbf{Y}_{.R}\,\right)$$

where $\otimes$ denotes the Kronecker product.

The $vec(.)$ operator and both Kronecker and Khatri-Rao products are related through the following properties:

$$vec(\mathbf{ABC})=\left(\mathbf{C}^T\otimes\mathbf{A}\right)vec(\mathbf{B}) \quad (2)$$

$$vec(\mathbf{A}diag(\mathbf{b})\mathbf{C})=\left(\mathbf{C}^T\odot\mathbf{A}\right)\mathbf{b} \quad (3)$$

Definition 2. The k-rank (Kruskal rank), $k_{\mathbf{X}}$, of an $I\times R$ matrix $\mathbf{X}$ is the maximal number k such that any set of k columns of $\mathbf{X}$ is linearly independent.

The rank and the k-rank of an $I\times R$ matrix $\mathbf{X}$ are related through the following inequality:

$$k_{\mathbf{X}}\le rank(\mathbf{X})\le\min\{I,R\}.$$

Now, let us recall the following lemma stated in Sidiropoulos et al. (2000):

Lemma 1. Consider the matrices $\mathbf{X}\in\Re^{I\times R}$ and $\mathbf{Y}\in\Re^{J\times R}$ with respective k-rank $k_{\mathbf{X}}$ and $k_{\mathbf{Y}}$. The khatri-Rao product $\mathbf{X}\odot\mathbf{Y}$ results on a full rank matrix if $k_{\mathbf{X}}+k_{\mathbf{Y}}\ge R+1$.

2. ON OBSERVABILITY OF DISCRETE-TIME NETWORKED SYSTEMS

Let us consider a network of N distributed nodes. The interactions, or information exchanges, between these nodes are modeled with a connected undirected graph $\mathscr{G}=(\mathscr{V},\mathscr{E})$, where $\mathscr{V}=\{1,\cdots,N\}$ denotes the vertex set whereas $\mathscr{E}\subseteq\mathscr{V}\times\mathscr{V}$ is the set of edges. We denote by $\mathscr{N}_i$ the set of nodes that interact with the ith node. Its cardinality, denoted N_i, is called the degree of the ith node. On important matrix characterizing a graph is the so called graph Laplacian matrix $\mathbf{L}$, with entries l_{ij}, defined as,

$$l_{i,j}=\begin{cases}N_i & \text{if} \quad i=j\\ -1 & \text{if} \quad j\in\mathscr{N}_i\\ 0 & \text{elsewhere}\end{cases}$$

Suppose that each node i has some initial values organized in a M-length row vector $\mathbf{x}_i(0)$. We assume that $M\ge N$. At each time-step k, the nodes exchange and update their values following a linear iteration scheme, i.e.

$$\mathbf{x}_i(k+1)=w_{ii}\mathbf{x}_i(k)+\sum_{j\in\mathscr{N}_i}w_{ij}\mathbf{x}_j(k), \quad (4)$$

where the $w_{i,j}$ are the entries of a consensus matrix $\mathbf{W}$, which is consistent with the graph $\mathscr{G}$. $\mathbf{W}$ is assumed to be symmetric. In matrix form, equation (4) can be written as:

$$\mathbf{X}(k+1)=\mathbf{W}\mathbf{X}(k), \quad (5)$$

where $\mathbf{X}(k)$ denotes the $N\times M$ matrix with $\mathbf{x}_i(k)$, $i=1,\cdots,N$, as rows.

Now, let us define by $\mathbf{E}_i$ the $\overline{N}_i\times N$ row selection matrix such that $\mathbf{Y}_i(k)=\mathbf{E}_i\mathbf{X}(k)$ be the outputs or node values that are seen by node i at the kth time-step. Note that $\overline{N}_i$ is bounded as $0<\overline{N}_i\le N_i+1$. We get the following dynamic representation of the network as viewed by node i:

$$\begin{aligned}\mathbf{X}(k+1)&=\mathbf{W}\mathbf{X}(k)\\ \mathbf{Y}_i(k)&=\mathbf{E}_i\mathbf{X}(k).\end{aligned} \quad (6)$$

From observability theory, it is well known that the pair $(\mathbf{W},\mathbf{E}_i)$ is observable if and only if the observability matrix $\mathbf{O}_i\in\Re^{N\overline{N}_i\times N}$ defined as

$$\mathbf{O}_i=\begin{pmatrix}\mathbf{E}_i\\ \mathbf{E}_i\mathbf{W}\\ \mathbf{E}_i\mathbf{W}^2\\ \vdots\\ \mathbf{E}_i\mathbf{W}^{N-1}\end{pmatrix} \quad (7)$$

is full column rank, i.e. $rank(\mathbf{O}_i)=N$.

Owing to the symmetry of $\mathbf{W}$, considering its eigenvalue decomposition $\mathbf{W}=\mathbf{U}\mathbf{D}\mathbf{U}^T$, where the eigenvectors and the eigenvalues are respectively organized in the orthogonal matrix $\mathbf{U}$ and the diagonal matrix $\mathbf{D}$, we can rewrite the observability matrix as

$$\mathbf{O}_i=\overline{\mathbf{O}}_i\mathbf{U}^T, \quad \text{with} \quad \overline{\mathbf{O}}_i=\begin{pmatrix}\mathbf{V}_i\\ \mathbf{V}_i\mathbf{D}\\ \mathbf{V}_i\mathbf{D}^2\\ \vdots\\ \mathbf{V}_i\mathbf{D}^{N-1}\end{pmatrix} \quad \text{and} \quad \mathbf{V}_i=\mathbf{E}_i\mathbf{U}.$$

One can note that $rank(\mathbf{O}_i)=rank(\overline{\mathbf{O}}_i)$. Thereferore, observability properties of the system (6) can be derived by studying the matrix $\overline{\mathbf{O}}_i$.

Indeed, from the definition of the Khatri-Rao product (1), we can rewrite the matrix $\overline{\mathbf{O}}_i$ as the Khatri-Rao product of a Vandermonde matrix

$$\mathbf{\Delta}=\left(\,\mathrm{vecd}(\mathbf{D}^0)^T\ \ \mathrm{vecd}(\mathbf{D})^T\ \ \mathrm{vecd}(\mathbf{D}^2)^T\ \cdots\ \mathrm{vecd}(\mathbf{D}^{N-1})^T\,\right)^T$$

with $\mathbf{V}_i$, i.e. $\overline{\mathbf{O}}_i = \mathbf{\Delta} \odot \mathbf{V}_i$. Therefore, we can rewrite the observability matrix $\mathbf{O}_i$ as

$$\mathbf{O}_i = (\mathbf{\Delta} \odot \mathbf{V}_i)\mathbf{U}^T. \tag{8}$$

The direct consequence of the above rewriting of the observability matrix is stated in the following lemma:

Lemma 2. The pair $(\mathbf{W}, \mathbf{E}_i)$ is observable if and only if the Khatri-Rao product $\mathbf{\Delta} \odot \mathbf{V}_i$ is full column rank.

Now, from lemmas 1 and 2, we can deduce the following theorem:

Theorem 1. Consider the system (6) where all the eigenvalues of $\mathbf{W}$ are simple and nonzero. The pair $(\mathbf{W}, \mathbf{E}_i)$ is observable if the matrix $\mathbf{E}_i\mathbf{U}$, with $\mathbf{U}$ the matrix of eigenvectors of $\mathbf{W}$, has no column with all zero elements.

Proof: From Lemma 2, the pair $(\mathbf{W}, \mathbf{E}_i)$ is observable if $\mathbf{\Delta} \odot \mathbf{V}_i$ is full column rank. From Lemma 1, we also know that a sufficient condition to get a full rank Khratri-Rao product is to have $k_{\mathbf{\Delta}} + k_{\mathbf{V}_i} \geq N + 1$. Since all the eigenvalues of $\mathbf{W}$ are all simple and nonzero the Vandermonde matrix $\mathbf{\Delta}$ is not only full column rank but also full k-rank, i.e. $k_{\mathbf{\Delta}} = N$. Therefore it suffices to have $k_{\mathbf{V}_i} \geq 1$ to ensure a full rank Khatri-Rao product. Such a condition is fulfilled if all the columns of $\mathbf{V}_i$ are nonzero. ■

We can also derive the following corollaries for the specific case where each node does not observe the values of its neighbors. In this case, the observation matrix $\mathbf{E}_i$ is restricted to a vector $\mathbf{e}_{i,N}^T$, which is the transpose of the ith vector of the canonical basis of $\Re^N$.

Corollary 1. Consider the system (6) where all the eigenvalues of $\mathbf{W}$ are simple and nonzero. The system (6) is observable from a single node i if and only if the ith row of eigenvector matrix of $\mathbf{W}$, i.e. $\mathbf{e}_{i,N}^T\mathbf{U}$, has no zero elements.

Corollary 2. Consider the system (6). If all the eigenvalues of $\mathbf{W}$ are simple and nonzero and if all the entries of the eigenvector matrix are nonzero then the system is observable from any single node.

The results above give some sufficient conditions ensuring observability in graphs where the matrix $\mathbf{W}$ has only simple eigenvalues. That is the case of paths. Indeed, it can be shown that the eigenvalues of a path are all simple. Therefore, the matrix $\mathbf{W} = \mathbf{I} - \varepsilon\mathbf{L}$, with ε an arbitrary nonzero scalar, has nonzero simple eigenvalues.

Example 1. Let us consider a consensus protocol through a path with 5 nodes. The consensus matrix being $\mathbf{W} = \mathbf{I} - 0.2\mathbf{L}$. The corresponding eigenvalues are 0.2764; 0.4764; 0.7236; 0.9236; 1.0000. The corresponding matrix of eigenvectors is given by:

$$\mathbf{U} = \begin{pmatrix} 0.1954 & -0.3717 & -0.5117 & -0.6015 & 0.4472 \\ -0.5117 & 0.6015 & 0.1954 & -0.3717 & 0.4472 \\ 0.6325 & 0 & 0.6325 & 0 & 0.4472 \\ -0.5117 & -0.6015 & 0.1954 & 0.3717 & 0.4472 \\ 0.1954 & 0.3717 & -0.5117 & 0.6015 & 0.4472 \end{pmatrix}$$

Applying Corrollary 1, we can deduce that the third node cannot observe the system using only its own observations. However, if observations of at least one of its neighbors are available then applying Theorem 1, the system is now observable. In Fig. 1, a circle represents a node that is able to observe the system from its own observations while a circle with a square inside represents a node that is unable to observe the system without the observations of at least one of its neighbors.

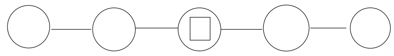

Fig. 1. Observability in a path with five nodes.

Note that necessary and sufficient observability conditions of paths have been recently proposed by Parlangeli and Notarstefano (2012) for systems whose state matrix is given by the graph Laplacian. Herein, for a more general state matrix we get similar results.

Now, let us consider the case of state matrices having at least one eigenvalue with multiplicity higher than 1. We state the following necessary condition:

Theorem 2. The system (6) with matrix $\mathbf{W}$ having D distinct nonzero eigenvalues is observable from node i only if

$$D\overline{N}_i \geq N, \tag{9}$$

with $0 < \overline{N}_i \leq N_i + 1$.

Proof: Knowing that a Khatri-rao product corresponds to a columnwise Kronecker product, it exists a column selection matrix $\mathbf{S}_{N^2,N}$ such that $\mathbf{\Delta} \odot \mathbf{V}_i = (\mathbf{\Delta} \otimes \mathbf{V}_i)\mathbf{S}_{N^2,N}$. As a consequence $rank(\mathbf{O}_i) \leq \min(rank(\mathbf{\Delta})rank(\mathbf{V}_i), rank(\mathbf{S}_{N^2,N}))$. One can note that the rank of the Vandermonde matrix $\mathbf{\Delta}$ is equal to the number of distinct generators, or equivalently the number of distinct entries in $\mathbf{D}$. Hence, $rank(\mathbf{\Delta}) = D$. We also have $rank(\mathbf{V}_i) = rank(\mathbf{E}_i) = \overline{N}_i$ and $rank(\mathbf{S}_{N^2,N}) = N$. If the observability matrix is full rank, then $N \leq min(D\overline{N}_i, N)$ that yields (9). ■

As shown in Parlangeli and Notarstefano (2012), we can conclude that if $\mathbf{W}$ has at least one eigenvalue with multiplicity higher or equal to 2, then the graph is not observable from a single node. Indeed, studying observability from a single node implies $\overline{N}_i = 1$. Thus condition (9) yields $D \geq N$; condition that is violated if at least one eigenvalue has multiplicity higher than one.

An application of theorem 2 concerns strongly regular graphs. Recall that a strongly regular graph (SRG) with parameters (n, k, a, c) is a graph on n vertices which is regular with valency (degree) k and has the following properties:

- any two adjacent vertices have exactly a common neighbors;
- any two nonadjacent vertices have exactly c common neighbors.

It is well known that a SRG has exactly 3 distinct eigenvalues, Godsil and Royle (2001). Therefore, the necessary condition (9) becomes:

$$k + 1 \geq \frac{n}{3}.$$

For some families of SRGs we can draw the following conclusions:

- A $n \times n$ *Rook's graph* (also known as two-dimensional Hamming graph or Latin square), a SRG with parameters $(n^2, 2n-2, n-2, 2)$ is not observable from any node if $n \geq 6$.
- The following SRGs are not observable: Brouwer-Haemers (SRG(81,20,1,6)), Higman-Sims (SRG(100,22,0,6)), M22 (SRG(77,16,0,4)), Hoffman-Singleton (SRG(50,7,0,1)),

Sims-Gewirtz (SRG(56,10,0,2)). (see Godsil and Royle (2001) and Brouwer et al. (1989) for more descriptions of these graphs).

Now, based on observability properties of consensus systems, we derive in the two next sections new methods for computing the eigenvalues of a consensus matrix in a decentralized way. We first start on the crucial issue of estimating the Laplacian eigenvalues before considering a more general setup.

3. DECENTRALIZED ESTIMATION OF LAPLACIAN EIGENVALUES

In this section, we investigate solutions for a given node i to estimate the eigenvalues λ_i, $i=1,\cdots,N$, of the Laplacian matrix from its observations.

Recently Franceschelli et al. (2009) and Sahai et al. (2010) have proposed methods based on FFT using the observations associated with specific data exchange protocols. These methods lead to decentralized estimation of Laplacian eigenvalues. However, the main drawbacks of FFT based methods is the impossibility to estimate the multiplicity of the eigenvalues. In addition, these methods inherit of the drawbacks of FFT based methods concerning the resolution between two peaks. In what follows, we show that based on a consensus protocol, each node i can estimate the Laplacian eigenvalues and their multiplicities using local observations. For this purpose, we consider system (6) with a consensus matrix given by

$$\mathbf{W} = \mathbf{I} - \varepsilon\mathbf{L}.$$

Our aim is then to estimate the eigenvalues of the Laplacian matrix from the observations $\mathbf{Y}_i(k)$, $k=0,1,\cdots,K_i$, where K_i stands for the observability index of the pair $(\mathbf{W},\mathbf{E}_i)$. We define by $\mathbf{O}_{i,K_i} \in \Re^{(K_i+1)\overline{N}_i \times N}$ the following sub-matrix of the observability matrix $\mathbf{O}_i$:

$$\mathbf{O}_{i,K_i} = \begin{pmatrix} \mathbf{E}_i \\ \mathbf{E}_i\mathbf{W} \\ \mathbf{E}_i\mathbf{W}^2 \\ \vdots \\ \mathbf{E}_i\mathbf{W}^{K_i} \end{pmatrix}. \quad (10)$$

By considering the eigenvalue decomposition of the Laplacian matrix, we get the following factorization: $\mathbf{L} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$, where $\mathbf{U}$ and $\mathbf{\Lambda}$ denote respectively the orthogonal matrix of eigenvectors and the diagonal matrix of eigenvalues ordered in ascending order. Therefore we get:

$$\mathbf{O}_{i,K_i} = \begin{pmatrix} \mathbf{V}_i \\ \mathbf{V}_i\mathbf{D} \\ \mathbf{V}_i\mathbf{D}^2 \\ \vdots \\ \mathbf{V}_i\mathbf{D}^{K_i} \end{pmatrix} \mathbf{U}^T = \overline{\mathbf{O}}_{i,K_i}\mathbf{U}^T, \quad (11)$$

with $\mathbf{V}_i = \mathbf{E}_i\mathbf{U}$ and $\mathbf{D} = \mathbf{I} - \varepsilon\mathbf{\Lambda}$.

Let us now construct the matrices $\overline{\mathbf{Y}}_i$ and $\overline{\overline{\mathbf{Y}}}_i$ with the available observations $\{\mathbf{Y}_i(k)\}$, $k=0,1,\cdots,K_i+1$:

$$\overline{\mathbf{Y}}_i = \begin{pmatrix} \mathbf{Y}_i(0) \\ \mathbf{Y}_i(1) \\ \vdots \\ \mathbf{Y}_i(K_i) \end{pmatrix} \quad \overline{\overline{\mathbf{Y}}}_i = \begin{pmatrix} \mathbf{Y}_i(1) \\ \mathbf{Y}_i(2) \\ \vdots \\ \mathbf{Y}_i(K_i+1) \end{pmatrix} \quad (12)$$

We can show that:

$$\overline{\mathbf{Y}}_i = \overline{\mathbf{O}}_{i,K_i}\mathbf{C}^T$$

and

$$\overline{\overline{\mathbf{Y}}}_i = \overline{\mathbf{O}}_{i,K_i}\mathbf{D}\mathbf{C}^T,$$

with $\mathbf{C}^T = \mathbf{U}^T\mathbf{X}(0)$.

Let us state the following theorem:

Theorem 3. Consider the observations $\{\mathbf{Y}_i(k)\}_{k=0,1,\cdots,K_i+1}$, at node i, organized in matrices $\overline{\mathbf{Y}}_i$ and $\overline{\overline{Y}}_i$ defined in (12), and the matrix $\widetilde{\mathbf{U}}$ of left singular vectors of the matrix $(\overline{\mathbf{Y}}_i^T \quad \overline{\overline{Y}}_i^T)^T$. If the pair $(\mathbf{W},\mathbf{E}_i)$, with $\mathbf{W} = \mathbf{I} - \varepsilon\mathbf{L}$, is observable with K_i as observability index and if $\mathbf{C}$ is full column rank then node i can compute all the eigenvalues of the Laplacian matrix $\mathbf{L}$ as $\mathbf{\Lambda} = \frac{1}{\varepsilon}(\mathbf{I}-\mathbf{D})$, where $\mathbf{D}$ results on the eigenvalues decomposition of the matrix $\left(\widetilde{\mathbf{U}}_1^T\widetilde{\mathbf{U}}_2\right)\left(\widetilde{\mathbf{U}}_1^T\widetilde{\mathbf{U}}_1\right)^{-1}$, with $\widetilde{\mathbf{U}} = \left(\widetilde{\mathbf{U}}_1^T \ \ \widetilde{\mathbf{U}}_2^T\right)^T$, the two sub-matrices $\widetilde{\mathbf{U}}_1$ and $\widetilde{\mathbf{U}}_2$ having the same number of rows.

Proof: From observability theory, if the pair $(\mathbf{W},\mathbf{E}_i)$ is observable with K_i as observability index, it is well known that $\mathbf{O}_{i,K_i}$ is full column rank. According to the results of section 2, $\overline{O}_{i,K_i}$ is also full column rank. Therefore, if $\mathbf{C}$ is also full column rank, knowing that $\overline{\mathbf{Y}}_i = \overline{\mathbf{O}}_{i,K_i}\mathbf{C}^T$, we can can conclude that the block matrix $\left(\overline{\mathbf{Y}}_i^T \ \overline{\overline{\mathbf{Y}}}_i^T\right)^T$ is full column rank. Now, let us consider the singular value decomposition (SVD) of the above defined matrix:

$$\begin{pmatrix} \overline{\mathbf{Y}}_i \\ \overline{\overline{\mathbf{Y}}}_i \end{pmatrix} = \begin{pmatrix} \overline{\mathbf{O}}_{i,K_i} \\ \overline{\mathbf{O}}_{i,K_i}\mathbf{D} \end{pmatrix}\mathbf{C}^T = \widetilde{\mathbf{U}}\mathbf{\Sigma}\widetilde{\mathbf{V}}^T.$$

$\mathbf{C}$ being full column rank then $span(\widetilde{\mathbf{U}}) = span(\begin{pmatrix} \overline{\mathbf{O}}_{i,K_i} \\ \overline{\mathbf{O}}_{i,K_i}\mathbf{D} \end{pmatrix})$. As a consequence, there exists a nonsingular matrix $\mathbf{T}$ such that:

$$\widetilde{\mathbf{U}} = \begin{pmatrix} \overline{\mathbf{O}}_{i,K_i} \\ \overline{\mathbf{O}}_{i,K_i}\mathbf{D} \end{pmatrix}\mathbf{T}^T$$

Let us denote by $\widetilde{\mathbf{U}}_1$ and $\widetilde{\mathbf{U}}_2$ respectively the top and the bottom parts of $\widetilde{\mathbf{U}}$ corresponding to: $\widetilde{\mathbf{U}}_1 = \overline{\mathbf{O}}_{i,K_i}\mathbf{T}^T$ and $\widetilde{\mathbf{U}}_2 = \overline{\mathbf{O}}_{i,K_i}\mathbf{D}\mathbf{T}^T$. We can also construct the matrices: $\mathbf{R}_1 = \widetilde{\mathbf{U}}_1^T\widetilde{\mathbf{U}}_1$ and $\mathbf{R}_2 = \widetilde{\mathbf{U}}_1^T\widetilde{\mathbf{U}}_2$. By defining $\mathbf{G} = \mathbf{T}\overline{\mathbf{O}}_{i,K_i}^T\overline{\mathbf{O}}_{i,K_i}$, we get:

$$\mathbf{R}_1 = \mathbf{G}\mathbf{T}^T \text{ and } \mathbf{R}_2 = \mathbf{G}\mathbf{D}\mathbf{T}^T \quad (13)$$

Since by construction $\mathbf{R}_1$ is nonsingular, we can define the matrix $\mathbf{R} = \mathbf{R}_2\mathbf{R}_1^{-1}$. By replacing $\mathbf{R}_1$ and $\mathbf{R}_2$ by their definitions, we get $\mathbf{R}\mathbf{G} = \mathbf{G}\mathbf{D}$, which is a standard eigenvalue problem. In other words, the consensus matrix and the matrix $\mathbf{R}$ built from the observations have exactly the same spectrum. ■

As shown in Sundaram and Hadjicostis (2008), the observability index of node i is upper-bounded by $N - N_i$. Now, based on the constructive proof of Theorem 3, the proposed procedure for computing the Laplacian eigenvalues is described in algorithm 1.

Example 2. Let us consider the graph with 20 nodes depicted in figure 2. The corresponding graph Laplacian matrix has the following eigenvalues: $eig(\mathbf{L}) = \{0; 0.4601; 1.0056; 1.7629; 2.9291; 3.5859; 3.8435; 5.0246; 5.7592; 6.2389; 6.9647; 7.5740; 8.0000; 8.2203; 8.4202; 9.0000; 9.4384; 10.0000; 10.3097; 11.4631\}$. Applying results of section 2, we note that only the nodes with a red asterisk can observe the system. As a consequence, these nodes can compute exactly the eigenvalues of the consensus

Algorithm 1 :Decentralized Laplacian eigenvalues estimation

Given the observations $\{\mathbf{Y}_i(k)\}_{k=0,1,\cdots,K_i+1}$, resulting from the linear dynamical system (6) with $\mathbf{W} = \mathbf{I} - \varepsilon\mathbf{L}$, the Laplacian eigenvalues can be computed by following the steps below:

(1) Construct the matrices $\overline{\mathbf{Y}}_i$ and $\overline{\overline{\mathbf{Y}}}_i$ as in (12).

(2) Compute the matrix $\widetilde{\mathbf{U}}$ of left singular vectors of the matrix $\left(\overline{\mathbf{Y}}_i^T \;\; \overline{\overline{\mathbf{Y}}}_i^T\right)^T$.

(3) Partition $\widetilde{\mathbf{U}}$ as $\widetilde{\mathbf{U}} = \left(\widetilde{\mathbf{U}}_1^T \;\; \widetilde{\mathbf{U}}_2^T\right)^T$, the two blocks having the same number of rows.

(4) Compute the matrices $\mathbf{R}_1 = \widetilde{\mathbf{U}}_1^T\widetilde{\mathbf{U}}_1$, $\mathbf{R}_2 = \widetilde{\mathbf{U}}_1^T\widetilde{\mathbf{U}}_2$, and $\mathbf{R} = \mathbf{R}_2\mathbf{R}_1^{-1}$.

(5) Compute the matrix $\mathbf{D}$ of the eigenvalues of the Laplacian based consensus matrix by computing the eigenvalues of $\mathbf{R}$.

(6) Deduce the Laplacian eigenvalues as $\mathbf{\Lambda} = \frac{1}{\varepsilon}(\mathbf{I} - \mathbf{D})$.

matrix and then the Laplacian eigenvalues. Running algorithm 1 at each of these nodes we get exactly the expected result.

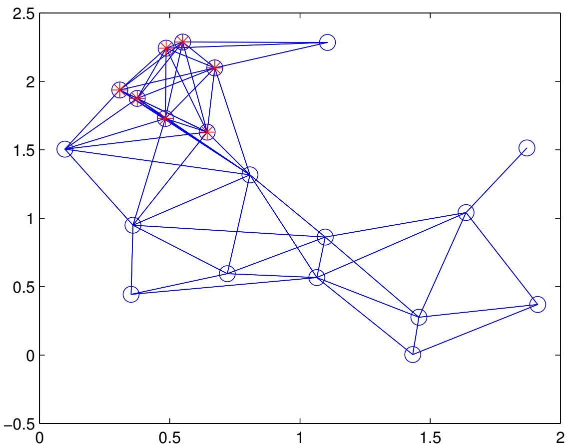

Fig. 2. Network with 20 nodes.

4. DECENTRALIZED ESTIMATION OF ARBITRARY NETWORK MATRICES EIGENVALUES AND CALCULATION OF AVERAGE CONSENSUS

In this section, we consider an arbitrary network matrix $\mathbf{W}$. Obviously, its eigenvalues can be estimated using the approach proposed in the previous section. Indeed, by solving the eigenvalue problem, each node i obtain both matrices $\mathbf{D}$ and $\mathbf{G}$. Therefore, from (13) we can deduce that

$$\mathbf{T}^T = \mathbf{G}^{-1}\mathbf{R}_1.$$

Knowing that $\widetilde{\mathbf{U}}_1 = \overline{\mathbf{O}}_{i,K_i}\mathbf{T}^T$, we can deduce that

$$\overline{\mathbf{O}}_{i,K_i} = \mathbf{R}_1^{-1}\mathbf{G}\widetilde{\mathbf{U}}_1 \tag{14}$$

and

$$\mathbf{C}^T = \mathbf{G}^{-1}\mathbf{R}_1\overline{\mathbf{Y}}_1. \tag{15}$$

Therefore, each node, from its observations, can estimate both the eigenvalues of the network matrix $\mathbf{W}$ and a linear combination of the initial values of the network. The following connection could be done with the work in Sundaram and Hadjicostis (2008):

Lemma 3. Node i can compute the linear functionals $\mathbf{QU}^T\mathbf{X}(0)$ from its observations $\mathbf{Y}_i$, $i = 0, 1, \cdots, K_i$, if and only if the row space of $\mathbf{Q}$ is contained in the row space of the matrix $\mathbf{R}_1^{-1}\mathbf{G}\widetilde{\mathbf{U}}_1$.

Proof:The proof of this lemma is similar to that of the Lemma 1 in Sundaram and Hadjicostis (2008). We have just replaced the observability matrix by its estimate. ■

Lemma 4. If the pair $(\mathbf{W}, \mathbf{E}_i)$ is observable, if $\mathbf{C}$ is full column rank and if $\mathbf{W}$ admits $\mathbf{1}/\sqrt{N}$ as an eigenvector associated with the simple eigenvalue 1 then the average consensus on initial conditions, $\overline{\mathbf{X}} = \frac{1}{N}\mathbf{1}\mathbf{1}^T\mathbf{X}(0)$, can be computed in finite-time as

$$\overline{\mathbf{X}} = \left(\frac{1}{\sqrt{N}}\mathbf{1} \;\; \mathbf{0}_{N\times(N-1)}\right)\mathbf{G}^{-1}\mathbf{R}_1\overline{\mathbf{Y}}_1 \tag{16}$$

Proof: If $\mathbf{1}/\sqrt{N}$ is an eigenvector of $\mathbf{W}$ associated with the simple eigenvalue 1 then $\mathbf{U}$ can be partitioned as $\mathbf{U} = \left(\frac{1}{\sqrt{N}}\mathbf{1} \;\; \overline{\mathbf{U}}\right)$ with $\mathbf{1}^T\overline{\mathbf{U}} = \mathbf{0}$ and $\overline{\mathbf{U}}^T\overline{\mathbf{U}} = \mathbf{I}$. Recalling that $\mathbf{C}^T = \mathbf{U}^T\mathbf{X}(0)$, we can rewrite the right hand side of this equation as

$$\begin{pmatrix} \frac{1}{\sqrt{N}}\mathbf{1}^T\mathbf{X}(0) \\ \overline{\mathbf{U}}^T\mathbf{X}(0) \end{pmatrix}.$$

Then, pre-multiplying $\mathbf{C}^T$ by $\left(\frac{1}{\sqrt{N}}\mathbf{1} \;\; \mathbf{0}_{N\times(N-1)}\right)$ and using (15) we get the desired result (16). ■

5. COLLABORATIVE NETWORK TOPOLOGY IDENTIFICATION

In the previous sections, we have shown that each node can compute both the eigenvalues of the network matrix and some linear functionals of the initial condition of the network state. Such computations are made using local observations. In addition, they can be easily implemented in anonymous network. The only important information being the number of nodes in the network. Knowing the network matrix spectrum, an important question is how inferring the overall network structure? To do so, the estimation of the network matrix eigenvectors is crucial. For this purpose, let us recall that for node i the observations at the time-step k i given by

$$\mathbf{Y}_i(k) = \mathbf{E}_i\mathbf{U}\mathbf{D}^k\mathbf{C}^T.$$

Assuming that the nodes know their IDs and those of their neighbors, we can conclude that, in addition of $\mathbf{C}$ and $\mathbf{D}$, the matrix $\mathbf{E}_i$ is also known by node i. However, these observations are not sufficient for recovering all the eigenvectors. Hence the necessity of resorting to a collaborative approach.

Let us reorganize the data in $\mathbf{Y}_i(k)$ in a vector by using the $vec(.)$ operator. We get:

$$vec(\mathbf{Y}_i(k)) = \left(\mathbf{C}\mathbf{D}^k \otimes \mathbf{E}_i\right) vec(\mathbf{U}).$$

Stacking these observations yields:

$$\mathbf{\Phi}_i vec(\mathbf{U}) = \boldsymbol{\psi}_i \tag{17}$$

where

$$\mathbf{\Phi}_i = \begin{pmatrix} \mathbf{C}\otimes\mathbf{E}_i \\ \mathbf{C}\mathbf{D}\otimes\mathbf{E}_i \\ \vdots \\ \mathbf{C}\mathbf{D}^{K_i}\otimes\mathbf{E}_i \end{pmatrix} \quad \boldsymbol{\psi}_i = \begin{pmatrix} vec(\mathbf{Y}_i(0)) \\ vec(\mathbf{Y}_i(1)) \\ \vdots \\ vec(\mathbf{Y}_i(K_i)) \end{pmatrix} \tag{18}$$

Since the node i is not connected to all the remaining nodes in the network, then one can check that $\mathbf{\Phi}$ is rank deficient. As a consequence node i can only carry out a partial estimation of the network eigenvectors. If all the local measurements were available at a given point then estimating the eigenvectors should resort to solving the least-squares problem

$$\begin{pmatrix} \mathbf{\Phi}_1 \\ \mathbf{\Phi}_2 \\ \vdots \\ \mathbf{\Phi}_N \end{pmatrix} vec(\mathbf{U}) = \begin{pmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \vdots \\ \boldsymbol{\psi}_N \end{pmatrix} \tag{19}$$

The least squares solution is obtained by solving

$$\mathbf{\Phi} vec(\mathbf{U}) = \boldsymbol{\psi}$$

where

$$\mathbf{\Phi} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{\Phi}_i^T \mathbf{\Phi}, \quad \boldsymbol{\psi} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{\Phi}_i^T \psi_i.$$

Obviously, $\mathbf{\Phi}$ and $\boldsymbol{\psi}$ can be computed as average values of local quantities. Such a computation can be carried out by means of average consensus algorithms. In particular, one can make use of the finite-time average consensus algorithm recently introduced in Kibangou (2011a, 2012b). In these algorithms, exact average consensus is achieved using a set of Graph Laplacian based matrices parameterized by the Laplacian eigenvalues. Consensus is achieved in a number of steps equal to the number of distinct nonzero graph Laplacian eigenvalues. These methods are particularly suitable for estimating $\mathbf{\Phi}$ and $\boldsymbol{\psi}$ in a distributed way since we have shown how estimating the Laplacian eigenvalues in the previous section.

The proposed collaborative network topology identification method is summarized in algorithm 2.

Algorithm 2 : Network topology identification

For each node i, given the matrices $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}_i$, $\mathbf{Y}_i(k)$

(1) Build the matrices $\mathbf{\Phi}_i$ and the vectors $\boldsymbol{\psi}_i$ as defined in (18).

(2) Compute the local quantities $\mathbf{\Phi}_i^T \mathbf{\Phi}_i$ and $\mathbf{\Phi}_i^T \boldsymbol{\psi}_i$.

(3) Run the average consensus algorithm to get $\mathbf{\Phi} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{\Phi}_i^T \mathbf{\Phi}$ and $\boldsymbol{\psi} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{\Phi}_i^T \psi_i$.

(4) Compute the matrix $\mathbf{U}$ by solving $\mathbf{\Phi} vec(\mathbf{U}) = \boldsymbol{\psi}$.

(5) Each node can then estimate the matrix topology as $\mathbf{W} = \mathbf{U}\mathbf{D}\mathbf{U}^T$.

6. CONCLUSION

In this paper, we have first stated some observability conditions for distributed systems modeled with graphs, then we have proposed some algorithms for estimating the eigenvalues of the consensus matrix, including the Laplacian eigenvalues, and some linear functionals of initial conditions. Eventually, we have shown how estimating the network matrix in a collaborative way. The observability conditions derived herein are essentially based on spectral properties of the state matrix. In particular, we have shown that observability depends on the rank of a Khatri-Rao product of two matrices. The first one is a Vandermonde matrix built with the eigenvalues of the state matrix whereas the second one is the eigenvector matrix. Therefore, based on the properties of the Khatri-Rao product, we have derived some sufficient observability conditions when all the eigenvalues are simple and nonzero. Then a new necessary condition has been proposed for a more general case.

Based on the observability properties sharing by some nodes, we have shown how estimating the Laplacian eigenvalues in a decentralized way. Unlike FFT based methods recently proposed in the literature, in the approach considered herein, we are also able to estimate the multiplicities of the eigenvalues. Several linear functionals of the initial condition can then be computed in a finite number of steps. That is the case of the average consensus on initial conditions. Once the consensus matrix eigenvalues have been estimated in a decentralized way, the nodes have to collaborate for estimating the eigenvectors in order to reconstitute the overall network topology. In the future, we intend to study the impact of imperfect communications in the estimation of both Laplacian eigenvalues and network matrix identification.

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