A proof of Pillai conjecture

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Abstract

(MSC=11D04) More than one century after its formulation by the Belgian mathematician Eugene Catalan, Preda Mihailescu has solved the open problem. But, is it all ? Mihailescu's solution utilizes computation on machines, we propose here not really a proof as it is entended classically, but a resolution of an equation like the resolution of the polynomial equations of third and fourth degrees. This solution is totally algebraic and does not utilize, of course, computers or any kind of calculation. Then, we generalize the proof to Pillai conjecture.

(Keywords : Diophantine equations ; Catalan equation ; Pillai conjecture ; Algebraic resolution)

Introduction

Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers stricly greater than 1, $X > 1$ and $Y > 1$, for which with exponants strictly greater than 1, $p > 1$ and $q > 1$,

$$Y^p = X^q + 1$$

but for $(X, Y, p, q) = (2, 3, 2, 3)$. We can verify that

$$3^2 = 2^3 + 1$$

Euler has proved that the equation $X^3+1 = Y^2$ has this only solution. We propose in this study a general solution. The particular cases already solved concern $p = 2$, solved by Ko Chao in 1965, and $q = 3$ which has been solved in 2002. The case $q = 2$ has been solved by Lebesgue in 1850. We solve here the equation for the general case. We generalize the proof to Pillai conjecture

$$Y^p = X^q + a$$

And prove that it has always a finite number of solutions for a fixed a.

The approach

Let

$$c = \frac{X^p - 1}{Y^{\frac{p}{2}}}, \quad c' = \frac{7 - X^p}{Y^{\frac{p}{2}}}$$

And

$$d = X^p - Y^{\frac{p}{2}}, \quad d' = X^p + Y^{\frac{p}{2}}$$

We have

$$Y^{\frac{p}{2}} = X^p - d = d' - X^p = \frac{d' - d}{2}$$

And

$$X^p = \frac{d' + d}{2}$$

But

$$(c + c')Y^{\frac{p}{2}} = X^p - 1 + 7 - X^p = 6 \Rightarrow Y^{\frac{p}{2}} = \frac{6}{c + c'} = \frac{d' - d}{2}$$

And

$$X^p = cY^{\frac{p}{2}} + 1 = \frac{6c}{c + c'} + 1 = \frac{7c + c'}{c + c'} = \frac{d' + d}{2}$$

We deduce

$$d = \frac{7c + c' - 6}{c + c'}, \quad d' = \frac{7c + c' + 6}{c + c'}$$

And

$$X^q = Y^p - 1 = \frac{36 - (c + c')^2}{(c + c')^2}$$

We have

$$(c + c')Y^{\frac{p}{2}} = 6 > 0$$

Thus $c + c' > 0$. Also

$$cY^{\frac{p}{2}} = X^p - 1 > 0$$

Thus $c > 0$. But $X^p \geq 4$, hence

$$cY^{\frac{p}{2}} = X^p - 1 \geq 7 - X^p = c'Y^{\frac{p}{2}}$$

hence $c \geq c'$

$$7 - X^p = 7 - \frac{7c + c'}{c + c'} = \frac{6c'}{c + c'}$$

If

$$c' > 0$$

Thus $X^p < 7$ and $X^p = 4$, it means that $c' < 0$ and

$$(c + c')Y^{\frac{p}{2}} = 6 > 0$$

$$(c + c')X^p = 7c + c' > 0$$

thus $c + c' > 0$ and

$$c' + 1 > 0$$

And

$$Y^{\frac{p}{2}} = \frac{6}{c + c'} \geq 3 \Rightarrow c + c' \leq 2$$

main results What we must retain is
1)$c' < 0$
2)$0 < c + c' < 2$
3)$0 < 7c + c'$
And we will discuss two cases
$I)c^2 < 1$
$II)c^2 \geq 1$

case $c^2 < 1$ $c < 1$ means that $(c-1)Y^{\frac{p}{2}} = d - 1 < 0$, we deduce

$$(c'+1)Y^{\frac{p}{2}} = 7 - d > 0$$

Thus $c' > -1$. But

$$3X^p(c+c') - 4Y^{\frac{p}{2}}(c+c') = 3(7c+c') - 24 = 21(c-1) + 3(c'-1) < 0$$

And

$$2X^q - X^{2p} = \frac{72 - 2(c+c')^2 - (7c+c')^2}{(c+c')^2}$$
$$= \frac{72 - 51c^2 - 3c'^2 - 18cc'}{(c+c')^2}$$
$$= \frac{51(1-c^2) - 3c'(c+c') - 15cc' + 21}{(c+c')^2} > 0$$

Thus

$$2X^q > X^{2p} \quad (1)$$

And

$$c^2X^q - X^{2p} = \frac{36c^2 - c^2(c+c')^2 - (7c+c')^2}{(c+c')^2}$$
$$= \frac{36c^2 - c^4 - c^2c'^2 - 2c^3c' - 49c^2 - c'^2 - 14cc'}{(c+c')^2}$$
$$= \frac{-13c(c+c') - c'c(1 + (c'+2c)c) - c^4 - c'^2}{(c+c')^2}$$
$$= \frac{c(-13c - 14c' - 2c^2c' - cc'^2 - c^3) - c'^2}{(c+c')^2}$$
$$= \frac{c(-13c - 13c' - c'c(c+c') - c^2(c'+c)) - c'(c+c')}{(c+c')^2}$$
$$< \frac{c(-12c - 12c') - (c+c')^2 - c(c+c')^2}{(c+c')^2} < 0$$

Thus

$$c^2X^q < X^{2p} \quad (2)$$

Also

$$100c^2X^q - 99X^{2p} = \frac{3600c^2 - 100c^2(c+c')^2 - 99(7c+c')^2}{(c+c')^2}$$

$$= \frac{(60c - \sqrt{99}(7c + c')(60c + \sqrt{99}(7c + c') - 100c^2(c + c')^2}{(c + c')^2}$$

$$= \frac{(-9.64c - 9.94c')(129.64c + 9.94c') - 100c^2(c + c')^2}{(c + c')^2}$$

$$< \frac{-1249.73c^2 - 98.8c'^2 - 1384.24cc'}{(c + c')^2}$$

$$= \frac{-67.26c'(c + 1.1c') - 1249.73c(c + c') - 24.81c'(c' + 2.44c)}{(c + c')^2}$$

$$= \frac{c'(-127.91c - 98.8c') - 9.77c(127.91c + 98.8c') - 29.11cc'}{(c + c')^2}$$

$$= \frac{-(127.91c + 98.8c')(9.77c + c') - 29.11cc'}{(c + c')^2} < 0$$

because

$$127.91c + 98.8 > -29.11c' > 0$$

$$9.77c + c' > c > 0$$

We deduce

$$\frac{c^2}{2} X^{2p} < c^2 X^q < X^{2p} \quad (3)$$

And

$$c^2 X^q < \frac{99}{100} X^{2p} \quad (4)$$

But

$$X^{2p-q-1} < 2X^{-1} \leq 1$$

Thus

$$2p \leq q + 1$$

We will give two proofs : the first : as $c^2 - 1 < 0$ and $Y^p \geq 9$, let

$$9(c^2 - 1) = 9X^p(X^p - 2) - 9c^2 X^q \geq (c^2 - 1)Y^p = X^p(X^p - 2) - X^q$$

Thus

$$8X^p(X^p - 2) \geq (9c^2 - 1)X^q$$

First case $9c^2 - 1 \geq 2$ then

$$8X^p(X^p - 2) \geq 2X^q$$

And

$$4X^{2p-q} \geq 1 + 8X^{p-q} > 1$$

Thus

$$4X^{2p-q+1} > X \geq 4$$

Consequently

$$q + 1 \geq 2p \geq q$$

As p and q do not have the same parity, we deduce that $q+1=2p$. Then

$$4X^q = 4X^{2p-1} \geq 4X^p(X^p-2) \geq X^q = X^{2p-1}$$

$$1 \geq X - 2X^{1-p}$$

$$X-1 \leq 2X^{1-p} \leq 1$$

And $X=2$, but

$$Y^p - 1 = (Y^{\frac{p}{2}}-1)(Y^{\frac{p}{2}}+1) = 2^q$$

If $Y^{\frac{p}{2}} = 2v+1$

$$4v(v+1) = 2^q \geq 2^3$$

It is possible if $v=1$, then $Y=3$ and $p=2$ and $q=3$ or $X=2$. Second case $3c^2<1$

$$0 > c^2-1 = X^{2p} - 2X^p - c^2X^q > X^{2p} - 2X^p - \frac{99}{100}X^{2p} = \frac{1}{100}X^p(X^p-200) > 0$$

It means that $0 \leq c^2-1 \leq 0$, or $c^2=1$ and

$$X^p - 2 = X^{q-p} \geq 2$$

$X^{p-1} - X^{q-p-1} = \frac{2}{X}$ is an integer, then $X=2$. And

$$2^{p-1} = 1 + 2^{q-p-1}$$

In one side an even number, in the other an odd one, the solution is $q=p+1$, then $2^{p-1}=2$, or $p=2$ and $Y=\pm 3$. Another proof : We have $1+\frac{1}{99} < X^\epsilon = \frac{X^{2p-q}}{c^2} < \frac{X}{c^2}$

$$-\epsilon \log(X) > (2p-q-\epsilon)\log(X) = 2\log(c)$$

$$\epsilon \log(X) = (2p-q)\log(X) - 2\log(c) = \log(\frac{X^{2p-q}}{c^2}) > 1$$

Thus $X^\epsilon > 1$

$$2 > 2p-q+1 = \frac{\log(c^2)}{\log(X)} + 1 > 0$$

because

$$\frac{\log(c^2)}{\log(X)} > -1$$

or

$$c^2 > X^{-1}$$

or

$$c^2 - 1 > X^{-1} - 1 > \frac{1-X}{X}$$

or

$$(c^2-1)X > 1-X$$

or

$$(c^2-1)X = X(X^{2p} - 2X^p - c^2X^q) > 1-X$$

and

$$X(X^{2p} - 2X^p - c^2X^q) > 1-X$$

and

$$X(X^p-1)^2 - c^2X^{q+1} > 1$$

And with $X^\epsilon > 1$

$$X(X^p-1)^2 - c^2X^{q+1} = X(X^p-1)^2 - X^{2p-\epsilon+1} = X(X^{2p} - X^{2p-\epsilon} - 2X^p + 1) > 1$$

because

$$X^p \geq 2X^{p-\epsilon}$$

$$X^\epsilon > 1 + \frac{1}{99}$$

$$X^p > 200$$

And

$$\frac{99X^p}{100} = X^{p-\epsilon}$$

$$\frac{X^p}{100} > 2$$

Thus we always have

$$q \leq 2p \leq q+1$$

and as p and q do not have the same parity, we have

$$q+1 \leq 2p \leq q+1$$

And $2p = q+1$. Then

$$0 \geq (c^2-1)Y^p = X^{2p} - 2X^p - X^q = X^{2p} - 2X^p - X^{2p-1} = X^p(X^{p-1}(X-1)-2) \geq 0$$

Thus $c^2 = 1$

$$X^p - X^{p-1} = 2$$

$$X^{p-1} - X^{p-2} = \frac{2}{X}$$

This expression is an integer, hence $X = 2$ and

$$2^{p-1} = 1 + 2^{p-2}$$

In one side an even number, in the other an odd one, it is possible if $p = 2$, thus $q = 2p - 1 = 3$

Case $c \geq 1$ We must retain here that
$1)c^2 > 1$
$2)c + c' < 2$
$3)c' < 0$
We have

$$2c^2X^q - X^{2p} = \frac{72c^2 - 2c^2(c+c')^2 - (7c+c')^2}{(c+c')^2}$$

$$= \frac{72c^2 - 49c^2 - c'^2 - 14cc' - 2c^2(c^2 + c'^2 + 2cc')}{(c+c')^2}$$

$$= \frac{c^2(23-2(c+c')^2) - c'(c+c') - 13cc'}{(c+c')^2}$$

$$> \frac{15c^2 - c'(c+c') - 13cc'}{(c+c')^2} > 0$$

Or

$$2c^2X^q > X^{2p} > 4X^p \Rightarrow c^2X^q > 2X^p \quad (5)$$

And

$$c^2X^q - X^{2p} = \frac{36c^2 - c^2(c+c')^2 - (7c+c')^2}{(c+c')^2}$$

$$= \frac{36c^2 - 50c^2 - 2c'^2 - 16cc' + (1-c^2)(c+c')^2}{(c+c')^2}$$

$$= \frac{-2(7c+c')(c+c') + (c+c')^2(1-c^2)}{(c+c')^2} < 0$$

We deduce

$$\frac{1}{2}X^{2p} < c^2X^q < X^{2p} < c^2X^{2p} \quad (6)$$

$$X^{2p-q} > c^2 > 1$$

And

$$2p \geq q+1$$

There are two cases again :
$I)c'^2 < 1$
$II)c'^2 > 1$

Case $c'^2 < 1$ We have with $u = q-p$

$$\frac{X^{2p} - 7X^p + 49}{X^q + X^p} < c'^2 = \frac{(X^p-7)^2}{X^q+1} < 1 = \frac{X^p}{X^{q-u}}$$

$u = q-p$, We have

$$X^{2p+q-u} - 14X^{p+q-u} + 49X^{q-u} \leq X^{p+q} + X^{2p}$$

$$X^{2p} - 14X^p + 49 \leq X^{p+u} + X^{2p-q+u}$$

$$X^{2p} - X^{p+u} - X^{2p-q+u} \leq 14X^p - 49$$

$$X^p - X^u - X^{p+u-q} \leq 14 - 49X^{-p}$$

$$X^p - X^u - 14 \leq X^{p+u-q} - 49X^{-p} = -48 + 49 - 49X^{-p} < -48 + 49 = 1$$

$$0 < X^p - X^{p-1} < X^p - X^{q-p} \leq 15$$

There are 6 possibilities $2k \in \{2, 4, 6, 8, 10, 12, 14\}$ and $X^{p-1} - X^{q-p-1} = \frac{2k}{X}$ is an integer thus $X = 2m \in \{2, 4, 6, 8, 10, 12, 14\}$

$$(2m)^{p-1} = 2v + 1 + (2m)^{q-p-1}$$

when $m \in 2, 6, 10, 12, 14$ in one side an even number, in the other an odd one, the solution is $q = p + 1$ which leads to

$$X^{p-1} = 2 \Rightarrow (X, p) = (2, 2)$$

And

$$(2m)^{p-2} = \frac{2v}{X} + (2m)^{q-p-2}$$

If $q \geq p+2$, it means $X = 2v \in 2, 4$ and it is impossible! Thus $X = 2$ and $(p, q) = (2, 3)$ or $Y = \pm 3$

Case $c'^2 > 1$ In this case

$$X^q = \frac{36 - (c + c')^2}{(c + c')^2} > \frac{36 - 4}{(c + c')^2} = \frac{32}{36}\left(\frac{36}{(c + c')^2}\right) = \frac{8}{9}Y^p > \frac{8}{9}X^p$$

More generally, we have

$$9(c^2 - 1) = 9X^{2p} - 18X^p - 9c^2X^q \leq (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q$$

$$8X^{2p} - 16X^p = 8c^2Y^p - 8 < (9c^2 - 1)X^q$$

We deduce

$$X^{q-2p} > \frac{8X^p(X^p - 2)}{(9c^2 - 1)X^{2p}}$$

But

$$18X^{2p} - 36X^p - 9c^2X^{2p} = 18c^2Y^p - 18 - 9c^2Y^p + 9 - 18X^p$$

$$= 9c^2Y^p - 9 - 18X^p = 9c^2X^q + 9(c^2 - 1) - 18X^p > 9(c^2 - 1) > 0 > X^{2p} - 2X^p - X^{2p}$$

In consequence

$$17X^p(X^p - 2) > (9c^2 - 1)X^{2p}$$

Or

$$X^{q-2p} > \frac{8X^p(X^p - 2)}{(9c^2 - 1)X^{2p}} > \frac{8}{17}$$

And

$$X^{q+1-2p} > \frac{8X}{17} \geq 1$$

Thus

$$2p \geq q + 1 \geq 2p$$

Hence $q + 1 = 2p$ and

$$X = \frac{X^{2p}}{X^q} \leq \frac{X^{2p}}{\frac{8}{17}X^{2p}} = \frac{17}{8}$$

And $X = 2$, it means

$$Y^p = 2^{2p-1}+1 \Rightarrow (Y^{\frac{p}{2}} - 1)(Y^{\frac{p}{2}} + 1) = 4k(k + 1) = 2^{2p-1} \Rightarrow k = 1 \Rightarrow q = 2p - 1 = 3$$

And $Y = \pm 3$

$$0 \leq (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q < X^{2p} - 2X^p - X^{2p} < 0$$

We have simultaneously $q+1=2p$ and

$$(c^2-1)Y^pX^{-p-1}=0=X^{p-1}-X^{p-2}-\frac{2}{X}=0$$

We deduce $(X,p)=(2,2)$ or Catalan solution.

Generalization to Pillai equation

Pillai equation is $Y^p=X^q+a$. We will define Pillai numbers : we note them : b_a they depend of a and heve the same proprieties than b for $a=1$. Thus, we write $1_a+7_a=8_a$ and $7_a-1_a=6_a$: our goal is to follow the approach as we have done for Catalan equation. Of course, we have $1_a=a_1$ et $0_a=0$. The equation becomes $Y^p=X^q+1_a$. Let us pose for p,q,X,Y,a integers stricly greater to 1. Let

$$c=\frac{X^p-1_a}{Y^{\frac{p}{2}}},\quad c'=\frac{7_a-X^p}{Y^{\frac{p}{2}}}$$

Also

$$d=X^p-Y^{\frac{p}{2}},\quad d'=X^p+Y^{\frac{p}{2}}$$

We have then

$$Y^{\frac{p}{2}}=X^p-d=d'-X^p=\frac{d'-d}{2}$$

Or

$$X^p=\frac{d'+d}{2}$$

And

$$(c+c')Y^{\frac{p}{2}}=X^p-1_a+7_a-X^p=6_a\Rightarrow Y^{\frac{p}{2}}=\frac{6_a}{c+c'}=\frac{d'-d}{2}$$

So

$$X^p=cY^{\frac{p}{2}}+1_a=\frac{6_ac}{c+c'}+1_a=\frac{7_ac+1_ac'}{c+c'}=\frac{d'+d}{2}$$

Thus

$$d=\frac{7_ac+1_ac'-6_a}{c+c'},\quad d'=\frac{7_ac+1_ac'+6_a}{c+c'}$$

And

$$X^q=Y^p-1_a=\frac{36_a1_a-1a(c+c')^2}{(c+c')^2}$$

Also

$$(c+c')Y^{\frac{p}{2}}=6_a>0$$

Thus $c+c'>0$. And

$$cY^{\frac{p}{2}}=X^p-1_a>0$$

So $c>0$. But $X^p\geq 4_a$, then

$$cY^{\frac{p}{2}}=X^p-1_a\geq 7_a-X^p=c'Y^{\frac{p}{2}}$$

And $c\geq c'$

$$7_a-X^p=7_a-\frac{7_ac+1_ac'}{c+c'}=\frac{6_ac'}{c+c'}$$

If

$$c' > 0$$

Then $X^p < 7_a$ and we have a finite number of solutions, we have $c' < 0$, with

$$(c+c')Y^{\frac{p}{2}} = 6_a > 0$$

$$(c+c')X^p = 7_a c + 1_a c' > 0$$

with $c + c' > 0$. And

$$Y^{\frac{p}{2}} = \frac{6_a}{c+c'} \geq 3_a \Rightarrow c + c' \leq 2 < 2_a$$

Main results We retain
$1)c' < 0$
$2)0 < c + c' < 2_a$
$3)0 < 7_a c + 1_a c'$
And we will discuss two cases
$I)c^2 < 1_a$
$II)c^2 \geq 1_a$

case $c^2 < 1_a$ We have

$$3_a X^p(c+c') - 4_a Y^{\frac{p}{2}}(c+c') = 3_a(7_a c + 1_a c') - 24_a 1_a = 21_a 1_a (c - 1_a) + 3_a 1_a (c' - 1_a) < 0$$

Let

$$2X^q - X^{2p} = \frac{72_a 1_a - 2_a(c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2}$$

$$= \frac{72_a 1_a - 1_a(49_a + 2)c^2 - 1_a(1_a + 2)c'^2 - 2_a(7_a + 1)cc'}{(c+c')^2}$$

$$= \frac{70_a 1_a - 49_a 1_a c^2 + 2_a(1_a - c^2) - (2_a + 1_a 1_a)c'(c+c') - 13_a 1_a cc'}{(c+c')^2} > 0$$

In reality $2_a X^q - X^p > 0$

$$2_a X^q > X^{2p} \quad (7)$$

And also

$$c^2 X^q - X^{2p} = \frac{36_a 1_a c^2 - 1_a c^2 (c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2}$$

$$= \frac{36_a 1_a c^2 - 1_a c^4 - 1_a c^2 c'^2 - 2_a c^3 c' - 49_a 1_a c^2 - 1_a 1_a c'^2 - 14_a 1_a cc'}{(c+c')^2}$$

$$= \frac{-13_a 1_a c(c+c') - 1_a c' c(1_a + (c' + 2c)c) - 1_a c^4 - 1_a 1_a c'^2}{(c+c')^2}$$

$$= \frac{c(-13_a 1_a c - 14_a 1_a c' - 2_a c^2 c' - 1_a cc'^2 - 1_a c^3) - 1_a 1_a c'^2}{(c+c')^2}$$

$$= \frac{c(-13_a 1_a c - 13_a 1_a c' - 1_a c' c(c+c') - 1_a c^2(c'+c)) - 1_a 1_a c'(c+c')}{(c+c')^2}$$

$$< \frac{c(-12_a 1_a c - 12_a 1_a c') - 1_a 1_a (c+c')^2 - 1_a c(c+c')^2}{(c+c')^2} < 0$$

So

$$c^2 X^q < X^{2p} \quad (8)$$

Let us try

$$100_a c^2 X^q - 99 X^{2p} = \frac{3600_a 1_a c^2 - 100_a c^2 (c+c')^2 - 99(7_a c + 1_a c')^2}{(c+c')^2}$$

$$= \frac{(60_a c - \sqrt{99}(7_a c + 1_a c')(60_a c + \sqrt{99}(7_a c + 1_a c') - 100_a c^2 (c+c')^2}{(c+c')^2}$$

$$= \frac{(-9.64_a c - 9.94_a c')(129.64_a c + 9.94_a c') - 100_a c^2 (c+c')^2}{(c+c')^2}$$

$$< \frac{-1249.73_a 1_a c^2 - 98.8_a 1_a c'^2 - 1384.24_a 1_a cc'}{(c+c')^2}$$

$$= \frac{-67.26_a c'(1_a c + 1.1_a c') - 1249.73_a 1_a c(c+c') - 24.81_a c'(1_a c' + 2.44_a c)}{(c+c')^2}$$

$$= \frac{1_a c'(-127.91_a c - 98.8_a c') - 9.77_a c(127.91_a c + 98.8_a c') - 29.11_a 1_a cc'}{(c+c')^2}$$

$$= \frac{-(127.91_a c + 98.8_a c')(9.77_a c + 1_a c') - 29.11_a 1_a cc'}{(c+c')^2} < 0$$

Because

$$127.91_a c + 98.8_a c' > -29.11_a c' > 0$$

$$9.77_a c + 1_a c' > 1_a c > 0$$

Thus, we have

$$\frac{c^2}{2_a} X^{2p} < c^2 X^q < X^{2p} \quad (9)$$

And

$$c^2 X^q < \frac{99}{100_a} X^{2p} \quad (10)$$

Or

$$X^{2p-q-1} < 2_a X^{-1} \leq 1$$

And evidently

$$2p - q - 1 \leq 0$$

We will give two proofs. The first : as $c^2 - 1_a < 0$ and $Y^p \geq 9_a$, so

$$9_a(c^2 - 1_a) = 9X^p(X^p - 2_a) - 9c^2 X^q \geq (c^2 - 1_a) Y^p = X^p(X^p - 2_a) - 1_a X^q$$

Or

$$8X^p(X^p - 2_a) \geq (9c^2 - 1_a) X^q$$

First case : $9c^2 - 1_a \geq 2_a$

$$8X^p(X^p - 2_a) \geq 2_a X^q$$

And

$$8X^{2p-q} \geq 2_a + 16_a X^{p-q} > 2$$

Hence

$$8X^{2p-q+1} > 2X \geq 8$$

In consequence : we have

$$X^{q+1-2p} = 1$$

It means

$$4X^q = 4X^{2p-1} \geq 4X^p(X^p - 2_a) \geq X^q = X^{2p-1}$$

Or

$$1 \geq X - 2_a X^{1-p}$$

And

$$X - 1 \leq 2_a X^{1-p} \leq 1$$

It means a finite number of solutions. Second case $3c^2 < 1_a$

$$0 > 1_a(c^2 - 1_a) = X^{2p} - 2_a X^p - c^2 X^q > X^{2p} - 2_a X^p - \frac{99}{100_a} X^{2p} = \frac{100_a - 99}{100_a} X^p (X^p - \frac{200_a 1_a}{100_a - 99}) > 0$$

So

$$c^2 > 1_a$$

Hence $0 \leq c^2 - 1_a \leq 0$, or $c^2 = 1_a$ and then

$$X^p - 2_a = X^{q-p} \geq 2$$

$X^{p-1} - X^{q-p-1} = \frac{2+2_a}{X}$ is an integer and there is a finite number of solutions. We always have

$$X^{q+1} \leq X^{2p} \leq X^{q+1}$$

Or

$$0 \geq (c^2 - 1_a)Y^p = X^{2p} - 2_a X^p - 1_a X^q = X^{2p} - 2_a X^p - 1_a X^{2p-1} = X^p(X^{p-1}(X - 1_a) - 2_a) \geq 0$$

And $c^2 = 1_a$

$$X^p - X^{p-1} = 2_a$$

And there is a finite number of solutions !

Cas $c^2 \geq 1_a$ We will retain
$1) c^2 > 1_a$
$2) c + c' < 2_a$
$3) c' < 0$
As precendently :

$$2c^2 X^q - X^{2p} = \frac{72_a 1_a c^2 - 2_a c^2 (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2}$$

$$= \frac{72_a 1_a c^2 - 49_a 1_a c^2 - 1_a 1_a c'^2 - 14_a 1_a cc' - 2_a c^2(c^2 + c'^2 + 2cc')}{(c+c')^2}$$

$$= \frac{c^2(23_a 1_a - 2_a(c+c')^2) - 1_a 1_a c'(c+c') - 13_a 1_a cc'}{(c+c')^2}$$

$$> \frac{15_a 1_a c^2 - 1_a 1_a c'(c+c') - 13_a 1_a cc'}{(c+c')^2} > 0$$

And

$$2c^2 X^q > X^{2p} \quad (11)$$

And

$$2c^2 X^q > X^{2p} > 4X^p \Rightarrow c^2 X^q > 2X^p \quad (11')$$

Also

$$c^2 X^q - X^{2p} = \frac{36_a 1_a c^2 - 1_a c^2(c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2}$$

$$< \frac{36_a 1_a c^2 - 50_a 1_a c^2 - 2_a 1_a c'^2 - 16_a 1_a cc' + (1_a 1_a - 1_a c^2)(c+c')^2}{(c+c')^2}$$

$$= \frac{-21_a(7_a c + 1_a c')(c+c') + (c+c')^2(1_a 1_a - 1_a c^2)}{(c+c')^2} < 0$$

Thus

$$\frac{1}{2} X^{2p} < c^2 X^q < X^{2p} < c^2 X^{2p} \quad (12)$$

Or

$$X^{2p-q} > c^2 > 1_a > 1$$

We have here

$$2p \geq q + 1$$

Two cases :
$I) c'^2 < 1$
$II) c'^2 > 1$

Case $c'^2 < 1$ We have, with $u = q - p$

$$\frac{X^{2p} - 7_a X^p + 49_a 1_a}{X^q + X^p} < c'^2 = \frac{(X^p - 7_a)^2}{X^q + 1_a} < 1 = \frac{X^p}{X^{q-u}}$$

$u = q - p$, thus

$$X^{2p+q-u} - 14_a X^{p+q-u} + 49_a 1_a X^{q-u} \leq X^{p+q} + X^{2p}$$

$$X^{2p} - 14_a X^p + 49_a 1_a \leq X^{p+u} + X^{2p-q+u}$$

$$X^{2p} - X^{p+u} - X^{2p-q+u} \leq 14_a X^p - 49_a 1_a$$

$$X^p - X^u - X^{p+u-q} \leq 14_a - 49_a 1_a X^{-p}$$

$$X^p - X^u - 14_a \leq X^{p+u-q} - 49_a 1_a X^{-p} = -48_a 1_a + 49_a 1_a - 49_a 1_a X^{-p} < -48_a 1_a + 49_a 1_a = 1_a 1_a$$

$$0 < X^p - X^{p-1} < X^p - X^{q-p} \leq 15_a + 1_a 1_a$$

And there is a finite number of de solutions !

Case $c'^2 > 1$ We have

$$X^q = \frac{36_a 1_a - 1_a(c+c')^2}{(c+c')^2} > \frac{36_a 1_a - 4_a}{(c+c')^2} = \frac{36_a 1_a - 4_a}{36_a 1_a}(\frac{36_a 1_a}{(c+c')^2})$$

$$= \frac{36_a 1_a - 4_a}{36_a 1_a} Y^p > \frac{36_a 1_a - 4_a}{36_a 1_a} X^p$$

More generally

$$9_a(c^2 - 1_a) = 9X^{2p} - 18_a X^p - 9c^2 X^q \leq (c^2 - 1_a)Y^p = X^{2p} - 2_a X^p - 1_a X^q$$

$$8X^{2p} - 16_a X^p = 8c^2 Y^p - 8_a 1_a < (9c^2 - 1_a)X^q$$

We deduce

$$X^{q-2p} > \frac{8X^p(X^p - 2_a)}{(9c^2 - 1_a)X^{2p}}$$

But

$$18X^{2p} - 36_a X^p - 9c^2 X^{2p} = 18c^2 Y^p - 18_a 1_a - 9c^2 Y^p + 9_a 1_a - 18_a X^p$$

$$= 9c^2 Y^p - 9_a 1_a - 18_a X^p = 9c^2 X^q + 9_a(c^2 - 1_a) - 18X^p > 9_a(c^2 - 1_a) > 0 > X^{2p} - 2_a X^p - 1_a X^{2p}$$

because of $(11')$. In consequence

$$17X^p(X^p - 2_a) > (9c^2 - 1_a)X^{2p}$$

Or

$$X^{q-2p} > \frac{8X^p(X^p - 2_a)}{(9c^2 - 1_a)X^{2p}} > \frac{8}{17}$$

And

$$X^{q+1-2p} > \frac{8X}{17} \geq 1$$

And

$$q > 2p$$

Thus

$$0 < (c^2 - 1_a)Y^p = X^{2p} - 2_a X^p - 1_a X^q < X^{2p} - 2_a X^p - 1_a X^{2p} < 0$$

It means that we have simultaneously $q + 1 = 2p$ and

$$(c^2 - 1_a)Y^p X^{-p-1} = 0 = X^{p-1} - 1_a X^{p-2} - \frac{2_a}{X} = 0$$

We deduce a finite number of solutions. In all cases, Pillai equation has a finite number of solutions, it is the formulation of the conjecture. Thus, we have proved it.

Conclusion

Catalan equation is solved, an original solution exists ! We have generalized the approach to Pillai equation and proved that it always has a finite number of solutions. It is the proof of Pillai conjecture. It seems that many problems of number theory can be solved like this.

Références

[1] P. MIHAILESCU A class number free criterion for catalan's conjecture, *Journal of Number theory* , **99** (2003).