On Necessary and Sufficient Conditions for Differential Flatness

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Abstract

This paper is devoted to the characterization of differentially flat nonlinear systems. Implicit representations of nonlinear systems, where the input variables are eliminated, are studied in the differential geometric framework of jets of infinite order. In this context, flatness may be seen as a generalization of the property of *uniformization* of Hilbert's 22nd problem. The notion of Lie-Bäcklund equivalence, introduced in [16], is adapted to this implicit setting. Lie-Bäcklund isomorphisms associated to a flat system, called *trivializations*, can be locally characterized in terms of polynomial matrices of the indeterminate $\frac{d}{dt}$. Such polynomial matrices are useful to compute the ideal of differential forms generated by the differentials of all possible trivializations. We introduce the notion of a *strongly closed ideal* of differential forms, and prove that flatness is equivalent to the strong closedness of the latter ideal. Various examples and consequences are presented.

Keywords. Nonlinear system, implicit system, manifold of jets of infinite order, Hilbert's 22nd problem, polynomial matrices, ideals, differential forms, differential flatness, flat output.

Introduction

Differential flatness, or more shortly, flatness, is a system property introduced more than ten years ago in [26, 14].

Let us briefly state an informal definition, that will be made more precise later: given a nonlinear system

$$\dot{x} = f(x, u) \tag{1}$$

where $x = (x_1, \ldots, x_n)$ is the state (belonging to a given smooth n-dimensional manifold) and $u = (u_1, \ldots, u_m)$ the control vector, $m \leq n$, the system (1) is said to be locally (differentially) flat if and only if there exists a vector $y = (y_1, \ldots, y_m)$ such that

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- y and its successive time derivatives $\dot{y}, \ddot{y}, \ldots$, are locally independent,
- y is locally a function of x, u and a finite number of time derivatives of the components of u,
- x and u can be locally expressed as functions of the components of y and a finite number of their derivatives: $x = \varphi_0(y, \dot{y}, \ldots, y^{(\alpha)})$, $u = \varphi_1(y, \dot{y}, \ldots, y^{(\alpha+1)})$, for some multi-integer $\alpha = (\alpha_1, \ldots, \alpha_m)$, and with the notation $y^{(\alpha)} = (\frac{d^{\alpha_1} y_1}{dt^{\alpha_1}}, \ldots, \frac{d^{\alpha_m} y_m}{dt^{\alpha_m}})$.

The vector y is called a *flat output*.

This concept has inspired an important literature. See [13, 27] for surveys on flatness and its applications. Let us just mention that flatness provides significant simplifications for the motion planning problem or for several aspects of feedback design.

Various formalisms have been introduced: finite dimensional differential geometric approaches [7, 17, 42, 43], differential algebra and related approaches [15, 2, 21], infinite dimensional differential geometry of jets and prolongations [16, 29, 34, 31, 38].

Note that flatness may be traced back to Hilbert and Cartan [19, 5]. In fact, using the definition presented in this paper, with, in place of (1), the set of $n - m$ implicit equations (2) where the control variables u are eliminated, flatness may be seen as a generalization in the framework of manifolds of jets of infinite order of the *uniformization of analytic functions* of Hilbert 's 22nd problem [18], solved by Poincaré [32] in 1907 (see [3] for a modern presentation of this subject and recent extensions and results). This problem consists, roughly speaking, given a set of complex polynomial equations in one complex variable, in finding an open dense subset D of the complex plane $\mathbb{C}$ and a holomorphic function s from D to $\mathbb{C}$ such that s is surjective and $s(p)$ identically satisfies the given equations for all values of the "parameter" $p \in D$. In our setting, $\mathbb{C}$ is replaced by a (real) manifold of jets of infinite order, a flat output $y_1, \ldots, y_m$ plays the role of the parameter p and s is the associated Lie-Bäcklund isomorphism $s = (\varphi_0, \varphi_1, \dot{\varphi}_1, \ddot{\varphi}_1, \ldots)$ with φ_0 and φ_1 defined above.

In the framework of linear finite or infinite dimensional systems, the notions of flatness and *parametrization* coincide as remarked by [36, 37], and in the behavioral approach of [33], flat outputs correspond to *latent variables* of *observable image representations* [44] (see also [12] for a module theoretic interpretation of the behavioral approach).

The characterization of differentially flat systems has aroused many contributions [2, 7, 9, 17, 21, 28, 30, 35, 38, 39, 42, 43]. Though general necessary and sufficient conditions exist (see e.g. [2, 9, 30]), they don't provide a fully computable set of conditions. More precisely, [2] gives an algorithm to compute a basis of the cotangent module, called infinitesimal Brunovsky form, and further integrability conditions are needed to deduce flat outputs. Recently, [9] has improved this result but still requiring the use of deformations, thus assuming that the system depends on a small parameter. In [30] the author proposes to express all the differential relations between the system variables x, u and a candidate flat output y and use Cartan-Kähler theory.

We adopt here the formalism of manifolds of jets of infinite order [1, 16, 24, 34, 45] and, as previously mentioned, in place of systems in explicit form (1), we consider (equivalent) implicit systems obtained from (1) by eliminating the input vector u. This representation has the major advantage to be naturally invariant by *endogeneous dynamic feedback* (see [16]). We adapt the notions of Lie-Bäcklund equivalence and Lie-Bäcklund isomorphism in this context and show, after restricting to the category of meromorphic functions, that the flatness property is naturally described in terms of polynomial matrices and differential forms deduced from the variational system equations. For a detailed presentation of polynomial rings and non commutative algebra, the reader may refer to [10, 23] and for exterior differential systems to [4].

Though our results show some parallelism with those of [2, 9], in particular concerning the study of variational properties, they differ from the latter by the fact that, as previously announced, they are invariant by endogeneous dynamic feedback. In addition, they exploit different ideas that may be seen as an extension of the linear approach presented in [25] to nonlinear systems and they provide effective flatness conditions as attested by the examples of the last section.

The paper is organized as follows: the first Section is devoted to the basic description of implicit control systems on manifolds of jets of infinite order. In Section 2, we extend the notions of Lie-Bäcklund equivalence and Lie-Bäcklund isomorphism to the implicit system framework. Section 3 deals with the presentation of some variational properties of flat systems. The necessary and sufficient conditions for flatness are stated in Theorems 2 and 3 of Section 4 and some consequences are presented. Section 5 is then devoted to examples and some concluding remarks are given. An appendix on polynomial matrices and their Smith decomposition is provided.

1 Implicit control systems on manifolds of jets of infinite order

Given an infinitely differentiable manifold X of dimension n, we denote its tangent space at an arbitrary point $x \in X$ by $\mathrm{T}_x X$, and its tangent bundle by $\mathrm{T}X = \bigcup_{x\in X} \mathrm{T}_x X$ (identified with the vector bundle $\mathrm{T}X \stackrel{\mathcal{P}}{\to} X$). Let F belong to $C^\infty(\mathrm{T}X; \mathbb{R}^{n-m})$, the set of C^∞ mappings from $\mathrm{T}X$ to $\mathbb{R}^{n-m}$. In the sequel, we will call *smooth* any function of class C^∞ in all its variables.

We consider an underdetermined implicit system of the form

$$F(x, \dot{x}) = 0 \tag{2}$$

regular in the sense that $\mathsf{rank}\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m$ in a suitable open subset of $\mathrm{T}X$.

Remark 1 *Note that any explicit system of the form $\dot{x} = f(x, u)$ with f smooth, $f(x, u) \in \mathrm{T}_x X$ for every $x \in X$ and u in an open subset U of $\mathbb{R}^m$, and $\mathsf{rank}\left(\frac{\partial f}{\partial u}\right) = m$ in a suitable open subset of $X \times U$, can be locally transformed into (2): permuting the lines of f, if necessary, such that the m last lines of f are locally independent functions of u, and still noting x the permuted vector (this abuse of notations being unambiguous), thanks to the implicit function Theorem one gets $u = \mu(x, \dot{x}_{n-m+1}, \dots, \dot{x}_n)$, μ smooth, and $\dot{x}_i = f_i(x, \mu(x, \dot{x}_{n-m+1}, \dots, \dot{x}_n))$ for $i = 1, \dots, n - m$. Thus, setting $F_i(x, \dot{x}) = \dot{x}_i - f_i(x, \mu(x, \dot{x}_{n-m+1}, \dots, \dot{x}_n))$, $i = 1, \dots, n - m$, the system is in the implicit form (2), and since $\frac{\partial F}{\partial \dot{x}} = \mathsf{diag}\{I_{n-m}, G\}$, G being the matrix made of the entries $\sum_{k=1}^m \frac{\partial f_i}{\partial u_k}\frac{\partial \mu_k}{\partial \dot{x}_j}$ for $i, j = n-m+1, \dots, n$, which don't depend on $(\dot{x}_1, \dots, \dot{x}_{n-m})$, we have $\mathsf{rank}\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m$.*

Conversely, (2) can be transformed into the explicit system $\dot{x} = f(x, u)$ with $f(x, u) \in \mathrm{T}_x X$ for every $x \in X$ a and every u in an open subset U of $\mathbb{R}^m$ with $\mathsf{rank}\left(\frac{\partial f}{\partial u}\right) = m$: the rank condition $\mathsf{rank}\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m$ and the implicit function Theorem yield $\dot{x}_i = f_i(x, \dot{x}_{n-m+1}, \dots, \dot{x}_n)$, $i = 1, \dots, n - m$, with f_i smooth for $i = 1, \dots n - m$, and, setting $\dot{x}_{n-m+j} = u_j$ for $j = 1, \dots, m$ (thus $u = (u_1, \dots, u_m) \in \mathbb{R}^m$), we finally get $\dot{x} = f(x, u)$ with $f_{n-m+j}(x, u) = u_j$ for $j = 1, \dots, m$, *and, by definition, f is a smooth vector field on X for every u in an open subset U of $\mathbb{R}^m$, with $\frac{\partial f}{\partial u} = I_m$ the identity matrix of $\mathbb{R}^m$.*

Clearly, the *underdetermined* character of (2) is expressed by the rank condition $\mathsf{rank}\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m$, which means that the system effectively depends on m independent control variables.

A smooth vector field f that depends, for every $x \in X$, on m independent variables $u \in \mathbb{R}^m$ with $\mathsf{rank}\left(\frac{\partial f}{\partial u}\right) = m$ in a suitable open set of $X \times \mathbb{R}^m$ and that satisfies $F(x, f(x,u)) = 0$ for every $u \in U$, is called *compatible* with (2).

We also introduce the following definition:

Definition 1 *We say that system (1) admits the equivalent representation (2) at (x_0, u_0), if and only if every smooth integral curve of (1) is a smooth integral curve of (2) passing through $(x_0, \dot{x}_0)$, with $\dot{x}_0 = f(x_0, u_0)$.*
System (1) admits the locally equivalent representation (2) if and only if (1) is equivalent to (2) at every point (x_0, u_0) of an open subset of $X \times \mathbb{R}^m$.

In [16] (see also [34] where a similar approach has been developed independently), infinite systems of coordinates $(x, \overline{u}) = (x, u, \dot{u}, \ldots)$ have been introduced to deal with prolonged vector fields $\overline{f}(x, \overline{u}) = \sum_{i=1}^{n} f_i(x,u)\frac{\partial}{\partial x_i} + \sum_{j=1}^{m}\sum_{k\geq 0} u_j^{(k+1)}\frac{\partial}{\partial u_j^{(k)}}$, for explicit systems $\dot{x} = f(x,u)$.

Here, we adopt an external description[1] of the prolonged manifold containing the solutions of (2): we consider the infinite dimensional manifold $\mathfrak{X} \stackrel{\text{def}}{=} X \times \mathbb{R}^n_\infty \stackrel{\text{def}}{=} X \times \mathbb{R}^n \times \mathbb{R}^n \times \ldots$ made of an infinite (but countable) number of copies of $\mathbb{R}^n$. To endow $\mathfrak{X}$ with a suitable topology, we define the continuity and differentiability of functions from $\mathfrak{X}$ to $\mathbb{R}$ as follows:

Definition 2 *We say that a function φ from $\mathfrak{X}$ to $\mathbb{R}$ is* continuous *(resp.* differentiable*) if φ depends only on a finite (but otherwise arbitrary) number of variables and is continuous (resp. differentiable) with respect to these variables.*

Thus $\mathfrak{X}$ is endowed with the coarsest topology that makes projections to products of X and a finite number, k, of copies of $\mathbb{R}^n$, namely $X \times \mathbb{R}^{kn}$, continuous for any $k \in \mathbb{N}$. This topology can be identified with the product topology of the infinite product $X \times \mathbb{R}^n \times \mathbb{R}^n \times \ldots$

C^∞ or analytic or meromorphic functions from $\mathfrak{X}$ to $\mathbb{R}$ are then defined as in the usual finite dimensional case since they only depend on a finite number of variables.

We assume that we are given the infinite set of global coordinates of $\mathfrak{X}$:

$$\overline{x} = (x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \ddot{x}_1, \ldots, \ddot{x}_n, \ldots, x_1^{(k)}, \ldots, x_n^{(k)}, \ldots) \tag{3}$$

and we endow $\mathfrak{X}$ with the so-called trivial Cartan vector field [24, 45]

$$\tau_{\mathfrak{X}} = \sum_{i=1}^{n}\sum_{j\geq 0} x_i^{(j+1)}\frac{\partial}{\partial x_i^{(j)}}. \tag{4}$$

[1]in the sense that the system manifold, namely the set of points $\overline{x} = (x, \dot{x}, \ddot{x}, \ldots)$ such that $F(x, \dot{x}) = 0$, is described by means of the larger manifold $\mathfrak{X} = X \times \mathbb{R}^n_\infty$.

so that $x_i^{(k)} = \frac{d^k x_i}{dt^k}$ for every $i = 1, \ldots, n$ and $k \geq 1$, with the convention $x_i^{(0)} = x_i$, where $\frac{d}{dt} x_i^{(j)} \stackrel{\text{def}}{=} L_{\tau_{\mathfrak{X}}} x_i^{(j)} = \dot{x}_i^{(j)} = x_i^{(j+1)}$, and where

$$L_{\tau_{\mathfrak{X}}} \varphi = \sum_{i=1}^{n} \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial \varphi}{\partial x_i^{(j)}}$$

is the Lie derivative of a function $\varphi \in C^\infty(\mathfrak{X}, \mathbb{R})$ along the vector field $\tau_{\mathfrak{X}}$ (this series having only a finite number of non zero terms according to Definition 2). Therefore the Cartan vector field acts on coordinates as a shift to the right and $\mathfrak{X}$ is called *manifold of jets of infinite order* (see [1, 24, 45]).

From now on, $\overline{x}$ $\overline{y}$, ... stand for the sequence of jets of infinite order of x, y,...

The implicit representation (2), as opposed to (1), is invariant by endogeneous dynamic feedback (see [16, Section IV] for a precise definition):

Proposition 1 *Consider the explicit system (1) with the endogeneous dynamic feedback*

$$u = a(x, z, v), \quad \dot{z} = b(x, z, v). \tag{5}$$

If the system (1) admits the locally equivalent implicit representation (2), then the closed-loop system

$$\dot{x} = f(x, a(x, z, v)), \quad \dot{z} = b(x, z, v) \tag{6}$$

also admits the locally equivalent implicit representation (2). In other words, system (2) is invariant by endogeneous dynamic feedback.

Proof. We assume that (1) admits the locally equivalent implicit representation (2) and that the endogeneous dynamic feedback (5) is given, with z remaining in Z, a given finite dimensional smooth manifold. By definition of an endogeneous dynamic feedback, the closed-loop system (6) is Lie-Bäcklund equivalent to (1). Thus, there exist finite integers α and β and locally onto smooth mappings Φ and Ψ such that

$$(x, z, v) = \Phi(x, u, \ldots, u^{(\alpha)}), \quad (x, u) = \Psi(x, z, v, \ldots, v^{(\beta)}). \tag{7}$$

According to Remark 1, there exists a locally defined smooth mapping μ such that (1) is locally equivalent to (2) with $u = \mu(x, \dot{x})$. Denoting by $u^{(k)} = \mu^{(k)}(x, \dot{x}, \ldots, x^{(k+1)})$ and $\tilde{\Phi}(x, \ldots, x^{(\alpha+1)}) = \Phi(x, \mu(x, \dot{x}), \ldots, \mu^{(\alpha)}(x, \ldots, x^{(\alpha+1)}))$, we immediately deduce that for every smooth local integral curve $t \mapsto x(t)$ of (2) passing through an arbitrary point $(x_0, \dot{x}_0)$, using the first relation of (7), $t \mapsto (x(t), z(t))$, given by $(x(t), z(t), v(t)) = \tilde{\Phi}(x(t), \ldots, x^{(\alpha+1)}(t))$, is also a smooth local integral curve of the closed-loop system (6) passing through (x_0, z_0) such that $(x_0, z_0, v_0) = \tilde{\Phi}(x_0, \ldots, x_0^{(\alpha+1)})$.

Conversely, if $t \mapsto (x(t), z(t), v(t))$ is a smooth local integral curve of (6) passing through (x_0, z_0), by the second relation of (7), we get that $(x(t), u(t)) = \Psi(x(t), z(t), v(t), \ldots, v^{(\beta)}(t))$ is a smooth local integral curve of (1) and, according to the assumption, $t \mapsto x(t)$ is also a local smooth integral curve of (2) passing through the corresponding point $(x_0, \dot{x}_0)$, which achieves the proof. ∎

Definition 3 *A regular implicit control system is defined as a triple* $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ *where* $\mathfrak{X} = X \times \mathbb{R}^n_\infty$, $\tau_{\mathfrak{X}}$ *is the trivial Cartan field on* $\mathfrak{X}$, *and* $F \in C^\infty(\mathrm{T}X; \mathbb{R}^{n-m})$ *satisfies* $\mathsf{rank}\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m$ *in a suitable open dense subset of* $\mathrm{T}X$.

Note that if $t \mapsto (x(t), \dot{x}(t))$ is an integral curve of (2), then $t \mapsto \overline{x}(t)$ is an integral curve of $L^k_{\tau_{\mathfrak{X}}} F = 0$ for every k. Therefore, every integral curve of the system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ lies in the set of $\overline{x}$ such that $L^k_{\tau_{\mathfrak{X}}} F = 0$ for every k.

The system $(\mathbb{R}^m_\infty, \tau_m, 0)$, where we have noted τ_m the trivial Cartan field of $\mathbb{R}^m_\infty$, is called the *trivial system* of dimension m since it corresponds to the void (unconstrained) implicit system $0 = 0$. Note that since $n - m = 0$, the rank condition on $\frac{\partial F}{\partial \dot{x}}$, which in this case is the matrix whose entries are all identically 0, is globally satisfied. The absence of equations relating the coordinates of $\mathbb{R}^m_\infty$ makes this system also an explicit one and, in the notations of [16], it corresponds to the trivial explicit system $(\mathbb{R}^m_\infty, \tau_m)$, which justifies its name.

2 Lie-Bäcklund equivalence for implicit systems

Let us slightly adapt the notion of Lie-Bäcklund equivalence[2] of [16] in our implicit control system context:

Let us consider two regular implicit control systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$, with $\mathfrak{X} = X \times \mathbb{R}^n_\infty$, $\dim X = n$, $\tau_{\mathfrak{X}}$ the associated trivial Cartan field and $\mathsf{rank}\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m$, and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$, with $\mathfrak{Y} = Y \times \mathbb{R}^p_\infty$, $\dim Y = p$, $\tau_{\mathfrak{Y}}$ the associated trivial Cartan field and $\mathsf{rank}\left(\frac{\partial G}{\partial \dot{y}}\right) = p - q$.

Set $\mathfrak{X}_0 = \{\overline{x} \in \mathfrak{X} | L^k_{\tau_{\mathfrak{X}}} F(\overline{x}) = 0,\ \forall k \geq 0\}$ and $\mathfrak{Y}_0 = \{\overline{y} \in \mathfrak{Y} | L^k_{\tau_{\mathfrak{Y}}} G(\overline{y}) = 0,\ \forall k \geq 0\}$. They are endowed with the topologies and differentiable structures induced by $\mathfrak{X}$ and $\mathfrak{Y}$ respectively.

Definition 4 *We say that two regular implicit control systems* $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ *and* $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ *are* Lie-Bäcklund equivalent *(or shortly L-B equivalent) at* $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathfrak{Y}_0$ *if and only if:*

(i) there exist neighborhoods $\mathcal{X}_0$ *and* $\mathcal{Y}_0$ *of* $\overline{x}_0$ *in* $\mathfrak{X}_0$ *and* $\overline{y}_0$ *in* $\mathfrak{Y}_0$ *respectively and a one-to-one mapping* $\Phi = (\varphi_0, \varphi_1, \ldots) \in C^\infty(\mathcal{Y}_0; \mathcal{X}_0)$ *satisfying* $\Phi(\overline{y}_0) = \overline{x}_0$ *and such that the trivial Cartan fields are* Φ*-related, namely* $\Phi_* \tau_{\mathfrak{Y}} = \tau_{\mathfrak{X}}$;

(ii) there exists a one-to-one mapping $\Psi = (\psi_0, \psi_1, \ldots) \in C^\infty(\mathcal{X}_0; \mathcal{Y}_0)$ *such that* $\Psi(\overline{x}_0) = \overline{y}_0$ *and* $\Psi_* \tau_{\mathfrak{X}} = \tau_{\mathfrak{Y}}$.

The mappings Φ *and* Ψ *are called* mutually inverse Lie-Bäcklund isomorphisms *at* $(\overline{x}_0, \overline{y}_0)$.

The two systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ *and* $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ *are called* locally L-B equivalent *if they are L-B equivalent at every pair* $(\overline{x}, \Psi(\overline{x})) = (\Phi(\overline{y}), \overline{y})$ *of an open dense subset* $\mathcal{Z}$ *of* $\mathfrak{X}_0 \times \mathfrak{Y}_0$, *with* Φ *and* Ψ *mutually inverse Lie-Bäcklund isomorphisms on* $\mathcal{Z}$.

[2]In the terminology of [16], different names have been introduced, which, in the author's opinion, are poorly matched: *differential equivalence* (corresponding to *endogeneous transformations* and Φ*-related Cartan fields*) and Lie-Bäcklund equivalence (including time scalings into the previous endogeneous transformations by replacing Cartan fields by Cartan distributions). In order to stress that they are two faces of the same coin, we propose to use *Lie-Bäcklund equivalence* (resp. *orbital Lie-Bäcklund equivalence*) in place of *differential equivalence* (resp. *Lie-Bäcklund equivalence*) and *Lie-Bäcklund isomorphisms* (resp. *orbital Lie-Bäcklund isomorphisms*) in place of *endogeneous transformations* (resp. *Lie-Bäcklund isomorphisms*).

It is easily seen that two systems $\dot{x} = f(x,u)$ and $\dot{y} = g(y,v)$ with f compatible with $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ and g compatible with $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$, are *differentially equivalent* in the sense of [16] (or L-B equivalent as proposed in footnote [2]) at the pair $((x_0, u_0, \dot{u}_0, \ldots), (y_0, v_0, \dot{v}_0, \ldots))$, with u_0 such that $\dot{x}_0 = f(x_0, u_0)$ and v_0 such that $\dot{y}_0 = g(y_0, v_0)$, if and only if $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ are L-B equivalent, in the sense of Definition 4, at $(\overline{x}_0, \overline{y}_0)$ with $\overline{x}_0 = (x_0, \dot{x}_0, \ldots)$ and $\overline{y}_0 = (y_0, \dot{y}_0, \ldots)$, using the construction of Remark 1: let Φ and Ψ satisfy *(i)* and *(ii)*. For every $\overline{g}$ compatible with $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ and for $\overline{y} \in \mathcal{Y}_0$ we have $G(y, g(y,v)) = 0$ for all v in a suitable open subset of $\mathbb{R}^q$. Since, by assumption, $\overline{x} = \Phi(\overline{y}) \in \mathcal{X}_0 \subset \mathfrak{X}_0$, with $\Phi = (\varphi_0, \varphi_1, \ldots)$, we have $x = \varphi_0(y, g(y,v), \frac{dg}{dt}(y, v, \dot{v}), \ldots) \stackrel{\Delta}{=} \tilde{\varphi}_0(y, \overline{v})$, and $\overline{x}$ satisfies $L^k_{\tau_{\mathfrak{X}}} F(\overline{x}) = 0$ for all $k \geq 0$. Let $\overline{v}_0$ be such that $\dot{y}_0 = g(y_0, v_0)$. Thus $\overline{x}_0 = \Phi(y_0, g(y_0, v_0), \frac{d}{dt} g(y_0, v_0, \dot{v}_0), \ldots)$. For every $\overline{f}$ compatible with $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ one has $\dot{x} = f(x,u)$ for $u = \mu(x, \dot{x})$, or $u = \mu(\tilde{\varphi}_0(y, \overline{v}), \frac{d}{dt} \tilde{\varphi}_0(y, \overline{v})) \stackrel{\Delta}{=} \tilde{\varphi}_1(y, \overline{v})$. Using $\Phi_* \tau_{\mathfrak{Y}} = \tau_{\mathfrak{X}}$ yields

$$
\begin{aligned}
& f(x,u)_{|(\tilde{\varphi}_0(y,\overline{v}), \tilde{\varphi}_1(y,\overline{v}))} = L_{\tau_{\mathfrak{X}}} x_{|(\tilde{\varphi}_0(y,\overline{v}), \tilde{\varphi}_1(y,\overline{v}))} = L_{\tau_{\mathfrak{Y}}} \varphi_0(y, L_{\overline{g}} y, L^2_{\overline{g}} y, \ldots) \\
&= \sum_{j \geq 0} \sum_{i=1}^{p} \frac{\partial \varphi_0}{\partial y_i^{(j)}} L_{\overline{g}} \left(L^j_{\overline{g}} y_i \right) = \sum_{j \geq 0} \sum_{i,k=1}^{p} g_k \frac{\partial \varphi_0}{\partial y_i^{(j)}} \frac{\partial L^j_{\overline{g}} y_i}{\partial y_k} + \sum_{j,l \geq 0} \sum_{k=1}^{q} \sum_{i=1}^{p} v_k^{(l+1)} \frac{\partial \varphi_0}{\partial y_i^{(j)}} \frac{\partial L^j_{\overline{g}} y_i}{\partial v_k^{(l)}} \\
&= \sum_{k \geq 0} g_k \frac{\partial \tilde{\varphi}_0}{\partial y_k} + \sum_{l \geq 0} \sum_{k=1}^{q} v_k^{(l+1)} \frac{\partial \tilde{\varphi}_0}{\partial v_k^{(l)}} = (\tilde{\varphi}_0)_* \overline{g}(y, \overline{v}).
\end{aligned}
$$

Analogously, $\dot{u} = \frac{d}{dt} u = L_{\tau_{\mathfrak{X}}} \mu(\overline{x}) = \frac{d}{dt} \tilde{\varphi}_1(y, \overline{v}) = \sum_{i=1}^{p} g_i(y,v) \frac{\partial \tilde{\varphi}_1}{\partial y_i} + \sum_{j \geq 0} \sum_{i=1}^{q} v_i^{(j+1)} \frac{\partial \tilde{\varphi}_1}{\partial v_i^{(j)}} = (\tilde{\varphi}_1)_* \overline{g}$, which proves that $(x, \overline{u}) = \tilde{\Phi}(y, \overline{v})$, with $\tilde{\Phi} = (\tilde{\varphi}_0, \tilde{\varphi}_1, \ldots)$, and $\overline{f} = \tilde{\Phi}_* \overline{g}$, defining $\overline{u}_0$ by $(x_0, \overline{u}_0) = \tilde{\Phi}(y_0, \overline{v}_0)$. Symmetrically, we have $(y, \overline{v}) = \tilde{\Psi}(x, \overline{u})$ with $\tilde{\Psi} = (\tilde{\psi}_0, \tilde{\psi}_1, \ldots)$ and $\tilde{\psi}_0(x, \overline{u}) = \psi_0(x, f(x,u), \frac{df}{dt}(x, u, \dot{u}), \ldots)$, $\tilde{\psi}_1(x, \overline{u}) = \nu(\tilde{\psi}_0(x, \overline{u}), \frac{d}{dt} \tilde{\psi}_0(x, \overline{u}))$, $v = \nu(x, \dot{x})$, and thus $\tilde{\Phi}$'s inverse is $\tilde{\Psi}$ with $(y_0, \overline{v}_0) = \tilde{\Psi}(x_0, \overline{u}_0)$, and $\overline{g} = \tilde{\Psi}_* \overline{f}$, which proves that the explicit systems $(X \times \mathbb{R}^m_\infty, \overline{f})$ and $(Y \times \mathbb{R}^q_\infty, \overline{g})$, for every Cartan fields $\overline{f}$ compatible with $F = 0$ and $\overline{g}$ compatible with $G = 0$, are locally L-B equivalent at $(x_0, \overline{u}_0)$, $(y_0, \overline{v}_0)$ for $\overline{u}_0$ and $\overline{v}_0$ suitably chosen, their choice depending on f and g. The proof of the converse follows the same lines.

An easy consequence of this definition is that L-B equivalence preserves equilibrium points, namely points $\overline{\tilde{y}} = (t, \tilde{y}, 0, 0, \ldots)$ (resp. $\overline{\tilde{x}} = (t, \tilde{x}, 0, 0, \ldots)$) such that $G(\tilde{y}, 0) = 0$ (resp. $F(\tilde{x}, 0) = 0$).

The following result is easily adapted from [16]:

Proposition 2 *If two regular implicit control systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ are locally L-B equivalent, they have the same coranks, namely $m = q$.*

3 Flatness and variational properties

First recall from [16] that a system in explicit form is flat if and only if it is L-B equivalent to a trivial system. The reader may easily check that this definition is just a concise and precise restatement of the definition given in the Introduction. The adaptation of this definition in our context is obvious:

Definition 5 *The implicit system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is* flat *at $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}^m_\infty$ if and only if it is L-B equivalent, at $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}^m_\infty$, to the m-dimensional trivial implicit system $(\mathbb{R}^m_\infty, \tau_m, 0)$. In this case, the mutually inverse L-B isomorphisms Φ and Ψ are called* inverse trivializations, *(or* uniformizations, *in reference to Hilbert's 22nd problem).*

Otherwise stated, (2) is flat at $(\overline{x}_0, \overline{y}_0)$ if the local integral curves $t \mapsto \overline{x}(t)$ of (2) around $\overline{x}_0$ are images by a smooth one-to-one mapping Φ, satisfying $\overline{x}_0 = \Phi(\overline{y}_0)$, of arbitrary curves in the coordinates $\overline{y} = (y_1, \ldots, y_m, \dot{y}_1, \ldots, \dot{y}_m, \ldots, y_1^{(k)}, \ldots, y_m^{(k)}, \ldots,)$ around $\overline{y}_0$. In other words, for every curve $t \mapsto \overline{y}(t)$ in a suitable time interval $\mathfrak{I}$, $\overline{x}(t) = (x(t), \dot{x}(t), \ddot{x}(t), \ldots) = \Phi(\overline{y}(t)) = (\varphi_0(\overline{y}(t)), \varphi_1(\overline{y}(t)), \varphi_2(\overline{y}(t)), \ldots)$ belongs to $\mathfrak{X}_0$ for all $t \in \mathfrak{I}$ and thus $F(x(t), \dot{x}(t)) = 0$. Conversely, if $t \mapsto x(t)$ is an integral curve of $F(x, \dot{x}) = 0$, there exists a curve $t \mapsto y(t)$ in $C^\infty(\mathfrak{I}; \mathbb{R}^m)$ such that $\overline{y}(t) = (y(t), \dot{y}(t), \ldots) = \Psi(\overline{x}(t)) = (\psi_0(\overline{x}(t)), \psi_1(\overline{x}(t)), \ldots)$ for all $t \in \mathfrak{I}$, namely $L_{\tau_{\mathfrak{Y}}} \overline{y}(t) = L_{\tau_{\mathfrak{X}}} \Psi(\overline{x}(t))$ for all $t \in \mathfrak{I}$. Recall that Φ (resp. Ψ) depend only on a finite number of derivatives of y (resp. x).

The extension of this definition to local flatness is straightforward.

Trivializations may be characterized in terms of the differential of F. A basis of the tangent space $\mathrm{T}_{\overline{x}}\mathfrak{X}$ of $\mathfrak{X}$ at a point $\overline{x} \in \mathfrak{X}$ consisting of the set of vectors $\{\frac{\partial}{\partial x_i^{(j)}} | i = 1, \ldots, n, j \geq 0\}$, a basis of the cotangent space $\mathrm{T}^*_{\overline{x}}\mathfrak{X}$ at $\overline{x}$ is therefore given by $\{dx_i^{(j)} | i = 1, \ldots, n, j \geq 0\}$ with $< dx_i^{(j)}, \frac{\partial}{\partial x_k^{(l)}} >= \delta_{i,k}\delta_{j,l}$, $\delta_{i,k}$ being the Kronecker symbol (i.e. $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise). The differential of F_i is thus given by

$$dF_i = \sum_{j=1}^{n} \left(\frac{\partial F_i}{\partial x_j} dx_j + \frac{\partial F_i}{\partial \dot{x}_j} d\dot{x}_j \right), \quad i = 1, \ldots, n - m. \tag{8}$$

Since smooth functions depend on a finite number of variables, their differential contain only a finite number of non zero terms. Accordingly, we define a 1-form on $\mathcal{X}$, an open dense subset of $\mathfrak{X}$, as a *finite* linear combination of the $dx_i^{(j)}$, with coefficients in $C^\infty(\mathcal{X}; \mathbb{R})$, or equivalently as a local C^∞ section of $\mathrm{T}^*(X \times \mathbb{R}^n)$ (see e.g. [45]). The set of 1-forms on $\mathcal{X}$ is denoted by $\Lambda^1(\mathcal{X})$. Clearly, the dF_i's are elements of $\Lambda^1(\mathfrak{X})$.

Note that the shift property of $\frac{d}{dt} = L_{\tau_{\mathfrak{X}}}$ on coordinates extends to differentials by defining $\frac{d}{dt} dx_i$ as: $\frac{d}{dt} dx_i = d\dot{x}_i = d\frac{d}{dt} x_i$. More generally, we define the Lie-derivative of a 1-form ω along a vector field $\overline{v}$ on $\mathfrak{X}$, which is a 1-form on $\mathfrak{X}$, denoted by $L_{\overline{v}}\omega$, as in the finite dimensional case by the Leibnitz rule:

$$\langle L_{\overline{v}}\omega, \overline{w} \rangle = L_{\overline{v}} \langle \omega, \overline{w} \rangle - \langle \omega, [\overline{v}, \overline{w}] \rangle$$

for every vector field $\overline{w}$ on $\mathfrak{X}$, where $[\overline{v}, \overline{w}]$ is the Lie-bracket of $\overline{v}$ and $\overline{w}$. Clearly, if $\omega = dx_i$, $\overline{v} = \tau_{\mathfrak{X}}$ and $\overline{w} = \frac{\partial}{\partial \dot{x}_i}$, we recover the previous formula.

If Φ is a smooth mapping from $\mathfrak{Y}$ to $\mathfrak{X}$, the definition of the image by Φ of a 1-form is the same as in the finite dimensional context: if $\omega \in \Lambda^1(\mathfrak{X})$, $\omega(\overline{x}) = \sum_{\substack{j \geq 0 \\ \mathit{finite}}} \sum_{i=1}^{n} \omega_{i,j}(\overline{x}) dx_i^{(j)}$, its (backward) image $\Phi^*\omega$ is the 1-form on $\mathfrak{Y}$ defined by

$$\Phi^*\omega(\overline{y}) = \sum_{k,l} \sum_{i,j} \omega_{i,j}(\Phi(\overline{y})) \frac{\partial \varphi_{i,j}}{\partial y_k^{(l)}}(\overline{y}) dy_k^{(l)} \tag{9}$$

where $\varphi_{i,j}$ is the i,j-th component of Φ, namely $x_i^{(j)} = \varphi_{i,j}(\overline{y})$.

Note again that, since the functions $\varphi_{i,j}$ depend on a finite number of variables, the 1-form $\Phi^*\omega$ contains only a finite number of non zero terms and, according to $x_i^{(j)} = \frac{d^j x_i}{dt^j}$, we have

$$\begin{aligned} dx_i^{(j)} &= \sum_{k,l} \frac{\partial \varphi_{i,j}}{\partial y_k^{(l)}} dy_k^{(l)} = \frac{d^j}{dt^j} dx_i = L_{\tau_{\mathfrak{X}}}^j dx_i = L_{\tau_{\mathfrak{Y}}}^j \left(\sum_{k,l} \frac{\partial \varphi_{i,0}}{\partial y_k^{(l)}} dy_k^{(l)} \right) \\ &= \sum_{k,l} \sum_{r=0}^{j} \frac{j!}{r!(j-r)!} \left(L_{\tau_{\mathfrak{Y}}}^r \left(\frac{\partial \varphi_{i,0}}{\partial y_k^{(l)}} \right) \right) dy_k^{(l+j-r)}. \end{aligned} \tag{10}$$

Theorem 1 *The system $(X \times \mathbb{R}_\infty^n, \tau_X, F)$ is flat at $(\overline{x}_0, \overline{y}_0)$ with $\overline{x}_0 \in \mathfrak{X}_0$ and $\overline{y}_0 \in \mathbb{R}_\infty^m$ if and only if there exists a locally smooth invertible mapping Φ from $\mathbb{R}_\infty^m$ to $\mathfrak{X}_0$, with smooth inverse, satisfying $\Phi(\overline{y}_0) = \overline{x}_0$, and such that*

$$\Phi^* dF = 0. \tag{11}$$

Proof. Necessity: If the system $(X \times \mathbb{R}_\infty^n, \tau_X, F)$ is flat at $(\overline{x}_0, \overline{y}_0)$, we have $F(\varphi_0(\overline{y}), \varphi_1(\overline{y})) = 0$ for all $\overline{y}$ in a neighborhood of $\overline{y}_0$ in $\mathbb{R}_\infty^m$. For every λ in a given interval $[0, \lambda_0[$ of $\mathbb{R}$ and a sufficiently small time interval $\mathfrak{I}$ containing 0, consider a smooth trajectory $t \mapsto y_\lambda(t)$ in a bounded neighborhood of y_0 in $\mathbb{R}^m$, such that $\sup\{\|y_\lambda^{(j)}(t)\| | j \geq 0, t \in \mathfrak{I}, \lambda \in [0, \lambda_0[\}$ is finite, where $\|\cdot\|$ denotes the Euclidian norm of $\mathbb{R}^m$, and set $\frac{\partial y_\lambda}{\partial \lambda}(t)_{|\lambda=0} = \zeta(t)$, which exists on $\mathfrak{I}$ by Ascoli's Theorem. Next, we set $\overline{x}_\lambda(t) = \Phi(\overline{y}_\lambda(t))$ for $t \in \mathfrak{I}$ and $\lambda \in [0, \lambda_0[$. We indeed have $F(\varphi_0(\overline{y}_\lambda(t)), \varphi_1(\overline{y}_\lambda(t))) = 0$ for all t and λ in their respective intervals and thus, differentiating with respect to λ at $\lambda = 0$, we get $\frac{\partial F}{\partial x} \frac{\partial \varphi_0}{\partial \overline{y}}(\overline{y}_0(t))\overline{\zeta}(t) + \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi_1}{\partial \overline{y}}(\overline{y}_0(t))\overline{\zeta}(t) = 0$ in $\mathfrak{I}$. At time $t = 0$, noting $y_\lambda(0) = y_{\lambda,0}$, we have $\frac{\partial F}{\partial x} \frac{\partial \varphi_0}{\partial \overline{y}}(\overline{y}_{0,0})\overline{\zeta}(0) + \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi_1}{\partial \overline{y}}(\overline{y}_{0,0})\overline{\zeta}(0) = 0$, and this expression is valid for every $\overline{y}_{0,0}$ in a neighborhood of $\overline{y}_0$ and $\overline{\zeta}(0) \in \mathrm{T}_{\overline{y}_{0,0}}\mathbb{R}_\infty^m$. We have thus proved that the 1-form $\frac{\partial F}{\partial x} \frac{\partial \varphi_0}{\partial \overline{y}} d\overline{y} + \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi_1}{\partial \overline{y}} d\overline{y} = \Phi^* dF$ vanishes on $\mathrm{T}_{\overline{y}_{0,0}}\mathbb{R}_\infty^m$ for every $\overline{y}_{0,0}$ in a neighborhood of y_0, and therefore is identically zero in a neighborhood of y_0, which proves (11).

Let us prove the sufficiency. Assuming that there exists a locally smooth invertible mapping $\Phi = (\varphi_0, \varphi_1, \ldots) \in C^\infty(\mathbb{R}_\infty^m; \mathfrak{X})$ satisfying (11) with $\Phi(\overline{y}_0) = \overline{x}_0$, the 1-forms $\Phi^* dF_i$ are obviously closed since they are the differentials of the functions $F_i \circ \Phi$, $i = 1, \ldots, n-m$. Thus (11) implies that $F_i(\varphi_0(\overline{y}), \varphi_1(\overline{y})) = c_i$, $i = 1, \ldots, n-m$, with c_i arbitrary constants. But since $\overline{x}_0 \in \mathfrak{X}_0$ and $\overline{x}_0 = \Phi(\overline{y}_0)$, we have $F(x_0, \dot{x}_0) = 0$ and $c_i = F_i(\varphi_0(\overline{y}_0), \varphi_1(\overline{y}_0)) = F_i(x_0, \dot{x}_0) = 0$, for $i = 1, \ldots, n-m$. Then, setting $\overline{x} = \Phi(\overline{y}) = (\varphi_0(\overline{y}), \varphi_1(\overline{y}), \ldots)$, we get that $x = \varphi_0(\overline{y})$ (which depends on y and a finite number of its derivatives) satisfies $F_i(x, \dot{x}) = 0$, $i = 1, \ldots, n-m$ and that $\dot{x} = L_{\tau_{\mathfrak{X}}} x = L_{\tau_{\mathfrak{Y}}} \varphi_0(\overline{y}) = \varphi_1(\overline{y})$. Following the same lines for all derivatives of x, we have proved that $\Phi_* \tau_{\mathfrak{Y}} = \tau_{\mathfrak{X}}$. Finally, since Φ is invertible with C^∞ inverse $\Psi = (\psi_0, \psi_1, \ldots)$, it is immediately seen that $\overline{y} = \Psi(\overline{x})$ (and therefore $y = \psi_0(\overline{x})$ only depends on a finite number of derivatives of x) and that $\Psi_* \tau_{\mathfrak{X}} = \tau_{\mathfrak{Y}}$, which proves the sufficiency and the proof is complete. ■

4 Flatness necessary and sufficient conditions

We now analyze condition (11) in more details with the (mild) restriction that F is *meromorphic* on $\mathrm{T}X$. This restriction is motivated by the use of algebraic properties of polynomial matrices and of modules over a principal ideal ring of polynomials, this ring being itself formed over the field of meromorphic functions, as will be made clear immediately.

We also restrict the inverse trivializations Φ and Ψ of definition 5 to the class of meromorphic functions.

In matrix notations and using indifferently $\frac{d}{dt}$ for $L_{\tau_{\mathfrak{X}}}$ or $L_{\tau_{\mathfrak{Y}}}$ (the context being unambiguous), according to (10), we have:

$$\begin{aligned}
\Phi^* dF &= \sum_{j\geq 0}\sum_{i=1}^{m}\left(\frac{\partial F}{\partial x}\frac{\partial \varphi_0}{\partial y_i^{(j)}} + \frac{\partial F}{\partial \dot{x}}\frac{\partial \varphi_1}{\partial y_i^{(j)}}\right) dy_i^{(j)} \\
&= \sum_{j\geq 0}\sum_{i=1}^{m}\left(\frac{\partial F}{\partial x}\frac{\partial \varphi_0}{\partial y_i^{(j)}} dy_i^{(j)} + \frac{\partial F}{\partial \dot{x}}\frac{d}{dt}\left(\frac{\partial \varphi_0}{\partial y_i^{(j)}} dy_i^{(j)}\right)\right) \\
&= \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}}\frac{d}{dt}\right)_{|\overline{x}=\Phi(\overline{y})}\left(\sum_{i=1}^{m}\sum_{j\geq 0}\frac{\partial \varphi_0}{\partial y_i^{(j)}}\frac{d^j}{dt^j} dy_i\right).
\end{aligned}$$

Introducing the following polynomial matrices where the indeterminate is the differential operator $\frac{d}{dt}$:

$$P(F) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}}\frac{d}{dt}, \quad P(\varphi_0) = \sum_{j\geq 0}\frac{\partial \varphi_0}{\partial y^{(j)}}\frac{d^j}{dt^j} \tag{12}$$

with $P(F)$ (resp. $P(\varphi_0)$) of size $(n-m)\times n$ (resp. $n\times m$), (11) reads:

$$\Phi^* dF_{|\overline{y}} = P(F)_{|\Phi(\overline{y})} P(\varphi_0)_{|\overline{y}}\, dy = 0. \tag{13}$$

or equivalently

$$\Phi^* dF_{|\Psi(\overline{x})} = P(F)_{|\overline{x}} P(\varphi_0)_{|\Psi(\overline{x})} dy = 0. \tag{14}$$

Clearly, the entries of these matrices are polynomials of the differential operator $\frac{d}{dt}$ whose coefficients are meromorphic functions from $\mathfrak{X}$ to $\mathbb{R}$. We denote by $\mathfrak{K}$ the field of meromorphic functions from $\mathfrak{X}$ to $\mathbb{R}$ and by $\mathfrak{K}[\frac{d}{dt}]$ the principal ideal ring of polynomials of $\frac{d}{dt} = L_{\tau_{\mathfrak{X}}}$ with coefficients in $\mathfrak{K}$.

We may also consider the field of meromorphic functions from $\mathfrak{Y} = \mathbb{R}^m_\infty$ to $\mathbb{R}$. In this case, the notations $\mathfrak{K}_{\mathfrak{Y}}$ and $\mathfrak{K}_{\mathfrak{Y}}[\frac{d}{dt}]$, with $\frac{d}{dt} = L_{\tau_{\mathfrak{Y}}}$, will replace the previous ones.

Recall that $\mathfrak{K}[\frac{d}{dt}]$ is non commutative, even if $n = 1$, as shown by the following example: denoting, with abusive notations, by x the 0th order operator $a \mapsto xa$ for every $a \in \mathfrak{K}$, we have, for $a \neq 0$, $\left(\frac{d}{dt}\cdot x - x\cdot\frac{d}{dt}\right)(a) = \dot{x}a + x\dot{a} - x\dot{a} = \dot{x}a \neq 0$, or $\frac{d}{dt}\cdot x - x\cdot\frac{d}{dt} = \dot{x}$.

For arbitrary integers p and q, let us denote by $\mathcal{M}_{p,q}[\frac{d}{dt}]$ the module of $p\times q$ matrices over $\mathfrak{K}[\frac{d}{dt}]$ (see e.g. [10, 23] for a detailed presentation of modules over non commutative rings). Recall that, since in general the inverse of a polynomial is not a polynomial, for arbitrary $p \in \mathbb{N}$, the inverse of a square invertible matrix of $\mathcal{M}_{p,p}[\frac{d}{dt}]$ doesn't generally belong to $\mathcal{M}_{p,p}[\frac{d}{dt}]$. Matrices whose inverse belong to $\mathcal{M}_{p,p}[\frac{d}{dt}]$ are called *unimodular matrices* and their set is denoted by $\mathcal{U}_p[\frac{d}{dt}]$. It forms a normal[3] subgroup of the group of invertible matrices of $\mathcal{M}_{p,p}[\frac{d}{dt}]$.

Here, $P(F) \in \mathcal{M}_{n-m,n}[\frac{d}{dt}]$ and, according to the Annex A (see also [10]) it admits a *Smith decomposition* (or diagonal reduction), given by

$$VP(F)U = (\Delta, 0_{n-m,m}) \tag{15}$$

with $0_{n-m,m}$ the $(n-m) \times m$ matrix whose entries are all zeros, $V \in \mathcal{U}_{n-m}[\frac{d}{dt}]$, $U \in \mathcal{U}_n[\frac{d}{dt}]$ and $\Delta \in \mathcal{M}_{n-m,n-m}[\frac{d}{dt}]$ a diagonal matrix whose entries $d_{i,i}$ divide $d_{j,j}$ for all $0 \leq i \leq j \leq n-m$. Moreover, Δ is unique.

Consider a point $\overline{x} \in \mathfrak{X}$ and its projection x on the original manifold X. Let us denote by $\xi = (\xi_1, \ldots, \xi_n)$ a basis of the tangent space $\mathrm{T}_x X$. We also denote by $[\xi]$ the $\mathfrak{K}[\frac{d}{dt}]$-module generated by ξ, by $[P(F)\xi]$ the submodule generated by the lines of $P(F)\xi$ and by $\mathcal{M}$ the quotient module $[\xi]/[P(F)\xi]$. We call $\mathcal{M}$ the *variational module* of (2) at $\overline{x}$.

Clearly, for the trivial system $\mathfrak{Y} = \mathbb{R}^m_\infty$, its variational module $\mathcal{M}_{\mathfrak{Y}}$ at an arbitrary point $\overline{y} \in \mathbb{R}^m_\infty$ reduces to the free $\mathfrak{K}_{\mathfrak{Y}}[\frac{d}{dt}]$-module $[\zeta]$ generated by $\zeta = (\zeta_1, \ldots, \zeta_m)$, a basis of the tangent space $\mathrm{T}_y\mathbb{R}^m$ at y.

According to [11], $\mathcal{M}$ can be uniquely decomposed into the following direct sum: $\mathcal{M} = \mathcal{T} \oplus \mathcal{F}$, where $\mathcal{T}$ is torsion and $\mathcal{F}$ is free. It is immediate to verify that $\mathcal{F} = \{\xi \in \mathcal{M} | \exists \zeta = (\zeta_1, \ldots, \zeta_{n-m}) \text{ s.t. } \xi = \begin{pmatrix} 0_{m,n-m} \\ I_{n-m} \end{pmatrix} \zeta\} = \mathrm{im}\ \pi_{n-m}$ where π_{n-m} is the projection operator $\pi_{n-m}(\xi_1, \ldots, \xi_n) = (0, \ldots, 0, \xi_{m+1}, \ldots, \xi_n)$ and $\mathcal{T} = \{\xi \in \mathcal{M} | \, (I_m, 0_{m,n-m})\, \xi \neq 0 \text{ and } \Delta\xi = 0\} = \ker \Delta \cap \ker \pi_{n-m}$. Therefore we have

$$\dim_{\mathfrak{K}}\ \mathcal{M} = n - m + \dim_{\mathfrak{K}} (\ker \Delta \cap \ker \pi_{n-m}) \, .$$

We define the rank of $P(F)$ by $\mathsf{rank}\,(P(F)) = \dim_{\mathfrak{K}}\ \mathcal{M} = n - m + \dim_{\mathfrak{K}} (\ker \Delta \cap \ker \pi_{n-m})$.

Definition 6 *Given a matrix $M \in \mathcal{M}_{p,q}[\frac{d}{dt}]$, we say that M is* hyper-regular *if and only if its Smith decomposition leads to $(I_p, 0_{p,q-p})$ if $p < q$, to I_p if $p = q$ and to $\begin{pmatrix} I_q \\ 0_{p-q,q} \end{pmatrix}$ if $p > q$.*

Clearly, the rank of an hyper-regular matrix is equal to $\min\{p, q\}$. Moreover, a square matrix $M \in \mathcal{M}_{p,p}[\frac{d}{dt}]$ is hyper-regular if and only if it is unimodular: $M \in \mathcal{U}_p[\frac{d}{dt}]$.

4.1 Flatness and controllability

Recall from [11] that the variational module of (2) at $\overline{x}$ is said to be *controllable* if and only if it is free.

Proposition 3 *If system (2) is locally flat at $\overline{x}_0$, its variational module at every point of a neighborhood of $\overline{x}_0$ is controllable and $P(F)$ is hyper-regular in this neighborhood.*

[3] $A^{-1}\mathcal{U}_p[\frac{d}{dt}]A = \mathcal{U}_p[\frac{d}{dt}]$ for every invertible $A \in \mathcal{M}_{p,p}[\frac{d}{dt}]$

Proof. Let $(\overline{x}, \overline{y})$ belong to a neighborhood of $(\overline{x}_0, \overline{y}_0)$ in $\mathfrak{X}_0 \times \mathbb{R}^m_\infty$ with $\overline{x} = \Phi(\overline{y})$, Φ being a trivialization. We denote as before φ_0 the first component of Φ, namely $x = \varphi_0(\overline{y})$, and we consider the variational module $\mathcal{M} = [\xi]/[P(F)\xi]$ at $\overline{x}$. Since Φ is a trivialization, it is not difficult to check that the Jacobian matrix $\frac{\partial \varphi_0}{\partial \overline{y}}$ has rank n in the considered neighborhood and that for every element $\tilde{\zeta} \in \mathcal{M}_{\mathfrak{Y}}$, there exists $\tilde{\xi} = P(\varphi_0)\tilde{\zeta} \in \mathcal{M}$, the converse being also true since the inverse trivialization Ψ enjoys the same properties. Moreover, since $L^k_{\tau_y} d\varphi_0 = d\left(L^k_{\tau_y}\varphi_0\right) = d\varphi_k$ for every k, it results that $\mathcal{M}$ is isomorphic to $\mathcal{M}_{\mathfrak{Y}}$ which is free. Hence $\mathcal{M}$ is controllable.

The variational module $\mathcal{M}$ at $\overline{x}$ being controllable, we have $\mathcal{T} = \ker \Delta \cap \ker \pi_{n-m} = \{0\}$, Δ being the diagonal matrix resulting from the Smith decomposition of $P(F)$. But if $\xi \in \ker \Delta \cap \ker \pi_{n-m}$, according to the fact that the diagonal elements $d_{i,i}$ of Δ are polynomials of $\frac{d}{dt}$, ξ satisfies the set of differential equations $d_{i,i}\xi = 0$, $i = 1, \ldots, n-m$. And since $d_{1,1}$ divides $d_{i,i}$ for $i = 1, \ldots, n-m$, $d_{1,1}\xi = 0$ implies $\xi = 0$ if and only if $d_{1,1} \in \mathfrak{K}$. Assuming that $d_{i,i} \in \mathfrak{K}$ and applying the same argument to $d_{i+1,i+1}$, we obviously get that $d_{i,i} \in \mathfrak{K}$ for all $i = 1, \ldots, n-m$ which proves that all the diagonal elements of Δ belong to $\mathfrak{K}$. It is thus straightforward to modify the unimodular matrices U and V such that $d_{i,i} = 1$ for all i, which proves the Proposition. ■

4.2 Algebraic characterization of the differential of a trivialization

From now on, we assume that $P(F)$ is hyper-regular in a neighborhood $\mathfrak{X}_0$ of $\overline{x}_0 \in \mathfrak{X}_0$.

In other words, there exist V and U such that

$$VP(F)U = (I_m, 0_{n-m,m}). \tag{16}$$

U and V satisfying (16) are indeed non unique. We say that $U \in \mathsf{R}-\mathsf{Smith}\,(P(F))$ and $V \in \mathsf{L}-\mathsf{Smith}\,(P(F))$ if they are such that $VP(F)U = (I_m, 0)$.

Accordingly, if $M \in \mathcal{M}_{n,m}[\frac{d}{dt}]$, is hyper-regular with $\mathsf{rank}\,(M) = m \leq n$, we say that $V \in \mathsf{L}-\mathsf{Smith}\,(M)$ and $W \in \mathsf{R}-\mathsf{Smith}\,(M)$ if $V \in \mathcal{U}_n[\frac{d}{dt}]$ and $W \in \mathcal{U}_m[\frac{d}{dt}]$ satisfy $VMW = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$.

In this framework, the set of all polynomial matrices $P(\varphi_0)_{|\Psi(\overline{x})} \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ satisfying (13), or (14), can be completely characterized. We first solve the matrix equation:

$$P(F)\Theta dy = 0 \tag{17}$$

where the entries of $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ are not supposed to be gradients of some function φ_0:

Lemma 1 *The set of hyper-regular matrices $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ satisfying (17) is nonempty and given by*

$$\Theta = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} W \tag{18}$$

with $U \in \mathsf{R}-\mathsf{Smith}\,(P(F))$ and $W \in \mathcal{U}_m[\frac{d}{dt}]$ arbitrary.

Proof. Using Theorem 4, at any point $\overline{x} = \Phi(\overline{y})$ of a suitable neighborhood where $\mathsf{rank}\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m$, equation (17), by (16), is equivalent to

$$VP(F)UU^{-1}\Theta = (I_{n-m}, 0_{n-m,m})\, U^{-1}\Theta = 0 \tag{19}$$

from which we deduce that $U^{-1}\Theta = \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} W$, where W is an arbitrary $m \times m$ polynomial matrix, or $\Theta = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} W$. The hyper-regularity of Θ and the invertibility of Ψ immediately imply the hyper-regularity of W. But since W is square $m \times m$ and hyper-regular, it is unimodular.

■

We introduce the notation:

$$\hat{U} = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix}. \tag{20}$$

Lemma 2 *For every $Q \in \mathsf{L} - \mathsf{Smith}\left(\hat{U}\right)$ there exists $Z \in \mathfrak{U}_m[\frac{d}{dt}]$ such that*

$$Q\Theta = \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix} Z. \tag{21}$$

Moreover, for every $Q \in \mathsf{L} - \mathsf{Smith}\left(\hat{U}\right)$, the submatrix $\hat{Q} = (0_{n-m,m}, I_{n-m})\, Q$ is equivalent to $P(F)$, more precisely there exists a unimodular matrix L such that $P(F) = L\hat{Q}$.

Proof. The first part is an immediate consequence of Theorem 4 applied to $\hat{U}$, which, by (20), is hyper-regular: there exist $Q \in \mathfrak{U}_n[\frac{d}{dt}]$ and $R \in \mathfrak{U}_m[\frac{d}{dt}]$ such that $Q\hat{U}R = \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix}$. The result follows by setting $R^{-1}W = Z \in \mathfrak{U}_m[\frac{d}{dt}]$.

To prove the second part, we left multiply (21) by $(0_{n-m,m}, I_{n-m})$ and use (18) to get $\hat{Q}\hat{U} = 0$. Then comparing with (18), where $P(F)\hat{U} = 0$, and taking into account the hyper-regularity of $\hat{U}$, the existence of a unimodular matrix L such that $P(F) = L\hat{Q}$ is proven.

■

4.3 Integrability

Let us denote by $Q_{i,j} = \sum_{k\geq 0} Q_{i,j}^k \frac{d^k}{dt^k}$ the (i,j)-th polynomial entry of $Q \in \mathsf{L} - \mathsf{Smith}\left(\hat{U}\right)$ obtained from Lemma 2. We also denote by ω the m-dimensional vector 1-form defined by

$$\omega(\overline{x}) = \begin{pmatrix} \omega_1(\overline{x}) \\ \vdots \\ \omega_m(\overline{x}) \end{pmatrix} = (I_m, 0_{m,n-m})\, Q(\overline{x}) dx_{|\mathfrak{X}_0} = \begin{pmatrix} \sum_{j=1}^n \sum_{k\geq 0} Q_{1,j}^k(\overline{x}) dx_j^{(k)}{}_{|\mathfrak{X}_0} \\ \vdots \\ \sum_{j=1}^n \sum_{k\geq 0} Q_{m,j}^k(\overline{x}) dx_j^{(k)}{}_{|\mathfrak{X}_0} \end{pmatrix} \tag{22}$$

the restriction to $\mathfrak{X}_0$ meaning that $\overline{x} \in \mathfrak{X}_0$ satisfies $L^k_{\tau_{\mathfrak{X}}} F = 0$ for all k and that the $dx_j^{(k)}$ are such that $dL^k_{\tau_{\mathfrak{X}}} F = 0$ for all k in $\mathfrak{X}_0$.

Since Q is unimodular, $(I_m, 0_{m,n-m})\, Q$ is hyper-regular of rank m. Thus, the forms $\omega_1, \ldots, \omega_m$ are independent.

Let us also recall that, if $\tau_1, \ldots, \tau_r$ are given independent 1-forms in $\Lambda^1(\mathfrak{X}_0)$, the $\mathfrak{K}[\frac{d}{dt}]$-ideal $\mathfrak{T}$ generated by $\tau_1, \ldots, \tau_r$, for an arbitrary integer r, is the set of all combinations with coefficients in $\mathfrak{K}[\frac{d}{dt}]$ of forms $\eta \wedge \tau_i$ with η an arbitrary form on $\mathfrak{X}_0$ of arbitrary degree and $i = 1, \ldots, r$.

Note that if another set of independent 1-forms $\kappa_1, \ldots, \kappa_s$ of $\mathfrak{T}$ is a generator of $\mathfrak{T}$, then $s = r$ and there exists a unimodular matrix $H \in \mathfrak{U}_r[\frac{d}{dt}]$ such that $\tau = H\kappa$, where $\tau = (\tau_1, \ldots, \tau_r)^T$ (the superscript ^{T} means transposition) and $\kappa = (\kappa_1, \ldots, \kappa_r)^T$.

Definition 7 *We say that the $\mathfrak{K}[\frac{d}{dt}]$-ideal $\mathfrak{T}$ generated by $\tau_1, \ldots, \tau_r$ is* strongly closed *if there exists a matrix $M \in \mathfrak{U}_m[\frac{d}{dt}]$ such that $d\tau = -M^{-1} dM \wedge \tau$.*

This definition is independent of the choice of generators since if κ is another vector of generators of $\mathfrak{T}$, we have $\tau = H\kappa$ for $H \in \mathfrak{U}_r[\frac{d}{dt}]$, and since $d\tau = -M^{-1}dM \wedge \tau$ is equivalent to $d(M\tau) = 0$, we have $d(MH\kappa) = 0$, or $d(MH) \wedge \kappa + MHd\kappa = 0$, which proves that $d\kappa = -(MH)^{-1} d(MH) \wedge \kappa$, MH being obviously unimodular.

Theorem 2 *A necessary and sufficient condition for system (2) to be flat at $(\overline{x}_0, \overline{y}_0)$ is that there exist $U \in \mathsf{R} - \mathsf{Smith}\,(P(F))$ and $Q \in \mathsf{L} - \mathsf{Smith}\left(\hat{U}\right)$, with $\hat{U}$ given by (20), such that the $\mathfrak{K}[\frac{d}{dt}]$-ideal Ω generated by the 1-forms $\omega_1, \ldots, \omega_m$ defined by (22) is strongly closed in $\mathfrak{X}_0$.*

Proof. Necessity: If system (2) is flat at $(\overline{x}_0, \overline{y}_0)$, there exists $\Phi = (\varphi_0, \varphi_1, \ldots)$ meromorphic from a neighborhood $\mathfrak{Y}_0$ of $\overline{y}_0$ in $\mathbb{R}^m_\infty$ to a neighborhood $\mathcal{X}_0$ of $\overline{x}_0$ in $\mathfrak{X}_0$ and bijective, such that $\overline{x} = \Phi(\overline{y})$ implies $F(\varphi_0(\overline{y}), \varphi_1(\overline{y})) = 0$ and $L^k_{\tau_{\mathfrak{Y}}}(F \circ \Phi) = 0$ for all $k \geq 0$. According to Theorem 1, $P(\varphi_0)$ satisfies (13), or (14), and is hyper-regular: proceeding as in Proposition 3, its kernel $P(\varphi_0)^{-1}(\{0\})$ cannot contain torsion elements since otherwise it would contradict the flatness property of the tangent trivial system $(\mathbb{R}^m_\infty, 0)$. Thus, in virtue of Lemma 1, there exists $W \in \mathfrak{U}_m[\frac{d}{dt}]$ such that

$$P(\varphi_0)_{|\Psi(\overline{x})} = U_{|\overline{x}} \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} W_{|\overline{x}}. \tag{23}$$

By Lemma 2, there exist $Q \in \mathsf{L} - \mathsf{Smith}\left(\hat{U}\right)$ and $Z \in \mathfrak{U}_m[\frac{d}{dt}]$ such that

$$Q_{|\overline{x}} P(\varphi_0)_{|\Psi(\overline{x})} = \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix} Z_{|\overline{x}}. \tag{24}$$

Thus, left multiplying by $(I_m, 0_{n-m,m})$ and denoting by $\tilde{Q} = (I_m, 0_{m,n-m})\, Q$, we get

$$\tilde{Q}_{|\overline{x}} P(\varphi_0)_{|\Psi(\overline{x})} = Z_{|\overline{x}}. \tag{25}$$

Taking the exterior derivative of $x = \varphi_0(\overline{y})$ yields $dx = P(\varphi_0)dy$, and thus, according to (25), $\omega(\Phi(\overline{y})) = \tilde{Q}_{|\Phi(\overline{y})} dx = \tilde{Q}_{|\Phi(\overline{y})} P(\varphi_0)_{|\overline{y}} dy = Z_{|\Phi(\overline{y})} dy$, or

$$\left(Z_{|\overline{x}}\right)^{-1} \tilde{Q}_{|\overline{x}} dx = dy. \tag{26}$$

The forms $\omega_1, \ldots, \omega_m$ are independent, generate the ideal Ω, and setting $M = (Z)^{-1}$ in (26) we get $M\omega = dy$. Thus, taking the exterior derivative of both sides yields $d(M\omega) = 0$, which proves that Ω is strongly closed.

Sufficiency: Assume that there exist $U \in \mathsf{R} - \mathsf{Smith}\,(P(F))$ and $Q \in \mathsf{L} - \mathsf{Smith}\left(\hat{U}\right)$ such that Ω is strongly closed in a neighborhood $\mathcal{X}_0$ of $\overline{x}_0$ in $\mathfrak{X}_0$. Let $M \in \mathcal{U}_m[\frac{d}{dt}]$ be such that $d(M\omega) = 0$. Setting $\eta = M\omega$, the 1-forms $\eta_1, \ldots, \eta_m$ generate Ω, are independent in $\mathcal{X}_0$ since $Q \in \mathsf{L} - \mathsf{Smith}\left(\hat{U}\right)$, and, when expressed in the basis $dx_i^{(j)}$, depend on a finite number of terms whose coefficients depend on a finite number of derivatives of x. In the corresponding finite dimensional manifold, we have $d\eta_i = 0$, $i = 1, \ldots, m$, and, by Poincaré's Lemma, there locally exists a mapping $\psi_0 \in C^\infty(\mathcal{X}_0; \mathbb{R}^m)$ such that $d\psi_0 = \eta = M\omega = M\tilde{Q}dx$. In addition ψ_0 is a meromorphic function of its arguments since its differential is, according to the previous relation.

Denoting by $y = \psi_0(\overline{x})$ for all $\overline{x} \in \mathcal{X}_0$ and $\Psi = \left(\psi_0, \frac{d}{dt}\psi_0, \frac{d^2}{dt^2}\psi_0, \ldots\right) = (\psi_0, \psi_1, \psi_2, \ldots)$, we have to prove that Ψ is a trivialization.

Since $Q \in \mathsf{L} - \mathsf{Smith}\left(\hat{U}\right)$, writing as before $Q = \left(\begin{array}{c} \tilde{Q} \\ \hat{Q} \end{array}\right)$ with $\tilde{Q} = (I_m, 0_{m,n-m})\,Q$ and $\hat{Q} = (0_{n-m,m}, I_{n-m})\,Q$, there exists $R \in \mathcal{U}_m[\frac{d}{dt}]$ such that $\tilde{Q}\hat{U}R = I_m$ and $\hat{Q}\hat{U}R = 0_{n-m,m}$. Setting $W = RM^{-1}$, we get $\tilde{Q}\hat{U}W = M^{-1}$ and $\hat{Q}\hat{U}W = 0_{n-m,m}$, which, combined with $d\psi_0 = M\tilde{Q}dx$, yields $\tilde{Q}\hat{U}Wd\psi_0 = \tilde{Q}dx$ and $\hat{Q}\hat{U}Wd\psi_0 = 0_{n-m}$. It results that there exists an element $dz \in \ker(\tilde{Q})$ such that $\hat{U}Wd\psi_0 = dx + dz$ on $\mathcal{X}_0$. But the latter relation, with the fact that dx is a basis of $\mathrm{T}_x^* X_0$ and that, by the second part of Lemma 2, we have $\hat{Q}dx = L^{-1}P(F)dx = 0$, implies $\hat{Q}dz = 0$, which means that $dz \in \ker(\tilde{Q}) \cap \ker(\hat{Q}) = \{0\}$. We have thus proved that $d\psi_0$ satisfies $dx = \hat{U}Wd\psi_0$ with $W = RM^{-1}$. If we denote by σ_i the highest polynomial degree of the entries of the ith column of $\hat{U}W$, we must have $n \leq m + \sigma_1 + \ldots + \sigma_m$ since otherwise this would contradict the surjectivity of $\hat{U}W$, considered as the matrix Ξ whose entries are the $(\hat{U}W)_{i,j}^k$'s, mapping an open subset of $\mathbb{R}^{\sigma_1+1} \times \ldots \times \mathbb{R}^{\sigma_m+1}$ to an open subset of $\mathbb{R}^n$. In addition, if we note $\sigma = \max(\sigma_i | i = 1, \ldots, m)$ and $\overline{y}^\sigma = \left(y_1^{(0)}, \ldots, y_1^{(\sigma_1)}, \ldots, y_m^{(0)}, \ldots, y_m^{(\sigma_m)}\right)$, $\Xi = \frac{\partial x}{\partial \overline{y}^\sigma}$ has rank n. Then the implicit system

$$\begin{array}{l} y = \psi_0(\overline{x}) \\ \dot{y} = \psi_1(\overline{x}) \\ \vdots \\ y^{(\sigma)} = \psi_\sigma(\overline{x}) \end{array} \tag{27}$$

has rank n with respect to x since its Jacobian matrix is a pseudo-inverse of Ξ. Hence, by the implicit function Theorem, a local solution to (27) is given by $x = \varphi_0(y, \ldots, y^{(\sigma)}, \dot{x}, \ldots, x^{(\rho)})$ for a suitable $\rho \in \mathbb{N}$. But differentiating φ_0, using the fact that $dF = 0$, or equivalently $P(F)dx = 0$, and comparing with (18), we find that φ_0 is independent of $(\dot{x}, \ldots, x^{(\rho)})$, or $x = \varphi_0(\overline{y})$. It results that $\Phi = (\varphi_0, \frac{d}{dt}\varphi_0, \ldots)$ is the inverse trivialization of Ψ which achieves to prove the Theorem. ■

Remark 2 *The strong closedness condition may be expressed in terms of distributions of vector fields by introducing the distribution* $\mathcal{D} = \{f \in \mathrm{T}\mathfrak{X} | f \lrcorner\, dF = 0, f \lrcorner\, \Omega = 0\}$*, the symbol* $\lrcorner$ *denoting the interior product between vector fields and forms. This may be the subject of future works.*

A characterization of the strong closedness condition is given by the next

Theorem 3 *The $\mathfrak{K}[\frac{d}{dt}]$-ideal Ω generated by the 1-forms $\omega_1, \ldots, \omega_m$ defined by (22) is strongly closed in $\mathfrak{X}_0$ if and only if there exists an $m \times m$ matrix μ, whose entries are polynomials of $\frac{d}{dt}$ with coefficients in $\Lambda^1(\mathfrak{X})$, and a matrix $M \in \mathfrak{U}_m[\frac{d}{dt}]$ such that*

$$d\omega = \mu \wedge \omega, \quad d\mu = \mu \wedge \mu, \quad dM = -M\mu \tag{28}$$

where we have denoted by ω the vector of 1-forms $(\omega_1, \ldots, \omega_m)^T$.

In addition, if (28) holds true, a flat output y is obtained by integration of $dy = M\omega$.

Proof. If the $\mathfrak{K}[\frac{d}{dt}]$-ideal Ω generated by the 1-forms $\omega_1, \ldots, \omega_m$ defined by (22) is strongly closed in $\mathfrak{X}_0$, by definition there exists $M \in \mathfrak{U}_m[\frac{d}{dt}]$ such that $d(M\omega) = 0$, or $d\omega = -M^{-1}dM \wedge \omega$. Setting $\mu = -M^{-1}dM$, we have $d\omega = \mu \wedge \omega$ and $M\mu = -dM$. Taking the exterior derivative of this latter expression, we get $Md\mu - M\mu \wedge \mu = M\,(d\mu - \mu \wedge \mu) = -d^2M = 0$. Since M is unimodular, this implies that $d\mu = \mu \wedge \mu$, which proves (28).

Conversely, since there exists $M \in \mathfrak{U}_m[\frac{d}{dt}]$ and μ satisfying (28), $dM = -M\mu$ implies that $\mu = -M^{-1}dM$ and thus $d\omega = -M^{-1}dM \wedge \omega$, or $Md\omega + dM \wedge \omega = d(M\omega) = 0$ which proves the strong closedness.

Finally, if there exists a matrix $M \in \mathfrak{U}_m[\frac{d}{dt}]$ such that $d(M\omega) = 0$, By Poincaré's Lemma, there exist m functions $y_1, \ldots, y_m$ such that $dy = M\omega$, which achieves the proof. ■

Remark 3 *Condition (28) may be seen as a generalization in the framework of manifolds of jets of infinite order of the well-known moving frame structure equations (see e.g. [8]).*

Remark 4 *For **non flat systems**, since for every unimodular matrix M, we have $d(M\omega) \neq 0$, there must exist a non zero 2-form $\tau(M)$ such that $d(M\omega) = \tau(M)$. Thus, setting $\varpi = M^{-1}\tau(M)$, it is clear that the 2-form ϖ is uniquely determined modulo Ω: if M_1 and M_2 are two unimodular matrices and if we note $\tau_i = d(M_i\omega)$, $\varpi_i = M_i^{-1}\tau_i$, $i = 1, 2$, we have $d\omega = -M_1^{-1}dM_1 \wedge \omega + \varpi_1 = -M_2^{-1}dM_2 \wedge \omega + \varpi_2$, and thus $\varpi_1 = \varpi_2$ modulo Ω. Moreover, it is easily seen that ω and ϖ satisfy:*

$$d\omega = \mu \wedge \omega + \varpi, \quad d\varpi = \mu \wedge \varpi, \quad d\mu = \mu \wedge \mu, \quad dM = -M\mu$$

for all unimodular matrix M, which indeed characterizes non flat systems. Therefore, one may introduce the equivalence relation "two systems are equivalent if and only if they correspond to the same 2-form ϖ and ideal Ω", which suggests that it is possible to classify the non flat systems. In particular, a system has a non zero defect (see [16]) if and only if it admits a non zero 2-form ϖ, thus describing a generalized curvature (as opposed to flatness).

4.4 Some easy consequences

We now show how several classical results of static feedback linearization [22, 20], or in the case $m = 1$ [6, 7, 34, 42, 43] can be recovered as consequences of Theorem 2.

If Ω is strongly closed, let us define σ_i as the maximum degree of the entries of the ith column of $P(\varphi_0)$, $i = 1, \ldots, m$, which, according to Theorem 2, locally yields:

$$x = \varphi_0\left(y_1, \ldots, y_1^{(\sigma_1)}, \ldots, y_m, \ldots, y_m^{(\sigma_m)}\right). \tag{29}$$

Definition 8 *We say that a flat output y is* minimal *if $N = \sum_{i=1}^{m} \sigma_i$ is minimal over all possible choices of $\hat{U}$, Q and Z.*

Obviously, a minimal N always exists for flat systems.

Corollary 1 *A necessary and sufficient condition for system (2) to be static feedback linearizable (see [22, 20]) is that the strong closednes condition of Theorem 2 holds true and that $n = m + N$ for a minimal N.*

Proof. It is easily seen that (2) is static feedback linearizable if and only if (2) is L-B equivalent to a trivial system, the trivialization Φ being such that φ_0 is a local diffeomorphism, which means that x is diffeomorphic to $\left(y_1, \ldots, y^{(\sigma_1)}, \ldots, y_m, \ldots, y^{(\sigma_m)}\right)$ for a minimal y, i.e. $n = m + N$. ■

Remark 5 *This Corollary generalizes the results of Jakubczyk and Respondek [22] and Hunt, Su and Meyer [20] in a twofold manner: first, it is not restricted to affine systems and second it applies not only in a neighborhood of an equilibrium point but of any trajectory around which the variational system is controllable.*

Corollary 2 *If $m = 1$, a necessary and sufficient condition for flatness is that Ω is closed in the ordinary sense, i.e. $d\omega_1 = \tau \wedge \omega_1$ for some 1-form τ. Furthermore, the system is flat if and only if it is static feedback linearizable.*

Proof. If Ω is generated by a single 1-form ω_1, the unimodular matrix M must be a non zero element of $\mathfrak{K}$ and the strong closedness of Ω reduces to ordinary closedness. Thus the system is flat if and only if Ω is closed. Assuming closedness, by Frobenius' Theorem, we deduce that there exists a scalar function ψ_0 such that $d\psi_0 = M\omega_1$ and that (29) holds true with $N = n - 1$. Using Corollary 2, the system is static feedback linearizable. The converse is trivial since every static feedback linearizable system is flat. ■

5 Examples

5.1 Non holonomic car

Consider the 3 dimensional system in the $x - y$ plane, representing a vehicle of length l, whose orientation is given by the angle θ, the coordinates (x, y) standing for the position of the middle of the rear axle, and controlled by the velocity modulus u and the angular position of the front wheels φ.

$$\begin{aligned} \dot{x} &= u \cos\theta \\ \dot{y} &= u \sin\theta \\ \dot{\theta} &= \tfrac{u}{l} \tan\varphi \end{aligned} \tag{30}$$

Since $n = 3$ and $m = 2$, $n - m = 1$ and (30) is equivalent to the single implicit equation obtained by eliminating the inputs u and φ:

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = \dot{x} \sin\theta - \dot{y} \cos\theta = 0 \tag{31}$$

We immediately have:

$$\begin{aligned} P(F) &= \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt}, \quad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial \dot{y}} \frac{d}{dt}, \quad \frac{\partial F}{\partial \theta} + \frac{\partial F}{\partial \dot{\theta}} \frac{d}{dt} \right) \\ &= \left(\sin\theta \frac{d}{dt}, \quad -\cos\theta \frac{d}{dt}, \quad \dot{x}\cos\theta + \dot{y}\sin\theta \right). \end{aligned} \tag{32}$$

Setting $E = \dot{x}\cos\theta + \dot{y}\sin\theta$, we apply the Smith decomposition algorithm of the Appendix: moving the last column (of degree zero) to the first place by a permutation with the two others of degree one in any order, we see that $\mathsf{rank}\,(P(F)) = 1$, that $P(F)$ is hyper-regular and we get

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{1}{E} & \frac{\cos\theta}{E}\frac{d}{dt} & -\frac{\sin\theta}{E}\frac{d}{dt} \end{pmatrix}.$$

Thus

$$\hat{U} = U \begin{pmatrix} 0_{1,2} \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \frac{\cos\theta}{E}\frac{d}{dt} & -\frac{\sin\theta}{E}\frac{d}{dt} \end{pmatrix}$$

with I_2 the identity matrix of $\mathbb{R}^2$. Again, computing $Q \in \mathsf{L-Smith}\left(\hat{U}\right)$ yields

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{\sin\theta}{E}\frac{d}{dt} & -\frac{\cos\theta}{E}\frac{d}{dt} & 1 \end{pmatrix}.$$

Multiplying Q by the vector $\begin{pmatrix} dx \\ dy \\ d\theta \end{pmatrix}$, the last line reads $\frac{1}{E}\left(\sin\theta d\dot{x} - \cos\theta d\dot{y} + (\dot{x}\cos\theta + \dot{y}\sin\theta)d\theta\right) = \frac{1}{E}d(\dot{x}\sin\theta - \dot{y}\cos\theta)$ and, by (31), identically vanishes on $\mathfrak{X}_0$.

The remaining part of the system, namely $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} dx \\ dy \\ d\theta \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$ is trivially strongly closed with $M = I_2$, which finally gives the flat output $y_1 = y$ and $y_2 = x$. We have thus recovered the flat output originally obtained in [41, 40], up to a permutation of the components of y.

5.2 The pendulum

We consider a pendulum in the vertical plane studied in [16], of length l and inertia J, whose mass m is concentrated at its end point C. It is controlled by the two components of the force F applied to the opposite end point A of the pendulum. Introducing an inertial frame $(0, x, z)$, it is modelled by

$$\begin{aligned} &\ddot{x} = u_1 \\ &\ddot{z} = u_2 \\ &a\ddot{\theta} = -u_1\cos\theta + (u_2 + 1)\sin\theta \end{aligned} \tag{33}$$

with $x = \frac{x_C}{g}$, $z = \frac{z_C}{g}$, (x_C, z_C) being the coordinates of C, θ being the angle between the pendulum and the vertical axis, $u_1 = \frac{F_x}{mg}$, $u_2 = \frac{F_z}{mg} - 1$, (F_x, F_z) being the components of the force F, and $a = \frac{J}{mgl}$.

An implicit model is given by:

$$F(x, \dot{x}, \ddot{x}, z, \dot{z}, \ddot{z}, \theta, \dot{\theta}, \ddot{\theta}) = a\ddot{\theta} + \ddot{x}\cos\theta - (\ddot{z} + 1)\sin\theta = 0. \tag{34}$$

Though this is a second order system, it can be easily transformed into a first order one by setting $\dot{x} = v_x$, $\dot{z} = v_z$ and $\dot{\theta} = v_\theta$. We thus obtain the 4 dimensional implicit system

$$\begin{array}{l} \dot{x} - v_x = 0 \\ \dot{z} - v_z = 0 \\ \dot{\theta} - v_\theta = 0 \\ a\dot{v}_\theta + \dot{v}_x\cos\theta - (\dot{v}_z + 1)\sin\theta = 0. \end{array} \tag{35}$$

However, it can be easily verified that all the results of this paper can be extended word for word to higher order systems. Thus, because of its smaller dimension, we prefer using (34) instead of (35).

The variational system corresponding to (34) is given by

$$\left(\cos\theta\frac{d^2}{dt^2}, -\sin\theta\frac{d^2}{dt^2}, a\frac{d^2}{dt^2} - b\right)\begin{pmatrix} dx \\ dz \\ d\theta \end{pmatrix} = 0 \tag{36}$$

where $b = \ddot{x}\sin\theta + (\ddot{z} + 1)\cos\theta$.

Using the identity $\left(\cos\theta\frac{d^2}{dt^2}\right)(-a\cos\theta) - \left(\sin\theta\frac{d^2}{dt^2}\right)(a\sin\theta) + \left(a\frac{d^2}{dt^2} - b\right)(1) = a\dot{\theta}^2 - b$ and setting $E = a\dot{\theta}^2 - b$, we have

$$\left(\cos\theta\frac{d^2}{dt^2}, -\sin\theta\frac{d^2}{dt^2}, a\frac{d^2}{dt^2} - b\right)\begin{pmatrix} -a\cos\theta & 0 & 1 \\ a\sin\theta & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{E} & \frac{\sin\theta}{E}\frac{d^2}{dt^2} & -\frac{\cos\theta}{E}\frac{d^2}{dt^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (1, 0, 0)$$

which proves the hyper-regularity of $P(F)$ and we get

$$U = \begin{pmatrix} -\frac{a\cos\theta}{E} & -\frac{a\sin\theta\cos\theta}{E}\frac{d^2}{dt^2} & \frac{a\cos^2\theta}{E}\frac{d^2}{dt^2} + 1 \\ \frac{a\sin\theta}{E} & \frac{a\sin^2\theta}{E}\frac{d^2}{dt^2} + 1 & -\frac{a\sin\theta\cos\theta}{E}\frac{d^2}{dt^2} \\ \frac{1}{E} & \frac{\sin\theta}{E}\frac{d^2}{dt^2} & -\frac{\cos\theta}{E}\frac{d^2}{dt^2} \end{pmatrix}, \quad \hat{U} = \begin{pmatrix} -\frac{a\sin\theta\cos\theta}{E}\frac{d^2}{dt^2} & \frac{a\cos^2\theta}{E}\frac{d^2}{dt^2} + 1 \\ \frac{a\sin^2\theta}{E}\frac{d^2}{dt^2} + 1 & -\frac{a\sin\theta\cos\theta}{E}\frac{d^2}{dt^2} \\ \frac{\sin\theta}{E}\frac{d^2}{dt^2} & -\frac{\cos\theta}{E}\frac{d^2}{dt^2} \end{pmatrix}.$$

Now left decomposing $\hat{U}$, we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\sin\theta}{E}\frac{d^2}{dt^2} & \frac{\cos\theta}{E}\frac{d^2}{dt^2} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -a\sin\theta \\ 1 & 0 & a\cos\theta \\ 0 & 0 & 1 \end{pmatrix}\hat{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence,

$$Q = \begin{pmatrix} 0 & 1 & -a\sin\theta \\ 1 & 0 & a\cos\theta \\ \frac{\cos\theta}{E}\frac{d^2}{dt^2} & -\frac{\sin\theta}{E}\frac{d^2}{dt^2} & \frac{a}{E}\frac{d^2}{dt^2} - \frac{b}{E} \end{pmatrix}$$

and

$$Q\begin{pmatrix} dx \\ dz \\ d\theta \end{pmatrix} = \begin{pmatrix} dz - a\sin\theta d\theta \\ dx + a\cos\theta d\theta \\ \frac{1}{E}\left(\cos\theta d\ddot{x} - \sin\theta d\ddot{z} + ad\ddot{\theta} - bd\theta\right) \end{pmatrix} = \begin{pmatrix} dz - a\sin\theta d\theta \\ dx + a\cos\theta d\theta \\ 0 \end{pmatrix} = \begin{pmatrix} d(z + a\cos\theta) \\ d(x + a\sin\theta) \\ 0 \end{pmatrix}$$

Thus, the strong closedness condition holds true: setting $M = I_2$, we obtain

$$d(z + a\cos\theta) = dy_1, \quad d(x + a\sin\theta) = dy_2$$

or

$$y_1 = z + a\cos\theta, \quad y_2 = x + a\sin\theta$$

which represents, up to a permutation of y_1 and y_2, the coordinates of the Huygens oscillation center, already found in [26, 16].

6 Concluding remarks

We have proved that flatness is equivalent to the strong closedness of the ideal of 1-forms representing the differentials of all possible trivializations, using an approach which is invariant by endogeneous dynamic feedback. Moreover, we have separated the algebraic characterization of their differentials (Lemmas 1 and 2) and the integrability aspects, that may be seen as a generalization, in the framework of manifolds of jets of infinite order, of the well-known moving frame structure equations. The computation of flat outputs in non trivial examples show the applicability of our results. Note that the solutions to these examples were already known since long. However, some other classes of examples, whose solutions are not presently known, are in preparation.

We have chosen to present our results in terms of differential forms, in contrast with a large part of the control literature in this domain, where vector fields and Lie brackets are prefered. The main argument in favor of the former language is that the results are easy to state and are constructive: flat outputs are directly computed by integration of the strongly closed ideal Ω. It might be possible however to express them in terms of distributions of vector fields and this could be the subject of future works.

Finally, our necessary and sufficient conditions might open a way to a classification of non flat systems by the introduction of a 2-form that may be interpreted as a generalized curvature (see Remark 4).

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Appendix

A The Smith decomposition algorithm

We consider matrices of size $p \times q$, for arbitrary integers p and q, over the principal ideal domain $\mathfrak{K}[\frac{d}{dt}]$, here the non commutative ring of polynomials of $\frac{d}{dt}$ with coefficients in the field $\mathfrak{K}$ of meromorphic functions on a suitable time interval $\mathcal{I}$. The set of all such matrices is denoted by $\mathcal{M}_{p,q}[\frac{d}{dt}]$. For arbitrary $p \in \mathbb{N}$, the set $\mathcal{U}_p[\frac{d}{dt}]$ of unimodular matrices of size $p \times p$ is the subgroup of $\mathcal{M}_{p,p}[\frac{d}{dt}]$ of invertible elements, namely the set of invertible polynomial matrices whose inverse is also polynomial.

The following fundamental result on the transformation of a polynomial matrix over a principal ideal domain to its Smith form (or diagonal reduction) may be found in [10, Chap.8]):

Theorem 4 *Given a $(\mu \times \nu)$ polynomial matrix A over the non commutative ring $\mathfrak{K}[\frac{d}{dt}]$, with $\mu \leq \nu$, there exist matrices $V \in \mathcal{U}_\mu[\frac{d}{dt}]$ and $U \in \mathcal{U}_\nu[\frac{d}{dt}]$ such that $VAU = (\Delta, 0)$ where Δ is a $\mu \times \mu$ (resp. $\nu \times \nu$) diagonal matrix whose diagonal elements, $(\delta_1, \ldots, \delta_\sigma, 0, \ldots, 0)$, are such that δ_i is a non zero $\frac{d}{dt}$-polynomial for $i = 1, \ldots, \sigma$, and is a divisor of δ_j for all $\sigma \geq j \geq i$.*

The group of unimodular matrices admits a finite set of generators corresponding to the following *elementary right and left actions*:

- *right actions* consist of permuting two columns, right multiplying a column by a non zero function of $\mathfrak{K}$, or adding the jth column right multiplied by an arbitrary polynomial to the ith column, for arbitrary i and j;

- *left actions* consist, analogously, of permuting two lines, left multiplying a line by a non zero function of $\mathfrak{K}$, or adding the jth line left multiplied by an arbitrary polynomial to the ith line, for arbitrary i and j.

Every elementary action may be represented by an *elementary unimodular matrix* of the form $T_{i,j}(p) = I_\nu + 1_{i,j}p$ with $1_{i,j}$ the matrix made of a single 1 at the intersection of line i and column j, $1 \leq i, j \leq \nu$, and zeros elsewhere, with p an arbitrary polynomial, and with $\nu = m$ for right actions and $\nu = n$ for left actions. One can easily prove that:

- right multiplication $AT_{i,j}(p)$ consists of adding the ith column of A right multiplied by p to the jth column of A, the remaining part of A remaining unchanged,

- left multiplication $T_{i,j}(p)A$ consists of adding the jth line of A left multiplied by p to the ith line of A, the remaining part of A remaining unchanged,

- $T_{i,j}^{-1}(p) = T_{i,j}(-p)$,

- $T_{i,j}(1)T_{j,i}(-1)T_{i,j}(1)A$ (resp. $AT_{i,j}(1)T_{j,i}(-1)T_{i,j}(1)$) is the permutation matrix replacing the jth line of A by the ith one and replacing the jth one of A by the ith one multiplied by -1, all other lines remaining unchanged (resp. the permutation matrix replacing the ith column of A by the jth one multiplied by -1 and replacing the jth one by the ith one, all other columns remaining unchanged).

Every unimodular matrix V (left) and U (right) may be obtained as a product of such elementary unimodular matrices, possibly with a diagonal matrix $D(\alpha) = \mathsf{diag}\{\alpha_1, \ldots, \alpha_\nu\}$ with $\alpha_i \in \mathfrak{K}$, $\alpha_i \neq 0$, $i = 1, \ldots, \nu$, at the end since $T_{i,j}(p)D(\alpha) = D(\alpha)T_{i,j}(\frac{1}{\alpha_i}p\alpha_j)$.

In addition, every unimodular matrix U is obtained by such a product: its decomposition yields $VU = I$ with V finite product of the $T_{i,j}(p)$'s and a diagonal matrix. Thus, since the inverse of any $T_{i,j}(p)$ is of the same form, namely $T_{i,j}(-p)$, and since the inverse of a diagonal matrix is diagonal, it results that $V^{-1} = U$ is a product of elementary matrices of the same form, which proves the assertion.

The algorithm of decomposition of the matrix A consists first in permuting columns (resp. lines) to put the element of lowest degree in upper left position, denoted by $a_{1,1}$, or creating this element by euclidian division of two or more elements of the first line (resp. column) by suitable right actions (resp. left actions). Then right divide all the other elements $a_{1,k}$ (resp. left divide the $a_{k,1}$) of the new first line (resp. first column) by $a_{1,1}$. If one of the rests is non zero, say $r_{1,k}$ (resp. $r_{k,1}$), subtract the corresponding column (resp. line) to the first column (resp. line) right multiplied (resp. left) by the corresponding quotient $q_{1,k}$ defined by the right Euclidian division $a_{1,k} = a_{1,1}q_{1,k} + r_{1,k}$ (resp. $q_{k,1}$ defined by $a_{k,1} = q_{k,1}a_{1,1} + r_{k,1}$). Then right multiplying all the columns by the corresponding quotients $q_{1,k}$, $k = 2, \ldots, \nu$ (resp. left multiplying lines by $q_{k,1}$, $k = 2, \ldots, \mu$), we iterate this process with the transformed first line (resp. first column) until it becomes $(a_{1,1}, 0, \ldots, 0)$ (resp. $(a_{1,1}, 0, \ldots, 0)^T$ where ^{T} means transposition). We then apply the same algorithm to the second line starting from $a_{2,2}$ and so on. To each transformation of lines and columns correspond a left or right elementary unimodular matrix and the unimodular matrix V (resp. U) is finally obtained as the product of all left (resp. right) elementary unimodular matrices so constructed.

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