Geometric, Algebraic and Topological Methods for Quantum Field Theory. Geometry of closed strings, A and B side of Witten. Part I : A side and enumerative geometry

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Geometric, Algebraic and Topological Methods for Quantum Field Theory
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1 Introduction

The intimate connection that links the modern geometry and physics is discovered by Einstein from the elegant formulation of general relativity. The main actors are Lorentzian or Riemannian manifolds in four dimension, its tangent bundle (geometry) and the energy-momentum tensor (Physics). The tangent bundle has the structure of a vector bundle, we can associate a principal bundle : the bundle of orthonormal Lie group $O(n)$ with maximal holonomy. All this was taken in the years 50 to give rise to geometrical theories of electromagnetism : the Yang-Mills theory from the theory of bundles and connections developed ten years earlier in mathematics [Steenrod, Ehresmann]. The advent of string theory has enriched this setting, we can truly say that modern geometry in all its forms has an entrance ticket for physics. The proof of a conjecture in the last century, in enumerative geometry has even found its inspiration in the physical field theory [Kontsevich 1990]. The advent of supersymmetric theories in physics has led to the theory of superstrings. It also led to extended particles live in "more realistic" space of ten topological dimensions. One can indeed define a "perfectly free" theory, by replacing a particle moving in classic space-time (1.3) with vibrating "bosonic" string undemanding nature of the geometric space which it operates, the classical Lorentzian spacetime $(1, D)$ but with $D = 26$ to exorcise the space of ghosts : these negative mass to the ground state, in this scenario the geometric setting is almost exactly that of general relativity. However, the constraints of supersymmetry are very strong, they lead to replace a spacetime (1.9) by a fibration on the classical spacetime (1.3) in varieties known as Calabi-Yau. The six remaining dimensions, to do so, must be properly compactified to generate our universe (and this is not so simple). At this price you can define complex manifolds (Kahler) very special holonomy which is lower (holonomy $SU(n)$ which imposes a zero Ricci curvature). The simplest example of such a variety is provided by the torus complex : an elliptic curve or complex manifold of dimension 1. In this talk, we will initially discuss the A side string theory, namely the of Gromov-Witten invariants. In a next talk, we discuss mirror symmetry B-model which generalizes the T duality in string theory and to calculate the invariants through the "mirror"

2 The Physics of field theory and string

2.1 Example of classical fields

The concept of field is fundamental in physics. A field φ is a function of a world sheet (source space) into a target space, M (space physics) with a sufficient number of dimensions. So given a "package" (Σ, M, φ) and a classical action : S where : Σ is the source space, often a manifold : for The classical mechanic of the point is the time axis (world line). For the conformal field théories like strings theories : Riemann surface ...
The Lagrangian density is a function on one or more fields and its first derivatives :

$$\mathcal{L} = \mathcal{L}(\varphi_1, \varphi_2, ..., \partial_\mu \varphi_1, \partial_\mu \varphi_2 ...).$$

The classical action is the integral of the classical Lagrangian density on the parameter space $S = \int \mathcal{L} d^{n+1}x$
Principle of least action : The minimization of the action ($\delta S = 0$, leads to each field noted just φ to the Euler-Lagrange equation gives the equations of motion of the particle $\dfrac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu (\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}) = 0$

2.2 Example of classical fields

The free particle : For a free particle, the field is simply the parameterized curve that describes the trajectory of the particle in free space :$x(t)$. In this case, you can take to Lagrangian density
$\mathcal{L} = \mathcal{L}(x(t), \dot{x}(t)) = \frac{1}{2} m \dot{x}^2$, Euler Lagrange equation is :
$\dfrac{\partial \mathcal{L}}{\partial x} - \partial_t (\dfrac{\partial \mathcal{L}}{\partial(\dot{x})}) = -\partial_t (\dfrac{\partial \mathcal{L}}{\partial(\dot{x})}) = 0$ solution is the uniform motion.

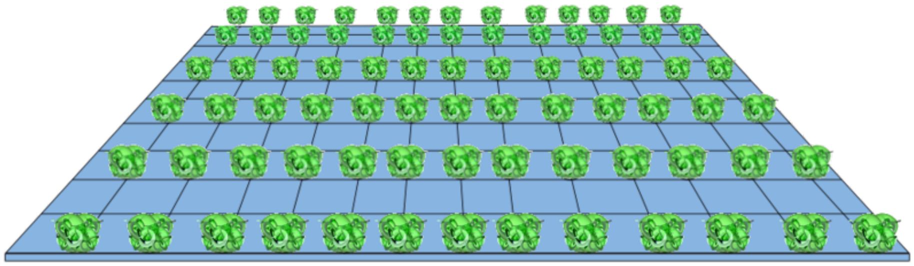

Figure 1 – Les dimensions enroulées

The free string : For the free string, the field is simply the function that describes the trajectory of the string in the target space : $X(\tau, \sigma)$. Its Lagrangian density is contained in the Nambu-Goto action :
$S = -T \int_{\tau_1}^{\tau_1} d\tau \int_0^l d\sigma \sqrt{\dot{X}^2 X'^2 - (\dot{X}X')^2}$ with $\dot{X}^2 = \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}, \dot{X}X' = \dot{X}^\mu X'^\nu \eta_{\mu\nu}$
Euler Lagrange equation is then : $\partial_\tau \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} + \partial_\sigma \frac{\partial \mathcal{L}}{\partial X'^\mu} = 0$ and the solution is the equation of vibrating strings.

2.3 Noether Symmetries

Symmetry of the action : the role of symmetry in physics is essential. We want, for example, such action invariant through a transformation like translation, rotation ... : if $\varphi \rightarrow \varphi + \delta\varphi$ alors $S \rightarrow S + \delta S$
Noether's theorem : through any symmetry, the action is the same : $\delta S = 0$

Example translation : $x \longrightarrow x + \epsilon$ is taken up the free particle, ϵ small, independent of time,
$\delta S = \int (m\dot{x}\dot{\epsilon})dt = \epsilon m\dot{x}|_{t_0}^{t_1} - \int (m\ddot{x})\epsilon dt$
and as $\ddot{x} = 0$ we get : The symmetry by translation is equivalent to the conservation of momentum $p = m\dot{x}$

Notation : In physics, symmetry $x \rightarrow x + \epsilon$ is denoted $\delta x = \epsilon$.

2.4 Complexification of the bosonic string

We can define the Polyakov action, much better than ***Nambu-Goto*** action described above which is difficult to quantify : that consist to add a metric field on the world sheet ($2d$- gravity). It can be show that these two action are equivalents :
$S = -T/2 \int_\Sigma \sqrt{-g} g^{\alpha\beta}(\sigma, \tau) \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} d\sigma d\tau$

Parameterized curve :
by setting $z = \tau + i\sigma$ the $(\tau, \sigma) \rightarrow X(\tau, \sigma)$, that is to say a real field is a field complex $z \longrightarrow f(z)$: a parameterized (complex) curve.

Particle string :
The evolution of a point particle is described by a parameterized curve, for a string it is a complex curve ,among some will have a privileged status, holomorphic curves are A model instanton of Witten and were studied by Gromov in symplectic geometry.

2.5 Quantum fields, QFT

Path integral :

Uncertainty on the position or momentum in quantum mechanic led to replace the classical solution (least action) by the partition function or the set of all possible solutions : It is the ***path integral*** $\mathcal{Z} = \int_{\Sigma \rightarrow M} e^{-S(\varphi)} \mathcal{D}\varphi$:

Correlation Functions :

Similarly, one can calculate correlation functions, or functions with n points.
$< \varphi_1(x_1), ..., \varphi_n(x_n) >= \int_{\Sigma \rightarrow M} \varphi_1(x_1)...\varphi_n(x_n) e^{-S(\varphi)} \mathcal{D}\varphi$

We can apply this machinery to the supersymmetric sigma model and define a new quantum field theory : the topological field theory TFT .The program developed by Witten is to calculate the correlation functions, by replacing each value by a cohomology class called $BRST$ cohomology. That requires, introducing fermionic variables ***invariant*** under this ***generalized Noether symmetries***.

These tools are ***supersymmetry***.

2.6 Supersymmetry

We can define a supersymmetric field theory $\Sigma \rightarrow M$ by adding fermionic variables, that is to say sections of some vector bundle E on Σ. A good image of a fermionic field is $\psi(x) = \Sigma f_i(x) dx_i$, a 1-form equipped with a wedge-product . We have the theorem :
Localization theorem : the path integral is localized around field configurations where fermionic variables stay invariant under supersymmetric

transformations. Supersymmetric transformation is infinitesimal transformation of the action, which transforms bosons into fermions and vice versa.

2.7 Calculus supersymmetric

We can define a supersymmetric Calculus :

Algebraic Computation : Let ψ_1, ψ_2 two fermions $\psi_1\psi_2 = -\psi_2\psi_1$ we deduce $\psi\psi = 0$ Let a bosonic variable $Xboson$,$\psi X = X\psi$

Calculus : $\int(a+b\psi)d\psi = b$, $\int \psi d\psi = 1$, $\int \psi_1\psi_2...\psi_n)d\psi_1 d\psi_2...d\psi_n = 1$, $\int d\psi = 0$

Change of variables : We have : $\int \widetilde{\psi} d\widetilde{\psi} = \int \psi d\psi = 1$

2.8 Example zero-dimensional supersymmetry

A "Toy" model is to make space for starting $\Sigma = \{P\}$ and target $M = \mathbb{R}$ the real line. In this context, a field is simply the variable x, the path integral is just $\mathcal{Z} = \int_M e^{-S(x)}dx$
A supersymmetric action is given by :
$S(x, \psi_1, \psi_2) = \frac{h'(x)^2}{2} - h"(x)\psi_1\psi_2$.
hence the partition function :
$\mathcal{Z} = \int e^{\frac{-h'(x)^2}{2} + h"(x)\psi_1,\psi_2} dx d\psi_1 d\psi_2$
by developing in power series fermionic part, we get :
$\mathcal{Z} = \int e^{\frac{-h'(x)^2}{2}}(1 + h"(x)\psi_1\psi_2) dx d\psi_1 d\psi_2$, but $\int d\psi = 0$, hence the first integral is zero, then :
$\mathcal{Z} = \int_M h"(x) e^{\frac{-h'(x)^2}{2}} dx \int \psi_1 d\psi_1 \int \psi_2 d\psi_2$,
and as $\int \psi d\psi = 1$ (fermionic integration) we get :

$\mathcal{Z} = \int_M h"(x) e^{\frac{-h'(x)^2}{2}} dx$

2.9 Supersymmetric transformations

For the example above, we can define supersymmetric transformations that respect this action.
$\delta x = \epsilon_1\psi_1 + \epsilon_2\psi_2$
$\delta\psi_1 = h'(x)\epsilon_2$
$\delta\psi_2 = -h'(x)\epsilon_1$

We show that $\delta S = 0$, the fermionic variables are invariant for supersymmetric transformation iff $h'(x) = 0$. If $h'(x) \neq 0$, the change of variables $(x, \psi_1, \psi_2) \to (x - \frac{\psi_1\psi_2}{h'(x)}, \psi_1, \psi_2)$ shows that the partition function is zero outside the critical points. By expanding to second order near the critical point x_c

$(h(x) = h(x_c) + \frac{h"(x_c)}{2}(x - x_c)^2)$:

$\mathcal{Z} = \int_M h"(x) e^{\frac{-h'(x)^2}{2}} dx$

$\mathcal{Z} = \sum_{h'(x_c)=0} h"(x_c) \int_M exp(-\frac{(h"(x_c)(x-x_c))^2}{2}) dx$,

with change of variables $y = |h"(x_c)|(x - x_c)$:

$\mathcal{Z} = \sum_{h'(x_c)=0} \sqrt{\pi} \frac{h"(x_c)}{|h"(x_c)|}$

Abstract

> Supersymmetry : We just define an action for a supersymmetric field theory of dimension 0 by adding fermions, supersymmetry variables.
>
> Invariance : This action is invariant under supersymmetric transformations.
>
> Location : The associated path integral is localized on the fields for which the fermions are invariant under supersymmetry.
>
> Towards a generalization : This suggests defining an operator that vanishes on the fermionic fields. A fermionic field is associated to a differential form, there is an idea of cohomology below.

2.10 Application : A model of Witten "A side of the mirror"

L, the supersymmetric lagrangian of a super-string is given by :

$L = 2t \int_\Sigma (\frac{1}{2} g_{IJ} \partial_z \phi^I \partial_{\overline{z}} \phi^J) d^2z + 2t \int_\Sigma (i\psi_z^{\overline{i}} D_{\overline{z}} \chi^i g_{\overline{i}i} + i\psi_z^i D_z \chi^{\overline{i}} g_{\overline{i}i} - R_{i\overline{i}j\overline{j}} \psi_{\overline{z}}^i \psi_z^{\overline{i}} \chi^j \chi^{\overline{j}}) d^2z$

The beginning of integral is the bosonic part of the action, the last , the fermionic part : fields are sections of bundles on Σ :

Party fermionic

•$\chi(z)$ a section $\mathcal{C}^\infty$ de $f^*TX \otimes \mathbb{C}$

•$\psi_z(z)$ a section $\mathcal{C}^\infty$ de $(T^{10}\Sigma)^* \otimes f^*T^{01}X$

•$\psi_{\overline{z}}$, a section $\mathcal{C}^\infty$ de $(T^{01}\Sigma)^* \otimes T^{10}X$

Transformation supersymmetric preserving action

$\delta x^I = \eta\chi^I \qquad \delta\chi^I = 0$

$\delta\psi_{\overline{z}}^i = \eta\partial_{\overline{z}}\phi_i \quad \delta\psi_z^{\overline{i}} = \eta\partial_z\overline{\phi_i}$

• If $\delta\psi_{\overline{z}}^i = \delta\psi_z^{\overline{i}} = 0$, we recognize the conditions of Cauchy-Riemann ! : The instantons of this model are curves "minimum energy" according to Gromov : holomorphic curves.

2.11 BRST Cohomology

At previous fermionic transformations one associates an operator Q (for charge), the terminology come from electromagnetism : charge is the integration of a "current". Mathematically, the operator Q has the properties of ordinary differential form (they will have an isomorphism between $BRST$ cohomology with that of De Rham :
we give now the main properties of this operator
Properties of the operator
• $Q(x^I) = \chi^I \; Q(\chi^I) = 0$
• Q is a linear operator.
• $Q(fg) = Q(f)g + fQ(g)$: Q is a derivation.
BRST cohomology
• We note that $Q^2 = 0$
• $H^p_{BRST} = \frac{KerQ:\mathcal{H}_p \to \mathcal{H}_{p+1}}{ImQ:\mathcal{H}_{p-1} \to \mathcal{H}_p}$ is the p- th cohomology group BRST
endsubsection

2.12 Correlation functions BRST

In correlation functions fields are replaced by their cohomology classes, so they are defined modulo an exact term by :

Correlation Functions
Correlation functions of topological field theory will be given by :
$< [\Phi_1(x_1)], ..., [\Phi_n(x_n)] >= \int_{\Sigma \to M} \Phi_1(x_1)...\Phi_n(x_n) e^{-S} \mathcal{D}x\mathcal{D}g\mathcal{D}\chi\mathcal{D}\psi$
• they do not depend on the selected points on the Riemann surface.

Correlations functions of side A of the mirror
• Let $\omega_1, ...\omega_n$ forms on M,
$< [\omega_1], ..., [\omega_n] >= \int_{\Sigma \to M} \omega_1 ... \omega_n e^{-(S_B(f)+S_F(f))} \mathcal{D}x\mathcal{D}g\mathcal{D}\chi\mathcal{D}\psi$
• For the theorem location : the path intgral is localized around holomorphic curves $\tilde{f}$: $< [\omega_1], ..., [\omega_n] >= \int_{\Sigma \to M} \omega_1 ... \omega_n e^{-(S_B(\tilde{f}))} \mathcal{D}x\mathcal{D}g\mathcal{D}\chi\mathcal{D}\psi$
• $e^{-(S_B(\tilde{f}))} = e^{-\int_\Sigma \tilde{f}^* \omega}$ is a topological invariant gives the "degree" of application $\tilde{f}$.

2.13 Relationship with enumerative geometry

The path integral above can be rewritten :
$< [\omega_1], ..., [\omega_n] >= \sum_{\beta \in H_2(M,\mathbb{Z})} e^{-\int_\Sigma \tilde{f}^* \omega} \int_{\tilde{f}(\Sigma)\in\beta} \omega_1 ... \omega_n \mathcal{D}x\mathcal{D}g\mathcal{D}\chi\mathcal{D}\psi$
Here $\beta \in H_2(X,\mathbb{Z})$ is a cohomology class, "specifically" the degree of $\tilde{f}$.

Counting curves
we can hope the integral :
$\int_{f(\Sigma)\in\beta} \omega_1 ... \omega_n \mathcal{D}x\mathcal{D}g\mathcal{D}\chi\mathcal{D}\psi$
taken on a moduli space $\mathcal{M}$ to define properly, can provide an integer. This will be the case if the dimension of this moduli space is related to the number of fields $[\omega_i]$ Gromov Witten invariants These integrals, which give integers in good cases are Gromov Witten invariants.Their knowledge provides a means of calculating correlation functions from a topological viewpoint and hope to understand better the physics !

3 The Mathematics of Gromov and Grothendieck

The global study of schemes or varieties is very difficult. Grothendieck already started studying moduli of curves, more malleable, as that means study object through its sections which is much easier. Gromov noted that. There are very few holomorphic functions on a complex manifold or almost-complex like symplectic manifolds.
The big idea of Gromov is to consider the study of a symplectic manifold through its sections that are : holomorphic curves

Curves, parametric curves
In mathematics the worldsheet is a curve (Riemann surface), the evolution of a bosonic string that is a field of $\Sigma \to X$ is a parameterized curve.

Symplectic Geometry or algebraic geometry
We can present the Gromov-Witten invariants by one of these two views Symplectic Geometry (Ruan, Tuan), Algebraic Geometry (Pandaripande, Katz).

3.1 Symplectic geometry and pseudoholomorphic curves

Recall that a symplectic manifold is a differential manifold equipped with a 2-closed form $\omega : (M,\omega)$. The simplest example is $\mathbb{R}^{2n}$ and $\omega = \sum dx_i \wedge dy_i$, We can also consider the space $\mathbb{C}^n$ or create new complex structures J which are all integrated. Thus $\mathbb{R}^{2n}$ can be endowed with a structure of analytic manifold which is not always the case.

Parameterized curve A parameterized curve is a map :
varphi : $(\Sigma, j) \rightarrow (M, Y)$, where j is an almost complex structure on Σ, J be an almost complex structure on the target.

Pseudoholomorphic Curve :
A parametric curve is pseudoholomorphe if its differential verifies equations of Cauchy-Riemann : $J \circ d\varphi = d\varphi \circ j$, (the differential is $\mathbb{C}$ linear).

3.2 Riemann-Roch formula moduli space of curves

For now we place ourselves in the more general context of the geometry. We give here without proof the Riemann Roch formula for a curve, is the case of Riemann surfaces which interests us here, there is $\mathcal{M}_g$, $\mathcal{M}_{g,n}$ respectively The moduli space of curves of genus g, and the moduli space of curves of genus g with n marked points :

Riemann Roch formula for a curve
The Riemann-Roch formula gives :

$$dim_{\mathbb{C}} H^0(T\Sigma) - dim_{\mathbb{C}} H^1(T\Sigma) = \int_{\Sigma} ch(T\Sigma) td(T\Sigma)$$

Specifically, $dim_{\mathbb{C}} H^0(T\Sigma)$, consider the infinitesimal automorphisms, $dim_{\mathbb{C}} H^1(T\Sigma)$ is the complex dimension of moduli space of curves. For a Riemann surface of genus g yields :

Dimension of $\mathcal{M}g$:

$$dim_{\mathbb{C}} H^0(T\Sigma) - dim_{\mathbb{C}} H^1(T\Sigma) = 3 - 3g$$
$$dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3 + \dim_{\mathbb{C}} H^0(T\Sigma)$$

3.3 Riemann-Roch formula for a map, curves, stable maps

Make first application of the formula above :
$H^0(T\Sigma)$ count the number of marked points need to stabilize the curve thus :

$g \geq 2 : dim(H^0(T\Sigma)) = 0) : dim(H^1(T\Sigma)) = dim(\mathcal{M}_g) = 3g - 3$
si $g = 1 : H^0(T\Sigma) = \mathbf{C} : dim(H^1(T\Sigma)) = dim(\mathcal{M}_g) = 1$

si $g=0$: $dim(H^0(T\Sigma))=3$: $H^1(T\Sigma)=\mathcal{M}_g$ is a point.

We will therefore focus our attention on stable scurves (and maps) . Let $\phi:\Sigma\rightarrow X$ be a holomorphic curve

Riemann Roch formula for a map :
The Riemann-Roch formula gives in this case :
$dim_{\mathbb{C}}H^0(\phi^*TX)-dim_{\mathbb{C}}H^1(\phi^*TX)=\int_\Sigma ch(\phi^*TX)td(\Sigma)$
$=n(1-g)+\int_\Sigma \phi^*c_1(TX)$

3.4 Curves, stable maps, moduli spaces

In genus 0, an automorphism of $P^1(\mathbb{C})$ is determined by the images of three distinct points, that is to say automorphisms which fix two points form a no discrete subgroup $PGL(2,\mathbb{C})$,
we will therefore infinitesimal automorphisms. To get rid of infinitesimal automorphisms : stabilize the curve, we must add "marked points" doing the curve is stabilized.
A stable curve is a curve on which we added enough marked points to kill the infinitesimal automorphisms.

Stable map
A stable map is a parametric curve $(\Sigma,(x_1,...,x_k),\varphi)$ for which Σ is stable. We can now give the definition of the moduli space of stable maps with k marked points : $\mathcal{M}_{g,n}(X,\beta)$.

Definition
We define the space :
$\mathcal{M}_{g,n}(X,\beta)=\{(\Sigma,(x_1,...,x_n),\varphi),\varphi:\Sigma\rightarrow X,\varphi_*(\Sigma)=\beta\}/\sim$ where $\sim$ are quotient by the group of reparameterization.

The space considered above is not compact in general, we denote $\overline{\mathcal{M}}_{g,n}(X,\beta)$ its compactification, this amounts to add singular curves .

3.5 (Virtual) dimension space of holomorphic curves

From the two previous formula (Riemann Roch for curve and parametric curve, we can deduce the virtual dimension of moduli space of holomorphic curves. For this we can reason using the exact sequences Consider the long

exact sequence in cohomology associated to the exact sequence :
$0 \to T_\Sigma \to f^* T_X \to N_{\Sigma/X} \to 0$
For details see [Pandharipande]
Dimension of $\mathcal{M}_g(X)$
By combining the two forms of the Riemann-Roch previous :
$dim_{virt}\overline{\mathcal{M}}_{g,n}(X,\beta) = (dim X)(1-g) + \int_{f_*(\Sigma)} c_1(TX) + 3g - 3 + n$
We could find directly this result in symplectic case (varieties that are treated are thus symplectic Kähler) relying on the index of a Fredholm operator for an elliptic complex adhoc.

4 Gromov-Witten invariants

We use the notation [.] To denote the fundamental class in $H_k(\overline{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q})$: We can now properly define from a mathematical point of view the Gromov-Witten invariants. Indeed, if $[\omega_1], ..., [\omega_n]$ cohomology class in $H^*_{DR}(X)$ such that $\Sigma_{i=1}^n deg[\omega_i] = k$ "integration on the space module will then be a non-zero number, So we can expect count something.

Definition : Gromov-Witten invariants
is called Gromov-Witten invariant quantity :

$< [\omega_1], ...[\omega_n] >_\beta = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]} ev^*([\omega_1]) \wedge ... \wedge ev^*([\omega_n])$

In this script we used an evaluation map :

$$\begin{array}{rcccc} ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X & : & (\Sigma, x_1, ..., x_n, \varphi) & \mapsto & \varphi(x_i) \\ \mathrm{ev}_i^* : H^*(X) \to H^*(\overline{\mathcal{M}}_{g,n}(X,\beta)) & : & [\omega_i] & \mapsto & ev_i^*([\omega_i]) \end{array}$$

This can be seen more explicitly by introducing the Poincare dual of C_i associated with $[\omega_i]$: the invariant is seen as an intersection of cycles :
Gromov-Witten invariant
is called Gromov-Witten invariant quantity :
$< [\omega_1], ...[\omega_n] >_\beta = \sharp(ev_1^{-1}(C_1) \cap ... \cap ev_n^{-1}(C_n)) \cap [\overline{\mathcal{M}}_{g,n}(X,\beta)])$

In this script we used an evaluation map :

$$\begin{array}{rcccccc} ev_i & : & \overline{\mathcal{M}}_{g,n}(X,\beta) \to X & : & (\Sigma, x_1, ..., x_n, \varphi) & \mapsto & \varphi(x_i) \\ ev_i^{-1} & : & X \to \overline{\mathcal{M}}_{g,n}(X,\beta) & : & C_i & \mapsto & ev_i^{-1}(C_i) \end{array}$$

We must understand that we will count the number of curves φ of class β cycles that meet C_i in points x_i ($\varphi(x_i) \in C_i$)

4.1 Example 1 : applications of $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$

Recall that $\int_{\mathbb{P}^1} \varphi^*(T\mathbb{P}^2) = d.3$, where d is the degree of application. Also remind that in this simple case the degree of application is less abstract than the class β for $d = 0, 1, 2...$ applications were constants, the class of straight, tapered ...

<u>Dimension of $\overline{\mathcal{M}}_{0,n}(X, d)$</u>
We have the formula : $dim(\overline{\mathcal{M}}_{0,n}(X, d)) = 3d - 1 + n$
If $d = 1$: $dim(\overline{\mathcal{M}}_{0,n}(X, d)) = 2 + n$: Class of lines
Si $d = 2$: $dim(\overline{\mathcal{M}}_{0,n}(X, d)) = 5 + n$: Class of conics
A requirement that the invariant $< [pt_1], ..., [pt_n] >_d$ is nontrivial it : it takes $\Sigma_i deg([pt_i]) = dim(\overline{\mathcal{M}}_{0,n}(X, d))$, but we know that
$deg([pt_i]) = codim[pt_i] = 2$
For $d = 1$ we find $n = 2$, $< [pt_1], [pt_2] >_1 = 1$: For two points passes only one line !

4.2 Formula iterative Kontsevich

You can find $n = 5$, $< [PT_1], ..., [pt_5] >= 1$. We find that by 5 points can be passed 1 tapered. Drawing diagram Feymann fours Maxim Kontsevich showed iterative formula below was then a conjecture enumerative geometry !
<u>Kontsevich formula</u> :
There n_d the number of curves of degree d through the number of desired point
$N_d = \sum_{d=d_1+d_2} N_{d_1} N_{d_2} (d_1^2 d_2^2 C_{3d_1-2}^{3d-4} - d_1^3 d_2 C_{3d_1-1}^{3d-4})$
By applying this formula, we find : $N_2 = 1$, $N_3 = 12$, $N_4 = 620$

4.3 Example 2 : Back physics !

We return now to our case for the supersymmetric A model of Witten, take X Calabi-Yau 3 included in $\mathbb{P}^4(\mathbb{C})$

<u>Dimension of $\overline{\mathcal{M}}_{g,n}(X, \beta)$</u>
It is known that Calabi Yau Ricci-flat is that means : $\int_\Sigma \varphi^*(c_1(X)) = 0$ in these conditions,
$dim(\overline{\mathcal{M}}_{g,n}(X, \beta)) = -3 + 3g + dimX(1 - g) + n$
If $g = 0$ and $dimX = 3$ we get :
$dim(\overline{\mathcal{M}}_{0,n}(X, \beta)) = n$
In $\overline{\mathcal{M}}_{0,n}(X, \beta))$ as $deg([H]) = codim[H] = 1$ it takes n hyperplanes in $< [H], ..., [H] >_\beta$ to make this non-trivial correlation function.

FIGURE 2 – Une variété de Calabi Yau

For $n = 3$, we have the three points correlation function or "pant" : $n = 3$, $< [H], [H], [H] >_\beta$, basic "lego" of topological fields theory .

4.4 Example 2 : Calculating the correlation function at three points : the problems, the mirror symmetry

The calculation of correlation functions by direct methods namely 'A'side of the mirror is not easy : if you expand you have :
$< [H], [H], [H] >= \sum_\beta < [H], [H], [H] >_\beta e^{-\int_\beta \omega}$.
A conjecture of Clemens said we can not calculate this number due to a problem of multiple coatings, which contradicts the expected dimension of 0 for space applications module with 0 runs scored.
To survive we must work on the other side of the mirror.

mirror symmetry
The objects we work with in enumerative geometry are forms of Khler and parameterize deformations of the geometry. The Mirror symmetry Mirror symmetry says that one can express the same physical setting deformations of complex structure on a variety mirror and identifying the correlation functions from the two models.

but That is for the next talk !

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