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KUREPA-TREES AND NAMBA FORCING

BERNHARD KÖNIG AND YASUO YOSHINOBU

Abstract. We show that compact cardinals and MM are sensitive to λ-closed forcings for arbitrarily large λ. This is done by adding ‘regressive’ λ-Kurepa-trees in either case. We argue that the destruction of regressive Kurepa-trees with MM requires the use of Namba forcing.

1. Introduction

Say that a tree $T$ of height $\lambda$ is $\gamma$-regressive if for all limit ordinals $\alpha < \lambda$ with $\text{cf}(\alpha) < \gamma$ there is a function $f_\alpha : T_\alpha \to T_{<\alpha}$ which is regressive, i.e. $f_\alpha(x) <_T x$ for all $x \in T_\alpha$ and if $x, y \in T_\alpha$ are distinct then $f_\alpha(x)$ or $f_\alpha(y)$ is strictly above the meet of $x$ and $y$. We give a summary of the main results of this paper:

5 Theorem. For all uncountable regular $\lambda$ there is a $\lambda$-closed forcing $K^{\lambda}_{\text{reg}}$ that adds a $\lambda$-regressive $\lambda$-Kurepa-tree.

This is contrasted in Section 4:

7 Theorem. Assume that $\kappa$ is a compact cardinal and $\lambda \geq \kappa$ is regular. Then there are no $\kappa$-regressive $\lambda$-Kurepa-trees.

Theorems 5 and 7 establish that compact cardinals are sensitive to $\lambda$-closed forcings for arbitrarily large $\lambda$. This should be compared with the well-known result that a supercompact cardinal $\kappa$ can be made indestructible by $\kappa$-directed-closed forcings [10]. These results drive a major wedge between the notions of $\lambda$-closed and $\lambda$-directed-closed. Another contrasting known result is that a strong cardinal $\kappa$ can be made indestructible by $\kappa^+$-closed forcings [3]. In Section 7 we prove

13 Theorem. Under MM, there are no $\omega_1$-regressive $\lambda$-Kurepa-trees for any uncountable regular $\lambda$.

This shows that MM is sensitive to $\lambda$-closed forcings for arbitrarily large $\lambda$, thus answering a question from both [7] and [8]. Note that MM is indestructible by $\omega_2$-directed-closed forcings [8], so again we

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find a remarkable gap between the notions of $\omega_2$-closed and $\omega_2$-directed-closed. Interestingly enough though, $\omega_2$-closed forcings can only violate a very small fragment of MM. To see this, let us denote by $\Gamma_{\text{cov}}$ the class of posets that preserve stationary subsets of $\omega_1$ and have the covering property, i.e. every countable set of ordinals in the extension can be covered by a countable set in the ground model. Then we have the following result from [7, p.302]:

1 Theorem. The axioms PFA, $\text{MA}(\Gamma_{\text{cov}})$ and $\text{MA}^+(\Gamma_{\text{cov}})$ are all indestructible by $\omega_2$-closed forcings respectively.\footnote{See below for a definition of the axioms $\text{MA}(\Gamma)$ and $\text{MA}^+(\Gamma)$.}

So Theorem 5 gives

2 Corollary. If $\lambda \geq \omega_2$ is regular, then $\text{MA}^+(\Gamma_{\text{cov}})$ is consistent with the existence of a $\lambda$-regressive $\lambda$-Kurepa-tree.

Again, compare this with Theorem 13. It is interesting to add that $\text{MA}^+(\Gamma_{\text{cov}})$ in particular implies the axioms PFA$^+$ and SRP. The typical example of a forcing that preserves stationary subsets of $\omega_1$ but does not have the covering property is Namba forcing and the proofs confirm that Namba forcing plays a crucial role in this context. It has already been established in [9] and [11] that $\text{MA}(\Gamma_{\text{cov}})$ can be preserved in an $(\omega_1, \infty)$-distributive forcing extension in which the Namba-fragment of MM fails. In our case though, the failure of MM is obtained with a considerably milder forcing, i.e. $\lambda$-closed for arbitrarily large $\lambda$.

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The reader requires a strong background in set-theoretic forcing, a good prerequisite would be [4]. We give some definitions that might not be in this last reference or because we defined them in a slightly different fashion. If $\Gamma$ is a class of posets then $\text{MA}(\Gamma)$ denotes the statement that whenever $P \in \Gamma$ and $D_\xi (\xi < \omega_1)$ is a collection of dense subsets of $P$ then there exists a filter $G$ on $P$ such that $D_\xi \cap G \neq \emptyset$ for all $\xi < \omega_1$. The stronger $\text{MA}^+(\Gamma)$ denotes the statement that whenever $P \in \Gamma$, $D_\xi (\xi < \omega_1)$ are dense subsets of $P$, and $\dot{S}$ is a $P$-name such that

\[ \Vdash_P \dot{S} \text{ is stationary in } \omega_1 \]

then there exists a filter $G$ on $P$ such that $D_\xi \cap G \neq \emptyset$ for all $\xi < \omega_1$, and

\[ \dot{S}[G] = \{ \gamma < \omega_1 : \exists q \in G (q \Vdash_P \gamma \in \dot{S}) \} \]

is stationary in $\omega_1$. In particular, PFA is $\text{MA}(\text{proper})$ and MM is $\text{MA}(\text{preserving stationary subsets of } \omega_1)$. The interested reader is referred to [1] and [2] for the history of these forcing axioms.
A partial order is \( \lambda \)-closed if it is closed under descending chains of length less than \( \lambda \). It is \( \lambda \)-directed-closed if it is closed under directed subsets of size less than \( \lambda \). [7] proves that PFA is preserved by \( \omega_2 \)-closed forcings and [8] that MM is preserved by \( \omega_2 \)-directed-closed forcings.

Namba forcing is denoted by \( Nm \): conditions are trees \( t \subseteq \omega \omega \) with a trunk \( \text{tr}(t) \) such that \( t \) is linear below \( \text{tr}(t) \) and has splitting \( \aleph_2 \) everywhere above the trunk. Smaller trees contain more information.

It is known that Namba forcing preserves stationary subsets of \( \omega_1 \). If \( t \in Nm \) and \( x \in t \) then the last element of \( x \) is also called the tag of \( x \), denoted as \( \text{tag}(x) \), and we define \( \text{Suc}_t(x) \) to be the set of tags of all immediate successors of \( x \) in \( t \). So \( \text{Suc}_t(x) \) is an unbounded subset of \( \omega_2 \). In an abuse of notation, a sequence is sometimes confused with its tag. We write \([t]\) for the set of infinite branches through \( t \).

2. Stationary limits

For a tree \( T \) and an ordinal \( \alpha \), let \( T_\alpha \) denote the \( \alpha \)th level of \( T \) and \( T_{<\alpha} = \bigcup_{\xi<\alpha} T_\xi \). If \( X \) is a set of ordinals, we write \( T \upharpoonright X \) for the subtree \( \bigcup_{\xi \in X} T_\xi \). The expression \( \text{ht}(T) \) denotes the height of \( T \). We only consider trees of functions. If \( T \) is a tree and \( B \) a collection of cofinal branches through \( T \) then we call \( B \) non-stationary over \( T \) if there is a function \( f : B \to T \) which is regressive, i.e. \( f(b) \in b \) for all \( b \in B \) and if \( b, b' \in B \) are distinct then \( f(b) \) or \( f(b') \) is strictly above \( b \cap b' \). Otherwise we call \( B \) stationary over \( T \). A tree \( T \) of height \( \kappa \) is called \( \gamma \)-regressive if \( T_\alpha \) is non-stationary over \( T_{<\alpha} \) for every limit ordinal \( \alpha < \kappa \) of cofinality less than \( \gamma \). The following is easy to check:

3 Remark. Assume that \( A \subseteq \alpha \) is cofinal in \( \alpha \). Then \( T_\alpha \) is stationary over \( T_{<\alpha} \) iff \( T_\alpha \) is stationary over \( T \upharpoonright A \).

The \( \omega \)-cofinal limits will figure prominently when dealing with \( \omega_1 \)-regressive trees, so we prove a useful Lemma about these. For simplicity we only consider trees of height \( \omega \). The reader will notice that the following observations are applicable in Section 6. If \( T \) is of height \( \omega \) and \( B \) a collection of infinite branches then for any subset \( S \subseteq T \) we let

\[
\overline{S} = \{ b \in B : b \cap S \text{ is infinite} \}.
\]

If \( S \) is countable and \( \overline{S} \) uncountable then we call \( S \) a Cantor-subtree of \( T \). The class \( N(T,B) \subseteq [H_\theta]^{|\aleph_0|} \) (for some large enough regular \( \theta \)) is defined by letting \( N \in N(T,B) \) if and only if there is \( b \in B \) such that \( b \subseteq N \) but \( b \notin N \). We have the following

4 Lemma. Assume that \( T \) has height \( \omega \) and size \( \aleph_1 \) and that \( B \) is a collection of infinite branches. Then the following are equivalent:
(1) \( B \) is stationary over \( T \).

(2) (a) Either there is a Cantor-subtree \( S \subseteq T \) or
    (b) if we identify \( T \) with \( \omega_1 \) by any enumeration then
        \[ E_B = \{ \alpha < \omega_1 : \sup(b) = \alpha \text{ for some } b \in B \} \]
    is stationary in \( \omega_1 \).

(3) \( N(T, B) \) is stationary in \( [H_\theta]^{\aleph_0} \).

Proof. The equivalence of (1) and (3) can be found in [6, p.112] and
the implication (2) \( \implies \) (1) is easy.

For (3) \( \implies \) (2), assume \( \neg(2) \) and show \( \neg(3) \): pick an enumeration
\( e : \omega_1 \to T \) such that \( E_B \) is nonstationary if we identify nodes with
countable ordinals via the enumeration \( e \). Pick a structure \( N \prec H_\theta \)
such that \( e, T, B \in N \) and set \( \gamma = N \cap \omega_1 \), so we have \( \gamma \notin E_B \). Let
\( b \in B \) be such that \( b \subseteq N \). Then \( \sup(b) < \gamma \) holds. Now define
\[ \mathcal{A} = \{ c \in B : \sup(c) = \sup(b) \} \]
Note that \( \mathcal{A} \in N \) and \( \mathcal{A} \) is countable since we know by \( \neg(2)(a) \) that
\( \sup(b) \) is countable. So \( \mathcal{A} \subseteq N \), therefore \( b \in N \). This shows that
\( N \notin N(T, B) \) and \( N(T, B) \) is non-stationary. \( \square \)

Note that the equivalence of (1) and (3) is to some extent already in
[1, p.955] but our result differs slightly from this last reference as we
have a stronger notion of non-stationarity. See also [6] for variations of
Lemma 4 in uncountable heights.

3. Creating regressive Kurepa-trees

Let \( \lambda \) be a regular uncountable cardinal throughout this section. We
describe the natural forcing \( K^\lambda_{\text{reg}} \) to add a \( \lambda \)-regressive \( \lambda \)-Kurepa-tree
and show that this forcing is \( \lambda \)-closed. We may assume the cardinal
arithmetic \( 2^{<\lambda} = \lambda \), otherwise a preliminary Cohen-subset of \( \lambda \) could
be added. Conditions of \( K^\lambda_{\text{reg}} \) are pairs \( (T, h) \), where

(1) \( T \) is a tree of height \( \alpha + 1 \) for some \( \alpha < \lambda \) and each level has
    size \( < \lambda \).

(2) \( T \) is \( \lambda \)-regressive, i.e. if \( \xi \leq \alpha \) then \( T_\xi \) is non-stationary over \( T_{<\xi} \).

(3) \( h : T_\alpha \to \lambda^+ \) is 1-1.

The condition \( (T, h) \) is stronger than \( (S, g) \) if

- \( S = T \upharpoonright \text{ht}(S) \).
- \( \text{rng}(g) \subseteq \text{rng}(h) \).
- \( g^{-1}(\nu) \leq_T h^{-1}(\nu) \) for all \( \nu \in \text{rng}(g) \).
A generic filter $G$ for $\mathcal{K}_{\text{reg}}^\lambda$ will produce a $\lambda$-regressive $\lambda$-tree $T_G$ in the first coordinate and the sets 

$$ b_\nu = \{ x \in T_G : \text{there is } (T, h) \in G \text{ such that } h(x) = \nu \} $$

for $\nu < \lambda^+$ form a collection of $\lambda^+$-many mutually different $\lambda$-branches through the tree $T_G$. Notice also that the standard arguments for $\lambda^+\text{-cc}$ go through here as we assumed $2^{<\lambda} = \lambda$.

So we are done once we show that $\mathcal{K}_{\text{reg}}^\lambda$ is $\lambda$-closed. To this end, let $(T^\xi, h^\xi) \ (\xi < \gamma)$ be a descending chain of conditions of length less than $\lambda$. We can obviously assume that $\gamma$ is a limit ordinal. If the height of $T^\xi$ is $\alpha^\xi + 1$, let $\alpha^\gamma = \sup{\xi<\gamma} \alpha^\xi$. We want to extend the tree

$$ T^* = \bigcup_{\xi<\gamma} T^\xi, $$

so we have to define the $\alpha^\gamma$th level: whenever $\nu \in \text{rng}(h^\xi)$ for some $\xi < \gamma$, then there is exactly one $\alpha^\gamma$-branch $c_\nu$ that has color $\nu$ on a final segment. Now define

$$ T^\alpha_{\alpha^\gamma} = \{ c_\nu : \nu \in \text{rng}(h^\xi) \text{ for some } \xi < \gamma \} $$

and let $T_{\alpha^\gamma}$ be the tree $T^*$ with the level $T^\alpha_{\alpha^\gamma}$ on top. The 1-1 function $h^\gamma : T^\alpha_{\alpha^\gamma} \to \lambda^+$ is defined by letting

$$ h^\gamma(c_\nu) = \nu. $$

We claim that $(T^\gamma, h^\gamma)$ is a condition: the only thing left to check is that $T^\alpha_{\alpha^\gamma}$ is non-stationary over $T^*$. But this is witnessed by the function

$$ f(c_\nu) = \text{the } <_T \text{-least } x \in c_\nu \text{ such that there is } \xi < \gamma \text{ with } h^\xi(x) = \nu. $$

Notice that $f$ is regressive: if

$$ f(c_\nu) \leq_T f(c_\mu) \leq_T c_\nu \cap c_\mu, $$

let $\xi$ witness that $f(c_\mu) = x$, i.e. $h^\xi(x) = \mu$. Then $h^\xi(x)$ must be color $\nu$ as well since $f(c_\nu) \leq_T x$ has color $\nu$. Thus, $\nu = h^\xi(x) = \mu$.

But $(T^\gamma, h^\gamma)$ extends the chain $(T^\xi, h^\xi) \ (\xi < \gamma)$, so we just showed

5 Theorem. $\mathcal{K}_{\text{reg}}^\lambda$ is a $\lambda$-closed forcing that adds a $\lambda$-regressive $\lambda$-Kurepa-tree.

We emphasize again that the forcing $\mathcal{K}_{\text{reg}}^\lambda$ is not $\omega_2$-directed-closed but the reader can check that the usual forcing to add a plain $\lambda$-Kurepa-tree (see e.g. [4]) actually is $\lambda$-directed-closed.
4. Destroying regressive Kurepa-trees above a compact cardinal

If $\lambda$ is a regular uncountable cardinal then a tree $T$ is called a weak $\lambda$-Kurepa-tree if

- $T$ has height $\lambda$,
- each level has size $\leq \lambda$ and
- $T$ has $\lambda^+$-many cofinal branches.

6 Lemma. Suppose that $\lambda$ is a regular uncountable cardinal and there is an elementary embedding $j : V \rightarrow M$ such that $\eta = \sup(j''\lambda) < j(\lambda)$ and $\text{cf}^M(\eta) < j(\kappa)$. Then there are no $\kappa$-recessive weak $\lambda$-Kurepa-trees.

Proof. Suppose that $T$ is a $\kappa$-recessive weak $\lambda$-Kurepa-tree and $j$ as above. Then there is a regressive function $f_\eta$ defined on the level $(jT)_\eta$. If $b$ is a cofinal branch through $T$, then we find $\alpha_b < \lambda$ such that

$$f_\eta(jb \upharpoonright \eta) \leq jT_jb \upharpoonright j(\alpha_b) = j(b \upharpoonright \alpha_b).$$

Note that if $b$ and $b'$ are two distinct branches through $T$ then $jb$ and $jb'$ must disagree below $\eta$. Moreover, $j(b \upharpoonright \alpha_b) \neq j(b' \upharpoonright \alpha_{b'})$ holds because $f_\eta$ is regressive. Then the assignment $b \mapsto b \upharpoonright \alpha_b$ must be 1-1, which is a contradiction to the fact that $T$ has $\lambda^+$-many branches. \hfill $\square$

Recall that a cardinal $\kappa$ is $\lambda$-compact if there is a fine ultrafilter on $P_\kappa \lambda$. If $\lambda$ is regular, the elementary embedding $j : V \rightarrow M$ with respect to such a fine ultrafilter has the following properties:

- the critical point of $j$ is $\kappa$,
- there is a discontinuity at $\lambda$, i.e. $\eta = \sup(j''\lambda) < j(\lambda)$ and
- $\text{cf}^M(\eta) < j(\kappa)$.

(see [5, §22] for more details). A cardinal $\kappa$ is said to be compact if it is $\lambda$-compact for all $\lambda$, so it follows from Lemma 6 and the above definition:

7 Theorem. Assume that $\kappa$ is a compact cardinal and $\lambda \geq \kappa$ is regular. Then there are no $\kappa$-recessive weak $\lambda$-Kurepa-trees.

Using Theorem 5, we have

8 Corollary. Compact cardinals are sensitive to $\lambda$-closed forcings for arbitrarily large $\lambda$. 
It was known before that adding a slim $\kappa$-Kurepa-tree destroys the ineffability of $\kappa$ and that slim $\kappa$-Kurepa-trees can be added with $\kappa$-closed forcing. But note that our notion of regressive is more universal: slim Kurepa-trees can exist above compact or even supercompact cardinals.

5. Oscillating branches

Now assume that $T$ is an $\omega_2$-tree: we enumerate each level by letting
\begin{equation}
T_\alpha = \{\tau(\alpha, \xi) : \xi < \omega_1\} \text{ for all } \alpha < \omega_2.
\end{equation}
In this situation we identify branches with functions from $\omega_2$ to $\omega_1$ that are induced by the enumerations of the levels. If $A \subseteq \omega_2$ is unbounded and $b : \omega_2 \to \omega_1$ is an $\omega_2$-branch through $T$ then we say that $b$ oscillates on $A$ if for all $\alpha < \omega_2$ and all $\zeta < \omega_1$ there is $\beta > \alpha$ in $A$ and $\xi > \zeta$ such that $b(\beta) = \xi$.

9 Lemma. Assume that $T$ is an $\omega_2$-Kurepa-tree with an enumeration $\tau(\alpha, \xi)$ ($\alpha < \omega_2, \xi < \omega_1$) as in (5.1) and $A_\iota$ ($\iota < \omega_2$) are $\aleph_2$-many unbounded subsets of $\omega_2$. Then there is an $\omega_2$-branch $b$ through $T$ that oscillates on every $A_\iota$ ($\iota < \omega_2$).

Proof. Assume not, then for every $\omega_2$-branch $b$ there is $\iota_b < \omega_2$ and there are $\alpha_b < \omega_2, \zeta_b < \omega_1$ such that
\[ b \upharpoonright (A_{\iota_b} \setminus \alpha_b) \subseteq \{\tau(\alpha, \xi) : \alpha \in A_{\iota_b} \setminus \alpha_b, \xi < \zeta_b\}. \]
By a cardinality argument we can find $\aleph_3$-many branches $b$ such that $\iota_0 = \iota_b$, $\alpha_0 = \alpha_b$ and $\zeta_0 = \zeta_b$. But then each of these branches is a different branch through the tree
\[ T_0 = \{\tau(\alpha, \xi) : \alpha \in A_{\iota_0} \setminus \alpha_0, \xi < \zeta_0\}. \]
$T_0$ has countable levels but $\aleph_3$-many branches, a contradiction. \qed

6. Destroying regressive Kurepa-trees with MM

We introduce a simplified notation for the following arguments: if $f : t \to \omega_1$ for some $t \in \text{Nm}$ and $\pi \in [t]$ then we let
\[ \sup_{n<\omega}^f(\pi) = \sup_{n<\omega} f(\pi \upharpoonright n). \]
If $b : \omega_2 \to \omega_1$ is an $\omega_2$-branch and $x \in \omega_2^{<\omega}$ then $b(x)$ really denotes the countable ordinal $b(\text{tag}(x))$.

\begin{footnote}{A $\kappa$-Kurepa-tree $T$ is called slim if $|T_\alpha| \leq |\alpha|$ for all $\alpha < \kappa$.}
\end{footnote}
10 Lemma. Assume that $T$ is an $\omega_2$-Kurepa-tree and $\mathcal{B}$ is the set of branches. Let $\tau(\alpha, \xi) (\alpha < \omega_2, \xi < \omega_1)$ be an enumeration as in (5.1). Then in the Namba extension $V^{Nm}$ there is a sequence

$$\Delta_G = \langle \delta^G_n : n < \omega \rangle$$

cofinal in $\omega V^2$ such that

$$\dot{\mathcal{E}}_{\mathcal{B}} = \{ \sup(b)(\Delta_G) : b \in \mathcal{B} \}$$

is stationary relative to every stationary $S \subseteq \omega_1$ in $V$, i.e. $\dot{\mathcal{E}}_{\mathcal{B}} \cap S$ is stationary for all stationary $S \subseteq \omega_1$ in the ground model.

Proof. Assume that $\dot{\mathcal{C}}$ is an Nm-name for a club in $\omega_1$, $S \subseteq \omega_1$ is a stationary set in $V$ and $t_0$ a condition in Nm. Our goal is to find a condition $t_3 \leq t_0$ and an ordinal $\xi_0 \in S$ such that $\dot{\mathcal{E}}_{\mathcal{B}} \cap \dot{\mathcal{C}}$ is stationary relative to every stationary $S \subseteq \omega_1$ in $V^{Nm}$, i.e. $\dot{\mathcal{E}}_{\mathcal{B}} \cap S$ is stationary for all stationary $S \subseteq \omega_1$ in the ground model.

(1) if $v \subseteq x$ are elements of $t_1$ then $f(v) < f(x)$.

(2) if the height of $x$ in $t_1$ is odd and $x$ is above the trunk then there is $\zeta < \omega_1$ such that

$$|\{ x \upharpoonright \beta \in t_1 \mid f(x \upharpoonright \beta) = \xi \}| = \aleph_2 \text{ for all } \xi > \zeta,$$

i.e. each ordinal in a final segment of $\omega_1$ has $\aleph_2$-many preimages in the set $\text{Suc}_{t_1}(x)$.

(3) if $G \subseteq \text{Nm}$ is generic with $t_1 \in G$ and $\pi : \omega \rightarrow \omega V^1$ is the corresponding Namba-sequence then $\sup(\langle \pi \rangle) \in \dot{\mathcal{C}}[G]$.

Given the condition $t_1$, we apply Lemma 9 to find a branch $b$ that oscillates on all sets $\text{Suc}_{t_1}(x)$ ($x \in t_1$). Using (1) and (2), we thin out again to get a condition $t_2 \leq t_1$ with the following property:

(4) if $v \subseteq x \subseteq y$ is a chain in $t_2$ above the trunk and the height of $x$ is odd then $f(v) < b(x) < f(y)$.

Note that in particular (2) can be preserved by passing to the condition $t_2$, so we may assume that $t_2$ has properties (1)-(4). Let us also assume for notational simplicity that the height of $\text{tr}(t_2)$ is even. The next step is to find $t_3 \leq t_2$ and $\xi_0 \in S$ such that

(5) $\sup(\langle \pi \rangle) = \xi_0$ for all branches $\pi$ in $[t_3]$.

To find $t_3$ and $\xi_0$, we define a game $\mathcal{G}(\gamma)$ for every limit $\gamma < \omega_1$. Fix a ladder sequence $l(\gamma) = (\gamma_n : n < \omega)$ for each such $\gamma$. The game $\mathcal{G}(\gamma)$ is played as follows:

$$\begin{array}{c|cccccc}
\text{I} & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \ldots \\
\text{II} & \beta_0 & \beta_1 & \beta_2 & \beta_3 & \ldots \\
\end{array}$$
where for all $n < \omega$

- $\alpha_n < \beta_n < \omega_2$,
- $s_n = \text{tr}(t_2)^\gamma(\beta_i : i \leq n) \in t_2$ and
- $f(s_n) \in (\gamma_n, \gamma)$ whenever $n$ is even.

II wins if he can make legal moves at each step, so the game is determined.

10.1 Claim. II wins $G(\gamma)$ for club many $\gamma$’s.

Proof. Assume not, then there is a stationary $U \subseteq \omega_1$ such that player I wins $G(\gamma)$ for each $\gamma \in U$ via the strategy $\sigma_\gamma$. Now pick a countable elementary $N$ such that $\xi = N \cap \omega_1 \in U$ and $t_2, f, l, U \in N$.

A ladder sequence $l(\xi) = (\xi_n : n < \omega)$ converging to $\xi$ is given and we define a sequence $(\beta_n : n < \omega)$ inductively as follows: let $\beta_n$ be the least

$$\beta > \sup_{\gamma \in U} \sigma_\gamma(\beta_i : i < n)$$

such that

- $s = \text{tr}(t_2)^\gamma(\beta_i : i < n)^\gamma \in t_2$ and
- $f(s) \in (\xi_n, \xi)$ whenever $n$ is even.

Such a $\beta$ exists in $N$ by (2) and elementarity. Note that $(\beta_n : n < \omega)$ is a possible record of moves for player II if player I goes along with the strategy $\sigma_\xi$. But II obviously wins the game $G(\xi)$ if the sequence $(\beta_n : n < \omega)$ is played, a contradiction. This proves the claim. □

Given the claim, pick $\xi_0 \in S$ above all $b(x)$ ($x \subseteq \text{tr}(t_2)$) such that II wins the game $G(\xi_0)$. Now we can easily find a condition $t_3 \leq t_2$ with property (5).

If we fix a generic $G \subseteq \text{Nm}$ with $t_3 \in G$ and let $\pi_G : \omega \to \omega_2^Y$ be the corresponding Namba-sequence, we can define $\delta^G_n = \pi_G(2n + 1)$ and $\Delta_G = (\delta^G_n : n < \omega)$. Then we have

- (6) $\sup^{(f)}(\Delta_G) \in \hat{C}[G]$ by (3),
- (7) $\sup^{(f)}(\Delta_G) = \xi_0$ by (5) and
- (8) $\sup^{(f)}(\Delta_G) = \sup_{n<\omega} b(\delta^G_n) = \sup^{(b)}(\Delta_G)$ by (4).

But this finishes the proof since

$$\xi_0 \in \hat{C}[G] \cap \hat{E}_B[G] \cap S.$$ □

11 Corollary. Assume that $T$ is an $\omega_2$-Kurepa-tree and $B$ the set of ground model branches through $T$. Then $B$ is stationary over $T$ in the Namba extension.
Finally we get the main result for $\omega_2$. We will prove a more general version of this in Theorem 13.

**12 Theorem.** There are no $\omega_1$-regressive $\omega_2$-Kurepa-trees under MM.

**Proof.** Assume that $T$ is an $\omega_1$-regressive $\omega_2$-Kurepa-tree and that

$$\tau(\alpha, \xi) \ (\alpha < \omega_2, \xi < \omega_1)$$

is an enumeration as in (5.1). Look at the iteration $\mathbb{P} = Nm \ast \text{CS}(\dot{E}_B)$, where $\text{CS}(\dot{E}_B)$ shoots a club through the set $\dot{E}_B$ from the statement of Lemma 10. The poset $\mathbb{P}$ preserves stationary subsets of $\omega_1$ by the fact that $\dot{E}_B$ is stationary relative to every stationary set in $V$. But we have that $\dot{E}_B$ is club in $V^{\mathbb{P}}$, so we can use MM to get a sequence $\Delta = \langle \delta_n : n < \omega \rangle$ converging to $\delta < \omega_2$ such that

$$\{\sup^{b}(\Delta) : b \text{ is a } \delta \text{-sequence in } T_\delta\}$$

is club in $\omega_1$. Using Lemma 4, we see that $T_\delta$ is definitely stationary over $T \upharpoonright \Delta$. So $T_\delta$ is stationary over $T_{<\delta}$ by Remark 3. Since $\text{cf}(\delta) = \omega$, this contradicts the fact that $T$ is $\omega_1$-regressive. \hfill $\square$

7. Larger heights

Starting from Theorem 12, we generalize the result to weak Kurepa-trees in all uncountable regular heights.

**13 Theorem.** Under MM, there are no $\omega_1$-regressive weak $\lambda$-Kurepa-trees for any uncountable regular $\lambda$.

**Proof.** Since PFA destroys weak $\omega_1$-Kurepa-trees (see [1]), we may assume that $\lambda$ is at least $\omega_2$. Now assume that $T$ is an $\omega_1$-regressive weak $\lambda$-Kurepa-tree and let $\mathcal{P} = \text{Col}(\omega_2, \lambda)$ be the usual $\omega_2$-directed collapse. Note that $\mathcal{P}$ has the $\lambda^+-\text{cc}$, because $\lambda^{\omega_1} = \lambda$ holds under MM (see [2]). So the tree $T$ has a cofinal subtree $T^*$ in $V^\mathcal{P}$ that is an $\omega_1$-regressive weak $\omega_2$-Kurepa-tree. By throwing away some nodes if necessary, we may assume that $T^*$ has the property that

$$T^*_x = \{y \in T^* : x \leq_T y\}$$

has $\aleph_3$-many branches for all $x \in T^*$.

Now we define an $\omega_2$-directed forcing $\mathcal{Q}$ in $V^\mathcal{P}$ that shoots an actual $\omega_2$-Kurepa-subtree through the tree $T^*$: conditions of $\mathcal{Q}$ are pairs of the form $(S, B)$, where

1. $S$ is a downward-closed subtree of $T^*$ of height $\alpha + 1$ for some ordinal $\alpha < \omega_2$,
2. $|S| \leq \omega_1$,
3. $B$ is a nonempty set of branches cofinal in $T^*$ and $|B| \leq \aleph_1$,
4. $b \upharpoonright (\alpha + 1) \subseteq S$ for all $b \in B$. 


We let $\langle S_0, B_0 \rangle \geq Q \langle S_1, B_1 \rangle$ if $S_0 = S_1 \upharpoonright \text{ht}(S_0)$ and $B_0 \subseteq B_1$.

If $X \subseteq Q$ is a set of mutually compatible conditions of size $\leq \aleph_1$ then we let $S_X$ and $B_X$ be the unions over the first respectively second coordinates of $X$. Now $S_X$ can be end-extended to a tree $\bar{S}_X$ of successor height by extending at least the cofinal branches in the non-empty set $B_X$. But then $(\bar{S}_X, B_X)$ is a condition stronger than every condition in $X$, hence $Q$ is $\omega_2$-directed-closed. An easy cardinality argument shows that $Q$ has the $\aleph_3$-$\text{cc}$ because $2^{\aleph_1} = \aleph_2$ holds in $V^P$. It is now straightforward that a generic filter $H \subseteq Q$ will produce an $\omega_2$-tree in the first coordinate which is $\omega_1$-regressive since it is a subtree of the original tree $T$ and notice that $P$ and $Q$ both preserve uncountable cofinalities. On the other hand, a density argument using (7.1) shows that the set

$$\mathcal{B} = \bigcup\{B : \text{there is } S \text{ such that } (S, B) \in H\}$$

has cardinality $\aleph_3$, so $H$ induces an $\omega_1$-regressive $\omega_2$-Kurepa-tree. The composition of two $\omega_2$-directed-closed forcings is again $\omega_2$-directed-closed and it was mentioned in the introduction that $\omega_2$-directed-closed forcings preserve MM, so we have the situation:

- $V^{P\ast Q} \models \text{MM}$
- $V^{P\ast Q} \models \text{"there is an } \omega_1\text{-regressive } \omega_2\text{-Kurepa-tree."}$

But this contradicts Theorem 12. \hfill \Box

Using Theorem 5, we have

**14 Corollary.** $\text{MM is sensitive to } \lambda\text{-closed forcing algebras for arbitrarily large } \lambda.$

**References**


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