CHARACTERISTIC POLYHEDRA OF SINGULARITIES WITHOUT COMPLETION

Vincent COSSART and Olivier PILTANT
cossart@math.uvsq.fr, piltant@math.uvsq.fr

Laboratoire de Mathématiques LMV UMR 8100
Université de Versailles
45 avenue des États-Unis
78035 VERSAILLES Cedex (France)

Dedicated to Heisuke Hironaka on his eightieth birthday.

INTRODUCTION

Let $(R, M, k := R/M)$ be a regular local ring, $f \in M$ and $(y, u_1, \ldots, u_d)$ be a regular system of parameters (r.s.p. for short) of R. Assume furthermore that

$$f \notin (u_1, \ldots, u_d).$$

To this situation, H. Hironaka [H] attaches a polyhedron $\Delta(f; u_1, \ldots, u_d; y) \subset \mathbb{R}^d_{\geqslant 0}$ and shows that there exists some $z \in \widehat{R}$ such that $(z, u_1, \ldots, u_d)$ is a r.s.p. of $\widehat{R}$ and

$$\Delta(f; u_1, \ldots, u_d; z) = \bigcap_{(\widehat{y}, u_1, \ldots, u_d)} \Delta(f; u_1, \ldots, u_d; \widehat{y}),$$

where the intersection runs over all r.s.p's of $\widehat{R}$ of the form $(\widehat{y}, u_1, \ldots, u_d)$. We prove here in theorem **II.3** that one can choose $z \in R$ whenever R is a G-ring.

See [C] for the case of analytic function ring, in a more general setup. The present result has been exposed by the first author at the conference "Resolution of singularities and related topics, 80th birthday of Heisuke Hironaka", Tordesillas.

We emphasize that no assumption is made on the characteristic nor residue characteristic of R. Since Hironaka's polyhedron is an essential tool for defining resolution invariants of singularities, theorem **II.3** is a very useful tool in order to prove *semicontinuity results* for such invariants. Some applications in mixed characteristic will appear in [CP].

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I CHARACTERISTIC POLYHEDRON OF f

In this section, we briefly survey known material on F-subsets, characteristic polyhedra and graded algebras with respect to weighted monomial filtrations.

I.1 DEFINITION.
(i) An F-subset $\Delta \subset \mathbb{R}^e_{\geqslant 0}$ is a closed convex subset of $\mathbb{R}^e_{\geqslant 0}$ such that $\mathbf{v} \in \Delta$ implies $\mathbf{v} + \mathbb{R}^e_{\geqslant 0} \subseteq \Delta$.
(ii) A point $\mathbf{v} \in \Delta$ is called a vertex if there exists a positive linear form L on $\mathbb{R}^e$ (i.e. with positive coefficients) such that

$$\{\mathbf{v}\} = \Delta \cap \{\mathbf{a} \in \mathbb{R}^e_{\geqslant 0} | L(\mathbf{a}) = 1\}.$$

(iii) An F-subset Δ is called a rational polyhedron if there exists finitely many nonnegative rational linear forms $L_1, \ldots, L_n$ on $\mathbb{R}^e$ (i.e. with nonnegative rational coefficients) such that

$$\Delta = \{\mathbf{a} \in \mathbb{R}^e_{\geqslant 0} | L_i(\mathbf{a}) \geqslant 1, \ 1 \leqslant i \leqslant n\}.$$

Given a r.s.p. $(y, u_1, u_2, \ldots, u_d) =: (y, \mathbf{u})$ of a regular local ring R and $f \in R$, there exists a *finite* sum expansion

$$f = \sum_{\mathbf{a}, b} C_{\mathbf{a},b} y^b \mathbf{u}^{\mathbf{a}}, \ b \in \mathbb{N}, \ \mathbf{a} \in \mathbb{N}^d. \tag{1}$$

where each $C_{\mathbf{a},b}$ is a unit in R. This follows easily from the facts that R is Noetherian and the map $R \subseteq \widehat{R}$ is faithfully flat. We regard $\mathbf{u}$ as "fixed" parameters and y as "varying", which is reflected in the indexing below.

Assume furthermore that

$$f \in M, \ f \notin (u_1, \ldots, u_d). \tag{2}$$

We let $\overline{R} := R/(u_1, \ldots, u_d)$, $\overline{f} \in \overline{R}$ be the image of f and "ord" be the valuation of the discrete valuation ring $\overline{R}$. We let

$$m := \mathrm{ord}\overline{f} \geqslant 1. \tag{3}$$

Assumption (2) and notation (3) are maintained all along this article. It is immediately seen from (1) that $\Delta(f; \mathbf{u}; y)$ below is independent of the chosen expansion (1) (f, $u_1, \ldots, u_d$ and y being fixed) and that it is indeed a rational polyhedron. See Hironaka's theorem [H](4.8) (restated as proposition **I.4** below) about the characteristic polyhedron.

I.2 DEFINITION. *With notations as above:*
(i) The rational polyhedron $\Delta(f; \mathbf{u}; y) \subset \mathbb{R}^d_{\geqslant 0}$ is the smallest F-subset containing all points of

$$S(f) := \{v = \frac{\mathbf{a}}{m - b} | 0 \leqslant b < m\}.$$

(ii) The "characteristic polyhedron" $\Delta(f; \mathbf{u}) \subset \mathbb{R}^d_{\geqslant 0}$ is the F-subset defined by the formula

$$\Delta(f; \mathbf{u}) := \bigcap_{(\widehat{y}, u_1, \ldots, u_d)} \Delta(f; \mathbf{u}; \widehat{y}), \tag{4}$$

where the intersection runs over all r.s.p's of $\widehat{R}$ of the form $(\widehat{y}, u_1, \ldots, u_d)$.
(iii) Let $L: \ (x_1, x_2, \ldots, x_d) \mapsto L(x_1, x_2, \ldots, x_d) = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_d x_d$, $\lambda_1, \lambda_2, \ldots, \lambda_d \in \mathbb{Q}_{\geqslant 0}$ be a nonzero nonnegative linear form on $\mathbb{R}^d$. We define

$$l(f, \mathbf{u}, y) := \min\{L(\mathbf{a}) | \mathbf{a} \in \Delta(f; \mathbf{u}; y)\} \geqslant 0.$$

We define a monomial valuation $v_{L,\mathbf{u},y,f}$ *on* R *by setting*

$$I_\lambda := (\{y^b\mathbf{u}^{\mathbf{a}} | l(f,\mathbf{u},y)b + L(\mathbf{a}) \geqslant \lambda\}) \subseteq R,$$

for $\lambda \geqslant 0$ *and* $v_{L,\mathbf{u},y,f}(g) := \min\{\lambda \in \mathbb{Q} | g \in I_\lambda\}$ *for any nonzero* $g \in R$.

The following proposition is an easy but useful exercise left to the reader.

I.3 PROPOSITION. *Let* L *be a nonzero nonnegative linear form as above, and let*

$$I := \{i | \lambda_i > 0\},\ I' := \{i | \lambda_i = 0\} = \{1, \ldots, d\} \backslash I.$$

The graded algebra $\mathrm{gr}_{v_{L,\mathbf{u},y,f}}(R)$ *of* R *w.r.t.* $v_{L,\mathbf{u},y,f}$ *is given by*
(i) if $l(f,\mathbf{u},y) \neq 0$, *then*

$$\mathrm{gr}_{v_{L,\mathbf{u},y,f}}(R) = \frac{R}{(y, \{u_i\}_{i\in I})}[Y, \{U_i\}_{i\in I}];$$

(ii) if $l(f,\mathbf{u},y) = 0$, *then*

$$\mathrm{gr}_{v_{L,\mathbf{u},y,f}}(R) = \frac{R}{(\{u_i\}_{i\in I})}[\{U_i\}_{i\in I}].$$

In particular, we have $\mathrm{gr}_{v_{L,\mathbf{u},y,f}}(R) \simeq k[Y, U_1, U_2, \ldots, U_d]$ *whenever* L *is positive.*

The following is Hironaka's theorem [H](4.8).

I.4 PROPOSITION (Hironaka). *With notations as above, there exists* $\widehat{y} \in \widehat{R}$ *such that* $(\widehat{y}, u_1, \ldots, u_d)$ *is a r.s.p. of* $\widehat{R}$ *and* $\Delta(f;\mathbf{u};\widehat{y}) = \Delta(f;\mathbf{u})$. *In particular,* $\Delta(f;\mathbf{u})$ *is a rational polyhedron.*

II GETTING RATIONAL COORDINATES

We now introduce the G-ring assumption on R ([EGAIV] 7.3.13 w.r.t. that property **P** in 7.3.8 (iv)' defined in 6.7.6; see also [M] top of p.256): a Noetherian local ring A is said to be a G-ring if for every $P \in \mathrm{Spec}A$ the formal fiber ring $\widehat{A} \otimes_A \kappa(P)$ is geometrically regular. The assumption is only used through the following lemma:

II.1 LEMMA. *Assume that the regular local ring* (R, M, k) *is a G-ring. The following holds:*
(i) let $f \in M$, $f \neq 0$ *be such that* $R/(f)$ *is a domain. Then* $\widehat{R}/(f)$ *is reduced;*
(ii) any quotient of R *or localisation of the polynomial ring* $R[T]$ *at a maximal ideal is again a G-ring.*

Proof. Let $f = \widehat{\delta}\widehat{f}_1^{a_1} \cdots \widehat{f}_s^{a_s}$ be a decomposition of f into pairwise nonequivalent irreducible factors in the UFD $\widehat{R}$, where $\widehat{\delta} \in \widehat{R}$ is a unit. By the G-ring assumption, the formal fiber ring $\widehat{R} \otimes_R QF(R/(f))$ is reduced, so $a_1 = \cdots = a_s = 1$. For (ii), [EGAIV] theorem 7.4.4 (ii) and 7.3.8 (iv)'.

II.2 LEMMA. *Let* $(R, M, k := R/M)$ *be a regular local ring with r.s.p.* $(y, \mathbf{u})$ *which is a G-ring. Let* $f \in M$, $f \notin (u_1, \ldots, u_d)$ *and assume that* $\Delta(f;\mathbf{u}) = \varnothing$.

Then there exists $z \in R$ *such that* $(z, u_1, \ldots, u_d)$ *is a r.s.p. of* R *and*

$$\Delta(f;\mathbf{u};z) = \varnothing.$$

Proof. By Hironaka's theorem (proposition **I.4**), there exists a r.s.p. $(\widehat{y}, u_1, \ldots, u_d)$ of $\widehat{R}$ such that $\Delta(f; \mathbf{u}; \widehat{y}) = \varnothing$. By definition **I.2**.(i), this means that $f = \widehat{\delta}\widehat{y}^m$ for some unit $\widehat{\delta} \in \widehat{R}$. By lemma **II.1**.(i), $\sqrt{(f)} \subset R$ is a prime ideal $(z) \subset R$ such that $(z)\widehat{R}$ is again prime. Since $\mathrm{ord}\overline{f} = m$ (notations of (3)), we have $\mathrm{ord}\overline{z} = 1$, i.e. $(z, u_1, \ldots, u_d)$ is a r.s.p. of R.

II.3 THEOREM. *Let $(R, M, k := R/M)$ be a regular local ring with r.s.p. $(y, \mathbf{u})$ which is a G-ring, and $f \in M$, $f \notin (u_1, \ldots, u_d)$.*

Then there exists $z \in R$ such that $(z, u_1, \ldots, u_d)$ is a r.s.p. of R and

$$\Delta(f; \mathbf{u}; z) = \Delta(f; \mathbf{u}). \tag{5}$$

Proof. By lemma **II.2**, it can be assumed that $\Delta(f; \mathbf{u}) \neq \varnothing$. By Hironaka's theorem (proposition **I.4**), there exists a r.s.p. of the form $(\widehat{y}, u_1, \ldots, u_d)$ of $\widehat{R}$ such that $\Delta(f; \mathbf{u}; \widehat{y}) = \Delta(f; \mathbf{u})$. The rational polyhedron $\Delta(f; \mathbf{u})$ may then be defined by a formula

$$\Delta(f; \mathbf{u}) = \{\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d_{\geqslant 0} \mid L_j(x_1, x_2, \ldots, x_d) \geqslant 1,\ 1 \leqslant j \leqslant n\},$$

for some $n \geqslant 1$; here each $L_j :\ \mathbb{R}^d \to \mathbb{R}^d$ is a (nonzero) nonnegative rational linear form verifying

$$L_j(\Delta(f; \mathbf{u})) = [1, +\infty[. \tag{6}$$

We fix $A \geqslant 1$ such that for all j, $1 \leqslant j \leqslant n$, we have

$$L_j(x_1, x_2, \ldots, x_d) := \lambda_{j,1}x_1 + \lambda_{j,2}x_2 + \cdots + \lambda_{j,d}x_d,\ \lambda_{j,1}, \lambda_{j,2}, \ldots, \lambda_{j,d} \in \frac{1}{A}\mathbb{N}.$$

With notations as in **I.2**.(iii), we let

$$L_j(\Delta(f; \mathbf{u}; y)) =: [l_j(f, \mathbf{u}, y), \infty[,\ 1 \leqslant j \leqslant n.$$

Note that $L_j(x_1, x_2, \ldots, x_d) = l_j(f, \mathbf{u}, y)$ is the equation of a face of $\Delta(f; \mathbf{u}; y)$. We attach a rational number:

$$\Lambda(f, \mathbf{u}, y) := \sum_{j=1}^{n} (1 - l_j(f, \mathbf{u}, y)) \geqslant 0. \tag{7}$$

Following Hironaka [H] (2.6), we consider the initial form $\mathrm{in}_{v_j} f$ of f with respect to the valuation $v_j := v_{L_j, \mathbf{u}, y, f}$ of **I.2**.(iii).

Suppose equality (5) is not achieved with $z = y$. By (6), this means that $l_j(f, \mathbf{u}, y) < 1$ for *some* j, $1 \leqslant j \leqslant n$ which we fix now. In particular we have $\Lambda(f, \mathbf{u}, y) > 0$. We consider two cases:

Case 1: $l_j(f, \mathbf{u}, y) > 0$. With notations as in proposition **I.3**, we let $t_{i'} := \mathrm{in}_{v_j} u_{i'} \in \mathrm{gr}_{v_j}(R)_0 = R/(y, \{u_i\}_{i \in I})$ for each $i' \in I'$; let $t_i := U_i \in \mathrm{gr}_{v_j}(R)$ for $i \in I$.

Case 2: $l_j(f, \mathbf{u}, y) = 0$. Similarly, we let $t_{i'} := \mathrm{in}_{v_j} u_{i'} \in \mathrm{gr}_{v_j}(R)_0 = R/(\{u_i\}_{i \in I})$ for each $i' \in I'$; let $Y := \mathrm{in}_{v_j} y \in \mathrm{gr}_{v_j}(R)_0$; let finally $t_i := U_i \in \mathrm{gr}_{v_j}(R)$ for $i \in I$.

Then

$$S := \mathrm{gr}_{v_j}(R)_{(Y, t_1, \ldots, t_d)}$$

is a regular local ring with r.s.p. $(Y, t_1, \ldots, t_d) =: (Y, \mathbf{t})$ and residue field $S/N = k$, $N := (Y, t_1, \ldots, t_d)$. Note that S is canonically endowed with a monomial valuation (still denoted by v_j) which is induced from the graded structure of $\mathrm{gr}_{v_j}(R)$. The valuation v_j also canonically extends to the formal completion $\widehat{S}$ w.r.t. N. We have inclusions $S \subseteq \mathrm{gr}_{v_j}(\widehat{R})_{(Y,t_1,\ldots,t_d)} \subseteq \widehat{S}$, with an isomorphism $\widehat{S} \simeq \mathrm{gr}_{v_j}(\widehat{R})_0[[Y, \{U_i\}_{i\in I}]]$ in case 1 (resp. $\widehat{S} \simeq \mathrm{gr}_{v_j}(\widehat{R})_0[[\{U_i\}_{i\in I}]]$ in case 2). We have $\mathrm{in}_{v_j} f \in N$, $\mathrm{in}_{v_j} f \notin (t_1, \ldots, t_d)$, so $\mathrm{in}_{v_j} f \in S$ verifies assumptions (2) and (3).

Hironaka's construction [H] (vertex preparation, lemma (3.10)) implies the following:

(a) $\widehat{Y} := \mathrm{in}_{v_j} \widehat{y} \in \mathrm{gr}_{v_j}(\widehat{R})_{l_j(f,\mathbf{u},y)}$ and $(\widehat{Y}, t_1, \ldots, t_d)$ is a r.s.p. of $\widehat{S}$;

(b) $\Delta(\widehat{y}; \mathbf{u}; y) \subseteq \Delta(f; \mathbf{u}; y)$;

(c) $\Delta(\mathrm{in}_{v_j} f; \mathbf{t}; \widehat{Y}) = \varnothing$, that is: $\mathrm{in}_{v_j} f = \widehat{\delta}\widehat{Y}^m$ for some unit $\widehat{\delta} \in \widehat{S}$.

By lemma **II.1**.(ii), S is a G-ring; by (a), $(\widehat{Y}, t_1, \ldots, t_d)$ is a r.s.p. of $\widehat{S}$. Since $\Delta(\mathrm{in}_{v_j} f; \mathbf{t}; \widehat{Y}) = \varnothing$ by (c), lemma **II.1**.(i) applies to $\mathrm{in}_{v_j} f \in N$, $\mathrm{in}_{v_j} f \notin (t_1, \ldots, t_d)$. So there exists $Z \in N$ such that $(Z, t_1, \ldots, t_d)$ is a r.s.p. of S and

$$\Delta(\mathrm{in}_{v_j} f; \mathbf{t}; Z) = \varnothing. \tag{8}$$

We pick for Z a finite expression in the form (1):

$$Z = \sum_{\mathbf{a},b} C_{\mathbf{a},b} Y^b \mathbf{t}^{\mathbf{a}}, \ b \in \mathbb{N}, \ \mathbf{a} \in \mathbb{N}^d,$$

where each $C_{\mathbf{a},b} \in S$ is a unit. Since $\mathrm{in}_{v_j} f \in \mathrm{gr}_{v_j}(\widehat{R})$ is homogeneous for v_j, it can assumed w.l.o.g. that $Z \in \mathrm{gr}_{v_j}(R)$ and Z is homogeneous (of degree $l_j(f, \mathbf{u}, y)$); in particular $C_{\mathbf{a},b} \in \mathrm{gr}_{v_j}(R)_0$ for each value of $(b, \mathbf{a})$.

Again by (c), $(Z) = (\widehat{Y})$ in $\widehat{S}$, which in turn implies that

$$\Delta(Z; \mathbf{t}; Y) = \Delta(\widehat{Y}; \mathbf{t}; Y). \tag{9}$$

Since $\Delta(\widehat{Y}; \mathbf{t}; Y) \subseteq \Delta(\widehat{y}; \mathbf{u}; y)$, any set of preimages $\gamma_{\mathbf{a},b} \in R$ of the elements $C_{\mathbf{a},b} \in \mathrm{gr}_{v_j}(R)_0$ defines an element

$$z := \sum_{\mathbf{a},b} \gamma_{\mathbf{a},b} y^b \mathbf{u}^{\mathbf{a}} \in R.$$

Since for each value of $(b, \mathbf{a})$, $C_{\mathbf{a},b}$ is a unit, $\gamma_{\mathbf{a},b} \in R$ is a unit. This implies that $\Delta(z; \mathbf{u}; y) = \Delta(Z; \mathbf{t}; Y)$, wherefrom we conclude using (9) and (b) that

$$\Delta(z; \mathbf{u}; y) \subseteq \Delta(f; \mathbf{u}; y). \tag{10}$$

Now (10) implies that $\Delta(f; \mathbf{u}; z) \subseteq \Delta(f; \mathbf{u}; y)$, and in particular we get

$$l_k(f, \mathbf{u}, z) \geqslant l_k(f, \mathbf{u}, y), \ 1 \leqslant k \leqslant n. \tag{11}$$

On the other hand (8) is equivalent to

$$l_j(f, \mathbf{u}, z) > l_j(f, \mathbf{u}, y)$$

and we conclude from (7) that

$$0 \leqslant \Lambda(f, \mathbf{u}, z) < \Lambda(f, \mathbf{u}, y). \tag{12}$$

Since $\Lambda(f, \mathbf{u}, y') \in \frac{1}{m!A}\mathbb{N}$ for any r.s.p. $(y', \mathbf{u})$ of R, (12) cannot repeat infinitely many times, that is, $\Lambda(f, \mathbf{u}, z) = 0$ for some z and (5) eventually holds.

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