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► **To cite this version:**

Stavros Vakeroudis, Marc Yor. Integrability properties and Limit Theorems for the exit time from a cone of planar Brownian motion. 2011. hal-00654695

HAL Id: hal-00654695

<https://hal.archives-ouvertes.fr/hal-00654695>

Preprint submitted on 22 Dec 2011

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Integrability properties and Limit Theorems for the exit time from a cone of planar Brownian motion

S. Vakeroudis ^{*†} M. Yor ^{*‡}

December 22, 2011

Abstract

We obtain some integrability properties and some limit Theorems for the exit time from a cone of a planar Brownian motion, and we check that our computations are correct via Bougerol's identity.

Key words: Bougerol's identity, planar Brownian motion, skew-product representation, exit time from a cone.

MSC Classification (2010): 60J65, 60F05.

1 Introduction

We consider a standard planar Brownian motion[§] $(Z_t = X_t + iY_t, t \geq 0)$, starting from $x_0 + i0, x_0 > 0$, where $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two independent linear Brownian motions, starting respectively from x_0 and 0.

As is well known [ItMK65], since $x_0 \neq 0$, $(Z_t, t \geq 0)$ does not visit a.s. the point 0 but keeps winding around 0 infinitely often. In particular, the continuous winding process $\theta_t = \text{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ is well defined. A scaling argument shows that we may assume $x_0 = 1$, without loss of generality, since, with obvious notation:

$$\left(Z_t^{(x_0)}, t \geq 0 \right) \stackrel{(law)}{=} \left(x_0 Z_{(t/x_0^2)}^{(1)}, t \geq 0 \right). \quad (1)$$

*Laboratoire de Probabilités et Modèles Aléatoires (LPMA) CNRS : UMR7599, Université Pierre et Marie Curie - Paris VI, Université Paris-Diderot - Paris VII, 4 Place Jussieu, 75252 Paris Cedex 05, France. E-mail: stavros.vakeroudis@etu.upmc.fr

†Probability and Statistics Group, School of Mathematics, University of Manchester, Alan Turing Building, Oxford Road, Manchester M13 9PL, United Kingdom.

‡Institut Universitaire de France, Paris, France. E-mail: yormarc@aol.com

§When we simply write: Brownian motion, we always mean real-valued Brownian motion, starting from 0. For 2-dimensional Brownian motion, we indicate planar or complex BM.

Thus, from now on, we shall take $x_0 = 1$.

Furthermore, there is the skew product representation:

$$\log |Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i\gamma_u) \Big|_{u=H_t} = \int_0^t \frac{ds}{|Z_s|^2}, \quad (2)$$

where $(\beta_u + i\gamma_u, u \geq 0)$ is another planar Brownian motion starting from $\log 1 + i0 = 0$. Thus, the Bessel clock H plays a key role in many aspects of the study of the winding number process $(\theta_t, t \geq 0)$ (see e.g. [Yor80]).

Rewriting (2) as:

$$\log |Z_t| = \beta_{H_t}; \quad \theta_t = \gamma_{H_t}, \quad (3)$$

we easily obtain that the two σ -fields $\sigma\{|Z_t|, t \geq 0\}$ and $\sigma\{\beta_u, u \geq 0\}$ are identical, whereas $(\gamma_u, u \geq 0)$ is independent from $(|Z_t|, t \geq 0)$.

We shall also use Bougerol's celebrated identity in law [Bou83, ADY97] and [Yor01] (p. 200), which may be written as:

$$\text{for fixed } t, \quad \sinh(\beta_t) \stackrel{(law)}{=} \hat{\beta}_{A_t(\beta)} \quad (4)$$

where $(\beta_u, u \geq 0)$ is 1-dimensional BM, $A_u(\beta) = \int_0^u ds \exp(2\beta_s)$ and $(\hat{\beta}_v, v \geq 0)$ is another BM, independent of $(\beta_u, u \geq 0)$. For the random times $T_c^{|\theta|} \equiv \inf\{t : |\theta_t| = c\}$, and $T_c^{|\gamma|} \equiv \inf\{t : |\gamma_t| = c\}$, ($c > 0$) by using the skew-product representation (3) of planar Brownian motion [ReY99], we obtain:

$$T_c^{|\theta|} = A_{T_c^{|\gamma|}}(\beta) \equiv \int_0^{T_c^{|\gamma|}} ds \exp(2\beta_s) = H_u^{-1} \Big|_{u=T_c^{|\gamma|}}. \quad (5)$$

Moreover, it has been recently shown that, Bougerol's identity applied with the random time $T_c^{|\theta|}$ instead of t in (4) yields the following [Vak11]:

Proposition 1.1 *The distribution of $T_c^{|\theta|}$ is characterized by its Gauss-Laplace transform:*

$$E \left[\sqrt{\frac{2c^2}{\pi T_c^{|\theta|}}} \exp\left(-\frac{x}{2T_c^{|\theta|}}\right) \right] = \frac{1}{\sqrt{1+x}} \varphi_m(x), \quad (6)$$

for every $x \geq 0$, with $m = \frac{\pi}{2c}$, and:

$$\varphi_m(x) = \frac{2}{(G_+(x))^m + (G_-(x))^m}, \quad G_{\pm}(x) = \sqrt{1+x} \pm \sqrt{x}. \quad (7)$$

The remainder of this article is organized as follows: in Section 2 we study some integrability properties for the exit times from a cone; more precisely we obtain some new results concerning the negative moments of $T_c^{|\theta|}$ and of $T_c^{\theta} \equiv \inf\{t : \theta_t = c\}$. In Section 3 we state and prove some limit Theorems for these random times for $c \rightarrow 0$ and for $c \rightarrow \infty$ followed by several generalizations (for extensions of these works to more general planar processes, see e.g. [DoV12]). We use these results in order to obtain (see Remark 3.4) a

new simple non-computational proof of Spitzer's celebrated asymptotic Theorem [Spi58], which states that:

$$\frac{2}{\log t} \theta_t \xrightarrow[t \rightarrow \infty]{(law)} C_1, \quad (8)$$

with C_1 denoting a standard Cauchy variable (for other proofs, see also e.g. [Wil74, Dur82, MeY82, BeW94, Yor97, Vak11]). Finally, in Section 4 we use the Gauss-Laplace transform (6) which is equivalent to Bougerol's identity (4) in order to check our results.

2 Integrability Properties

Concerning the moments of $T_c^{|\theta|}$, we have the following (a more extended discussion is found in e.g. [MaY05]):

Theorem 2.1 *For every $c > 0$, $T_c^{|\theta|}$ enjoys the following integrability properties:*

(i) *for $p > 0$, $E \left[\left(T_c^{|\theta|} \right)^p \right] < \infty$, if and only if $p < \frac{\pi}{4c}$.*

(ii) *for any $p < 0$, $E \left[\left(T_c^{|\theta|} \right)^p \right] < \infty$.*

Corollary 2.2 *For $0 < c < d$, the random times $T_{-d,c}^\theta \equiv \inf\{t : \theta_t \notin (-d, c)\}$, $T_c^{|\theta|}$ and T_c^θ satisfy the inequality:*

$$T_c^\theta \geq T_{-d,c}^\theta \geq T_c^{|\theta|}. \quad (9)$$

Thus, their negative moments satisfy:

$$\text{for } p > 0, \quad E \left[\frac{1}{\left(T_c^\theta \right)^p} \right] \leq E \left[\frac{1}{\left(T_{-d,c}^\theta \right)^p} \right] \leq E \left[\frac{1}{\left(T_c^{|\theta|} \right)^p} \right] < \infty. \quad (10)$$

Proofs of Theorem 2.1 and of Corollary 2.2

(i) The original proof is given by Spitzer [Spi58], followed later by many authors [Wil74, Bur77, MeY82, Dur82, Yor85]. See also [ReY99] Ex. 2.21/page 196.

(ii) In order to obtain this result, we might use the representation $T_c^{|\theta|} = A_{T_c^{|\theta|}}$ together with a recurrence formula for the negative moments of A_t [Duf00], Theorem 4.2, p. 417 (in fact, Dufresne also considers $A_t^{(\mu)} = \int_0^t ds \exp(2\beta_s + 2\mu s)$, but we only need to take $\mu = 0$ for our purpose, and we note $A_t \equiv A_t^{(0)}$ [Vakth11]). However, we can also obtain this result by simply remarking that the RHS of the Gauss-Laplace transform (6) in Proposition 1.1 is an infinitely differentiable function in 0 (see also [VaY11]), thus:

$$E \left[\frac{1}{\left(T_c^{|\theta|} \right)^p} \right] < \infty, \quad \text{for every } p > 0. \quad (11)$$

Now, Corollary 2.2 follows immediately from Theorem 2.1 (ii). ■

3 Limit Theorems for $T_c^{|\theta|}$

3.1 Limit Theorems for $T_c^{|\theta|}$, as $c \rightarrow 0$ and $c \rightarrow \infty$

The skew-product representation of planar Brownian motion allows to prove the three following asymptotic results for $T_c^{|\theta|}$:

Proposition 3.1 a) For $c \rightarrow 0$, we have:

$$\frac{1}{c^2} T_c^{|\theta|} \xrightarrow[c \rightarrow 0]{(law)} T_1^{|\gamma|}. \quad (12)$$

b) For $c \rightarrow \infty$, we have:

$$\frac{1}{c} \log (T_c^{|\theta|}) \xrightarrow[c \rightarrow \infty]{(law)} 2|\beta|_{T_1^{|\gamma|}}. \quad (13)$$

c) For $\varepsilon \rightarrow 0$, we have:

$$\frac{1}{\varepsilon^2} \left(T_{c+\varepsilon}^{|\theta|} - T_c^{|\theta|} \right) \xrightarrow[\varepsilon \rightarrow 0]{(law)} \exp \left(2\beta_{T_c^{|\gamma|}} \right) T_1^{\gamma'}, \quad (14)$$

where γ' stands for a real Brownian motion, independent from γ , and $T_1^{\gamma'} = \inf\{t : \gamma'_t = 1\}$

Proof of Proposition 3.1:

We rely upon (5) for the three proofs. By using the scaling property of BM, we obtain:

$$T_c^{|\theta|} = A_{T_c^{|\gamma|}}(\beta) \stackrel{(law)}{=} A_u(\beta) \Big|_{u=c^2 T_1^{|\gamma|}}$$

thus:

$$\frac{1}{c^2} T_c^{|\theta|} \stackrel{(law)}{=} \int_0^{T_1^{|\gamma|}} dv \exp(2c\beta_v). \quad (15)$$

a) For $c \rightarrow 0$, the RHS of (15) converges to $T_1^{|\gamma|}$, thus we obtain part a) of the Proposition.

b) For $c \rightarrow \infty$, taking logarithms on both sides of (15) and dividing by c , on the LHS we obtain $\frac{1}{c} \log (T_c^{|\theta|}) - \frac{2}{c} \log c$ and on the RHS:

$$\frac{1}{c} \log \left(\int_0^{T_1^{|\gamma|}} dv \exp(2c\beta_v) \right) = \log \left(\int_0^{T_1^{|\gamma|}} dv \exp(2c\beta_v) \right)^{1/c},$$

which, from the classical Laplace argument: $\|f\|_p \xrightarrow{p \rightarrow \infty} \|f\|_\infty$, converges for $c \rightarrow \infty$, towards:

$$2 \sup_{v \leq T_1^{|\gamma|}} (\beta_v) \stackrel{(law)}{=} 2|\beta|_{T_1^{|\gamma|}}.$$

This proves part *b*) of the Proposition.

c)

$$\begin{aligned} T_{c+\varepsilon}^{|\theta|} - T_c^{|\theta|} &= \int_{T_c^{|\gamma|}}^{T_{c+\varepsilon}^{|\gamma|}} du \exp(2\beta_u) = \int_0^{T_{c+\varepsilon}^{|\gamma|} - T_c^{|\gamma|}} dv \exp\left(2\beta_{T_c^{|\gamma|} + v}\right) \exp\left(2\left(\beta_{v+T_c^{|\gamma|}} - \beta_{T_c^{|\gamma|}}\right)\right) \\ &= \exp\left(2\beta_{T_c^{|\gamma|}}\right) \int_0^{T_{c+\varepsilon}^{|\gamma|} - T_c^{|\gamma|}} dv \exp(2B_v), \end{aligned} \quad (16)$$

where $(B_s \equiv \beta_{s+T_c^{|\gamma|}} - \beta_{T_c^{|\gamma|}}, s \geq 0)$ is a BM independent of $T_c^{|\gamma|}$.

We study now $\tilde{T}_{c,c+\varepsilon}^{|\gamma|} \equiv T_{c+\varepsilon}^{|\gamma|} - T_c^{|\gamma|}$, the first hitting time of the level $c+\varepsilon$ from $|\gamma|$, starting from c . Thus, we define: $\rho_u \equiv |\gamma_u|$, starting also from c . Thus, $\rho_u = c + \delta_u + L_u$, where $(\delta_s, s \geq 0)$ is a BM and $(L_s, s \geq 0)$ is the local time of ρ at 0. Thus:

$$\begin{aligned} \tilde{T}_{c,c+\varepsilon}^{|\gamma|} &= \inf\{u \geq 0 : \rho_u = c + \varepsilon\} \equiv \inf\{u \geq 0 : \delta_u + L_u = \varepsilon\} \\ &\stackrel{u=\varepsilon^2 v}{=} \varepsilon^2 \inf\left\{v \geq 0 : \frac{1}{\varepsilon}\delta_{v\varepsilon^2} + \frac{1}{\varepsilon}L_{v\varepsilon^2} = 1\right\}. \end{aligned} \quad (17)$$

From Skorokhod's Lemma [ReY99]:

$$L_u = \sup_{y \leq u} ((-c - \delta_y) \vee 0)$$

we deduce:

$$\frac{1}{\varepsilon}L_{v\varepsilon^2} = \sup_{y \leq v\varepsilon^2} ((-c - \delta_y) \vee 0) \stackrel{y=\varepsilon^2 \sigma}{=} \sup_{\sigma \leq v} \left(\left(-c - \varepsilon \frac{1}{\varepsilon} \delta_{\sigma\varepsilon^2} \right) \vee 0 \right) = 0. \quad (18)$$

Hence, with γ' denoting a new BM independent from γ , (16) writes:

$$T_{c+\varepsilon}^{|\theta|} - T_c^{|\theta|} = \exp\left(2\beta_{T_c^{|\gamma|}}\right) \int_0^{\varepsilon^2 T_1^{\gamma'}} dv \exp(2B_v). \quad (19)$$

Thus, dividing both sides of (19) by ε^2 and making $\varepsilon \rightarrow 0$, we obtain part *c*) of the Proposition. \blacksquare

Remark 3.2 *The asymptotic result *c*) in Proposition 3.1 may also be obtained in a straightforward manner from (16) by analytic computations. Indeed, using the Laplace transform of the first hitting time of a fixed level by the absolute value of a linear Brownian motion $E\left[e^{-\frac{\lambda^2}{2}T_b^{|\gamma|}}\right] = \frac{1}{\cosh(\lambda b)}$ (see e.g. Proposition 3.7, p 71 in Revuz and Yor [ReY99]), we have that for $0 < c < b$, and $\lambda \geq 0$:*

$$E\left[e^{-\frac{\lambda^2}{2}(T_b^{|\gamma|} - T_c^{|\gamma|})}\right] = \frac{\cosh(\lambda c)}{\cosh(\lambda b)} \quad (20)$$

Using now $b = c + \varepsilon$, for every $\varepsilon > 0$, the latter equals:

$$\frac{\cosh\left(\frac{\lambda c}{\varepsilon}\right)}{\cosh\left(\frac{\lambda}{\varepsilon}(c + \varepsilon)\right)} \xrightarrow{\varepsilon \rightarrow 0} e^{-\lambda}.$$

The result follows now by remarking that $e^{-\lambda}$ is the Laplace transform (for the argument $\lambda^2/2$) of the first hitting time of 1 by a linear Brownian motion γ' , independent from γ .

3.2 Generalizations

Obviously we can obtain several variants of Proposition 3.1, by studying $T_{-bc,ac}^\theta$, $0 < a, b \leq \infty$, for $c \rightarrow 0$ or $c \rightarrow \infty$, and a, b fixed. We define $T_{-d,c}^\gamma \equiv \inf\{t : \gamma_t \notin (-d, c)\}$ and we have:

- $\frac{1}{c^2} T_{-bc,ac}^\theta \xrightarrow[c \rightarrow 0]{(law)} T_{-b,a}^\gamma$.
- $\frac{1}{c} \log(T_{-bc,ac}^\theta) \xrightarrow[c \rightarrow \infty]{(law)} 2|\beta|_{T_{-b,a}^\gamma}$.

In particular, we can take $b = \infty$, hence:

Corollary 3.3 a) For $c \rightarrow 0$, we have:

$$\frac{1}{c^2} T_{ac}^\theta \xrightarrow[c \rightarrow 0]{(law)} T_a^\gamma. \quad (21)$$

b) For $c \rightarrow \infty$, we have:

$$\frac{1}{c} \log(T_{ac}^\theta) \xrightarrow[c \rightarrow \infty]{(law)} 2|\beta|_{T_a^\gamma} \stackrel{(law)}{=} 2|C_a|, \quad (22)$$

where $(C_a, a \geq 0)$ is a standard Cauchy process.

Remark 3.4 (Yet another proof of Spitzer's Theorem)

Taking $a = 1$, from Corollary 3.3(b), we can obtain yet another proof of Spitzer's celebrated asymptotic Theorem stated in (8). Indeed, (22) can be equivalently stated as:

$$P(\log T_c^\theta < cx) \xrightarrow[c \rightarrow \infty]{(law)} P(2|C_1| < x). \quad (23)$$

Now, the LHS of (23) equals:

$$\begin{aligned} P(\log T_c^\theta < cx) &\equiv P(T_c^\theta < \exp(cx)) \equiv P\left(\sup_{u \leq \exp(cx)} \theta_u > c\right) \\ &= P(|\theta_{\exp(cx)}| > c) = P\left(|\theta_t| > \frac{\log t}{x}\right), \end{aligned} \quad (24)$$

with $t = \exp(cx)$. Thus, because $|C_1| \stackrel{(law)}{=} |C_1|^{-1}$, (23) now writes:

$$\text{for every } x > 0 \text{ given, } P\left(|\theta_t| > \frac{\log t}{x}\right) \xrightarrow[t \rightarrow \infty]{(law)} P\left(|C_1| > \frac{2}{x}\right), \quad (25)$$

which yields precisely Spitzer's Theorem (8).

3.3 Speed of convergence

We can easily improve upon Proposition 3.1 by studying the speed of convergence of the distribution of $\frac{1}{c^2} T_c^{|\theta|}$ towards that of $T_1^{|\gamma|}$, i.e.:

Proposition 3.5 *For any function $\varphi \in \mathcal{C}^2$, with compact support,*

$$\frac{1}{c^2} \left(E \left[\varphi \left(\frac{1}{c^2} T_c^{|\theta|} \right) \right] - E \left[\varphi \left(T_1^{|\gamma|} \right) \right] \right) \xrightarrow{c \rightarrow 0} E \left[\varphi' \left(T_1^{|\gamma|} \right) \left(T_1^{|\gamma|} \right)^2 + \frac{2}{3} \varphi'' \left(T_1^{|\gamma|} \right) \left(T_1^{|\gamma|} \right)^3 \right]. \quad (26)$$

Proof of Proposition 3.5:

We develop $\exp(2c\beta_v)$, for $c \rightarrow 0$, up to the second order term, i.e.:

$$e^{2c\beta_v} = 1 + 2c\beta_v + 2c^2\beta_v^2 + \dots$$

More precisely, we develop up to the second order term, and we obtain:

$$\begin{aligned} E \left[\varphi \left(\frac{1}{c^2} T_c^{|\theta|} \right) \right] &= E \left[\varphi \left(\int_0^{T_1^{|\gamma|}} dv \exp(2c\beta_v) \right) \right] \\ &= E \left[\varphi \left(T_1^{|\gamma|} \right) + \varphi' \left(T_1^{|\gamma|} \right) \int_0^{T_1^{|\gamma|}} (2c\beta_v + 2c^2\beta_v^2) dv \right] \\ &\quad + \frac{1}{2} E \left[\varphi'' \left(T_1^{|\gamma|} \right) 4c^2 \left(\int_0^{T_1^{|\gamma|}} \beta_v dv \right)^2 \right] + c^2 o(c). \end{aligned}$$

We then remark that $E \left[\int_0^t \beta_v dv \right] = 0$, $E \left[\int_0^t \beta_v^2 dv \right] = t^2/2$ and $E \left[\left(\int_0^t \beta_v dv \right)^2 \right] = t^3/3$, thus we obtain (26). ■

4 Checks via Bougerol's identity

So far, we have not made use of Bougerol's identity (4), which helps us to characterize the distribution of $T_c^{|\theta|}$ [Vak11]. In this Subsection, we verify that writing the Gauss-Laplace transform in (6) as:

$$E \left[\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\frac{1}{c^2} T_c^{|\theta|}}} \exp \left(-\frac{xc^2}{2T_c^{|\theta|}} \right) \right] = \frac{1}{\sqrt{1+xc^2}} \varphi_m(xc^2), \quad (27)$$

with $m = \pi/(2c)$, we find asymptotically for $c \rightarrow 0$ the Gauss-Laplace transform of $T_1^{|\gamma|}$. Indeed, from (27), for $c \rightarrow 0$, we obtain:

$$E \left[\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{T_1^{|\gamma|}}} \exp \left(-\frac{x}{2T_1^{|\gamma|}} \right) \right] = \lim_{c \rightarrow 0} \frac{2}{\left(\sqrt{1+xc^2} + \sqrt{xc^2} \right)^{\pi/2c} + \left(\sqrt{1+xc^2} - \sqrt{xc^2} \right)^{\pi/2c}}. \quad (28)$$

Let us now study:

$$\begin{aligned} (\sqrt{1+xc^2} + \sqrt{xc^2})^{\pi/2c} &= \exp\left(\frac{\pi}{2c} \log\left[1 + (\sqrt{1+xc^2} - 1) + \sqrt{xc^2}\right]\right) \\ &\sim \exp\left(\frac{\pi}{2c} \left[c\sqrt{x} + \frac{xc^2}{2}\right]\right) \xrightarrow{c \rightarrow 0} \exp\left(\frac{\pi\sqrt{x}}{2}\right). \end{aligned}$$

A similar calculation finally gives:

$$E \left[\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{T_1^{|\gamma|}}} \exp\left(-\frac{x}{2T_1^{|\gamma|}}\right) \right] = \frac{1}{\cosh\left(\frac{\pi}{2}\sqrt{x}\right)}, \quad (29)$$

a result which is in agreement with the law of $\beta_{T_1^{|\gamma|}}$, whose density is:

$$E \left[\frac{1}{\sqrt{2\pi T_1^{|\gamma|}}} \exp\left(-\frac{y^2}{2T_1^{|\gamma|}}\right) \right] = \frac{1}{2 \cosh\left(\frac{\pi}{2}y\right)}. \quad (30)$$

Indeed, the law of $\beta_{T_c^{|\gamma|}}$ may be obtained from its characteristic function which is given by [ReY99], page 73:

$$E \left[\exp(i\lambda\beta_{T_c^{|\gamma|}}) \right] = \frac{1}{\cosh(\lambda c)}.$$

It is well known that [Lev80, BiY87]:

$$\begin{aligned} E \left[\exp(i\lambda\beta_{T_c^{|\gamma|}}) \right] &= \frac{1}{\cosh(\lambda c)} = \frac{1}{\cosh(\pi\lambda\frac{c}{\pi})} = \int_{-\infty}^{\infty} e^{i(\frac{\lambda c}{\pi})y} \frac{1}{2\pi} \frac{1}{\cosh(\frac{y}{2})} dy \\ &\stackrel{x=\frac{cy}{\pi}}{=} \int_{-\infty}^{\infty} e^{i\lambda x} \frac{1}{2\pi} \frac{\frac{\pi}{c}}{\cosh(\frac{x\pi}{2c})} dx = \int_{-\infty}^{\infty} e^{i\lambda x} \frac{1}{2c} \frac{1}{\cosh(\frac{x\pi}{2c})} dx. \end{aligned} \quad (31)$$

So, the density $h_{-c,c}$ of $\beta_{T_c^{|\gamma|}}$ is:

$$h_{-c,c}(y) = \left(\frac{1}{2c}\right) \frac{1}{\cosh(\frac{y\pi}{2c})} = \left(\frac{1}{c}\right) \frac{1}{e^{\frac{y\pi}{2c}} + e^{-\frac{y\pi}{2c}}},$$

and for $c = 1$, we obtain (30).

We recall from Remark 3.2 that (see also [PiY03], where further results concerning the infinitely divisible distributions generated by some Lévy processes associated with the hyperbolic functions \cosh , \sinh and \tanh can also be found):

$$E \left[\exp\left(-\frac{\lambda^2}{2}T_c^{|\gamma|}\right) \right] = \frac{1}{\cosh(\lambda c)}, \quad (32)$$

thus, for $c = 1$ and $\lambda = \frac{\pi}{2}\sqrt{x}$, (29) now writes:

$$E \left[\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{T_1^{|\gamma|}}} \exp\left(-\frac{x}{2T_1^{|\gamma|}}\right) \right] = E \left[\exp\left(-\frac{x\pi^2}{8}T_1^{|\gamma|}\right) \right], \quad (33)$$

a result which gives a probabilistic proof of the reciprocal relation in [BPY01] (using the notation of this article, Table 1, p.442):

$$f_{C_1}(x) = \left(\frac{2}{\pi x}\right)^{3/2} f_{C_1}\left(\frac{4}{\pi^2 x}\right).$$

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