A Central Limit Theorem for a sequence of Brownian motions in the unit sphere in Rn

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A Central Limit Theorem for a sequence of Brownian motions in the unit sphere in $\mathbb{R}^n$

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Abstract

We use a Stochastic Differential Equation satisfied by Brownian motion taking values in the unit sphere $S_{n-1} \subset \mathbb{R}^n$ and we obtain a Central Limit Theorem for a sequence of such Brownian motions. We also generalize the results to the case of the n-dimensional Ornstein-Uhlenbeck processes.

Key words: Central Limit Theorem, Brownian motion in the unit sphere in $\mathbb{R}^n$, Ornstein-Uhlenbeck processes.

1 Introduction

This paper may be regarded as an extension to higher dimensions of the 2-dimensional study made in Vakeroudis et al. (2011).
We now consider a sequence of Brownian motions $\left(\Theta_t^{(k)}, t \geq 0\right)$, $k \in \mathbb{N}$ taking values in the unit sphere $S_{n-1}(\subset \mathbb{R}^n)$, all starting from the same point on the sphere. In Section 2, we introduce a general representation of $\Theta^{(k)}$ in terms of a Stochastic Differential Equation. Using this representation, we describe in detail in Section 3 the limit in law, as $K \to \infty$, for the renormalized sum:

$$Z_t^K \equiv \frac{1}{\sqrt{K}} \sum_{k=1}^{K} \left(\Theta_t^{(k)} - E\left[\Theta_t^{(k)}\right]\right)$$

of these processes, indexed by $t \geq 0$, and taking values in $\mathbb{R}^n$. Of course, one could invoke the classical Central Limit Theorem (CLT), at least for the finite dimensional marginals of $\left(Z_t^K, t \geq 0\right)$, as $K \to \infty$. However, with the help of stochastic calculus, there is much

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more to say about the description of the asymptotics. Finally, in this Section, we remark that the CLT can be generalized to Ornstein-Uhlenbeck processes taking values in the unit sphere $S_{n-1}(\subset \mathbb{R}^n)$. Three technical points are gathered in an Appendix.

Further extensions may also be obtained, by following e.g. Itô (1983) or Ochi (1985) and studying for which class of functions $f(\Theta)$ we can obtain a functional CLT such as (12) (see below) for $f(\Theta_t)$, instead of the unique function $f_*(\Theta) = \Theta$ which we study here.

2 A presentation of Brownian motion in the sphere S_{n-1}

As remarked in Stroock (1971) and Yor (1984) (eq. (4.j), p.34), Brownian motion $(\Theta_t, t \geq 0)$ in the unit sphere $S_{n-1} \subset \mathbb{R}^n$ may be viewed as the solution of a Stochastic Differential Equation:

$$\Theta_t = \Theta_0 + \int_0^t \sigma^{0,1}(\Theta_s) \cdot dB_s - \frac{n-1}{2} \int_0^t ds\ \Theta_s\ . \tag{1}$$

In (1), $B_t \equiv \left(B_t^{(i)}, i \leq n\right)$, $t \geq 0$, denotes a n-dimensional Brownian motion starting from $a \neq 0$, while $(\sigma^{0,1}(x), x \in S_{n-1})$ denotes the family of $n \times n$ matrices (see e.g. Krylov (1980)), defined by:

$$\sigma^{0,1}(x) = (\delta_{i,j} - x_i x_j)_{i,j \leq n}\,, \quad (x \in S_{n-1}) \tag{2}$$

and/or characterized by:

$$\sigma^{0,1}(x) \cdot x = 0, \quad \text{and} \quad \sigma^{0,1}(x) \cdot y = y, \quad \text{if } y \cdot x = 0. \tag{3}$$

Note that $\sigma^{0,1}(x)$ is symmetric and satisfies: $\sigma^{0,1}(x)\sigma^{0,1}(x) = \sigma^{0,1}(x)$.
Thus, from (3), we deduce that:

$$\sigma^{0,1}(x)m = m - (m \cdot x)\, x,\ m \in \mathbb{R}^n, \tag{4}$$

$$\left(\sigma^{0,1}(x)m\right) \cdot \left(\sigma^{0,1}(x)m'\right) = (m \cdot m') - (m \cdot x)\,(m' \cdot x)\,, m, m' \in \mathbb{R}^n. \tag{5}$$

3 A Central Limit Theorem for a sequence of Brownian motions in the sphere $S_{n-1}(\subset \mathbb{R}^n)$

Let $\Theta^{(1)}, \ldots, \Theta^{(k)}, \ldots$ be a sequence of such independent and identically distributed Brownian motions in the sphere S_{n-1}. We aim for a Central Limit Theorem concerning:

$$Z_t^K \equiv \frac{1}{\sqrt{K}} \sum_{k=1}^K \left(\Theta_t^{(k)} - E\left[\Theta_t^{(k)}\right]\right). \tag{6}$$

Adding K equations of the kind of (1) term by term, for $\left(\Theta_t^{(k)}, k \leq K\right)$, it is immediate that:

$$Z_t^K = M_t^K - \frac{n-1}{2} \int_0^t ds\ Z_s^K, \tag{7}$$

with

$$M_t^K = \frac{1}{\sqrt{K}} \sum_{k=1}^{K} \int_0^t \sigma^{0,1}(\Theta_s^{(k)}) \cdot dB_s^{(k)}. \tag{8}$$

Thus, from (7), we obtain:

$$Z_t^K = \exp\left(\frac{-(n-1)t}{2}\right) \int_0^t \exp\left(\frac{(n-1)s}{2}\right) dM_s^K. \tag{9}$$

Now, clearly, the Central Limit Theorem for $\left(Z_t^K\right)$, $K \to \infty$, which we are seeking, will follow from the limit in law of the martingales $\left(M_t^K, t \geq 0\right)$, as $K \to \infty$. We now state both limit results in the following:

Theorem 3.1 *a) The sequence of martingales* $\left(M_t^K, t \geq 0\right)$ *converges in law, as* $K \to \infty$*, towards:*

$$M_t^{(\infty)} = \sqrt{1 - \frac{1}{n}} \left\{ \Theta(0) \int_0^t \sqrt{1 - e^{-ns}} d\beta_s + \int_0^t \sqrt{1 + \frac{e^{-ns}}{n-1}} dB'_s \right\}, \tag{10}$$

where $(\beta_s, s \geq 0)$ *is a 1-dimensional BM and* $(B'_s, s \geq 0)$ *is a* $(n-1)-$*dimensional BM taking values in the hyperplane which is orthogonal to* $\Theta(0)$*, and* B' *is independent of* β*.*
b) Consequently, $\left(Z_t^K, t \geq 0\right)$ *converges in law, as* $K \to \infty$*, towards:*

$$Z_t^{(\infty)} = \exp\left(-\frac{n-1}{2} t\right) \int_0^t \exp\left(\frac{n-1}{2} s\right) dM_s^{(\infty)}. \tag{11}$$

Proof of Theorem 3.1: Using the Law of Large Numbers, it is not difficult to show that[††]:

$$\left(M_t^K, t \geq 0\right) \underset{K\to\infty}{\overset{(law)}{\longrightarrow}} \int_0^t \sqrt{E\left[\sigma^{0,1}(\Theta_s^{(1)})\right]} \cdot dB_s^{(1)} \equiv \left(M_t^{\infty}, t \geq 0\right), \tag{12}$$

where $Q(s) \equiv E\left[\sigma^{0,1}(\Theta_s^{(1)})\right]$ is a deterministic matrix, depending on s. The RHS of (12) is a centered Gaussian martingale in $\mathbb{R}^n$. Before computing the square root involved in (12), we shall first calculate (see (2) for the definition of $\sigma^{0,1}$):

$$E\left[\sigma^{0,1}(\Theta_s^{(1)})\right] = \left(\delta_{i,j} - E\left[\Theta_i(s)\Theta_j(s)\right]\right)_{i,j\leq n}. \tag{13}$$

In order to calculate $E\left[\Theta_s^{(i)}\Theta_s^{(j)}\right]$ as "naturally" as possible, we consider two generic vectors m and m' in $\mathbb{R}^n$, and we compute:

$$\varphi_{m,m'}(t) \equiv E\left[\left(m \cdot \Theta_t\right)\left(m' \cdot \Theta_t\right)\right]. \tag{14}$$

[††]In the Appendix A.1, a more general result, concerning $\frac{1}{\sqrt{K}} \int_0^t \sum_{k=1}^K H_s^{(k)} \cdot dB_s^{(k)}$ is presented, where $(B^{(k)}, k = 1, \ldots, K)$ are K independent BMs and $(H^{(k)}, B^{(k)})$ are $k \leq K$ iid random vectors.

Using (1) and the (special) properties of the matrices $\{\sigma^{0,1}(x)\}$, we easily deduce from Itô's formula, that:

$$\begin{aligned} E\left[(m\cdot\Theta_t)\,(m'\cdot\Theta_t)\right] &= (m\cdot\Theta_0)\,(m'\cdot\Theta_0)-(n-1)\int_0^t ds\; E\left[(m\cdot\Theta_s)\,(m'\cdot\Theta_s)\right] \\ &\quad+\int_0^t ds\; E\left[\left(\sigma^{0,1}(\Theta_s)m\right)\cdot\left(\sigma^{0,1}(\Theta_s)m'\right)\right]. \end{aligned} \tag{15}$$

Using (5), (15) simplifies as:

$$\begin{aligned} E\left[(m\cdot\Theta_t)\,(m'\cdot\Theta_t)\right] &= (m\cdot\Theta_0)\,(m'\cdot\Theta_0)-(n-1)\int_0^t ds\; E\left[(m\cdot\Theta_s)\,(m'\cdot\Theta_s)\right] \\ &\quad+\int_0^t ds\;\left(m\cdot m'-E\left[(m\cdot\Theta_s)\,(m'\cdot\Theta_s)\right]\right) \\ &= (m\cdot\Theta_0)\,(m'\cdot\Theta_0)+(m\cdot m')\,t-n\int_0^t ds\; E\left[(m\cdot\Theta_s)\,(m'\cdot\Theta_s)\right]. \end{aligned} \tag{16}$$

Consequently, the function $\varphi_{m,m'}(t)=E\left[(m\cdot\Theta_t)\,(m'\cdot\Theta_t)\right]$ is the solution of a first order linear differential equation, hence:

$$E\left[(m\cdot\Theta_t)\,(m'\cdot\Theta_t)\right]=e^{-nt}\left\{(m\cdot\Theta_0)\,(m'\cdot\Theta_0)+(m\cdot m')\int_0^t e^{ns}ds\right\}. \tag{17}$$

Now, taking $m=e_i$ and $m'=e_j$, where $(e_k;k\leq n)$ is the canonical basis of $\mathbb{R}^n$, the matrix $Q(s)$ has elements:

$$\text{for } i\neq j,\quad (Q(s))_{i,j}=-E\left[\Theta_i(s)\Theta_j(s)\right]=-\Theta_i(0)\Theta_j(0)e^{-ns}, \tag{18}$$

$$\begin{aligned} \text{for } i=j,\quad (Q(s))_{i,i} &= 1-E\left[\Theta_i(s)\Theta_i(s)\right] \\ &= 1-\left\{(\Theta_i(0))^2 e^{-ns}+e^{-ns}\left(\frac{e^{ns}-1}{n}\right)\right\} \\ &= \left(1-\frac{1}{n}\right)+e^{-ns}\left(\frac{1}{n}-(\Theta_i(0))^2\right). \end{aligned} \tag{19}$$

Finally:

$$\begin{aligned} Q(s) &= \left(1-\frac{1}{n}\right)Id+e^{-ns}\left(\frac{1}{n}\delta_{ij}-\Theta_i(0)\Theta_j(0)\right)_{i,j\leq n} \\ &\equiv \left(1-\frac{1}{n}\left(1-e^{-ns}\right)\right)Id-e^{-ns}\left(\Theta_i(0)\Theta_j(0)\right)_{i,j\leq n}. \end{aligned} \tag{20}$$

Using (20) in the RHS of (12), we obtain:

$$M_t^{(\infty)}\equiv\int_0^t\sqrt{Q(s)}\,dB_s. \tag{21}$$

Now $\sqrt{Q(s)} \equiv \Lambda(s)$, where (for the explicit calculation, see Appendix A.2):

$$\begin{aligned}\Lambda(s) &\equiv \sqrt{1-\frac{1}{n}}\sqrt{1-e^{-ns}}Id + \sqrt{1-\frac{1}{n}}\left(\sqrt{1+\frac{e^{-ns}}{n-1}} - \sqrt{1-e^{-ns}}\right)\sigma^{0,1}\left(\Theta(0)\right) \\ &= \sqrt{1-\frac{1}{n}}\left\{\sqrt{1-e^{-ns}}Id + \left(\sqrt{1+\frac{e^{-ns}}{n-1}} - \sqrt{1-e^{-ns}}\right)\sigma^{0,1}\left(\Theta(0)\right)\right\}. \quad (22)\end{aligned}$$

Thus, (21) now writes:

$$\begin{aligned}M_t^{(\infty)} &\equiv \int_0^t \Lambda(s)\, dB_s \\ &= \sqrt{1-\frac{1}{n}}\left\{\int_0^t \sqrt{1-e^{-ns}}dB_s + \int_0^t \left[\sqrt{1+\frac{e^{-ns}}{n-1}} - \sqrt{1-e^{-ns}}\right]\sigma^{0,1}\left(\Theta(0)\right) dB_s\right\}. \quad (23)\end{aligned}$$

We remark here that, with $\beta_s \equiv \Theta(0)\cdot B_s$,

$$B'_s = B_s - \Theta(0)\beta_s \equiv \sigma^{0,1}\left(\Theta(0)\right)B_s \quad (24)$$

is a $(n-1)$-dimensional BM taking values in the hyperplane which is orthogonal to $\Theta(0)$. Thus, from (23) we deduce (10).
From (9), letting $K \to \infty$, we obtain (11). ∎

Moreover, changing the variables $s = t-u$ and using the dominated convergence Theorem, we have:

$$\begin{aligned}Z_t^{(\infty)} &\overset{(law)}{=} \sqrt{1-\frac{1}{n}}\exp\left(-\frac{n-1}{2}t\right)\int_0^t \exp\left(\frac{n-1}{2}s\right) \times \\ &\qquad \times \left\{\Theta(0)\sqrt{1-e^{-ns}}d\beta_s + \sqrt{1+\frac{e^{-ns}}{n-1}}dB'_s\right\} \quad (25) \\ &\underset{(law)}{\overset{s=t-u}{=}} \sqrt{1-\frac{1}{n}}\int_0^t \exp\left(-\frac{n-1}{2}u\right)\left\{\sqrt{1-e^{-n(t-u)}}\Theta(0)d\beta_u + \sqrt{1+\frac{e^{-n(t-u)}}{n-1}}dB'_u\right\} \\ &\overset{t\to\infty}{\longrightarrow} \sqrt{1-\frac{1}{n}}\int_0^\infty \exp\left(-\frac{n-1}{2}u\right)\Theta(0)d\beta_u + \sqrt{1-\frac{1}{n}}\int_0^\infty \exp\left(-\frac{n-1}{2}u\right)dB'_u. \quad (26)\end{aligned}$$

Proposition 3.2 *The following asymptotic results hold:*
a)

$$Z_t^{(\infty)} \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} Z_\infty^{(\infty)}, \quad (27)$$

where:

$$Z_\infty^{(\infty)} \equiv \sqrt{1-\frac{1}{n}}\int_0^\infty \exp\left(-\frac{n-1}{2}u\right)dB_u\ . \quad (28)$$

b)

$$Z_t^{(\infty)} - \exp\left(-\frac{n-1}{2}t\right)\int_0^t \sqrt{1-\frac{1}{n}}\exp\left(\frac{n-1}{2}s\right)dB_s \xrightarrow[t\to\infty]{L^2} 0. \tag{29}$$

Part *a*) of Proposition 3.2 follows from the previous calculations, using (24). In order to prove part *b*), it suffices to use the expression (25) and the following Proposition, which reinforces the convergence in L^2 result in (29).

Proposition 3.3 *As $t \to \infty$, the Gaussian martingales:*

$$\left(G_t^{(0)}\Theta(0), t \geq 0\right) \equiv \Theta(0)\left(\int_0^t \sqrt{1-e^{-ns}}e^{\frac{n-1}{2}s}d\beta_s - \int_0^t e^{\frac{n-1}{2}s}d\beta_s, t\geq 0\right), \tag{30}$$

and

$$(G'_t, t\geq 0) \equiv \left(\int_0^t \sqrt{1+\frac{e^{-ns}}{n-1}}e^{\frac{n-1}{2}s}dB'_s - \int_0^t e^{\frac{n-1}{2}s}\, dB'_s, t\geq 0\right) \tag{31}$$

converge a.s. and in L^2, and the limit variables are Gaussian, with variances, respectively: $\left(\frac{\sqrt{\pi}\Gamma(-1+\frac{1}{n})}{n\Gamma(\frac{1}{2}+\frac{1}{n})} - \frac{n+1}{n-1}\right)$, *and* $\frac{2\,{}_2F_1\left(-\frac{1}{2},-1+\frac{1}{n};\frac{1}{n};\frac{1}{1-n}\right)-1}{n-1}$.

Proof of Proposition 3.3:
a) The increasing process of the real-valued Gaussian martingale $G_t^{(0)}$ is:

$$\int_0^t e^{(n-1)s}\left(\sqrt{1-e^{-ns}}-1\right)^2 ds,$$

which converges, as $t\to\infty$; thus:

$$G_t^{(0)} \xrightarrow[t\to\infty]{} \int_0^\infty \left(\sqrt{1-e^{-ns}}e^{\frac{n-1}{2}s} - e^{\frac{n-1}{2}s}\right)d\beta_s,$$

where the convergence holds both a.s. and in every L^p. Of course, the limit variable is Gaussian and its variance is given by (we change the variables $u = e^{-ns}$ and $B(a,b)$ denotes the Beta function with arguments a and b**):

$$\begin{aligned}
&\int_0^\infty ds\, e^{(n-1)s}\left(\sqrt{1-e^{-ns}}-1\right)^2 = \frac{1}{n}\int_0^1 du\, u^{-2+\frac{1}{n}}\left(\sqrt{1-u}-1\right)^2 \\
=\ & \frac{1}{n}\left[\int_0^1 du\, u^{-2+\frac{1}{n}}\left((1-u)-2\sqrt{1-u}+1\right)\right] \\
=\ & \frac{1}{n}\left\{B\left(-1+\frac{1}{n},2\right) - 2B\left(-1+\frac{1}{n},\frac{3}{2}\right) - \frac{n}{n-1}\right\} \\
=\ & \frac{\sqrt{\pi}\Gamma(-1+\frac{1}{n})}{n\Gamma(\frac{1}{2}+\frac{1}{n})} - \frac{n+1}{n-1}\ .
\end{aligned}$$

**We recall that $(\Gamma(x), x\geq 0)$ denotes the Gamma function and $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

To be rigorous, the integral $\int_0^1 du\, u^{-\alpha}\left(\sqrt{1-u}-1\right)^2$, which is well defined for $0<\alpha<1$, can be extended analytically for any complex α with $\mathrm{Re}(\alpha)<3$.
$b)$ Likewise, the "increasing process" of the vector-valued Gaussian martingale G'_t is:

$$\int_0^t e^{(n-1)s}\left(\sqrt{1+\frac{e^{-ns}}{n-1}}-1\right)^2 ds$$

which also converges as $t\to\infty$. The limit variable:

$$\int_0^\infty \left(\sqrt{1+\frac{e^{-ns}}{n-1}}e^{\frac{n-1}{2}s}-e^{\frac{n-1}{2}s}\right) dB'_s,$$

is also Gaussian and, by repeating the previous calculation, we easily compute its variance.

■

Proof of Proposition 3.2:
From Proposition 3.3, by multiplying both processes by $e^{\left(-\frac{(n-1)t}{2}\right)}\sqrt{1-\frac{1}{n}}$, we obtain (29).

■

Remark 3.4 ***(The Ornstein-Uhlenbeck case)***
In fact, for every process satisfying:

$$dZ_s = d\mathbb{B}_s + h(|Z_s|)Z_s ds, \tag{32}$$

where $(\mathbb{B}_t, t\geq 0)$ *is a* n*-dimensional Brownian motion (BM) and* $h:\mathbb{R}_+\to\mathbb{R}$ *is a bounded function, there is a CLT of the kind of Theorem 3.1. See Appendix A.3 for the proof.*

A Appendix

A.1 Generalization for a class of symmetric matrices

For K independent Brownian motions, and a class of symmetric matrices $H^{(k)}$ such that $(H^{(k)}, B^{(k)})_{k\leq K}$ are iid, we have:

$$\tilde{M}_t^{(K)} \equiv \frac{1}{\sqrt{K}}\int_0^t \sum_{k=1}^K H_s^{(k)}\cdot dB_s^{(k)} \underset{K\to\infty}{\overset{(law)}{\longrightarrow}} \int_0^t h_s dB_s, \tag{33}$$

with h_s a deterministic symmetric positive definite matrix and $(B_t, t\geq 0)$ a n-dimensional BM.
Indeed, using m a generic vector in $\mathbb{R}^n$, we have:

$$\begin{aligned} m\cdot\tilde{M}_t^{(K)} &\equiv m\cdot\frac{1}{\sqrt{K}}\int_0^t\sum_{k=1}^K H_s^{(k)}\cdot dB_s^{(k)} = \frac{1}{\sqrt{K}}\sum_{k=1}^K\int_0^t\left(H_s^{(k)}m\right)\cdot dB_s^{(k)} \\ &\underset{K\to\infty}{\overset{(law)}{\longrightarrow}} \mathcal{N}\left(0;\int_0^t dsE\left[|H_s^1\cdot m|^2\right]\right), \end{aligned} \tag{34}$$

and for the variance, we have:

$$\int_0^t dsE\left[|H_s^1 \cdot m|^2\right] = m \cdot \int_0^t dsE\left[H_s^1 \tilde{H}_s^1\right] m, \tag{35}$$

and

$$E\left[H_s^1\right] \equiv h_s^2. \tag{36}$$

Remark A.1 *In our case, we have:* $H_s^k \tilde{H}_s^k = H_s^k$.

A.2 Square root of Q_s

$$\begin{aligned} Q(s) &= \left(1 - \frac{1}{n}\right) Id + e^{-ns}\left(\frac{1}{n}\delta_{ij} - \Theta_i(0)\Theta_j(0)\right)_{(i,j\leq n)} \\ &\equiv \left(1 - \frac{1}{n}\right)\left(1 - e^{-ns}\right) Id + e^{-ns} \cdot \sigma^{0,1}\left(\Theta(0)\right) \ . \end{aligned} \tag{37}$$

We are searching for $a(s)$ and $b(s)$ such that:

$$\left(a(s)I + b(s)\sigma^{0,1}\left(\Theta(0)\right)\right)^2 = Q(s),$$

or equivalently:

$$\left(a(s)\right)^2 I + 2a(s)b(s)\sigma^{0,1}\left(\Theta(0)\right) + \left(b(s)\right)^2 \sigma^{0,1}\left(\Theta(0)\right) = Q(s).$$

We compare with (37) and we find:

$$\left(a(s)\right)^2 = \left(1 - \frac{1}{n}\right)\left(1 - e^{-ns}\right) \quad ; \quad 2a(s)b(s) + \left(b(s)\right)^2 = e^{-ns}.$$

Solving this system of equations, we easily obtain:

$$\begin{cases} a(s) = \sqrt{1 - \frac{1}{n}}\sqrt{1 - e^{-ns}} \\ b(s) = \sqrt{1 - \frac{1}{n}}\left(\sqrt{1 - \frac{e^{-ns}}{n-1}} - \sqrt{1 - e^{-ns}}\right). \end{cases} \tag{38}$$

A.3 The Ornstein-Uhlenbeck case

We consider the n-dimensional Ornstein-Uhlenbeck (OU) process:

$$Z_t = z_0 + \mathbb{B}_t - \lambda \int_0^t Z_s ds, \tag{39}$$

where $(\mathbb{B}_t, t \geq 0)$ is a n-dimensional Brownian motion (BM), $z_0 \in \mathbb{R}^n$ and $\lambda \geq 0$.

Proposition A.2 *The Ornstein-Uhlenbeck (OU) process* $\left(\tilde{\Theta}_t, t \geq 0\right)$ *in the unit sphere* S_{n-1} *is the solution of the Stochastic Differential Equation*

$$\Theta_t^Z = \Theta_0^Z + \int_0^t \sigma^{0,1}(\Theta_s^Z) \cdot d\hat{\mathbb{B}}_s - \left(\frac{n-1}{2} + \lambda\right) \int_0^t ds\ \Theta_s^Z, \tag{40}$$

where $\left(\hat{\mathbb{B}}_t, t \geq 0\right)$ *is a* n*-dimensional BM.*

Proof of Proposition A.2:
We shall study $\tilde{\varphi}_t \equiv \frac{Z_t}{|Z_t|}$. We remark that the Jacobi matrix and the Hessian matrix associated respectively to the functions $\Phi(x) \equiv \frac{x}{|x|}$, $(x \neq 0)$ and $g(x) \equiv |x|$ are given by:

$$\left(\frac{\partial}{\partial x_j}\Phi_i(x)\right) = \frac{1}{|x|}\sigma^{0,1}(x) \quad ; \quad \left(\frac{\partial^2 g(x)}{\partial x_i \partial x_j}\right) = \frac{1}{|x|}\sigma^{0,1}(x).$$

Hence, using (39), $\tilde{\varphi}$ satisfies the following Stochastic Differential Equation

$$\tilde{\varphi}_t = \tilde{\varphi}_0 + \int_0^t \frac{1}{|Z_s|}\sigma^{0,1}(\tilde{\varphi}_s)\cdot dZ_s - \frac{n-1}{2}\int_0^t \frac{ds}{|Z_s|^2}\tilde{\varphi}_s \tag{41}$$

$$= \tilde{\varphi}_0 + \int_0^t \frac{1}{|Z_s|}\sigma^{0,1}(\tilde{\varphi}_s)\cdot d\mathbb{B}_s - \int_0^t \left(\frac{n-1}{2}\frac{\tilde{\varphi}_s}{|Z_s|^2} + \frac{\lambda Z_s}{|Z_s|}\sigma^{0,1}(\tilde{\varphi}_s)\right) ds$$

$$= \tilde{\varphi}_0 + \int_0^t \frac{1}{|Z_s|}\sigma^{0,1}(\tilde{\varphi}_s)\cdot d\mathbb{B}_s - \int_0^t \tilde{\varphi}_s \left(\frac{n-1}{2}\frac{1}{|Z_s|^2} + \lambda\, \sigma^{0,1}(\tilde{\varphi}_s)\right) ds. \tag{42}$$

We can replace the BM $\mathbb{B}$ by another BM $\mathbb{B}^*$:

$$\mathbb{B}^*_t \equiv \int_0^t \left(\sigma^{0,1}(\tilde{\varphi}_s)\cdot d\mathbb{B}_s + \sigma^{1,0}(\tilde{\varphi}_s)\cdot dW_s\right),$$

where $(W_t, t \geq 0)$ is a BM independent from $\mathbb{B}$.
Thus, $\left(\gamma_t \equiv \int_0^t \frac{Z_s}{|Z_s|}\cdot d\mathbb{B}_s, t \geq 0\right)$ and $\mathbb{B}^*$ are two independent BMs, and from Knight's theorem (see e.g. Revuz and Yor (1999) and other references therein), γ is independent from the BM $\left(\hat{\mathbb{B}}_t, t \geq 0\right)$ obtained by changing the time scale of $\left(\int_0^t \frac{1}{|Z_s|} d\mathbb{B}^*_s\right)$ with the inverse of $\int_0^t \frac{ds}{|Z_s|^2}$. Finally, $\left(\Theta^Z_t, t \geq 0\right)$ may be obtained from $(\tilde{\varphi}_t, t \geq 0)$ by making the same change of time scale. ∎

Corollary A.3 *The angular part of a 2-dimensional Ornstein-Uhlenbeck process is equal to the angular part of a planar Brownian motion, considered under the time scale* $\alpha_t = \frac{e^{2\lambda t}-1}{2\lambda}$.

Remark A.4 *For further results concerning the case of a complex-valued OU process, see Vakeroudis (2011).*

Proof of Corollary A.3:
It follows easily from equation (39) for $n = 2$ by taking the angular part. For the new time scale, it suffices to remark that, with $< \cdot >$ denoting the quadratic variation of a martingale:

$$\alpha_t \equiv < \int_0^{\cdot} e^{\lambda s}\cdot d\mathbb{B}_s >_t = \int_0^t e^{2\lambda s} ds = \frac{e^{2\lambda t}-1}{2\lambda}.$$

∎

Remark A.5 *Proposition A.2 is easily generalized for every process of the kind:*

$$dZ_s = d\mathbb{B}_s + h(|Z_s|)Z_s ds, \tag{43}$$

for every bounded function $h : \mathbb{R}^n \to \mathbb{R}$.

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