DECIDABILITY OF THE HD0L ULTIMATE PERIODICITY PROBLEM

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ABSTRACT. In this paper we prove the decidability of the HD0L ultimate periodicity problem.

1. Introduction

1.1. **The HD0L ultimate periodicity problem.** In this paper we prove the decidability of the following problem :

Input: Two finite alphabets, a substitution $\sigma : A^* \to A^*$ prolongable on the letter a, a morphism $\phi : A^* \to B^*$.
Question: Do there exist two words u and v such that the morphic sequence $\phi(\sigma^\omega(a))$ is equal to uv^ω (*i.e.*, is ultimately periodic)?

We will refer to it as the *HD0L ultimate periodicity problem.*

Theorem 1. *The HD0L ultimate periodicity problem is decidable.*

This result was announced in [Durand preprint 2011]. While we were ending the redaction of this paper, I. Mitrofanov put on Arxiv [Mitrofanov preprint 2011] another solution of this problem.
This problem was open for about 30 years.
In 1986, positive answers were given independently for D0L systems (or purely substitutive sequences) in both [Harju and Linna 1986] and [Pansiot 1986], and, for automatic sequences (which are particular HD0L sequences) in [Honkala 1986]. Other proofs have been given for the D0L case in [Honkala 2008] and for automatic sequences in [Allouche, Rampersad and Shallit 2009].
Recently in [Durand preprint 2011] the primitive case has been solved.
In [Honkala and Rigo 2004] is given an equivalent statement of the HD0L ultimate periodicity problem in terms of recognizable sets of integers and abstract numeration systems. In fact, J. Honkala already gave a positive answer to this question in [Honkala 1986] but in the restricted case of the usual integer bases, *i.e.*, for k-automatic sequences or constant length substitutive sequences. Recently, in [Bell, Charlier, Fraenkel, and Rigo 2009], a positive answer has been given for a (large) class of numeration systems including for instance the Fibonacci numeration system.
Let us point out that the characterization of recognizable sets of integers for abstract numeration systems in terms of substitutions given in [Maes and Rigo 2002] (see also [Lecomte and Rigo 2010]), together with Theorem 1, provides a decision procedure to test whether a recognizable set of integers in some abstract numeration system is a finite union of arithmetic progressions.

1.2. **Organization of the paper.** In Section 2 are the classical definitions. The proof of the HD0L ultimate periodicity problem is in Section 3. It could be sketched as follows. First we recall some primitivity arguments about matrices and substitutions. The "best or easiest situation" is when we deal with growing substitutions and codings (letter-to-letter morphisms). It is known that we can always consider we are working with codings (see [Cobham 1968, Pansiot 1983, Allouche and Shallit 2003, Cassaigne and Nicolas 2003, Honkala 2009]). But from these proofs it is not clear that this could be algorithmically done. Thus we provide an algorithm using the proof of [Cassaigne and Nicolas 2003] where we replace some (non-algorithmic) arguments (Lemma 2, Lemma 3 and Lemme 4 of this paper) by algorithmic ones. Then we show with simple and algorithmic arguments that we can suppose we are working with non-erasing substitutions.
We treat separately growing and non-growing substitutions. For growing substitutions we look at their primitive components and we use the decidability result established in [Durand preprint 2011] about periodicity for primitive substitutions. Indeed, these primitive components should generate periodic sequences. Hence, we check it is the case (if not, then the sequence is not ultimately periodic). From there, Lemma 11 allows us to conclude.
For the non-growing case we use a result of Pansiot [Pansiot 1984] saying that we can either consider we are in the growing case or there are longer and longer periodic words with the same period in the sequence. From there, we again conclude with Lemma 11.

1.3. **Questions and comments.** We did not compute the complexity of the algorithm provided by our proof of the HD0L ultimate periodicity problem. Looking at Proposition 4 and the results in [Durand preprint 2011] we use here, our approach provides a high complexity.
Our result is for one-dimensional sequences. What can be said about multidimensional sequences generated by substitution rules ? or self-similar tilings ? It seems hopeless to generalize our method to tilings, although the main and key result we use to solve the HD0L ultimate periodicity problem (that is, the main result in [Durand 1998], see [Durand preprint 2011]) has been generalized to higher dimensions by N. Priebe in [Priebe 2000] (see also [Priebe and Solomyak 2001]). But observe that in [Leroux 2005] the author gives a polynomial time algorithm to know whether or not a Number Decision Diagram defines a Presburger definable set (see also [Muchnik 2003] where it was first proven but with a much higher complexity). From this result and [Cerný and Gruska 1986a, Salon 1986, Salon 1987] it is decidable to know whether a multidimensional automatic sequence (or fixed point of a multidimensional "uniform" substitution) has certain type of periodicity (see [Leroux 2005, Muchnik 2003]). From [Durand and Rigo preprint 2011] this type of periodicity is equivalent to a block complexity condition.

2. Words, morphisms, substitutive and HD0L sequences

In this section we recall classical definitions and notation. Observe that the notion of substitution we use below could be slightly different from other definitions in the litterature.

2.1. **Words and sequences.** An *alphabet* A is a finite set of elements called *letters*. Its cardinality is $|A|$. A *word* over A is an element of the free monoid generated

by A, denoted by A^*. Let $x = x_0x_1 \cdots x_{n-1}$ (with $x_i \in A$, $0 \le i \le n-1$) be a word, its *length* is n and is denoted by $|x|$. The *empty word* is denoted by ϵ, $|\epsilon| = 0$. The set of non-empty words over A is denoted by A^+. The elements of $A^{\mathbb{N}}$ are called *sequences*. If $x = x_0x_1 \cdots$ is a sequence (with $x_i \in A$, $i \in \mathbb{N}$) and $I = [k, l]$ an interval of $\mathbb{N}$ we set $x_I = x_kx_{k+1} \cdots x_l$ and we say that x_I is a *factor* of x. If $k = 0$, we say that x_I is a *prefix* of x. The set of factors of length n of x is written $\mathcal{L}_n(x)$ and the set of factors of x, or the *language* of x, is denoted by $\mathcal{L}(x)$. The *occurrences* in x of a word u are the integers i such that $x_{[i,i+|u|-1]} = u$. If u has an occurrence in x, we also say that u *appears* in x. When x is a word, we use the same terminology with similar definitions.
The sequence x is *ultimately periodic* if there exist a word u and a non-empty word v such that $x = uv^\omega$, where $v^\omega = vvv\cdots$. In this case v is called a *word period* and $|v|$ is called a *length period* of x. It is *periodic* if u is the empty word. A word u is *recurrent* in x if it appears in x infinitely many times.

2.2. **Morphisms and matrices.** Let A and B be two alphabets. Let σ be a *morphism* from A^* to B^*. When $\sigma(A) = B$, we say σ is a *coding*. We say σ is *erasing* if there exists $b \in A$ such that $\sigma(b)$ is the empty word. Such a letter is called *erasing letter*. If $\sigma(A)$ is included in B^+, it induces by concatenation a map from $A^{\mathbb{N}}$ to $B^{\mathbb{N}}$. This map is also denoted by σ. With the morphism σ is naturally associated its *incidence matrix* $M_\sigma = (m_{i,j})_{i\in B, j\in A}$ where $m_{i,j}$ is the number of occurrences of i in the word $\sigma(j)$.
Let σ be an endomorphism. We say it is *primitive* whenever its incidence matrix is primitive (*i.e.*, when it has a power with positive coefficients). We denote by $\mathcal{L}(\sigma)$ the set of words having an occurrence in some image of σ^n for some $n \in \mathbb{N}$.

2.3. **Substitutions and substitutive sequences.** We say that an endomorphism $\sigma : A^* \to A^*$ is *prolongable on* $a \in A$ if there exists a word $u \in A^+$ such that $\sigma(a) = au$ and moreover, if $\lim_{n\to+\infty} |\sigma^n(a)| = +\infty$. Prolongable endomorphisms are called *substitutions*.
We say a letter $b \in A$ is *growing* (w.r.t. σ) if $\lim_{n\to+\infty} |\sigma^n(b)| = +\infty$. We say σ is *growing* whenever all letters of A are growing.
Since for all $n \in \mathbb{N}$, $\sigma^n(a)$ is a prefix of $\sigma^{n+1}(a)$ and because $|\sigma^n(a)|$ tends to infinity with n, the sequence $(\sigma^n(aaa\cdots))_{n\ge0}$ converges (for the usual product topology on $A^{\mathbb{N}}$) to a sequence denoted by $\sigma^\omega(a)$. The endomorphism σ being continuous for the product topology, $\sigma^\omega(a)$ is a fixed point of σ: $\sigma(\sigma^\omega(a)) = \sigma^\omega(a)$. A sequence obtained in this way (by iterating a prolongable substitution) is said to be *purely substitutive* (w.r.t. σ). If $x \in A^{\mathbb{N}}$ is purely substitutive and $\phi : A^* \to B^*$ is a morphism then the sequence $y = \phi(x)$ is said to be a *morphic sequence* (w.r.t. (σ, ϕ)). When ϕ is a coding, we say y is *substitutive* (w.r.t. (σ, ϕ)).
The *language* of $\sigma : A^* \to A^*$, denoted by $\mathcal{L}(\sigma)$, is the set of words having an occurrence in $\sigma^n(b)$ for some $n \in \mathbb{N}$ and $b \in A$.

2.4. **D0L and HD0L sequences.** A *D0L system* is a triple $G = (A, h, u)$ where A is a finite alphabet, $\sigma : A^* \to A^*$ is an endomorphism and u is a word in A^*. A *HD0L system* is a 5-tuple $G = (A, B, \sigma, \phi, u)$ where (A, h, u) is a D0L system, B is a finite alphabet and $\phi : A^* \to B^*$ is a morphism.
If the sequence $(\sigma^n(uuu\cdots))_n$ (resp. $(\phi(\sigma^n(uuu\cdots))$ converges in $A^{\mathbb{N}}$ (resp. $B^{\mathbb{N}}$), we denote its limit by $w(G)$. It is not difficult to prove that $w(G)$ is a morphic sequence and that this can be algorithmically shown.

It is important to note that it is decidable to know whether $w(G)$ exists. This is left as an exercise. Thus, the HD0L ultimate periodicity problem also holds for the sequences $w(G)$ where G is a HD0L system.

3. Ultimate periodicity of HD0L sequences

In the sequel $\sigma : A^* \to A^*$ is a substitution prolongable on a, $\phi : A^* \to B^*$ is a morphism, $y = \sigma^\omega(a)$ and $x = \phi(y)$ is a sequence of $B^{\mathbb{N}}$. We have to find an algorithm telling if x is ultimately periodic or not.

3.1. Primitiveness assumption and sub-substitutions.

We recall that the HD0L ultimate periodicity problem is already known in the primitive case.

Theorem 2. [Durand preprint 2011] *The HD0L ultimate periodicity problem is decidable in the context of primitive substitutions. Moreover, a word period can be explicitly computed.*

Proof. The first part is Theorem 26 in [Durand preprint 2011]. The second part can be easily deduced from the proof of this theorem. □

The following lemma shows that is is decidable to check that a nonnegative matrix is primitive.

Lemma 3. [Horn and Johnson 1990] *The $n \times n$ nonnegative matrix M is primitive if and only if M^{n^2-2n+2} has positive entries.*

From Lemma 3, Section 4.4 and Section 4.5 in [Lind and Marcus 1995] we deduce following proposition.

Proposition 4. *Let $M = (m_{i,j})_{i,j\in A}$ be a matrix with non-negative coefficients. There exists three positive integer $p \neq 0$, q, l, where $q \leq l-1$, and a partition $\{A_i; 1 \leq i \leq l\}$ of A such that*

$$M^p = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_q \\ A_{q+1} \\ A_{q+2} \\ \vdots \\ A_l \end{array} \begin{array}{c} \begin{array}{cccccccc} A_1 & A_2 & \cdots & A_q & A_{q+1} & A_{q+2} & \cdots & A_l \end{array} \\ \begin{pmatrix} M_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ M_{1,2} & M_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ M_{1,q} & M_{2,q} & \cdots & M_q & 0 & 0 & \cdots & 0 \\ M_{1,q+1} & M_{2,q+1} & \cdots & M_{q,q+1} & M_{q+1} & 0 & \cdots & 0 \\ M_{1,q+2} & M_{2,q+2} & \cdots & M_{q,q+2} & 0 & M_{q+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{1,l} & M_{2,l} & \cdots & M_{q,l} & 0 & 0 & \cdots & M_l \end{pmatrix} \end{array},$$

where the matrices M_i have only positive entries or are equal to zero. Moreover, the partition and p can be algorithmically computed.

Thus we have the following corollary whose proof can be given without to use this proposition.

Corollary 5. *One can decide whether a substitution has erasing letters and non-growing letters.*

The following corollary will be helpful to change the representation of x (in terms of (σ, ϕ)) to a more convenient representation.

Corollary 6. *Let $\tau : A^* \to A^*$ be an endomorphism whose incidence matrix has the form of M^p in Proposition 4. Then for all $b \in A$ and all $j \geq 1$ the letters having an occurrence in $\left(\sigma^{|A|}\right)^j(b)$ or $\left(\sigma^{|A|}\right)^{j+1}(b)$ are the same.*

In what follows we keep the notation of Proposition 4. We will say that $\{A_i; 1 \leq i \leq l\}$ is a *primitive component partition of A (with respect to M)*, the A_i being the *primitive components*. If i belongs to $\{q+1, \cdots, l\}$ we will say that A_i is a *principal primitive component of A (with respect to M)*.
Let $\tau : A^* \to A^*$ be a substitution whose incidence matrix has the form of M^p in Proposition 4. Let $i \in \{q+1, \cdots, l\}$. We denote τ_i the restriction $\tau_{/A_i} : A_i^* \to A^*$ of τ to A_i^*. Because $\tau_i(A_i)$ is included in A_i^* we can consider that τ_i is an endomorphism of A_i^* whose incidence matrix is M_i. When it defines a substitution, we say it is a *sub-substitution of τ*. Moreover the matrix M_i has positive coefficients which implies that the substitution τ_i is primitive.
A non-trivial primitive endomorphism always has some power that is a substitution. For non-primitive endomorphisms we have the following corollary.

Corollary 7. *Let $\tau : A^* \to A^*$ be an endomorphism whose incidence matrix has the form of M^p in Proposition 4. Then, with the notation of Proposition 4, there exists $k \leq |A|^{|A|}$ satisfying : for all $i \geq q+1$ such that M_i is neither a null matrix nor the 1×1 matrix $[1]$, the endomorphism τ_i^k is a (primitive) substitution for some letter in A_i.*

The following lemma is easy to establish.

Lemma 8. *Let $x = \phi(\sigma^\omega(a))$. If $x = uv^\omega$, where v is not the empty word, then each sub-substitution σ' of σ such that $\mathcal{L}(\sigma') \subset \mathcal{L}(\sigma^\omega(a))$ verifies $\phi(\mathcal{L}(\sigma')) = \mathcal{L}(v^\omega)$.*

3.2. **Reduction of the problem.** It may happen, as for σ defined by $a \mapsto ab$, $b \mapsto a$, $c \mapsto c$, that some letter of the alphabet, here the alphabet is $\{a, b, c\}$, does not appear in $\sigma^\omega(a)$. It is preferable to avoid this situation. Corollary 6 enables us to avoid this algorithmically. Indeed, from Proposition 4, taking a power of σ (that can be algorithmically found) if needed, we can suppose

(P1) the incidence matrix of σ has the form of M^p in Proposition 4.

Then, consider $\sigma^{|A|}$ instead of σ. Hence, σ will continue to satisfy (P1) and, from Corollary, 6 we have

(P2) for all $b \in A$ and all $j \geq 1$ the letters having an occurrence in $\sigma^j(b)$ or $\sigma^{j+1}(b)$ are the same.

Let A' be the set of letters appearing in $\sigma^\omega(a)$. From (P2) it can be checked that $\sigma(A')$ is included in A'^* and that the set of letters appearing in $\sigma(a)$ is A'. Thus σ', the restriction of σ to A', defines a substitution prolongable on a satisfying $\sigma'^\omega(a) = \sigma^\omega(a)$ such that all letters of A' have an occurrence in $\sigma'^\omega(a)$ and all letters of A' occur in $\sigma'(a)$.
When we work with morphic sequences it is much simpler to handle non-erasing substitutions and even better to suppose that ϕ is a coding. Such a reduction is possible as shown in [Cassaigne and Nicolas 2003].

Theorem 9. *Let x be a morphic sequence. Then, x is substitutive with respect to a non-erasing substitution.*

This result was previously proven in [Cobham 1968] and [Pansiot 1983] (see also [Allouche and Shallit 2003] and [Honkala 2009]). But here as we are interested in decidable issues, we need more. The proof of J. Cassaigne and F. Nicolas is short and inspired by [Durand 1998]. This is the second part of this proof that corresponds to [Durand 1998]. It is clearly algorithmic. Whereas the first part (Lemma 2, Lemma 3 and Lemma 4 of [Cassaigne and Nicolas 2003]) is not because it uses the fact that from any sequence of integers, we can extract a subsequence that is either constant or strictly increasing. They use these lemmas to show the key point of their proof : we can always suppose that ϕ and σ fulfill the following :

$$|\phi(\sigma(a))| > |\phi(a)| > 0 \text{ and } |\phi(\sigma(b))| \geq |\phi(b)| \text{ for all } b \in A. \tag{3.1}$$

Below we show that this can be algorithmically realized. This provides an algorithm for Theorem 9.

First let us show that σ can be supposed to be non-erasing.

As σ satisfies (P2), each letter e is either erasing or, for all l, $\sigma^l(e)$ is not the empty-word. Let A' be the set of non-erasing letters and A'' the set of erasing letters. Let ψ be the morphism that sends the elements of A'' to the empty word and that is the identity for the other letters. Then we define σ' to be the unique endomorphism defined on A' satisfying $\psi\sigma = \sigma'\psi$. Observe that σ' is easily algorithmically definable and prolongable on a. Moreover we have $\sigma\psi = \sigma$. Let $z = \sigma'^{\omega}(a)$. Then, $\psi(y) = z$ and $\sigma(z) = y$.

If σ' is non-erasing (which can be checked using Proposition 4) then we are done. Otherwise, we proceed in the same way with σ'. As the cardinality of the alphabet strictly decreases we will end, in at least $|A|$ steps of the procedure, with a morphism φ and a non-erasing substitution $\tau : \tilde{A}^* \to \tilde{A}^*$ prolongable on a such that $x = \varphi(t)$ where $t = \tau^{\omega}(a)$.

Note that τ keeps properties (P1) and (P2) σ already had. Thus we can also consider

(P3) σ is non-erasing.

Consequently, from (P2), $|\phi \circ \sigma(\sigma(a))| > |\phi(\sigma(a))| > |\phi(a)|$, otherwise $\tau^{\omega}(a)$ would not be an infinite sequence, and $|\phi \circ \sigma(\sigma(b))| \geq |\phi(\sigma(b))|$ for all $b \in A$. Incidently, it implies that $|\phi(\sigma(a))| > 0$. Hence changing ϕ by $\phi \circ \sigma$ if needed, we can suppose the following inequalities holds :

$$|\phi(\sigma(a))| > |\phi(a)| > 0 \text{ and } |\varphi(\tau(b))| \geq |\varphi(b)| \text{ for all } b \in A.$$

Hence, together with the argument of the proof of Theorem 9 we obtain the algorithm we are looking for. This is summarized in the following theorem.

Theorem 10. *There exists an algorithm that given ϕ and σ compute a coding φ and a non-erasing substitution τ, prolongable on a, such that $x = \varphi(z)$ where $z = \tau^{\omega}(a)$.*

Thus, in the sequel we suppose ϕ is a coding and σ is a non-erasing substitution. We end this section with a technical lemma checking the ultimate periodicity.

Lemma 11. *Let $t \in A^{\mathbb{N}}$, φ be a coding defined on A^*, $z = \varphi(t)$, and, u and v be non-empty words. Then, $z = uv^{\omega}$ iff and only if*

(1) *for all recurrent words $B = b_1 b_2 \cdots b_{2|v|} \in \mathcal{L}(t)$, where the b_i's are letters, $\varphi(B) = s_B v^{r_B} p_B$ where r_B is a positive integer, s_B is a suffix of v and p_B a prefix of v;*

(2) *for all recurrent words* $BB' \in \mathcal{L}(t)$, *where* B *and* B' *are words of length* $2|v|$, $p_B s_{B'} = v$.

Proof. The proof is left to the reader. □

3.3. **The case of substitutive sequences with respect to growing substitutions.** In the sequel we suppose σ is a growing substitution. From Corollary 5 it is decidable to know whether we are in this situation.
We recall that from the previous section we can suppose ϕ is a coding and that σ satisfies (P1), (P2) and (P3).

Lemma 12. *Let* u *and* v *be two words. It is decidable to check whether or not* $\mathcal{L}(u^\omega)$ *is equal to* $\mathcal{L}(v^\omega)$.

Theorem 13. *The HD0L ultimate periodicity problem is decidable for substitutive sequences w.r.t. growing substitutions.*

Proof. Let us use the notation of Proposition 4. From Corollary 7, taking a power of σ (less that $|A|^{|A|}$) if needed, we can suppose that for all $i \geq q+1$ such that M_i is neither a null matrix nor the 1×1 matrix [1], the endomorphism τ_i defines a primitive sub-substitution. For such i, we denote by σ_i the sub-substitution corresponding to A_i.
Observe that for all $i \geq q+1$ and $b \in A_i$, the word $\sigma^n(b) = \sigma_i^n(b)$ is recurrent in $\sigma^\omega(a)$. Thus, to check the periodicity of x, we start checking with Theorem 2 that the languages $\phi(\mathcal{L}(\sigma'))$, where σ' is a sub-substitution of σ, are periodic (*i.e.,* equal to some $\mathcal{L}(w(\sigma')^\omega)$ where $w(\sigma')$ is a word period). We point out that when the language is periodic then $w(\sigma')$ can be computed. If for some σ' its language is not periodic then x is not ultimately periodic. Then, we check that all these periodic languages are equal using Lemma 12. From Lemma 8, if this checking fails, then x is not periodic.
Hence we suppose it is the case : There exists a word v that is algorithmically given by Theorem 2 such that $\phi(\mathcal{L}(\sigma')) = \mathcal{L}(v^\omega)$ for all sub-substitution σ' of σ. Consequently, we should check whether there exists u such that $x = uv^\omega$.
We note that for all n the words of length n can be algorithmically enumerated, using for example the substitution on the words of length n in [Queffèlec 1987]. We conclude using Lemma 11. □

3.4. **The case of substitutive sequences with respect to non-growing substitutions.** In the sequel we suppose that σ is a non-growing substitution. From Corollary 5 it is decidable to know whether we are in this situation. We recall that from the previous section we can suppose ϕ is a coding and that σ satisfies (P1), (P2) and (P3).

Lemma 14. [Pansiot 1984, Théorème 4.1] *The substitution* σ *satisfied exactly one one the following two statements.*

(1) *The length of words (occurring in* $\sigma^\omega(a)$*) consisting of non-growing letters is bounded.*
(2) *There exists a growing letter* $b \in A$, *occurring in* $\sigma^\omega(a)$, *such that* $\sigma(b) = vbu$ *(or* ubv*) with* $u \in C^* \setminus \{\epsilon\}$ *where* C *is the set of non-growing letters.*

Moreover, in the situation (1) *the sequence* $\sigma^\omega(a)$ *can be algorithmically defined as a substitutive sequence w.r.t. a growing substitution.*

Lemma 15. *It is decidable to know whether σ satisfies* (1) *or* (2) *of Lemma 14.*

Proof. It can be easily algorithmically checked whether we are in the situation (2) of Lemma 14. Thus it is decidable to know whether we are in situation (1) of Lemma 14. □

Theorem 16. *The HD0L ultimate periodicity problem is decidable for substitutive sequences w.r.t. non-erasing substitutions.*

Proof. From Theorem 13, Lemma 14 and Lemma 15 it remains to consider that σ satisfies (2) in Lemma 14 : Let b be a letter occurring in σ^ω such that $\sigma(b) = vbu$ (or ubv) with $u \in C^* \setminus \{\epsilon\}$ where C is the set of non-growing letters. Then $\mathcal{L}(u^\omega) \subset \mathcal{L}(\sigma)$. We conclude using Lemma 11. □

This ends the proof of Theorem 1.

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