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Frédéric Herau, Wei-Xi Li

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Frédéric Hérau [*] Wei-Xi Li [*][†]

Abstract

In this paper we study a Landau-type operator with an external force. It is a linear model of the Landau equation near Maxwellian distributions. Making use of multiplier method, we get the global hypoelliptic estimate under suitable assumptions on the external potential.

Keywords: global hypoellipticity, Landau equation, anisotropic diffusion, Wick quantization

2010 MSC: 35H10, 35H20, 35B65, 82C40.

1 Introduction and main results

In this article we are interested in the study of the regularity of solutions of some kinetic equations. In the case of linear or linearized equations, the corresponding collision operator may behave in some cases like a Laplacian - or at least a fractional power of a Laplacien, and we may hope for some improved smoothness when times goes on. This type of question concerns Fokker-Planck equations, Landau equations or Boltzmann equations without angular cut-off.

In the linear homogeneous case these equations have then a parabolic behavior, and the study of the local smoothing properties in the velocity variable is rather direct. In the non-homogeneous case, the regularization in space variable is not so easy, but occurs anyway thanks to the so-called hypoelliptic structure of the equation. This type of behavior is a subject of intensive recent research in kinetic theory, coming back to the first results of Hörmander concerning the Kolmogorov equation [12], and is also in the core of averaging lemmas (see e.g. [2]). In this article we are interested in global estimates of the following Landau-type operator

$$P = i(y \cdot D_x - \partial_x V(x) \cdot D_y) + D_y \cdot \nu(y) D_y + \big(y \wedge D_y\big) \cdot \mu(y)\big(y \wedge D_y\big) + F(y),$$

where $D_x = -i\partial_x$, $D_y = -i\partial_y$, and $x \in \mathbb{R}^3$ is the space variable and $y \in \mathbb{R}^3$ is the velocity variable, and $X \cdot Y$ stands for the standard dot-product on $\mathbb{R}^3$. The real-valued function $V(x)$ of space variable x stands for the macroscopic force, and the functions $\nu(y)$, $\mu(y)$ and $F(y)$ of the variable y in the diffusion are smooth and real-valued with the properties subsequently listed below.

(i) There exist a constant $c > 0$ such that

$$\forall\, y \in \mathbb{R}^3, \quad \nu(y) \geq c \langle y \rangle^{\gamma}, \quad \mu(y) \geq c \langle y \rangle^{\gamma} \quad \text{and} \quad F(y) \geq c \langle y \rangle^{2+\gamma}, \tag{1}$$

with $\gamma \in [0, 1]$ and $\langle y \rangle = \big(1 + |y|^2\big)^{1/2}$.

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(ii) For any $\alpha \in \mathbb{Z}_+^3$, there exists a constant C_α such that

$$\forall\ y \in \mathbb{R}^3, \quad |\partial^\alpha \nu(y)| + |\partial^\alpha \mu(y)| \le C_\alpha \langle y\rangle^{\gamma - |\alpha|}, \quad \text{and} \quad |\partial^\alpha F(y)| \le C_\alpha \langle y\rangle^{2+\gamma-|\alpha|}. \tag{2}$$

It is sometimes convenient to rewrite the operator as the form

$$P = i(y \cdot D_x - \partial_x V(x) \cdot D_y) + \big(B(y)D_y\big)^* \cdot B(y)D_y + F(y),$$

where the matrix $B(y)$ is given by

$$B(y) = \big(B_{jk}(y)\big)_{1\le j,k\le 3} = \begin{pmatrix} \sqrt{\lambda(y)} & -y_3\sqrt{\mu(y)} & y_2\sqrt{\mu(y)} \\ -y_3\sqrt{\mu(y)} & \sqrt{\lambda(y)} & -y_1\sqrt{\mu(y)} \\ -y_2\sqrt{\mu(y)} & y_1\sqrt{\mu(y)} & \sqrt{\lambda(y)} \end{pmatrix}, \tag{3}$$

and $\big(B(y)D_y\big)^* = D_y B(y)^T$, with B^T the transpose of B, is the formal adjoint of $B(y)D_y$. By (1) and (2) one has, for any $y, \eta \in \mathbb{R}^3$ and any $\alpha \in \mathbb{Z}_+^3$,

$$|\partial^\alpha B_{jk}(y)| \le C_\alpha \langle y\rangle^{1-|\alpha|+\gamma/2} \tag{4}$$

$$\text{and} \quad |B(y)\eta|^2 = \nu(y)\,|\eta|^2 + \mu(y)\,|y\wedge\eta|^2 \ge c\,|y|^\gamma\,\big(\,|\eta|^2 + |y\wedge\eta|^2\,\big). \tag{5}$$

As a result, observing $i(y\cdot D_x - \partial_x V(x)\cdot D_y)$ is skew-adjoint, we have

$$\begin{aligned} &\big\|\, \langle y\rangle^{1+\frac{\gamma}{2}}\, u\big\|_{L^2}^2 + \big\|\, \langle y\rangle^{\frac{\gamma}{2}}\, D_y u\big\|_{L^2}^2 + \big\|\, \langle y\rangle^{\frac{\gamma}{2}}\,(y\wedge D_y)u\big\|_{L^2}^2 \\ &\qquad\qquad \le c^{-1}\big\|B(y)D_y u\big\|_{L^2}^2 \le c^{-1}\mathrm{Re}\,(Pu,u)_{L^2}, \end{aligned} \tag{6}$$

where $(\cdot,\ \cdot)_{L^2}$ and $\big\|\cdot\big\|_{L^2}$ standing for the inner product and norm in $L^2(\mathbb{R}^6_{x,y})$.

Denoting by (ξ,η) the dual variables of (x,y), we notice that the diffusion only occurs in the variables (y,η), but not in the other directions; and that the cross product term $y\wedge D_y$ improves this diffusion in specific directions of the phase space where the variables y and η are orthogonal. In this work, we aim at proving that linear Landau-type operators are actually hypoelliptic despite this lack of diffusion in the spatial derivative D_x. More specifically, we shall be concerned in proving global hypoelliptic estimates with weights in both spatial and velocity derivatives whose structure is exactly related to the anisotropy of the diffusion. Our main results can be stated as follows.

Theorem 1.1 *Let $V \in C^2(\mathbb{R}^3;\ \mathbb{R})$ satisfy that*

$$\forall\ |\alpha| = 2, \ \ \exists\ C_\alpha > 0 \ \ \text{such that} \ \ \forall x \in \mathbb{R}^3, \quad |\partial_x^\alpha V(x)| \le C_\alpha \langle \partial_x V(x)\rangle^{2/3}. \tag{7}$$

Then there exists a constant C such that for any $u \in C_0^\infty(\mathbb{R}^6)$ one has

$$\begin{aligned} &\big\|\, \langle y\rangle^{\frac{\gamma}{6}}\, |\partial_x V(x)|^{2/3}\, u\big\|_{L^2} + \big\|\, \langle y\rangle^{2+\frac{5\gamma}{6}}\, u\big\|_{L^2} + \big\|\, |D_x|^{\frac{2}{3}}\, u\big\|_{L^2} + \big\|\, \langle y\rangle^{\frac{\gamma}{2}}\, |D_y|^2\, u\big\|_{L^2} \\ &\qquad\qquad + \big\|\, \langle y\rangle^{\frac{\gamma}{2}}\, |y\wedge D_y|^2\, u\big\|_{L^2} \le\ C\big(\,\|Pu\|_{L^2} + \|u\|_{L^2}\,\big). \end{aligned} \tag{8}$$

Moreover if V satisfies the condition that

$$C_0^{-1}\langle x\rangle^M \le \langle \partial_x V(x)\rangle \le C_0\,\langle x\rangle^M, \quad \text{and} \ \ \forall\ |\alpha| \ge 2, \ |\partial_x^\alpha V(x)| \le C_\alpha \langle \partial_x V(x)\rangle^{1/3}, \tag{9}$$

with M, C_0 two positive numbers and C_α a constant depending only on α, we have additionally

$$\big\|\big(\,\langle \partial_x V\wedge\eta + y\wedge\xi\rangle^{\frac{2}{3}}\,\big)^w u\big\|_{L^2} + \big\|\big(\,\langle \partial_x V\wedge\xi\rangle^{\frac{2}{5}}\,\big)^w u\big\|_{L^2} \le\ C\big(\,\|Pu\|_{L^2} + \|u\|_{L^2}\,\big), \tag{10}$$

where p^w stands for the Weyl quantization of the symbol p.

Estimates of the type given in Theorem 1.1 can be analyzed through different point of views. At first they give at least local regularity estimates in the velocity direction, according to the term $|D_y|^2$ appearing in (8). Now one of the goal of this article was to give global estimates in order to identify the good functional spaces associated to the problems: here we are able to prove that in the elliptic direction we have an estimate of type

$$\| \langle y \rangle^{2+\frac{5\gamma}{6}} u\|_{L^2} + \| \langle y \rangle^{\frac{\gamma}{2}} |D_y|^2 u\|_{L^2} + \| \langle y \rangle^{\frac{\gamma}{2}} |y \wedge D_y|^2 u\|_{L^2} \leq \; C \left\{ \|Pu\|_{L^2} + \|u\|_{L^2} \right\}.$$

A priori it does not seem to be optimal, and indeed when $V = 0$ a similar inequality was proven in [10] with an exponent γ there instead of $5\gamma/6$ and $\gamma/2$ here, but the study with $V \neq 0$ is harder. Similarly to [10] anyway, we recover here in (8) some intrinsic global anisotropy via a term of type $y \wedge D_y$ already appearing in the definition of the original operator.

The second main feature of this result is to reflect the regularizing effect in space variable x, thanks to the hypoelliptic structure, which leads to terms involving e.g. $|D_x|^{2/3}$. Recall that the exponent $2/3$ here is optimal , according to local estimates coming back to [12] (see also [1]). In this direction (i.e. concerning local optimal subelliptic estimates for kinetic models), we mention also the works [19, 20] and [17] on the Boltzmann operator without cut-off and the series of works on Gevrey regularity [5, 4, 3].

Now similarly to the case of elliptic directions, it may be interesting to get global weighted estimates in space direction. In [9], [8] the authors studied the Fokker-Planck case, in particular with a potential, following original ideas by [7] (see also [6]). In this direction the work [10] also gave a first subelliptic global (optimal) estimate, concerning the Landau operator in the case when there is no potential; the main feature of that work was to show that subellipticity in space direction occured with anisotropic weights of type $\langle y \rangle^{\gamma}\, y \wedge D_x$. In the present article we recover the same type of behavior, with additional terms - also involving wedges - linked with the potential V.

In order to prove the result, we use the same multiplier method as in [10], which allows to obtain a global hypoellipticity result with optimal loss of $4/3$ derivatives. This method has been first introduced in [11] for the Fokker-Planck equation. It has then been extended to more general doubly characteristic quadratic differential operators by K. Pravda-Starov in [21] to get optimal hypoelliptic estimates. The present work is a natural continuation of [10], and as there we will make a strong use of pseudodifferential and Wick calculus, following the presentation by Lerner in [16].

The plan of the article is the following. In the second section we introduce some notations and facts about the symbolic calculus. In the third section we prove some weighted estimates in space and velocity, without derivatives, needed later to complete the proof. In Section 4, We essentially work on the velocity side after a change of operator through a partial Wick quantization in (x, ξ). This allows to treat the space variable (and its dual) as parameters, and to get optimal velocity estimates with parameters. In the last two sections we go back to the original operator and complete the proof.

2 Notations and some basic facts on symbolic calculus

We firstly list some notations used throughout the paper. Denote respectively by $(\cdot,\ \cdot)_{L^2}$ and $\|\cdot\|$ the inner product and the norm in $L^2(\mathbb{R}^n)$. For a vector-valued functions $U = (u_1, \cdots, u_n)$ the norm $\|U\|_{L^2}$ stands for $\left(\sum_j \|u_j\|_{L^2}^2\right)^{1/2}$.

To simplify the notation, by $A \lesssim B$ we mean there exists a positive constant C, such that $A \leq CB$, and similarly for $A \gtrsim B$. While the notation $A \approx B$ means both $A \lesssim B$ and $B \lesssim A$ hold.

Now we introduce some notations of phase space analysis and recall some basic properties of symbolic calculus, and refer to [13] and [16] for detailed discussions. Throughout the paper let g be the admissible metric $|dz|^2+|d\zeta|^2$ and m be an admissible weight for g (see [13] and [16] for instance the definitions of admissible metric and weight). Given a symbol $p(z,\zeta)$, we say $p\in S\big(m,\ g\big)$ if

$$\forall\ \alpha,\ \beta\in\mathbb{Z}_+^n,\ \ \forall\ (z,\zeta)\in\mathbb{R}^{2n},\quad \Big|\partial_z^\alpha\partial_\zeta^\beta p(z,\zeta)\Big|\leq C_{\alpha,\beta}\ m(z,\zeta),$$

with $C_{\alpha,\beta}$ a constant depending only on α,β. For such a symbol p we may define its Weyl quantization p^w by

$$\forall u\in\mathcal{S}(\mathbb{R}^n),\quad p^w u(z)=\int e^{2i\pi(z-v)\cdot\zeta}p\left(\frac{z+v}{2},\zeta\right)u(v)dvd\zeta.$$

The L^2 continuity theorem in the class $S\big(1,\ g\big)$, which will be used frequently, says that if $p\in S\big(1,\ g\big)$ then

$$\forall\ u\in L^2,\quad \big\|p^w u\big\|_{L^2}\lesssim\big\|u\big\|_{L^2}.$$

We shall denote by $Op(S(m,g))$ the set of operators whose symbols are in the class $S(m,g)$. Finally let's recall some basic properties of the Wick quantization, and refer the reader to the works of Lerner [14, 15, 16] for thorough and extensive presentations of this quantization and some of its applications. Using the notation $Z=(z,\zeta)\in\mathbb{R}^{2n}$, the wave-packets transform of a function $u\in\mathcal{S}\big(\mathbb{R}^n\big)$ is defined by

$$Wu(Z)=\big(u,\varphi_Z\big)_{L^2(\mathbb{R}^n)}=2^{n/4}\int_{\mathbb{R}^n}u(v)e^{-\pi|v-z|^2}e^{2i\pi(v-z/2)\cdot\zeta}dv,$$

with

$$\varphi_Z(v)=2^{n/4}e^{-\pi|v-z|^2}e^{2i\pi(v-z/2)\cdot\eta},\quad v\in\mathbb{R}^n.$$

Then one can verify that W is an isometric mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$:

$$\big\|Wu\big\|_{L^2(\mathbb{R}^{2n})}=\big\|u\big\|_{L^2(\mathbb{R}^n)}.\tag{11}$$

Moreover the operator

$$\pi_{\mathcal{H}}=WW^*$$

with W^* the adjoint of W, is an orthogonal projection on a closed space in L^2, whose kernel is given by

$$K\big(Z,\tilde Z\big)=e^{-\frac{\pi}{2}\left(|z-\tilde z|^2+|\zeta-\tilde\zeta|^2\right)}e^{i\pi\left(z-\tilde z\right)\cdot\left(\zeta+\tilde\zeta\right)}\quad Z=(z,\zeta),\ \tilde Z=(\tilde z,\tilde\zeta).\tag{12}$$

We define the Wick quantization of any L^∞ symbol p as

$$p^{\text{Wick}}=W^*pW.$$

The main property of the Wick quantization is its positivity, i.e.,

$$p(Z)\geq 0\quad\text{for all } Z\in\mathbb{R}^{2n}\text{ implies}\quad p^{\text{Wick}}\geq 0.\tag{13}$$

According to Proposition 2.4.3 in [16], the Wick and Weyl quantizations of a symbol p are linked by the following identities

$$p^{\text{Wick}}=p^w+r^w\tag{14}$$

with

$$r(Z) = \int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta) p''(Z+\theta Y) Y^2 e^{-2\pi|Y|^2} 2^n \, dY d\theta.$$

We also recall the following composition formula obtained in the proof of Proposition 3.4 in [14]

$$p^{\text{Wick}} q^{\text{Wick}} = \big[pq - \frac{1}{4\pi} p' \cdot q' + \frac{1}{4i\pi}\{p,q\}\big]^{\text{Wick}} + T,$$

with T a bounded operator in $L^2(\mathbb{R}^{2n})$, when $p \in L^\infty(\mathbb{R}^{2n})$ and q is a smooth symbol whose derivatives of order ≥ 2 are bounded on $\mathbb{R}^{2n}$. The notation $\{p,q\}$ denotes the Poisson bracket defined by

$$\{p,\ q\} = \frac{\partial p}{\partial \zeta} \cdot \frac{\partial q}{\partial z} - \frac{\partial p}{\partial z} \cdot \frac{\partial q}{\partial \zeta}. \tag{15}$$

3 First part of the proof of Theorem 1.1: weighted estimates

In this section we are mainly concerned with the estimate in weighted L^2 norms, that is

Proposition 3.1 *Let $V(x) \in C^2(\mathbb{R}^3;\ \mathbb{R})$ satisfy the condition (7). Then*

$$\forall\ u \in C_0^\infty(\mathbb{R}^6), \quad \big\| \langle y\rangle^{\frac{\gamma}{6}} |\partial_x V(x)|^{\frac{2}{3}} u \big\|_{L^2} + \big\| \langle y\rangle^{2+\frac{5\gamma}{6}} u \big\|_{L^2} \lesssim \big\|Pu\big\|_{L^2} + \big\|u\big\|_{L^2}. \tag{16}$$

In order to prove this proposition, we begin with

Lemma 3.2 (Lemma 3.7 in [10]) *Let $p \in S(1, |dy|^2 + |d\eta|^2)$ and $B(y)$ be the matrix given in (3). We have*

$$\forall\ u \in C_0^\infty(\mathbb{R}^6), \quad |(F(y),\ p^w u)_{L^2}| + \big|\big((B(y)D_y)^* B(y) D_y u,\ p^w u\big)_{L^2}\big| \lesssim |(Pu,\ u)_{L^2}|. \tag{17}$$

Lemma 3.3 *For all $u \in C_0^\infty(\mathbb{R}^3)$ we have*

$$\begin{aligned}\big\| \langle y\rangle^{2+\frac{5\gamma}{6}} u\big\|_{L^2} + \big\| \langle y\rangle^{1+\frac{5\gamma}{6}} D_y u\big\|_{L^2} + \big\| \langle y\rangle^{1+\frac{5\gamma}{6}} \big(y\wedge D_y\big) u\big\|_{L^2} \\ \lesssim \big\| \langle y\rangle^{\frac{\gamma}{6}} \langle \partial_x V\rangle^{\frac{2}{3}} u\big\|_{L^2} + \big\|Pu\big\|_{L^2}.\end{aligned} \tag{18}$$

Proof. In the proof we let $u \in C_0^\infty(\mathbb{R}^{2n})$. The conclusion will follow if one could prove

$$\big\| \langle y\rangle^{2+\frac{5\gamma}{6}} u\big\|_{L^2}^2 + \big\| \langle y\rangle^{1+\frac{\gamma}{3}} B(y) D_y u\big\|_{L^2}^2 \lesssim \big\| \langle y\rangle^{\frac{\gamma}{6}} \langle \partial_x V\rangle^{\frac{2}{3}} u\big\|_{L^2}^2 + \big\|Pu\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2, \tag{19}$$

since by (4), one has

$$\langle y\rangle^{1+\frac{\gamma}{3}} |B(y) D_y u| \geq \langle y\rangle^{1+\frac{5\gamma}{6}} |D_y u| + \langle y\rangle^{1+\frac{5\gamma}{6}} \big|\big(y\wedge D_y\big) u\big|.$$

As a preliminary step , let's firstly show that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\begin{aligned}\Big|\Big(P \langle y\rangle^{1+\frac{\gamma}{3}} u,\ \langle y\rangle^{1+\frac{\gamma}{3}} u\Big)_{L^2}\Big| \lesssim \varepsilon\ \Big(\big\| \langle y\rangle^{2+\frac{5\gamma}{6}} u\big\|_{L^2}^2 + \big\| \langle y\rangle^{1+\frac{\gamma}{3}} B(y) D_y u\big\|_{L^2}^2\Big) \\ + C_\varepsilon \Big\{\big\| \langle y\rangle^{\frac{\gamma}{6}} \langle \partial_x V(x)\rangle^{\frac{2}{3}} u\big\|_{L^2}^2 + \big\|Pu\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2\Big\}.\end{aligned} \tag{20}$$

In fact, the estimate

$$\langle \partial_x V(x)\rangle \langle y\rangle^{1+2\gamma/3} \leq \varepsilon \langle y\rangle^{4+5\gamma/3} + C_\varepsilon \langle y\rangle^{\gamma/3} \langle \partial_x V(x)\rangle^{4/3}$$

yields

$$\Big(\langle y\rangle^{1+2\gamma/3}\,\langle\partial_x V(x)\rangle\,u,\ u\Big)_{L^2}\le\varepsilon\big\|\,\langle y\rangle^{2+5\gamma/6}\,u\big\|_{L^2}^2+C_\varepsilon\big\|\,\langle y\rangle^{\gamma/6}\,\langle\partial_x V(x)\rangle^{2/3}\,u\big\|_{L^2}^2. \tag{21}$$

Consequently, using (2) we compute

$$\Big|\big[P,\ \langle y\rangle^{1+\frac{\gamma}{3}}\,\big]u\Big|\lesssim|\partial_x V(x)|\,\langle y\rangle^{\frac{\gamma}{3}}\,|u|+\langle y\rangle^{1+\frac{5\gamma}{6}}\,|B(y)D_y u|\,,$$

and thus

$$\begin{aligned}
&\Big|\Big(\big[P,\ \langle y\rangle^{1+\frac{\gamma}{3}}\,\big]u,\ \langle y\rangle^{1+\frac{\gamma}{3}}\,u\Big)_{L^2}\Big|\\
&\lesssim\Big(|\partial_x V(x)|\,\langle y\rangle^{1+\frac{2\gamma}{3}}\,u,\ u\Big)_{L^2}+\Big(\langle y\rangle^{\frac{\gamma}{3}}\,|B(y)D_y u|,\ \langle y\rangle^{2+\frac{5\gamma}{6}}\,|u|\Big)_{L^2}\\
&\lesssim\ \varepsilon\,\big\|\,\langle y\rangle^{2+5\gamma/6}\,u\big\|_{L^2}^2+C_\varepsilon\big\|\,\langle y\rangle^{\gamma/6}\,\langle\partial_x V(x)\rangle^{2/3}\,u\big\|_{L^2}^2+C_\varepsilon\big\|\,\langle y\rangle^{\frac{\gamma}{3}}\,B(y)D_y u\big\|_{L^2}^2\\
&\lesssim\ \varepsilon\,\big\|\,\langle y\rangle^{2+5\gamma/6}\,u\big\|_{L^2}^2+\varepsilon\big\|\,\langle y\rangle^{1+\gamma/3}\,B(y)D_y u\big\|_{L^2}^2\\
&\quad+C_\varepsilon\big\|\,\langle y\rangle^{\gamma/6}\,\langle\partial_x V(x)\rangle^{2/3}\,u\big\|_{L^2}^2+C_{\varepsilon,\gamma}\big\|\,\langle y\rangle^{-1}\,B(y)D_y u\big\|_{L^2}^2\\
&\lesssim\ \varepsilon\,\big(\big\|\,\langle y\rangle^{2+5\gamma/6}\,u\big\|_{L^2}^2+\big\|\,\langle y\rangle^{1+\gamma/3}\,B(y)D_y u\big\|_{L^2}^2\big)\\
&\quad+C_\varepsilon\big(\big\|\,\langle y\rangle^{\gamma/6}\,\langle\partial_x V(x)\rangle^{2/3}\,u\big\|_{L^2}^2+\big\|Pu\big\|_{L^2}^2+\big\|u\big\|_{L^2}^2\big),
\end{aligned}$$

where the second inequality follows from (21), the third inequality holds because

$$\forall\ \tilde\varepsilon>0,\quad\big\|\,\langle y\rangle^{\frac{\gamma}{3}}\,B(y)D_y u\big\|_{L^2}\le\tilde\varepsilon\big\|\,\langle y\rangle^{1+\frac{\gamma}{3}}\,B(y)D_y u\big\|_{L^2}+C_{\tilde\varepsilon}\big\|\,\langle y\rangle^{-1}\,B(y)D_y u\big\|_{L^2},$$

and the last inequality follows from (6) since by (4),

$$\big\|\,\langle y\rangle^{-1}\,B(y)D_y u\big\|_{L^2}\le\big\|\,\langle y\rangle^{\gamma/2}\,D_y u\big\|_{L^2}.$$

As a result, observing

$$\Big|\Big(P\,\langle y\rangle^{1+\frac{\gamma}{3}}\,u,\ \langle y\rangle^{1+\frac{\gamma}{3}}\,u\Big)_{L^2}\Big|\le\Big|\Big(Pu,\ \langle y\rangle^{2+\frac{2\gamma}{3}}\,u\Big)_{L^2}\Big|+\Big|\Big(\big[P,\ \langle y\rangle^{1+\frac{\gamma}{3}}\,\big]u,\ \langle y\rangle^{1+\frac{\gamma}{3}}\,u\Big)_{L^2}\Big|$$

and

$$\Big|\Big(Pu,\ \langle y\rangle^{2+\frac{2\gamma}{3}}\,u\Big)_{L^2}\Big|\le\varepsilon\big\|\,\langle y\rangle^{2+\frac{5\gamma}{6}}\,u\big\|_{L^2}^2+C_\varepsilon\big\|Pu\big\|_{L^2}^2$$

due to the fact that $2\gamma/3\le5\gamma/6$ for $\gamma\ge0$, we obtain the inequality (20).

Now we prove (19). Let's firstly write

$$\begin{aligned}
&\big\|\,\langle y\rangle^{2+\frac{5\gamma}{6}}\,u\big\|_{L^2}^2+\big\|\,\langle y\rangle^{1+\frac{\gamma}{3}}\,B(y)D_y u\big\|_{L^2}^2\\
\lesssim\ &\big\|\,\langle y\rangle^{1+\frac{\gamma}{2}}\,\langle y\rangle^{1+\frac{\gamma}{3}}\,u\big\|_{L^2}^2+\big\|B(y)D_y\,\langle y\rangle^{1+\frac{\gamma}{3}}\,u\big\|_{L^2}^2+\big\|B(y)\big[D_y,\ \langle y\rangle^{1+\frac{\gamma}{3}}\,\big]u\big\|_{L^2}^2\\
\lesssim\ &\Big|\Big(P\,\langle y\rangle^{1+\frac{\gamma}{3}}\,u,\ \langle y\rangle^{1+\frac{\gamma}{3}}\,u\Big)_{L^2}\Big|+\big\|B(y)\big[D_y,\ \langle y\rangle^{1+\frac{\gamma}{3}}\,\big]u\big\|_{L^2}^2,
\end{aligned}$$

the last inequality using (6). For the last term, we have

$$\big\|B(y)\big[D_y,\ \langle y\rangle^{1+\frac{\gamma}{3}}\,\big]u\big\|_{L^2}^2\lesssim\big\|\,\langle y\rangle^{1+\frac{5\gamma}{6}}\,u\big\|_{L^2}^2\le\varepsilon\big\|\,\langle y\rangle^{2+\frac{5\gamma}{6}}\,u\big\|_{L^2}^2+C_\varepsilon\big\|u\big\|_{L^2}^2.$$

Then the desired estimate (19) follows from the above inequalities and (20), completing the proof of Lemma 3.3. □

Proof of Proposition 3.1. Let $\rho \in C^1(\mathbb{R}^{2n})$ be a real-valued function given by

$$\rho = \rho(x,y) = \frac{2\langle y\rangle^{\frac{\gamma}{3}}\,\partial_x V(x)\cdot y}{\langle \partial_x V(x)\rangle^{4/3}}\phi(x,y),$$

with

$$\phi = \chi\left(\frac{\langle y\rangle^{2+2\gamma/3}}{\langle \partial_x V(x)\rangle^{2/3}}\right),$$

where $\chi \in C_0^\infty(\mathbb{R};\ [0,1])$ such that $\chi = 1$ in $[-1,1]$ and $\mathrm{supp}\ \chi \subset [-2,2]$. We have, using the notation $Q = y\cdot D_x - \partial_x V(x)\cdot D_y$,

$$\mathrm{Re}\,(Pu,\ \rho u)_{L^2} = \mathrm{Re}\,(iQu,\ \rho u)_{L^2} + \big((B(y)D_y)^*\cdot B(y)D_y u,\ \rho u\big)_{L^2} + (F(y)u,\ \rho u)_{L^2}\,,$$

which along with (17) yields

$$\mathrm{Re}\,(iQu,\ \rho u)_{L^2} \lesssim |(Pu,\ u)_{L^2}| + |(Pu,\ \rho u)_{L^2}|\,.$$

Next we want to give a lower bound for the term on the left side. Direct computation shows that

$$\mathrm{Re}\,(iQu,\ \rho u)_{L^2} = \frac{i}{2}\big([\rho,\ Q]u,\ u\big)_{L^2} = \sum_{j=1}^{3}(\mathcal{A}_j u,\ u)_{L^2}\,, \tag{22}$$

with $\mathcal{A}_j$ given by

$$\begin{aligned}
\mathcal{A}_1 &= \frac{\langle y\rangle^{\frac{\gamma}{3}}\,|\partial_x V(x)|^2}{\langle \partial_x V(x)\rangle^{4/3}}\phi,\\
\mathcal{A}_2 &= \langle \partial_x V(x)\rangle^{-4/3}\big(\partial_x V(x)\cdot y\big)\partial_x V(x)\cdot\partial_y\Big[\langle y\rangle^{\frac{\gamma}{3}}\phi(x,y)\Big],\\
\mathcal{A}_3 &= -\langle y\rangle^{\frac{\gamma}{3}}\,y\cdot\partial_x\Big(\langle \partial_x V(x)\rangle^{-4/3}\big(\partial_x V(x)\cdot y\phi(x,y)\big)\Big).
\end{aligned}$$

We will proceed to treat the above three terms . Firstly one has

$$\begin{aligned}
\mathcal{A}_1 &= \langle y\rangle^{\gamma/3}\langle \partial_x V(x)\rangle^{2/3}\phi(x,y) - \frac{\langle y\rangle^{\gamma/3}}{\langle \partial_x V(x)\rangle^{4/3}}\phi(x,y)\\
&= \langle y\rangle^{\gamma/3}\langle \partial_x V(x)\rangle^{2/3} - \langle y\rangle^{\gamma/3}\langle \partial_x V(x)\rangle^{2/3}\big(1-\phi(x,y)\big) - \frac{\langle y\rangle^{\gamma/3}}{\langle \partial_x V(x)\rangle^{4/3}}\phi(x,y),
\end{aligned}$$

from which it follows that

$$(\mathcal{A}_1 u,\ u)_{L^2} \geq \Big(\langle y\rangle^{\gamma/3}\langle \partial_x V(x)\rangle^{2/3}u, u\Big)_{L^2} - \big\|\langle y\rangle^{1+\gamma/2}u\big\|_{L^2}^2. \tag{23}$$

Here we used the facts that

$$\frac{\langle y\rangle^{\gamma/3}}{\langle \partial_x V(x)\rangle^{4/3}} \leq 1$$

on the support of ϕ, and $\langle \partial_x V(x)\rangle^{2/3} \leq \langle y\rangle^{2+2\gamma/3}$ on the support of $1-\phi$. As for the term $\mathcal{A}_2$ we make use of the relation

$$\forall\ \sigma\in\mathbb{R},\quad \partial_x V(x)\partial_y\big(\langle y\rangle^\sigma\big) = \sigma\langle y\rangle^{\sigma-2}\,\partial_x V(x)\cdot y,$$

to compute

$$
\begin{aligned}
\mathcal{A}_2 &= \langle \partial_x V(x)\rangle^{-\frac{4}{3}} |\partial_x V(x)\cdot y|^2 \left[\frac{\gamma}{3}\langle y\rangle^{\frac{\gamma}{3}-2}\phi + \frac{(2+\gamma)\langle y\rangle^{\gamma}}{\langle \partial_x V(x)\rangle^{2/3}}\chi'\left(\frac{\langle y\rangle^{2+2\gamma/3}}{\langle \partial_x V(x)\rangle^{2/3}}\right)\right]\\
&\gtrsim -\langle \partial_x V(x)\rangle^{-4/3}|\partial_x V(x)\cdot y|^2\langle \partial_x V(x)\rangle^{-2/3}\langle y\rangle^{\gamma}\\
&\gtrsim -\langle y\rangle^{2+\gamma},
\end{aligned}
$$

the first inequality using the fact that $\gamma \geq 0$ and hence the term $\frac{\gamma}{2}\langle y\rangle^{\frac{\gamma}{2}-2}\phi$ is nonnegative. As a result we conclude

$$
(\mathcal{A}_2 u,\ u)_{L^2} \geq -\left(\langle y\rangle^{2+\gamma} u,\ u\right)_{L^2}. \tag{24}
$$

For the term $\mathcal{A}_3$, using (7) gives

$$
\mathcal{A}_3 \geq -\langle y\rangle^{2+\gamma/3} \geq -\langle y\rangle^{2+\gamma},
$$

and thus

$$
(\mathcal{A}_3 u,\ u)_{L^2} \geq -\left(\langle y\rangle^{2+\gamma} u,\ u\right)_{L^2}.
$$

This along with (22), (23) and (24) shows that

$$
\begin{aligned}
\left(\langle y\rangle^{\gamma/3}\langle \partial_x V(x)\rangle^{2/3} u, u\right)_{L^2} &\lesssim \|\langle y\rangle^{1+\gamma/2} u\|_{L^2}^2 + |(Pu,\ u)_{L^2}| + |(Pu,\ \rho u)_{L^2}|\\
&\lesssim |(Pu,\ u)_{L^2}| + |(Pu,\ \rho u)_{L^2}|.
\end{aligned}
$$

Now for any $u \in C_0^\infty(\mathbb{R}^{2n})$, we use the above estimate to the function $\langle \partial_x V(x)\rangle^{1/3} u$; this gives

$$
\left(\langle y\rangle^{\gamma/3}\langle \partial_x V(x)\rangle^{4/3} u,\ u\right)_{L^2} \lesssim \|\langle \partial_x V(x)\rangle^{-1/3} P\langle \partial_x V(x)\rangle^{1/3} u\|_{L^2}\|\langle \partial_x V(x)\rangle^{2/3} u\|_{L^2},
$$

which, together with the fact that $\gamma \geq 0$, implies

$$
\begin{aligned}
\|\langle y\rangle^{\gamma/6}\langle \partial_x V(x)\rangle^{2/3} u\|_{L^2} &\lesssim \|\langle \partial_x V(x)\rangle^{-1/3} P\langle \partial_x V(x)\rangle^{1/3} u\|_{L^2}\\
&\lesssim \|Pu\|_{L^2} + \|\langle \partial_x V(x)\rangle^{-1/3}\left[P,\ \langle \partial_x V(x)\rangle^{1/3}\right] u\|_{L^2}.
\end{aligned}
$$

Moreover in view of (7) we have

$$
\|\langle \partial_x V(x)\rangle^{-1/3}\left[P,\ \langle \partial_x V(x)\rangle^{1/3}\right] u\|_{L^2} \lesssim \|\langle y\rangle u\|_{L^2} \lesssim \|Pu\|_{L^2} + \|u\|_{L^2}.
$$

Then the desired inequality (16) follows, completing the proof of Proposition 3.1. □

4 Hypoelliptic estimates for the operator with parameters

In this section *we always consider* $X = (x, \xi) \in \mathbb{R}^6$ *as parameters*, and study the operator acting on the velocity variable y:

$$
P_X = iQ_X + \left(B(y)D_y\right)^* \cdot B(y)D_y + F(y), \tag{25}
$$

where $Q_X = y\cdot\xi - \partial_x V(x)\cdot D_y$ and $B(y)$ is the matrix given in (3).

Notations Throughout this section, we will use $\|\cdot\|_{L^2}$ and $\left(\cdot,\ \cdot\right)_{L^2}$ to denote respectively the norm and inner product in the space $L^2(\mathbb{R}^3_y)$. Given a symbol p, we use p^{Wick} and p^w to denote the Wick and Weyl quantization of p in the variables (y, η).

The main result of this section is the following proposition.

Proposition 4.1 *Let* λ *be defined by*

$$\lambda = \Big(1 + |\partial_x V \wedge \eta + y \wedge \xi|^2 + |\partial_x V(x)|^2 + |\xi|^2 + |y|^6 + |\eta|^6 + \langle \partial_x V(x) \wedge \xi \rangle^{6/5}\Big)^{1/2}. \tag{26}$$

Then the following estimate

$$\begin{aligned}&\Big(\langle \partial_x V(x)\rangle^{2/3} + \langle \xi \rangle^{2/3}\Big) \|u\|_{L^2} + \| \langle y \rangle^{\gamma/2} |D_y|^2 u\|_{L^2} + \| \langle y \rangle^{\gamma/2} |y \wedge D_y|^2 u\|_{L^2} \\ &\quad + \|(\lambda^{2/3})^w u\|_{L^2} \lesssim \|P_X u\|_{L^2} + \|u\|_{L^2}\end{aligned} \tag{27}$$

holds for all $u \in \mathcal{S}(\mathbb{R}^3_y)$*, uniformly with respect to* X*.*

We will make use of the multiplier method introduced in [10], to show the above proposition through the following subsections.

4.1 Lemmas

Before the proof of Proposition 4.1, we list some lemmas.

Lemma 4.2 *Let* λ *be defined in (26). Then*

$$\forall\ \sigma \in \mathbb{R}, \quad \lambda^\sigma \in S(\lambda^\sigma, |dy|^2 + |d\eta|^2) \tag{28}$$

uniformly with respect to X*. Moreover if* $\sigma \leq 1$ *then the inequality*

$$\forall\ |\alpha| + |\beta| \geq 1, \quad \Big|\partial_y^\alpha \partial_\eta^\beta (\lambda^\sigma)\Big| \lesssim \langle \partial_x V(x) \rangle^\sigma + \langle \xi \rangle^\sigma \tag{29}$$

holds uniformly with respect to X*, and thus*

$$(\lambda^\sigma)^{\mathrm{Wick}} = (\lambda^\sigma)^w + (\langle \partial_x V(x) \rangle^\sigma + \langle \xi \rangle^\sigma) r^w, \tag{30}$$

with $r \in S(1, |dy|^2 + |d\eta|^2)$ *uniformly with respect to* X*.*

Proof. By direct verification we see that for all $(y, \eta) \in \mathbb{R}^6$ and all $\alpha, \beta \in \mathbb{Z}^3_+$ one has

$$\Big|\partial_y^\alpha \partial_\eta^\beta (\lambda(y,\eta)^2)\Big| \leq \lambda(y,\eta)^2,$$

which implies (28). Moreover note that

$$\forall\ |\alpha| + |\beta| \geq 1, \quad \Big|\partial_y^\alpha \partial_\eta^\beta (\lambda(y,\eta)^2)\Big| \leq (\langle \partial_x V(x) \rangle + \langle \xi \rangle) \lambda(y,\eta),$$

and thus

$$\forall\ \sigma \in \mathbb{R}, \quad \Big|\partial_y^\alpha \partial_\eta^\beta (\lambda(y,\eta)^\sigma)\Big| \lesssim |\sigma|\, \lambda^{\sigma-1} (\langle \partial_x V(x) \rangle + \langle \xi \rangle).$$

Then we get (29) if $\sigma \leq 1$, and thus (30) in view of (14), completing the proof of Lemma 4.2. □

Lemma 4.3 *Let* λ *be given in (26). Then for all* $u \in \mathcal{S}(\mathbb{R}^3)$ *one has*

$$\| \langle y \rangle^{\gamma/2} |D_y|^2 u\|_{L^2} + \| \langle y \rangle^{\gamma/2} |y \wedge D_y|^2 u\|_{L^2} \lesssim \|P_X u\|_{L^2} + \|\Phi^{2/3} u\|_{L^2} + \|(\lambda^{2/3})^w u\|_{L^2}, \tag{31}$$

where Φ *is defined by*

$$\Phi = \Phi(X) = (1 + |\partial_x V(x)|^2 + |\xi|^2)^{1/2}. \tag{32}$$

Proof. Similar to (6), we have, for any $u \in \mathcal{S}(\mathbb{R}^3_y)$,

$$\left\| \langle y \rangle^{1+\gamma/2} u \right\|_{L^2}^2 + \left\| \langle y \rangle^{\gamma/2} D_y u \right\|_{L^2}^2 + \left\| \langle y \rangle^{\gamma/2} \big(y \wedge D_y\big) u \right\|_{L^2}^2 \lesssim \mathrm{Re}\,(P_X u, u)_{L^2}\,. \tag{33}$$

Using the above inequality to $D_{y_j} u$ gives

$$\begin{aligned}\sum_{j,k=1}^{n} \left\| \langle y \rangle^{\gamma/2} D_{y_k} \cdot D_{y_j} u \right\|_{L^2}^2 &\lesssim \sum_{j=1}^{n} \left| \big(P_X D_{y_j} u,\ D_{y_j} u\big)_{L^2} \right| \\ &\lesssim \left| (P_X u,\ D_y \cdot D_y u)_{L^2} \right| + \sum_{j=1}^{n} \left| \big(\big[P_X,\ D_{y_j}\big] u,\ D_{y_j} u \big)_{L^2} \right|,\end{aligned}$$

which with the fact that $\gamma \geq 0$ implies

$$\sum_{j,k=1}^{n} \left\| \langle y \rangle^{\gamma/2} D_{y_k} \cdot D_{y_j} u \right\|_{L^2}^2 \lesssim \left\| P_X u \right\|_{L^2}^2 + \sum_{j=1}^{n} \left| \big(\big[P_X,\ D_{y_j}\big] u,\ D_{y_j} u \big)_{L^2} \right|. \tag{34}$$

So we only need to handle the last term in the above inequality. Direct verification shows

$$\big[P_X,\ D_{y_j}\big] = \xi_j + \big(\big(D_{y_j} B(y)\big) D_y\big)^* \cdot B(y) D_y + \big(B(y) D_y\big)^* \cdot \big(D_{y_j} B(y)\big) D_y + \big(D_{y_j} F(y)\big).$$

This gives

$$\sum_{j=1}^{n} \left| \big(\big[P,\ D_{y_j}\big] u,\ D_{y_j} u \big)_{L^2} \right| \leq \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3, \tag{35}$$

with

$$\begin{aligned}\mathcal{B}_1 &= \sum_{j=1}^{n} \left| \big(\xi_j u,\ D_{y_j} u \big)_{L^2} \right|, \\ \mathcal{B}_2 &= \sum_{j=1}^{n} \Big(\left| \big(B(y) D_y u,\ \big(D_{y_j} B\big) D_y D_{y_j} u \big)_{L^2} \right| + \left| \big(\big(D_{y_j} B\big) D_y u,\ B(y) D_y D_{y_j} u \big)_{L^2} \right| \Big), \\ \mathcal{B}_3 &= \sum_{j=1}^{n} \left| \big(\big(D_{y_j} F(y)\big) u,\ D_{y_j} u \big)_{L^2} \right|.\end{aligned}$$

By Parseval's theorem, we may write, denoting by $\hat{u}$ the Fourier transform with respect to y,

$$\left| \big(\xi_j u,\ D_{y_j} u \big)_{L^2} \right| = \left| (\xi_j \hat{u},\ \eta_j \hat{u})_{L^2(\mathbb{R}^6_\eta)} \right|,$$

and hence

$$\mathcal{B}_1 \leq \varepsilon \left\| D_y \cdot D_y u \right\|_{L^2}^2 + C_\varepsilon \left\| \langle \xi \rangle^{2/3} u \right\|_{L^2}^2,$$

due to the inequality

$$|\xi_j \eta_j| \leq \varepsilon\, |\eta|^2 + C_\varepsilon \,\langle \xi \rangle^{4/3}.$$

From (4) it follows that

$$\begin{aligned}\mathcal{B}_2 + \mathcal{B}_3 &\leq \varepsilon \sum_{j,k=1}^{n} \left\| \langle y \rangle^{\gamma/2} D_{y_k} D_{y_j} u \right\|_{L^2}^2 + C_\varepsilon \left\| \langle y \rangle^{1+\gamma/2} D_y u \right\|_{L^2}^2 \\ &\leq \varepsilon \sum_{j,k=1}^{n} \left\| \langle y \rangle^{\gamma/2} D_{y_k} D_{y_j} u \right\|_{L^2}^2 + C_\varepsilon \big(\left\| P_X u \right\|_{L^2}^2 + \left\| \langle \partial_x V(x) \rangle^{2/3} u \right\|_{L^2}^2 \big),\end{aligned}$$

the last inequality using lemma 3.3. Due to the arbitrariness of the number ε, the above inequalities along with (34) and (35) give the desired upper bound for the first term on the left hand side of (31).

It remains to treat the second term. In the following discussion we use the notation

$$T = (T_1, \cdots, T_n) = y \wedge D_y. \quad A = (A_1, A_2, A_3) = y \wedge \xi + \partial_x V(x) \wedge D_y.$$

From (33) it follows that

$$\begin{aligned}
&\sum_{j,k=1}^{n} \left(\left\| \langle y \rangle^{\gamma/2} T_k \cdot T_j u \right\|_{L^2}^2 + \left\| D_{y_k} \cdot T_j u \right\|_{L^2}^2 + \left\| y_k \cdot T_j u \right\|_{L^2}^2 \right) \\
&\leq \sum_{j=1}^{n} \left| (P_X T_j u,\ T_j u)_{L^2} \right| \leq \left| (P_X u,\ T \cdot T u)_{L^2} \right| + \left| \big([P_X,\ T] u,\ T u \big)_{L^2} \right|,
\end{aligned}$$

which with the fact that $\gamma \geq 0$ implies

$$\begin{aligned}
&\sum_{j,k=1}^{n} \left(\left\| \langle y \rangle^{\gamma/2} T_k \cdot T_j u \right\|_{L^2}^2 + \left\| D_{y_k} \cdot T_j u \right\|_{L^2}^2 + \left\| y_k \cdot T_j u \right\|_{L^2}^2 \right) \\
&\lesssim \left\| P_X u \right\|_{L^2}^2 + \left| \big([P_X,\ T] u,\ T u \big)_{L^2} \right|.
\end{aligned} \tag{36}$$

In order to handle the last term in the above inequality, we write

$$\begin{aligned}
\big[P,\ T_j\big] &= -A_j + \big[D_y,\ T_j\big] \cdot \nu(y) D_y + D_y \cdot \nu(y) \big[D_y,\ T_j\big] + D_y \cdot \big(T_j \nu(y)\big) D_y \\
&\quad + \big[T,\ T_j\big] \cdot \mu(y) T + T \cdot (T_j \mu(y)) T + T \cdot \mu(y) \big[T,\ T_j\big] + \big(T_j F(y)\big).
\end{aligned}$$

This gives

$$\left| \big([P,\ T] u,\ T u \big)_{L^2} \right| \leq \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_4, \tag{37}$$

with

$$\begin{aligned}
\mathcal{N}_1 &= \left| (A u,\ T u)_{L^2} \right|, \\
\mathcal{N}_2 &= \sum_{j=1}^{n} \left| \Big(\big[D_y,\ T_j\big] \cdot \nu(y) D_y u + D_y \cdot \nu(y) \big[D_y,\ T_j\big] u + D_y \cdot \big(T_j \nu(y)\big) D_y u,\ T_j u \Big)_{L^2} \right|, \\
\mathcal{N}_3 &= \sum_{j=1}^{n} \left| \Big(\big[T,\ T_j\big] \cdot \mu(y) T u + T \cdot (T_j \mu(y)) T u + T \cdot \mu(y) \big[T,\ T_j\big] u,\ T_j u \Big)_{L^2} \right|, \\
\mathcal{N}_4 &= \sum_{j=1}^{n} \left| \big((\partial_{y_j} F(y)) u,\ D_{y_j} u \big)_{L^2} \right|.
\end{aligned}$$

Next we treat the above four terms. For the term $\mathcal{N}_1$ one has, with λ defined in (26),

$$\begin{aligned}
(A u,\ T u)_{L^2} &\leq \varepsilon \left\| \big(\lambda^{1/3}\big)^w T u \right\|_{L^2}^2 + C_\varepsilon \left\| \big(\lambda^{-1/3}\big)^w A u \right\|_{L^2}^2 \\
&\leq \varepsilon \left\| \big(\lambda^{1/3}\big)^w T u \right\|_{L^2}^2 + C_\varepsilon \left\| \big(\lambda^{2/3}\big)^w u \right\|_{L^2}^2,
\end{aligned}$$

the last inequality holding because

$$\big(\lambda^{-1/3}\big)^w A \big(\lambda^{-2/3}\big)^w \in \mathrm{Op}\big(S(1,\ |dy|^2 + |d\eta|^2)\big).$$

On the other hand,

$$\begin{aligned}\big\|(\lambda^{1/3})^w Tu\big\|_{L^2}^2 &\lesssim \Big|\Big((\lambda^{2/3})^w Tu,\ Tu\Big)_{L^2}\Big| \\ &\lesssim \Big|\Big((\lambda^{2/3})^w u,\ T\cdot Tu\Big)_{L^2}\Big| + \Big|\Big([(\lambda^{2/3})^w,\ T]u,\ Tu\Big)_{L^2}\Big| \\ &\lesssim \varepsilon\big\|T\cdot Tu\big\|_{L^2} + C_\varepsilon\big\|(\lambda^{2/3})^w u\big\|_{L^2} + \Big|\Big([(\lambda^{2/3})^w,\ T]u,\ Tu\Big)_{L^2}\Big|.\end{aligned}$$

Observing (28), symbolic calculus give that

$$\big[(\lambda^{2/3})^w,\ T\big] = \big[(\lambda^{2/3})^w,\ y\wedge D_y\big] = D_y b_1^w + y b_2^w + b_3^w,$$

with b_j, $1\le j\le 3$, belonging to $S\big(\lambda^{2/3}, |dy|^2+|d\eta|^2\big)$ uniformly with respect to X. This shows

$$\Big|\Big([(\lambda^{2/3})^w,\ T]u,\ Tu\Big)_{L^2}\Big| \lesssim \varepsilon\sum_{j,k=1}^n \Big(\big\|D_{y_k}\cdot T_j u\big\|_{L^2}^2 + \big\|y_k\cdot T_j u\big\|_{L^2}^2\Big) + C_\varepsilon\big\|(\lambda^{2/3})^w u\big\|_{L^2}^2.$$

Combining the above inequalities, we have

$$\mathcal{N}_1 \lesssim \varepsilon\sum_{j,k=1}^n \Big(\big\|\langle y\rangle^{\gamma/2} T_k\cdot T_j u\big\|_{L^2}^2 + \big\|D_{y_k}\cdot T_j u\big\|_{L^2}^2 + \big\|y_k\cdot T_j u\big\|_{L^2}^2\Big) + C_\varepsilon\big\|(\lambda^{2/3})^w u\big\|_{L^2}^2.$$

Direct verification shows

$$\big[T_j,\ D_{y_k}\big] = \sum_\ell a_{j,k}^\ell D_{y_\ell},\quad \big[T_1,\ T_2\big] = iT_3,\quad \big[T_3,\ T_1\big] = iT_2,\quad \big[T_2,\ T_3\big] = iT_1,$$

with $a_{j,k}^\ell\in\{0,-1,+1\}$, and thus

$$\begin{aligned}\mathcal{N}_2+\mathcal{N}_3 &\lesssim \varepsilon\sum_{j,k=1}^n \Big(\big\|\langle y\rangle^{\gamma/2} T_k\cdot T_j u\big\|_{L^2}^2 + \big\|\langle y\rangle^{\gamma/2} D_{y_k}\cdot T_j u\big\|_{L^2}^2 + \big\|\langle y\rangle^{\gamma/2} y_k\cdot T_j u\big\|_{L^2}^2\Big) \\ &\quad + C_\varepsilon\big(\big\|\langle y\rangle^{\gamma/2} Tu\big\|_{L^2}^2 + \big\|\langle y\rangle^{\gamma/2} D_y u\big\|_{L^2}^2 + \big\|\langle y\rangle^{\gamma/2} yu\big\|_{L^2}^2\big) \\ &\lesssim \varepsilon\sum_{j,k=1}^n \Big(\big\|\langle y\rangle^{\gamma/2} T_k\cdot T_j u\big\|_{L^2}^2 + \big\|\langle y\rangle^{\gamma/2} D_{y_k}\cdot T_j u\big\|_{L^2}^2 + \big\|\langle y\rangle^{\gamma/2} y_k\cdot T_j u\big\|_{L^2}^2\Big) \\ &\quad + C_\varepsilon\big(\big\|Pu\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2\big),\end{aligned}$$

the last inequality using (6). It remains to treat $\mathcal{N}_4$, and by (1) and (6) we have

$$\mathcal{N}_4 \lesssim \big\|\langle y\rangle^{1+\gamma/2} u\big\|_{L^2}^2 + \big\|\langle y\rangle^{\gamma/2} D_y u\big\|_{L^2}^2 \lesssim \big(\big\|P_X u\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2\big).$$

Combining the above estimates, we conclude

$$\begin{aligned}&\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_4+\mathcal{N}_4 \\ &\lesssim \varepsilon\sum_{j,k=1}^n \Big(\big\|\langle y\rangle^{\gamma/2} T_k\cdot T_j u\big\|_{L^2}^2 + \big\|\langle y\rangle^{\gamma/2} D_{y_k}\cdot T_j u\big\|_{L^2}^2 + \big\|\langle y\rangle^{\gamma/2} y_k\cdot T_j u\big\|_{L^2}^2\Big) \\ &\quad + C_\varepsilon\Big(\big\|P_X u\big\|_{L^2}^2 + \big\|(\lambda^{2/3})^w u\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2\Big).\end{aligned}$$

This along with (36) and (37) yields the desired upper bound for $\big\|\langle y\rangle^{\gamma/2}|y\wedge D_y|^2\big\|_{L^2}$, letting ε small enough. The proof of Lemma 4.3 is thus complete. □

Lemma 4.4 *Let $p \in S(1, |dy|^2 + |d\eta|^2)$ uniformly with respect to X, and let λ be defined in (26). Then for any $\varepsilon > 0$ there exists a constant C_ε, such that*

$$\Big(P_X\big(\lambda^{1/3}\big)^w u,\ p^w\big(\lambda^{1/3}\big)^w u\Big)_{L^2} \lesssim \varepsilon \big\|\big(\lambda^{2/3}\big)^w u\big\|_{L^2}^2 + C_\varepsilon\left\{\big\|P_X u\big\|_{L^2}^2 + \big\|\Phi^{2/3}u\big\|_{L^2}^2 + \big\|\left\langle \partial_x V(x)\wedge\xi\right\rangle^{2/5}u\big\|_{L^2}^2\right\}, \tag{38}$$

where Φ is given in (32).

Proof. As a preliminary step we firstly show that for any $\varepsilon, \tilde\varepsilon > 0$ there exists a constant $C_{\varepsilon,\tilde\varepsilon}$, such that

$$\big(\big[P_X,\ \big(\lambda^{1/3}\big)^w\big]u,\ a^w\big(\lambda^{1/3}\big)^w u\big)_{L^2} \lesssim\ \varepsilon\big(P_X\big(\lambda^{1/3}\big)^w u,\ \big(\lambda^{1/3}\big)^w u\big)_{L^2} \tag{39}$$

$$+\tilde\varepsilon\big\|\big(\lambda^{2/3}\big)^w u\big\|_{L^2}^2 + C_{\varepsilon.\tilde\varepsilon}\left\{\big\|\Phi^{2/3}u\big\|_{L^2}^2 + \big\|\left\langle\partial_x V(x)\wedge\xi\right\rangle^{2/5}u\big\|_{L^2}^2\right\}, \tag{40}$$

where a is an arbitrary symbol belonging to $S(1, |dy|^2 + |d\eta|^2)$ uniformly with respect to X. Observing (2) and (29), symbolic calculus (see for instance Theorem 2.3.8 in [16]) shows that the symbols of the commutators

$$\big[\nu(y),\ \big(\lambda^{1/3}\big)^w\big] \quad \text{and} \quad \big[\mu(y),\ \big(\lambda^{1/3}\big)^w\big]$$

belong to $S\big(\Phi^{1/3}, |dy|^2 + |d\eta|^2\big)$ uniformly with respect to X. As a result, using the notation

$$\begin{aligned}
\mathcal{Z}_1 &= \Big(D_y\cdot\big[\nu(y),\ \big(\lambda^{1/3}\big)^w\big]D_y u,\ a^w\big(\lambda^{1/3}\big)^w u\Big)_{L^2},\\
\mathcal{Z}_2 &= \Big(\big(y\wedge D_y\big)\cdot\big[\mu(y),\ \big(\lambda^{1/3}\big)^w\big]\big(y\wedge D_y\big)u,\ a^w\big(\lambda^{1/3}\big)^w u\Big)_{L^2},
\end{aligned}$$

we have

$$\begin{aligned}
\mathcal{Z}_1 + \mathcal{Z}_2 &\leq \varepsilon\big\|\left\langle D_y\right\rangle a^w\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2 + \varepsilon\big\|\big(y\wedge D_y\big)a^w\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2\\
&\quad + C_\varepsilon\big\|\left\langle D_y\right\rangle\Phi^{1/3}u\big\|_{L^2}^2 + C_\varepsilon\big\|\big(y\wedge D_y\big)\Phi^{1/3}u\big\|_{L^2}^2\\
&\leq \varepsilon\big\|\big(\left\langle D_y\right\rangle + \left\langle y\right\rangle\big)\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2 + \varepsilon\big\|\big(y\wedge D_y\big)\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2\\
&\quad + C_\varepsilon\Phi^{2/3}\big\|\left\langle D_y\right\rangle u\big\|_{L^2}^2 + C_\varepsilon\Phi^{2/3}\big\|\big(y\wedge D_y\big)u\big\|_{L^2}^2,
\end{aligned}$$

the last inequality holding because

$$\big[D_y,\ a^w\big],\quad \big[y\wedge D_y,\ a^w\big]\Big(\big(1+\left\langle y\right\rangle+\left\langle\eta\right\rangle\big)^{-1}\Big)^w \in \mathrm{Op}\big(S\big(1, |dy|^2 + |d\eta|^2\big)\big),$$

since $a \in S\big(1, |dy|^2 + |d\eta|^2\big)$ uniformly with respect to X. Moreover using (33) gives

$$\mathcal{Z}_1 + \mathcal{Z}_2 \leq \varepsilon\Big(P_X\big(\lambda^{1/3}\big)^w u,\ \big(\lambda^{1/3}\big)^w u\Big)_{L^2} + C_\varepsilon\left\{\big\|P_X u\big\|_{L^2}^2 + \big\|\Phi^{2/3}u\big\|_{L^2}^2\right\}. \tag{41}$$

Denote

$$\begin{aligned}
\mathcal{Z}_3 &= \Big(\big[D_y,\ \big(\lambda^{1/3}\big)^w\big]\cdot\nu(y)D_y u,\ a^w\big(\lambda^{1/3}\big)^w u\Big)_{L^2}\\
&\quad + \Big(D_y\cdot\nu(y)\big[D_y,\ \big(\lambda^{1/3}\big)^w\big]u,\ a^w\big(\lambda^{1/3}\big)^w u\Big)_{L^2},\\
\mathcal{Z}_4 &= \Big(\big[y\wedge D_y,\ \big(\lambda^{1/3}\big)^w\big]\cdot\mu(y)\big(y\wedge D_y\big)u,\ a^w\big(\lambda^{1/3}\big)^w u\Big)_{L^2}\\
&\quad + \Big(\big(y\wedge D_y\big)\cdot\mu(y)\big[y\wedge D_y,\ \big(\lambda^{1/3}\big)^w\big]u,\ a^w\big(\lambda^{1/3}\big)^w u\Big)_{L^2}.
\end{aligned}$$

Observing (29), symbolic calculus gives that

$$\big[D_y,\ (\lambda^{1/3})^w\big]=a_1^w,\quad \big[(y\wedge D_y),\ (\lambda^{1/3})^w\big]=a_2^w D_y+a_3^w y+a_4^w,$$

with a_j, $1\leq j\leq 4$, belonging to $S\big(\Phi^{1/3},|dy|^2+|d\eta|^2\big)$ uniformly with respect to X. It then follows that

$$\begin{aligned}\mathcal{Z}_3+\mathcal{Z}_4 \ \ \leq\ \ & \varepsilon\big\|\left\langle D_y\right\rangle a^w\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2+\varepsilon\big\|\left\langle y\right\rangle^{1+\gamma/2}a^w\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2\\ &+\varepsilon\big\|\left\langle y\right\rangle^{\gamma/2}\left\langle D_y\right\rangle a^w\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2+\varepsilon\big\|\left\langle y\right\rangle^{\gamma/2}\left\langle y\wedge D_y\right\rangle a^w\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2\\ &+C_\varepsilon\big\|\left\langle y\right\rangle^{\gamma/2}\big(\left\langle y\right\rangle+\left\langle D_y\right\rangle\big)\Phi^{1/3}u\big\|_{L^2}^2+C_\varepsilon\big\|\left\langle y\right\rangle^{\gamma/2}\big(y\wedge D_y\big)\Phi^{1/3}u\big\|_{L^2}^2.\end{aligned}$$

Using similar arguments as the treatment of $\mathcal{Z}_1$ and $\mathcal{Z}_2$, we conclude

$$\mathcal{Z}_3+\mathcal{Z}_4\leq\varepsilon\left(P_X\big(\lambda^{1/3}\big)^w u,\ \big(\lambda^{1/3}\big)^w u\right)_{L^2}+C_\varepsilon\left\{\big\|P_X u\big\|_{L^2}^2+\big\|\Phi^{2/3}u\big\|_{L^2}^2\right\}.$$

This along with (41) gives

$$\begin{aligned}&\left(\big[\big(B(y)D_y\big)^*B(y)D_y,\ \big(\lambda^{1/3}\big)^w\big]u,\ a^w\big(\lambda^{1/3}\big)^w u\right)_{L^2}\\ &\lesssim\ \varepsilon\ \left(P\big(\lambda^{1/3}\big)^w u,\ \big(\lambda^{1/3}\big)^w u\right)_{L^2}+C_\varepsilon\left\{\big\|P_X u\big\|_{L^2}^2+\big\|\Phi^{2/3}u\big\|_{L^2}^2\right\},\end{aligned}\tag{42}$$

since

$$\left(\big[\big(B(y)D_y\big)^*B(y)D_y,\ \big(\lambda^{1/3}\big)^w\big]u,\ a^w\big(\lambda^{1/3}\big)^w u\right)_{L^2}=\sum_{1\leq j\leq 4}\mathcal{Z}_j.$$

Moreover we have

$$\begin{aligned}&\left(\big[F(y),\ \big(\lambda^{1/3}\big)^w\big]u,\ a^w\big(\lambda^{1/3}\big)^w u\right)_{L^2}\\ &\lesssim\varepsilon\left(P\big(\lambda^{1/3}\big)^w u,\ \big(\lambda^{1/3}\big)^w u\right)_{L^2}+C_\varepsilon\left(\big\|P_X u\big\|_{L^2}^2+\big\|\Phi^{2/3}u\big\|_{L^2}^2\right),\end{aligned}\tag{43}$$

which can be deduced similarly as above, since by (2),

$$\big[F(y),\ \big(\lambda^{1/3}\big)^w\big]\in\mathrm{Op}\big(S\big(\left\langle y\right\rangle^{1+\gamma}\Phi^{1/3},|dy|^2+|d\eta|^2\big)\big)$$

uniformly with respect to X. Next we treat the commutator $\big[iQ_X,\ \big(\lambda^{1/3}\big)^w\big]$, whose symbol is

$$-\frac{\lambda^{1/3-2}}{6}\Big[2\big(\partial_x V(x)\wedge\eta+y\wedge\xi\big)\cdot\big(\partial_x V(x)\wedge\xi\big)+3\left|y\right|^4\partial_x V(x)\cdot y+3\left|\eta\right|^4\xi\cdot\eta\Big].$$

In view of (28) and (26), one could verify that the above symbol belongs to

$$S\left(\left\langle\partial_x V(x)\wedge\xi\right\rangle^{2/5}\lambda^{1/3}+\Phi^{2/3}\lambda^{1/3},\ |dy|^2+|d\eta|^2\right)$$

uniformly with respect to X. As a result, observing $\lambda^{1/3}\in S(\lambda^{1/3},\ |dy|^2+|d\eta|^2)$ uniformly with respect to X, we have

$$\big(\lambda^{-1/3}\big)^w a^w\big[iQ_X,\ \big(\lambda^{1/3}\big)^w\big]\in\mathrm{Op}\big(S\big(\left\langle\partial_x V(x)\wedge\xi\right\rangle^{2/5}+\Phi^{2/3},\ |dy|^2+|d\eta|^2\big)\big)$$

uniformly with respect to X, which implies, with $\tilde\varepsilon$ arbitrarily small,

$$\begin{aligned}&\left(\big[iQ_X,\ \big(\lambda^{1/3}\big)^w\big]u,\ a^w\big(\lambda^{1/3}\big)^w u\right)_{L^2}\\ &\qquad\lesssim\tilde\varepsilon\big\|\big(\lambda^{2/3}\big)^w\big\|_{L^2}^2+C_{\tilde\varepsilon}\left(\big\|\left\langle\partial_x V(x)\wedge\xi\right\rangle^{2/5}u\big\|_{L^2}^2+\big\|\Phi^{2/3}u\big\|_{L^2}^2\right).\end{aligned}$$

This along with (42) and (43) gives (39), since

$$\big[P_X,\ (\lambda^{1/3})^w\big] = \big[iQ_X,\ (\lambda^{1/3})^w\big] + \big[(B(y)D_y)^* B(y)D_y + F(y),\ (\lambda^{1/3})^w\big].$$

Next we prove (38). The relation

$$\begin{aligned} & \operatorname{Re}\Big(P(\lambda^{1/3})^w u,\ (\lambda^{1/3})^w u\Big)_{L^2} + \operatorname{Re}\Big(P(\lambda^{1/3})^w u,\ p^w(\lambda^{1/3})^w u\Big)_{L^2} \\ = \ & \operatorname{Re}\Big(Pu,\ (\lambda^{1/3})^w(\mathrm{Id}+p^w)(\lambda^{1/3})^w u\Big)_{L^2} + \operatorname{Re}\Big(\big[P,\ (\lambda^{1/3})^w\big]u,\ (\mathrm{Id}+p^w)(\lambda^{1/3})^w u\Big)_{L^2} \end{aligned}$$

gives, with $\tilde\varepsilon > 0$ arbitrary,

$$\begin{aligned} & \operatorname{Re}\Big(P(\lambda^{1/3})^w u,\ (\lambda^{1/3})^w u\Big)_{L^2} + \operatorname{Re}\Big(P(\lambda^{1/3})^w u,\ p^w(\lambda^{1/3})^w u\Big)_{L^2} \\ \lesssim \ & \tilde\varepsilon \big\|(\lambda^{2/3})^w\big\|_{L^2}^2 + C_{\tilde\varepsilon}\big\|P_X u\big\|_{L^2}^2 + \operatorname{Re}\Big(\big[P,\ (\lambda^{1/3})^w\big]u,\ (\mathrm{Id}+p^w)(\lambda^{1/3})^w u\Big)_{L^2}. \end{aligned}$$

We could apply (39) with $a = 1+p$ to control the last term in the above inequality; this gives, with $\varepsilon, \tilde\varepsilon > 0$ arbitrarily small,

$$\begin{aligned} & \operatorname{Re}\Big(P(\lambda^{1/3})^w u,\ (\lambda^{1/3})^w u\Big)_{L^2} + \operatorname{Re}\Big(P(\lambda^{1/3})^w u,\ p^w(\lambda^{1/3})^w u\Big)_{L^2} \\ & \lesssim \varepsilon\Big(P(\lambda^{1/3})^w u,\ (\lambda^{1/3})^w u\Big)_{L^2} + \tilde\varepsilon\big\|(\lambda^{2/3})^w\big\|_{L^2}^2 \\ & \quad + C_{\varepsilon,\tilde\varepsilon}\Big(\big\|P_X u\big\|_{L^2}^2 + \big\|\Phi^{2/3}u\big\|_{L^2}^2 + \big\|\left\langle \partial_x V(x)\wedge\xi\right\rangle^{2/5} u\big\|_{L^2}^2\Big). \end{aligned}$$

Letting ε small enough yields the desired estimate (38). The proof is thus complete. □

4.2 Proof of Proposition 4.1

In what follows, let h_N, with N a large integer, be a symbol defined by

$$h_N = h_N(y,\eta) = \frac{\partial_x V(x)\cdot y + \xi\cdot\eta + \big(\partial_x V\wedge\eta + y\wedge\xi\big)\cdot\big(y\wedge\eta\big)}{\lambda_N^{4/3}}\psi_N(y,\eta), \tag{44}$$

where

$$\lambda_N = \Big(1 + |\partial_x V\wedge\eta + y\wedge\xi|^2 + |\partial_x V(x)|^2 + |\xi|^2 + N^{-1}\left\langle\partial_x V(x)\wedge\xi\right\rangle^{6/5}\Big)^{1/2}, \tag{45}$$

and

$$\psi_N(y,\eta) \ = \ \chi\left(\frac{\big(\,|y\wedge\eta|^2 + |y|^{2+\gamma} + |\eta|^2\,\big)N^2}{\lambda_N^{2/3}}\right), \tag{46}$$

with $\chi \in C_0^\infty(\mathbb{R};\ [0,1])$ such that $\chi = 1$ in $[-1,1]$ and supp $\chi \subset [-2,2]$.

Lemma 4.5 *Let λ_N be given in (45). Then*

$$\forall\ \sigma\in\mathbb{R},\quad \lambda_N^\sigma \in S(\lambda_N^\sigma, |dy|^2 + |d\eta|^2) \tag{47}$$

uniformly with respect to X. Moreover if $\sigma \le 1$ then

$$\forall\ |\alpha|+|\beta|\ge 1,\quad \Big|\partial_y^\alpha\partial_\eta^\beta\big(\lambda_N^\sigma\big)\Big| \lesssim \left\langle\partial_x V(x)\right\rangle^\sigma + \left\langle\xi\right\rangle^\sigma. \tag{48}$$

Proof. The proof is the same as that of Lemma 4.2. □

Lemma 4.6 *The symbol h_N given in (44) belongs to $S(1, |dy|^2 + |d\eta|^2)$ uniformly with respect to X.*

Proof. It is just a straightforward verification by (47). □

Lemma 4.7 *Let λ_N and ψ_N be given in (45) and (46). Then for any $\sigma \in \mathbb{R}$ the following two inequalities*

$$\big|\big(\xi \cdot \partial_\eta + \partial_x V(x) \cdot \partial_y\big)\lambda_N^\sigma\big| \lesssim N\lambda_N^{\sigma+\frac{2}{3}} \tag{49}$$

and

$$\big|\big(\xi \cdot \partial_\eta + \partial_x V(x) \cdot \partial_y\big)\psi_N\big| \lesssim N^3\big(\, |y \wedge \eta|^2 + |y|^2 + |\eta|^2\,\big) \tag{50}$$

hold uniformly with respect to (x, ξ).

Proof. Using the inequality $\langle \partial_x V \wedge \xi \rangle \leq N^{5/6}\lambda_N^{5/3}$ due to (45), we can verify

$$\big|\xi \cdot \partial_\eta\big(\lambda_N^2\big)\big| + \big|\partial_x V(x) \cdot \partial_y\big(\lambda_N^2\big)\big| \lesssim \lambda_N \,\langle \partial_x V \wedge \xi \rangle \lesssim N\lambda_N^{2+2/3}.$$

Then for any $\sigma \in \mathbb{R}$ one has

$$\big|\xi \cdot \partial_\eta\big(\lambda_N^\sigma\big)\big| + \big|\partial_x V(x) \cdot \partial_y\big(\lambda_N^\sigma\big)\big| \lesssim N\lambda_N^\sigma\lambda_N^{2/3}.$$

Thus (49) follows. In order to show (50), we write

$$\big|\big(\xi \cdot \partial_\eta + \partial_x V(x) \cdot \partial_y\big)\psi_N\big| = \big(\mathcal{K}_1 + \mathcal{K}_2\big),$$

with

$$\mathcal{K}_1 = N^2\left|\lambda_N^{-\frac{2}{3}}\big(\xi \cdot \partial_\eta + \partial_x V(x) \cdot \partial_y\big)\big[\,|y \wedge \eta|^2 + |y|^2 + |\eta|^2\,\big]\chi'\Big((|y \wedge \eta|^2 + |y|^2 + |\eta|^2)N^2\lambda_N^{-2/3}\Big)\right|,$$

$$\mathcal{K}_2 = N^2\bigg|\big(\,|y \wedge \eta|^2 + |y|^2 + |\eta|^2\,\big)\big[\big(\xi \cdot \partial_\eta + \partial_x V(x) \cdot \partial_y\big)\lambda_N^{-2/3}\big] \chi'\Big((|y \wedge \eta|^2 + |y|^2 + |\eta|^2)N^2\lambda_N^{-2/3}\Big)\bigg|.$$

Using (49) shows

$$\mathcal{K}_2 \lesssim N^3\big(\,|y \wedge \eta|^2 + |y|^2 + |\eta|^2\,\big).$$

Moreover direct computation gives

$$\mathcal{K}_1 \lesssim N^2\lambda_N^{1/3}\big(\,|y \wedge \eta| + |y| + |\eta|\,\big)\chi'\Big((|y \wedge \eta|^2 + |y|^2 + |\eta|^2)N^2\lambda_N^{-2/3}\Big) \lesssim N^3\big(\,|y \wedge \eta|^2 + |y|^2 + |\eta|^2\,\big),$$

the last inequality following from the fact that $\lambda_N^{2/3} \lesssim \big(\,|y \wedge \eta|^2 + |y|^2 + |\eta|^2\,\big)N^2$ on the support of the function

$$\chi'\Big((|y \wedge \eta|^2 + |y|^2 + |\eta|^2)N^2\lambda_N^{-2/3}\Big).$$

Then the above inequalities yield the desired inequality (50). The proof of Lemma 4.7 is thus complete. □

Proof of Proposition 4.1. This will occupy the rest of this section. Since the proof is quite long, we divide it into three steps.

Step I) Let N be a large integer to be determined later and $H = h_N^{\rm Wick}$ be the Wick quantization of the symbol h_N given in (44). To simplify the notation we will use C_N to denote different suitable constants which depend only on N. In the following discussion, let $u \in \mathcal{S}(\mathbb{R}^3_y)$. By (14) and Lemma 4.6 we can find a symbol $\tilde h_N$ such that $H = \tilde h_N^w$ with $\tilde h_N \in S(1, |dy|^2 + |d\eta|^2)$ uniformly with respect to X. Then using (17) gives

$$\left|\big((B(y)D_y)^* B(y)D_y u,\ Hu\big)_{L^2} + (Fu,\ Hu)_{L^2}\right| \lesssim |(P_X u,\ u)_{L^2}|\,.$$

This together with the relation

$$\begin{aligned}\mathrm{Re}\ (iQ_X u,\ H\,u)_{L^2} = \mathrm{Re}\ (P_X u,\ H\,u)_{L^2} - \mathrm{Re}\ \big((B(y)D_y)^* B(y)D_y u,\ Hu\big)_{L^2}\\ - \mathrm{Re}\ (Fu,\ Hu)_{L^2}\end{aligned}$$

yields

$$\mathrm{Re}\ (iQ_X u,\ H\,u)_{L^2} \lesssim |(P_X u,\ u)_{L^2}| + |(P_X u, Hu)_{L^2}|\,. \tag{51}$$

Next we give a lower bound for the term on the left side. Observe the symbol of Q_X is a first order polynomial in y, η. Then

$$iQ_X = i\big(y\cdot\xi - \partial_x V(x)\cdot\eta\big)^{\rm Wick},$$

and hence

$$\mathrm{Re}\ (iQ_X u,\ H\,u)_{L^2} = \frac{1}{4\pi}\Big(\big\{h,\ y\cdot\xi - \partial_x V(x)\cdot\eta\big\}^{\rm Wick} u,\ u\Big)_{L^2}, \tag{52}$$

where $\{\cdot,\ \cdot\}$ is the Poisson bracket defined in (15). Direct calculus shows

$$\begin{aligned}
&\big\{h,\ y\cdot\xi - \partial_x V(x)\cdot\eta\big\}\\
&= \frac{|\partial_x V(x)\wedge\eta + y\wedge\xi|^2 + |\partial_x V(x)|^2 + |\xi|^2 + 2\big(\partial_x V\wedge\xi\big)\cdot\big(y\wedge\eta\big)}{\lambda_N^{4/3}}\psi_N\\
&\quad + \big(\partial_x V\wedge\eta + y\wedge\xi\big)\cdot\big(y\wedge\eta\big)\big[\big(\xi\cdot\partial_\eta + \partial_x V\cdot\partial_y\big)\big(\lambda_N^{-4/3}\psi_N\big)\big]\\
&= \lambda_N^{2/3}\psi_N - \frac{1 + N^{-1}\left\langle\partial_x V(x)\wedge y\right\rangle^{6/5}}{\lambda_N^{4/3}}\psi_N + \frac{2\big(\partial_x V\wedge\xi\big)\cdot\big(y\wedge\eta\big)}{\lambda_N^{4/3}}\psi_N\\
&\quad + \big(\partial_x V\wedge\eta + y\wedge\xi\big)\cdot\big(y\wedge\eta\big)\big[\big(\xi\cdot\partial_\eta + \partial_x V\cdot\partial_y\big)\big(\lambda_N^{-4/3}\psi_N\big)\big]\\
&\geq \lambda_N^{2/3} - \lambda_N^{2/3}(1-\psi_N) - \frac{1 + N^{-1}\left\langle\partial_x V\wedge\xi\right\rangle^{6/5}}{\lambda_N^{4/3}} - \frac{2\left|\big(\partial_x V\wedge\xi\big)\cdot\big(y\wedge\eta\big)\right|}{\lambda_N^{4/3}}\psi_N\\
&\quad - \Big|\big(\partial_x V\wedge\eta + y\wedge\xi\big)\cdot\big(y\wedge\eta\big)\big[\big(\xi\cdot\partial_\eta + \partial_x V\cdot\partial_y\big)\big(\lambda_N^{-4/3}\psi_N\big)\big]\Big|\\
&\geq \lambda_N^{2/3} - N^2\big(\,|y\wedge\eta|^2 + |y|^2 + |\eta|^2\,\big) - \frac{1 + N^{-1}\left\langle\partial_x V\wedge\xi\right\rangle^{6/5}}{\lambda_N^{4/3}}\\
&\quad - \frac{2\left|\big(\partial_x V\wedge\xi\big)\cdot\big(y\wedge\eta\big)\right|}{\lambda_N^{4/3}}\psi_N\\
&\quad - \Big|\big(\partial_x V\wedge\eta + y\wedge\xi\big)\cdot\big(y\wedge\eta\big)\big[\big(\xi\cdot\partial_\eta + \partial_x V\cdot\partial_y\big)\big(\lambda_N^{-4/3}\psi_N\big)\big]\Big|,
\end{aligned}$$

the last inequality holding because $\lambda_N^{2/3} \leq N^2\big(|y\wedge\eta|^2+|y|^2+|\eta|^2\big)$ on the support of $1-\psi_N$. Due to the positivity of the Wick quantization, the above inequalities, along with (51), (52) and the estimate

$$\Big(\big(|y\wedge\eta|^2+|y|^2+|\eta|^2\big)^{\rm Wick}u,\ u\Big)_{L^2} \lesssim |(P_X u,\ u)_{L^2}| + \|u\|_{L^2}^2 \tag{53}$$

due to (33), yield

$$\Big(\big(\lambda_N^{2/3}\big)^{\rm Wick}u,\ u\Big)_{L^2 R_y^3)} \lesssim \sum_{j=1}^{3}\Big(R_j^{\rm Wick}u,\ u\Big)_{L^2} + |(P_X u,\ u)_{L^2}| + |(P_X u,\ Hu)_{L^2}| + \|u\|_{L^2}^2, \tag{54}$$

where R_j are given by

$$\begin{aligned} R_1 &= \frac{1+N^{-1}\left\langle \partial_x V\wedge\xi\right\rangle^{6/5}}{\lambda_N^{4/3}},\\ R_2 &= \frac{2\big|\big(\partial_x V\wedge\xi\big)\cdot\big(y\wedge\eta\big)\big|}{\lambda_N^{4/3}}\psi_N,\\ R_3 &= \Big|\big(\partial_x V\wedge\eta+y\wedge\xi\big)\cdot\big(y\wedge\eta\big)\big[\big(\xi\cdot\partial_\eta+\partial_x V\cdot\partial_y\big)\big(\lambda_N^{-4/3}\psi_N\big)\big]\Big|. \end{aligned}$$

Step II) In this step we treat the above terms R_j, and show that there exists a symbol q, belonging to $S(1,|dy|^2+|d\eta|^2)$ uniformly with respect to X, such that

$$\sum_{j=1}^{3}\Big(R_j^{\rm Wick}u,\ u\Big)_{L^2} \leq N^{-1/3}\Big(\big(\lambda_N^{2/3}\big)^{\rm Wick}u,\ u\Big)_{L^2} \tag{55}$$

$$+C_N\Big\{|(P_X u,\ u)_{L^2}| + \Big|\Big(P_X u,\ q^{\rm Wick}u\Big)_{L^2}\Big| + \|u\|_{L^2}^2\Big\}. \tag{56}$$

For this purpose we define q by

$$q(y,\eta)=q_X(y,\eta)=\frac{\big(\partial_x V(x)\wedge\xi\big)\cdot\big(\partial_x V(x)\wedge\eta+y\wedge\xi\big)}{\left\langle\partial_x V(x)\wedge\xi\right\rangle^{8/5}}\varphi(y,\eta),$$

with

$$\varphi(y,\eta)=\chi\left(\frac{|\partial_x V(x)\wedge\eta+y\wedge\xi|^2+|\partial_x V(x)|^2+|\xi|^2}{\left\langle\partial_x V(x)\wedge\xi\right\rangle^{6/5}}\right).$$

Then one can verify that $q\in S(1,|dy|^2+|d\eta|^2)$ uniformly with respect to (x,ξ). Thus similar to (51) we conclude

$$\Big(iQ_X u,\ q^{\rm Wick}u\Big)_{L^2} \lesssim |(P_X u,\ u)_{L^2}| + \Big|\Big(P_X u,\ q^{\rm Wick}u\Big)_{L^2}\Big|. \tag{57}$$

On the other hand, it is just a direct computation of the Poisson bracket to see that

$$\begin{aligned} \Big(iQ_X u,\ q^{\rm Wick}u\Big)_{L^2} &= \frac{1}{4\pi}\Big(\big\{q(y,\eta),\ y\cdot\xi-\partial_x V(x)\cdot\eta\big\}^{\rm Wick}u,\ u\Big)_{L^2}\\ &= \frac{1}{4\pi}\Big(R_{1,1}^{\rm Wick}u,\ u\Big)_{L^2} + \frac{1}{4\pi}\Big(R_{1,2}^{\rm Wick}u,\ u\Big)_{L^2}, \end{aligned} \tag{58}$$

with

$$\begin{aligned} R_{1,1} &= \frac{2\,|\partial_x V(x)\wedge\xi|^2}{\left\langle\partial_x V(x)\wedge\xi\right\rangle^{8/5}}\varphi,\\ R_{1,2} &= \frac{\big(\partial_x V(x)\wedge\xi\big)\cdot\big(\partial_x V(x)\wedge\eta+y\wedge\xi\big)}{\left\langle\partial_x V(x)\wedge\xi\right\rangle^{8/5}}\big[\big(\xi\cdot\partial_\eta+\partial_x V(x)\cdot\partial_y\big)\varphi(y,\eta)\big]. \end{aligned}$$

Moreover we have

$$R_{1,2} \lesssim \big(\,|\partial_x V(x)\wedge\eta + y\wedge\xi|^2 + |\partial_x V(x)|^2 + |\xi|^2\,\big)^{1/3} \lesssim \lambda_N^{2/3}$$

due to the fact that

$$\langle \partial_x V(x)\wedge\xi\rangle^{2/5} \approx \big(\,|\partial_x V(x)\wedge\eta + y\wedge\xi|^2 + |\partial_x V(x)|^2 + |\xi|^2\,\big)^{1/3}$$

on the support of φ', and

$$\begin{aligned} R_{1,1} &= \langle \partial_x V(x)\wedge\xi\rangle^{2/5} - \frac{1}{\langle \partial_x V(x)\wedge\xi\rangle^{8/5}}\varphi - \langle \partial_x V(x)\wedge\xi\rangle^{2/5}\,(1-\varphi) \\ &\geq \langle \partial_x V(x)\wedge\xi\rangle^{2/5} - \frac{1}{\langle \partial_x V(x)\wedge\xi\rangle^{8/5}} - \big(\,|\partial_x V(x)\wedge\eta + y\wedge\xi|^2 + |\partial_x V(x)|^2 + |\xi|^2\,\big)^{1/3}, \end{aligned}$$

where the inequality holds because

$$\langle \partial_x V(x)\wedge\xi\rangle^{2/5} \leq \big(\,|\partial_x V(x)\wedge\eta + y\wedge\xi|^2 + |\partial_x V(x)|^2 + |\xi|^2\,\big)^{1/3}$$

on the support of $1-\varphi$. These inequalities, combining (58) and (57), yield

$$\begin{aligned} \Big(\big(\,\langle \partial_x V(x)\wedge\xi\rangle^{2/5}\,\big)^{\rm Wick} u,\ u\Big)_{L^2} &\lesssim \Big(\big(\lambda_N^{2/3}\big)^{\rm Wick} u,\ u\Big)_{L^2} \\ &\quad + |(P_X u,\ u)_{L^2}| + \Big|\Big(P_X u,\ q^{\rm Wick} u\Big)_{L^2}\Big| + \|u\|_{L^2}^2. \end{aligned}$$

Consequently, observing that

$$R_1 = \frac{1 + N^{-1}\langle \partial_x V\wedge\xi\rangle^{6/5}}{\lambda_N^{4/3}} \leq N^{-1/3}\langle \partial_x V(x)\wedge\xi\rangle^{2/5} + 1,$$

and

$$R_2 = \frac{\big|\big(\partial_x V\wedge\xi\big)\cdot(y\wedge\eta)\big|}{\lambda_N^{4/3}}\psi_N \lesssim \frac{N^{-1}\langle \partial_x V\wedge\xi\rangle}{\lambda}\,\frac{N\,|y\wedge\eta|}{\lambda_N^{1/3}}\psi_N \leq N^{-1/2}\langle \partial_x V(x)\wedge\xi\rangle^{2/5},$$

we get the desired upper bound for the terms R_1 and R_2.

It remains to handle R_3. By virtue of (49) and (50), we compute

$$R_3 \lesssim N\lambda_N^{1/3}\,|y\wedge\eta| + N^2\big(\,|y\wedge\eta|^2 + |y|^2 + |\eta|^2\,\big) \leq N^{-1}\lambda_N^{2/3} + C_N\big(\,|y\wedge\eta|^2 + |y|^2 + |\eta|^2\,\big).$$

As a result, the positivity of Wick quantization gives

$$\begin{aligned} \Big(R_3^{\rm Wick} u,\ u\Big)_{L^2} &\leq N^{-1}\Big(\big(\lambda_N^{2/3}\big)^{\rm Wick} u,\ u\Big)_{L^2} + C_N\Big(\big(\,|y\wedge\eta|^2 + |y|^2 + |\eta|^2\,\big)^{\rm Wick} u,\ u\Big)_{L^2} \\ &\leq N^{-1}\Big(\big(\lambda_N^{2/3}\big)^{\rm Wick} u,\ u\Big)_{L^2} + C_N\Big\{|(P_X u,\ u)_{L^2}| + \|u\|_{L^2}^2\Big\}, \end{aligned}$$

the last inequality using (53). Thus the desired estimate (55) follows.

Step III) Now we proceed the proof of Proposition 4.1. From (54) and (55), it follows that there exists a symbol $p \in S(1, |dy|^2 + |d\eta|^2)$ uniformly with respect to X, such that

$$\begin{aligned} \Big(\big(\lambda_N^{2/3}\big)^{\rm Wick} u, u\Big)_{L^2} &\leq N^{-1/3}\Big(\big(\lambda_N^{2/3}\big)^{\rm Wick} u,\ u\Big)_{L^2} \\ &\quad + C_N\Big\{|(P_X u,\ u)_{L^2}| + \Big|\Big(P_X u,\ p^{\rm Wick} u\Big)_{L^2}\Big| + \|u\|_{L^2}^2\Big\}, \end{aligned}$$

which allows us to choose an integer N_0 large enough, such that

$$\left((\lambda_{N_0}^{2/3})^{\mathrm{Wick}}u,\ u\right)_{L^2} \lesssim C_{N_0}\left\{|(P_X u,\ u)_{L^2}| + \left|\left(P_X u,\ p^{\mathrm{Wick}}u\right)_{L^2}\right| + \|u\|_{L^2}^2\right\}.$$

Consequently, observing that

$$\lambda^{2/3} \lesssim \lambda_{N_0}^{2/3} + |y|^2 + |\eta|^2$$

with λ defined in (26), we get, combining (53),

$$\left(\left(\lambda^{2/3}\right)^{\mathrm{Wick}}u,\ u\right)_{L^2} \lesssim |(P_X u,\ u)_{L^2}| + \left|\left(P_X u,\ p^{\mathrm{Wick}}u\right)_{L^2}\right| + \|u\|_{L^2}^2. \tag{59}$$

Note $\langle\partial_x V(x)\rangle^{2/3} + \langle\xi\rangle^{2/3} \le \lambda^{2/3}$, then the above inequality yields

$$\left(\left(\langle\partial_x V(x)\rangle^{2/3} + \langle\xi\rangle^{2/3}\right)u,\ u\right)_{L^2} \lesssim |(P_X u,\ u)_{L^2}| + \left|\left(P_X u,\ p^{\mathrm{Wick}}u\right)_{L^2}\right| + \|u\|_{L^2}^2.$$

Since $p \in S(1, |dy|^2 + |d\eta|^2)$ uniformly with respect to X, applying the above inequality to the function

$$\left(\langle\partial_x V(x)\rangle^{2/3} + \langle\xi\rangle^{2/3}\right)^{1/2} u$$

implies

$$\left(\langle\partial_x V(x)\rangle^{2/3} + \langle\xi\rangle^{2/3}\right)\|u\|_{L^2} \lesssim \|P_X u\|_{L^2} + \|u\|_{L^2}. \tag{60}$$

Similarly, since $\langle\partial_x V(x) \wedge \xi\rangle^{2/5} \le \lambda^{2/3}$, by virtue of (59) we have, repeating the above arguments,

$$\langle\partial_x V(x) \wedge \xi\rangle^{2/5}\|u\|_{L^2} \lesssim \|P_X u\|_{L^2} + \|u\|_{L^2}. \tag{61}$$

Now we apply (59) to the function $\left(\lambda^{1/3}\right)^w u$, to get

$$\begin{aligned}
&\left(\left(\lambda^{2/3}\right)^{\mathrm{Wick}}\left(\lambda^{1/3}\right)^w u,\ \left(\lambda^{1/3}\right)^w u\right)_{L^2} \\
\lesssim\ & \left|\left(P_X\left(\lambda^{1/3}\right)^w u,\ \left(\lambda^{1/3}\right)^w u\right)_{L^2}\right| + \left|\left(P_X\left(\lambda^{1/3}\right)^w u,\ p^{\mathrm{Wick}}\left(\lambda^{1/3}\right)^w u\right)_{L^2}\right| + \left\|\left(\lambda^{1/3}\right)^w u\right\|_{L^2}^2 \\
\le\ & \varepsilon\left\|\left(\lambda^{2/3}\right)^w u\right\|_{L^2}^2 + C_\varepsilon\left(\|\Phi^{2/3}u\|_{L^2}^2 + \|\langle\partial_x V(x) \wedge \xi\rangle^{2/5}u\|_{L^2}^2 + \|u\|_{L^2}^2\right),
\end{aligned}$$

where the last inequality follows from (38). Furthermore, using (30) implies

$$\left(\left(\lambda^{2/3}\right)^{\mathrm{Wick}}\left(\lambda^{1/3}\right)^w u,\ \left(\lambda^{1/3}\right)^w u\right)_{L^2} \gtrsim \left\|\left(\lambda^{2/3}\right)^w u\right\|_{L^2}^2 - \|\Phi^{2/3}u\|_{L^2}^2.$$

Combining the above inequalities, we have

$$\begin{aligned}
\left\|\left(\lambda^{2/3}\right)^w u\right\|_{L^2}^2 &\lesssim \varepsilon\left\|\left(\lambda^{2/3}\right)^w u\right\|_{L^2}^2 + C_\varepsilon\left(\|\Phi^{2/3}u\|_{L^2}^2 + \|\langle\partial_x V(x) \wedge \xi\rangle^{2/5}u\|_{L^2}^2 + \|u\|_{L^2}^2\right) \\
&\lesssim \varepsilon\left\|\left(\lambda^{2/3}\right)^w u\right\|_{L^2}^2 + C_\varepsilon\left(\|P_X u\|_{L^2}^2 + \|u\|_{L^2}^2\right),
\end{aligned}$$

the last inequality following from (60) and (61). Letting the number ε small enough yields

$$\left\|\left(\lambda^{2/3}\right)^w u\right\|_{L^2} \lesssim \|P_X u\|_{L^2} + \|u\|_{L^2}.$$

This, along with (31) and (60), gives the desired estimate (27), completing the proof of Proposition 4.1. □

5 Proof of Theorem 1.1: regularity estimates in all variables

In this section we show the hypoelliptic estimates in spatial and velocity variables for the original operator P. Throughout this section $\|\cdot\|_{L^2}$ stands for the norm in $L^2(\mathbb{R}^6_{x,y})$.

Proposition 5.1 *Let $V(x)$ be a C^2-function satisfying the assumption (7). Then for any $u \in C_0^\infty\big(\mathbb{R}^{2n}\big)$ one has*

$$\big\| \, |D_x|^{2/3} u \big\|_{L^2} + \big\| \, \langle y\rangle^{\gamma/2} \, |D_y|^2 \, u \big\|_{L^2} + \big\| \, \langle y\rangle^{\gamma/2} \, |y\wedge D_y|^2 \, u \big\|_{L^2} \lesssim \big\|Pu\big\|_{L^2} + \big\|u\big\|_{L^2}. \tag{62}$$

Proof. The proof is quite similar as that of Proposition 4.1 in [18]. So we only give a sketch here and refer to [18] for more detailed discussions. With each fixed $x_\mu \in \mathbb{R}^3$ we associate an operator

$$P_{x_\mu} = i\big(y\cdot D_x - \partial_x V(x_\mu)\cdot D_y\big) + \big(B(y)D_y\big)^* \cdot B(y)D_y + F(y).$$

Let P_{X_μ}, with $X_\mu = (x_\mu, \xi)$, be the operator defined in (25), i.e.,

$$P_{X_\mu} = i\big(y\cdot \xi - \partial_x V(x_\mu)\cdot D_y\big) + \big(B(y)D_y\big)^* \cdot B(y)D_y + F(y).$$

Observe

$$\mathcal{F}_x P_{x_\mu} = P_{X_\mu},$$

where $\mathcal{F}_x$ stands for the partial Fourier transform in the x variable. Suppose V satisfies the condition (7). Then performing the Fourier transform with respect to x, it follows from (27) that for all $u \in C_0^\infty\big(\mathbb{R}^6\big)$,

$$\big\| \, \langle D_x\rangle^{2/3} u \big\|_{L^2} + \big\| \, \langle y\rangle^{\gamma/2} \, |D_y|^2 \, u \big\|_{L^2} + \big\| \, \langle y\rangle^{\gamma/2} \, |y\wedge D_y|^2 \, u \big\|_{L^2} \lesssim \big\|P_{x_\mu} u\big\|_{L^2} + \big\|u\big\|_{L^2}. \tag{63}$$

Lemma 4.2 in [18] shows that the metric

$$g_x = \langle \partial_x V(x)\rangle^{2/3} \, |dx|^2, \quad x\in\mathbb{R}^3,$$

is slowly varying, i.e., we can find two constants $C_*, r_0 > 0$ such that if $g_x(x-y) \le r_0^2$ then

$$C_*^{-1} \le \frac{g_x}{g_y} \le C_*.$$

The main feature of a slowly varying metric is that it allows us to introduce some partitions of unity related to the metric (see for instance Lemma 18.4.4 of [13]). Precisely, we could find a constant $r > 0$ and a sequence $x_\mu \in \mathbb{R}^n, \mu \ge 1$, such that the union of the balls

$$\Omega_{\mu,r} = \big\{x\in\mathbb{R}^n; \quad g_{x_\mu}\big(x - x_\mu\big) < r^2\big\}$$

covers the whole space $\mathbb{R}^n$. Moreover there exists a positive integer N_r, depending only on r, such that the intersection of more than N_r balls is always empty. One can choose a family of nonnegative functions $\{\varphi_\mu\}_{\mu\ge1}$ in $S(1, g_x)$ such that

$$\text{supp } \varphi_\mu \subset \Omega_{\mu,r}, \quad \sum_{\mu\ge1}\varphi_\mu^2 = 1 \quad \text{and} \quad \sup_{\mu\ge1} |\partial_x\varphi_\mu(x)| \lesssim \langle \partial_x V(x)\rangle^{1/3}.$$

By Lemma 4.6 in [18] we see

$$\big\| \, \langle D_x\rangle^{2/3} u \big\|_{L^2}^2 \lesssim \sum_{\mu\ge1} \big\| \, \langle D_x\rangle^{2/3} \varphi_\mu u\big\|_{L^2}^2 + \big\|Pu\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2. \tag{64}$$

Using the notation

$$R_\mu = -y \cdot \partial_x \varphi_\mu(x) - \varphi_\mu\big(\partial_x V(x) - \partial_x V(x_\mu)\big) \cdot \partial_y,$$

we write

$$\varphi_\mu P u = P_{x_\mu} \varphi_\mu \, u + R_\mu u.$$

Then

$$\sum_{\mu\geq 1} \big\|P_{x_\mu}\varphi_\mu \, u\big\|_{L^2}^2 \leq 2\sum_{\mu\geq 1}\big(\big\|\varphi_\mu P u\big\|_{L^2}^2 + \big\|R_\mu u\big\|_{L^2}^2\big) \leq 2\big\|Pu\big\|_{L^2}^2 + 2\sum_{\mu\geq 1}\big\|R_\mu u\big\|_{L^2}^2.$$

On the other hand, by Lemma 4.9 in [18] we have

$$\sum_{\mu\geq 1}\big\|R_\mu u\big\|_{L^2}^2 \lesssim \big\|Pu\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2.$$

The above two inequalities yield

$$\forall\, u \in C_0^\infty\big(\mathbb{R}^{2n}\big), \quad \sum_{\mu\geq 1}\big\|P_{x_\mu}\varphi_\mu u\big\|_{L^2}^2 \lesssim \big\|Pu\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2.$$

Using (64) and (63), we have

$$\begin{aligned} \big\|\,\langle D_x\rangle^{\frac{2}{3}}\, u\big\|_{L^2}^2 \;&\lesssim\; \sum_{\mu\geq 1}\big\|\,\langle D_x\rangle^{\frac{2}{3}}\,\varphi_\mu\, u\big\|_{L^2}^2 + \big\|Pu\big\|_{L^2}^2 + C\big\|u\big\|_{L^2}^2 \\ &\lesssim\; \sum_{\mu\geq 1}\big\|P_{x_\mu}\varphi_\mu\, u\big\|_{L^2}^2 + \big\|Pu\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2, \end{aligned}$$

and

$$\begin{aligned} &\big\|\,\langle y\rangle^{\gamma/2}\,|D_y|^2\, u\big\|_{L^2}^2 + \big\|\,\langle y\rangle^{\gamma/2}\,|y\wedge D_y|^2\, u\big\|_{L^2}^2 \\ &\quad= \sum_{\mu\geq 1}\big\|\,\langle y\rangle^{\gamma/2}\,|D_y|^2\,\varphi_\mu u\big\|_{L^2}^2 + \sum_{\mu\geq 1}\big\|\,\langle y\rangle^{\gamma/2}\,|y\wedge D_y|^2\,\varphi_\mu u\big\|_{L^2}^2 \\ &\quad\lesssim \sum_{\mu\geq 1}\big\|P_{x_\mu}\varphi_\mu\, u\big\|_{L^2}^2 + \sum_{\mu\geq 1}\big\|\varphi_\mu\, u\big\|_{L^2}^2. \end{aligned}$$

As a result, combining these inequalities gives (62). The proof is then complete. □

6 End of the proof of Theorem 1.1: anisotropic estimates

In this section we prove the anisotropic estimate (10) in Theorem 1.1 under the condition (9). Starting from the estimates for operators with parameters given in Section 4, we firstly establish a estimate in Wick quantization, and then come back to the Weyl quantization from the Wick quantization.

Notations Throughout this section, $\big\|\cdot\big\|_{L^2}$ stands for the norm in $L^2(\mathbb{R}^6_{x,y})$. Given a symbol p, we use p^{Wick} to denote Wick quantization of p in all variable (x, y, ξ, η), while $p^{\mathrm{Wick(x)}}$ and $p^{\mathrm{Wick(y)}}$ to denote respectively the Wick quantization of p in (x,ξ) and in (y,η), and similarly for the Weyl quantization p^w, $p^{w(x)}$ and $p^{w(y)}$, and for the wave packets transform W, W_x and W_y.

Proposition 6.1 (Estimates in Wick quantization) *Let V satisfy the condition (9). Then*

$$\forall\ u \in \mathcal{S}(\mathbb{R}^6_{x,y}), \quad \big\|\big(\lambda^{2/3}\big)^{\rm Wick} u\big\|_{L^2} \leq \big\|\tilde{P}u\big\|_{L^2} + \big\|Pu\big\|_{L^2} + \big\|u\big\|_{L^2}, \tag{65}$$

where λ is given in (26) and $\tilde{P}$ is defined by

$$\tilde{P} = i\big(y\cdot\xi - \partial_x V(x)\cdot D_y\big)^{\rm Wick(x)} + \big(B(y)D_y\big)^* \cdot B(y)D_y + F(y). \tag{66}$$

Proof. We prove this proposition in three steps. In what follows let $u \in \mathcal{S}(\mathbb{R}^6_{x,y})$ and use the notation $X = (x,\xi)$.

Step 1) Let λ be given in (26). Then for any $f \in \mathcal{S}(\mathbb{R}^9_{x,\xi,y})$, we have, by (30),

$$\begin{aligned} &\big\|\big(\lambda^{2/3}\big)^{\rm Wick(y)} f(X,\cdot)\big\|_{L^2(\mathbb{R}^3_y)} \\ \lesssim\ &\big\|\big(\lambda^{2/3}\big)^{w(y)} f(X,\cdot)\big\|_{L^2(\mathbb{R}^3_y)} + \big\|\left\langle \partial_x V(x)\right\rangle^{2/3} f(X,\cdot)\big\|_{L^2(\mathbb{R}^3_y)} + \big\|\left\langle \xi\right\rangle^{2/3} f(X,\cdot)\big\|_{L^2(\mathbb{R}^3_y)}. \end{aligned}$$

This along with (27) gives for all $f \in \mathcal{S}(\mathbb{R}^9_{x,\xi,y})$,

$$\big\|\left\langle \partial_x V(x)\right\rangle^{2/3} f\big\|_{L^2(\mathbb{R}^3_y)} + \big\|\big(\lambda^{2/3}\big)^{\rm Wick(y)} f\big\|_{L^2(\mathbb{R}^3_y)} \lesssim \big\|P_X f\big\|_{L^2(\mathbb{R}^3_y)} + \big\|f\big\|_{L^2(\mathbb{R}^3_y)},$$

which holds uniformly with respect to (x,ξ). Integrating both sides of the above estimate over $\mathbb{R}^6_{x,\xi}$ yields for all $f \in \mathcal{S}(\mathbb{R}^9_{x,\xi,y})$,

$$\big\|\left\langle \partial_x V(x)\right\rangle^{2/3} f\big\|_{L^2(\mathbb{R}^9)} + \big\|\big(\lambda^{2/3}\big)^{\rm Wick(y)} f\big\|_{L^2(\mathbb{R}^9)} \lesssim \big\|P_X f\big\|_{L^2(\mathbb{R}^9)} + \big\|f\big\|_{L^2(\mathbb{R}^9)}.$$

IN particular, for any $u \in \mathcal{S}(\mathbb{R}^6_{x,y})$, applying the above inequality to the function $W_x u$, with W_x the wave packets transform only in the (x,ξ) variables, we have

$$\big\|\left\langle \partial_x V(x)\right\rangle^{2/3} W_x u\big\|_{L^2(\mathbb{R}^9)} + \big\|\big(\lambda^{2/3}\big)^{\rm Wick(y)} W_x u\big\|_{L^2(\mathbb{R}^9)} \lesssim \big\|P_X W_x u\big\|_{L^2(\mathbb{R}^9)} + \big\|W_x u\big\|_{L^2(\mathbb{R}^9)}.$$

Note the operator $\pi_{\mathcal{H}} = W_x W_x^*$ is an orthogonal projection on a closed space in L^2, then from the above inequality it follows that

$$\begin{aligned} \big\|W_x W_x^* \big(\lambda^{2/3}\big)^{\rm Wick(y)} W_x u\big\|_{L^2(\mathbb{R}^9)} &\lesssim \big\|\big(\lambda^{2/3}\big)^{\rm Wick(y)} W_x u\big\|_{L^2(\mathbb{R}^9)} \\ &\leq \big\|P_X W_x u\big\|_{L^2(\mathbb{R}^9)} + \big\|W_x u\big\|_{L^2(\mathbb{R}^9)}. \end{aligned}$$

On the other hand by (11) we see

$$\big\|W_x W_x^* \big(\lambda^{2/3}\big)^{\rm Wick(y)} W_x u\big\|_{L^2(\mathbb{R}^9)} = \big\|W_x^* \big(\lambda^{2/3}\big)^{\rm Wick(y)} W_x u\big\|_{L^2(\mathbb{R}^6)} = \big\|\big(\lambda^{2/3}\big)^{\rm Wick} u\big\|_{L^2(\mathbb{R}^6)},$$

where the last equality follows from the relation

$$W_x^* W_y^* \lambda^{2/3} W_y W_x = W^* \lambda^{2/3} W.$$

Then the above inequalities yield

$$\begin{aligned} &\big\|\left\langle \partial_x V(x)\right\rangle^{2/3} W_x u\big\|_{L^2(\mathbb{R}^9)} + \big\|\big(\lambda^{2/3}\big)^{\rm Wick} u\big\|_{L^2(\mathbb{R}^6)} \lesssim \big\|P_X W_x u\big\|_{L^2(\mathbb{R}^9)} + \big\|W_x u\big\|_{L^2(\mathbb{R}^9)} \\ \lesssim &\big\|\pi_{\mathcal{H}} P_X W_x u\big\|_{L^2(\mathbb{R}^9)} + \big\|(1-\pi_{\mathcal{H}}) P_X W_x u\big\|_{L^2(\mathbb{R}^9)} + \big\|u\big\|_{L^2(\mathbb{R}^6)}. \end{aligned} \tag{67}$$

Using (11) again, we have

$$\begin{aligned}&\left\|\pi_{\mathcal H}P_XW_xu\right\|_{L^2(\mathbb R^9)}=\left\|W_xW_x^*P_XW_xu\right\|_{L^2(\mathbb R^9)}=\left\|W_x^*P_XW_xu\right\|_{L^2(\mathbb R^6)}\\&=\left\|\left[W_x^*\big(iy\cdot\xi-i\partial_xV(x)\cdot D_y\big)W_x+W_x^*\big((B(y)D_y)^*B(y)D_y+F(y)\big)W_x\right]u\right\|_{L^2(\mathbb R^6)},\end{aligned}$$

and thus, with $\tilde P$ given in (66),

$$\left\|\pi_{\mathcal H}P_XW_xu\right\|_{L^2(\mathbb R^9)}=\left\|\tilde Pu\right\|_{L^2(\mathbb R^6)} \tag{68}$$

due to the relation

$$W_x^*\big((B(y)D_y)^*B(y)D_y+F(y)\big)W_x=(B(y)D_y)^*B(y)D_y+F(y),$$

since $W_x^*W_x=\mathrm{Id}$ and $(B(y)D_y)^*B(y)D_y+F(y)$ commutes with W_x. Moreover observe

$$\big(1-\pi_{\mathcal H}\big)\big((B(y)D_y)^*B(y)D_y+F(y)\big)W_x=0,$$

since $\big(1-\pi_{\mathcal H}\big)W_x=W_x(Id-W_x^*W_x)=0$. Thus

$$\begin{aligned}\big(1-\pi_{\mathcal H}\big)P_XW_x&=\big(iy\cdot\xi-i\partial_xV(x)\cdot D_y\big)W_x-W_xW_x^*\big(iy\cdot\xi-i\partial_xV(x)\cdot D_y\big)W_x\\&=-\big[W_xW_x^*,\ iy\cdot\xi-i\partial_xV(x)\cdot D_y\big]W_x.\end{aligned}$$

This along with (67) and (68) gives

$$\begin{aligned}&\left\|\langle\partial_xV(x)\rangle^{2/3}W_xu\right\|_{L^2(\mathbb R^9)}+\left\|\big(\lambda^{2/3}\big)^{\mathrm{Wick}}u\right\|_{L^2(\mathbb R^6)}\\&\lesssim\left\|\tilde Pu\right\|_{L^2(\mathbb R^6)}+\left\|u\right\|_{L^2(\mathbb R^6)}+\left\|\big[\pi_H,\ y\cdot\xi-\partial_xV(x)\cdot D_y\big]W_xu\right\|_{L^2(\mathbb R^9)}.\end{aligned} \tag{69}$$

Step 2) In this step we deal with the last term in (69), and show that for $\varepsilon>0$ there exists a constant C_ε such that

$$\left\|\big[\pi_{\mathcal H},\ y\cdot\xi\big]W_xu\right\|_{L^2(\mathbb R^9)}\lesssim\left\|Pu\right\|_{L^2(\mathbb R^6)}+\left\|u\right\|_{L^2(\mathbb R^6)} \tag{70}$$

and

$$\left\|\big[\pi_{\mathcal H},\ \partial_xV(x)\cdot D_y\big]W_xu\right\|_{L^2(\mathbb R^9)}\ \le\ \varepsilon\left\|\langle\partial_xV(x)\rangle^{2/3}W_xu\right\|_{L^2(\mathbb R^9)} \tag{71}$$

$$+C_\varepsilon\left(\left\|Pu\right\|_{L^2(\mathbb R^6)}+\left\|u\right\|_{L^2(\mathbb R^6)}\right). \tag{72}$$

Let's firstly prove (70). In view of (12) we see the kernel of the commutator $\big[\pi_{\mathcal H},\ y\cdot\xi\big]$ is given by

$$K_1\big(X,\tilde X\big)=K_1\big((x,\xi),(\tilde x,\tilde\xi)\big)=e^{-\frac{\pi}{2}\left(|x-\tilde x|^2+|\xi-\tilde\xi|^2\right)}e^{i\pi\left(x-\tilde x\right)\cdot\left(\xi+\tilde\xi\right)}\big(y\cdot\tilde\xi-y\cdot\xi\big).$$

Then

$$\left|K_1\big(X,\tilde X\big)\right|\le e^{-\frac{\pi}{2}\left(|x-\tilde x|^2+|\xi-\tilde\xi|^2\right)}\left|\xi-\tilde\xi\right||y|\,,$$

and thus

$$\sup_{X\in\mathbb R^6}\int_{\mathbb R^6}\left|K_1\big(X,\tilde X\big)\right|d\tilde X+\sup_{\tilde X\in\mathbb R^6}\int_{\mathbb R^6}\left|K_1\big(X,\tilde X\big)\right|dX\lesssim|y|\,.$$

Consequently using Schur criterion we have

$$\left\|\big[\pi_{\mathcal H},\ y\cdot\xi\big]W_xu(\cdot,y)\right\|_{L^2(\mathbb R^6_{x,\xi})}\lesssim|y|\left\|W_xu(\cdot,y)\right\|_{L^2(\mathbb R^6_{x,\xi})}=|y|\left\|u(\cdot,y)\right\|_{L^2(\mathbb R^3_x)},$$

the last equality following from (11). Integrating both sides with respect to y gives

$$\left\| \left[\pi_{\mathcal{H}},\ y\cdot\xi\right] W_x u\right\|_{L^2(\mathbb{R}^9)} \lesssim \left\| \langle y\rangle\, u\right\|_{L^2(\mathbb{R}^6)},$$

which along with (6) gives the desired estimate (70).

It remains to show (71). In view of (12) we see the kernel of the commutator $\left[\pi_{\mathcal{H}},\ \partial_x V(x)\cdot D_y\right]$ is given by

$$K_2\big(X,\tilde{X}\big)=\sum_{1\leq j\leq 3}\tilde{K}_{2,j}\cdot D_{y_j},$$

with

$$\tilde{K}_{2,j}\big(X,\tilde{X}\big)=e^{-\frac{\pi}{2}\left(|x-\tilde{x}|^2+|\xi-\tilde{\xi}|^2\right)}e^{i\pi\left(x-\tilde{x}\right)\cdot\left(\xi+\tilde{\xi}\right)}\big(\partial_{x_j}V(\tilde{x})-\partial_{x_j}V(x)\big).$$

Direct computation shows

$$\sup_{X\in\mathbb{R}^6}\int_{\mathbb{R}^6}\left|\tilde{K}_{2,j}\big(X,\tilde{X}\big)\right|\langle\partial_x V(x)\rangle^{-1/3}\,d\tilde{X}+\sup_{\tilde{X}\in\mathbb{R}^6}\int_{\mathbb{R}^6}\left|\tilde{K}_{2,j}\big(X,\tilde{X}\big)\right|\langle\partial_x V(x)\rangle^{-1/3}\,dX\lesssim C,$$

since

$$\begin{aligned}
|\partial_x V(\tilde{x})-\partial_x V(x)| &\lesssim |x-\tilde{x}|\sum_{j,k}\int_0^1\left|\partial_{x_j x_k}V\big(x+\theta(\tilde{x}-x)\big)\right|\,d\theta\\
&\lesssim |x-\tilde{x}|\int_0^1\langle x+\theta(\tilde{x}-x)\rangle^{M/3}\,d\theta\\
&\lesssim C_M\langle x-\tilde{x}\rangle^{1+M/3}\langle x\rangle^{M/3}\\
&\lesssim C_M\langle x-\tilde{x}\rangle^{1+M/3}\langle\partial_x V(x)\rangle^{1/3},
\end{aligned}$$

the second and the last inequalities using (9), and the third inequality holding because

$$\langle x+\theta(\tilde{x}-x)\rangle^{M/3}\leq C_M\langle\theta(\tilde{x}-x)\rangle^{M/3}\langle x\rangle^{M/3}$$

with C_M a constant depending only on M. Using again the Schur criterion for the kernel

$$\tilde{K}_{2,j}\big(X,\tilde{X}\big)\,\langle\partial_x V(x)\rangle^{-1/3}$$

implies

$$\begin{aligned}
\left\|\left[\pi_{\mathcal{H}},\ \partial_x V(x)\cdot D_y\right]W_x u\right\|_{L^2(\mathbb{R}^9)} &\lesssim \sum_{j=1}^{3}\left\|\tilde{K}_{2,j}D_{y_j}W_x u\right\|_{L^2(\mathbb{R}^9)}\\
&\lesssim \left\|\langle\partial_x V(x)\rangle^{1/3}\langle D_y\rangle W_x u\right\|_{L^2(\mathbb{R}^9)}.
\end{aligned}$$

On the other hand for any $\varepsilon>0$ we have

$$\left\|\langle\partial_x V(x)\rangle^{1/3}\langle D_y\rangle W_x u\right\|_{L^2(\mathbb{R}^9)}\leq\varepsilon\left\|\langle\partial_x V(x)\rangle^{\frac{2}{3}}W_x u\right\|_{L^2(\mathbb{R}^9)}+C_\varepsilon\left\|\langle D_y\rangle^2 W_x u\right\|_{L^2(\mathbb{R}^9)}$$

and moreover by (11) and (62),

$$\left\|\langle D_y\rangle^2 W_x u\right\|_{L^2(\mathbb{R}^9)}=\left\|\langle D_y\rangle^2 u\right\|_{L^2R^6)}\lesssim\|Pu\|_{L^2}+\|u\|_{L^2}.$$

Then (71) follows from the above inequalities.

Step 3) By virtue of (69), (70) and (71), the desired estimate (65) follows if we let the number ε in (71) be small enough. The proof of Proposition 6.1 is thus complete. □

End of the proof of Theorem 1.1. Now we are ready to prove the anisotropic estimate (10) in Theorem 1.1. This will occupy the rest of the section.

Step a) Using the inequality

$$\|(\lambda^{2/3})^w u\|_{L^2} \lesssim \|(\lambda^{2/3})^{\mathrm{Wick}} u\|_{L^2} + \|\left((\lambda^{2/3})^w - (\lambda^{2/3})^{\mathrm{Wick}}\right) u\|_{L^2},$$

we have, by (65),

$$\|(\lambda^{2/3})^w u\|_{L^2} \lesssim \|Pu\|_{L^2} + \|u\|_{L^2} + \mathcal{R}_1 + \|(P-\tilde{P})u\|_{L^2} \lesssim \|Pu\|_{L^2} + \|u\|_{L^2} + \mathcal{R}_1 + \mathcal{R}_2, \quad (73)$$

with

$$\begin{aligned} \mathcal{R}_1 &= \|\left((\lambda^{2/3})^w - (\lambda^{2/3})^{\mathrm{Wick}}\right) u\|_{L^2}, \\ \mathcal{R}_2 &= \|\left(\partial_x V(x) - (\partial_x V(x))^{\mathrm{Wick(x)}}\right) \cdot D_y u\|_{L^2}. \end{aligned}$$

Here we used the fact that

$$\begin{aligned} P - \tilde{P} &= i\big(y\cdot D_x - \partial_x V(x)\cdot D_y\big) - i\big(y\cdot\xi - \partial_x V(x)\cdot D_y\big)^{\mathrm{Wick(x)}} \\ &= -i\partial_x V(x)\cdot D_y + i\big(\partial_x V(x)\cdot D_y\big)^{\mathrm{Wick(x)}}, \end{aligned}$$

the last equality holding because $y\cdot D_x - \big(y\cdot\xi\big)^{\mathrm{Wick(x)}} = 0$ due to (14) since the symbol $y\cdot\xi$ is a first order polynomial in (x,ξ).

Step b) In this step we show that

$$\|\left((\lambda^{2/3})^w - (\lambda^{2/3})^{\mathrm{Wick}}\right) u\|_{L^2} \leq \|Pu\|_{L^2} + \|u\|_{L^2}. \quad (74)$$

By (14),

$$(\lambda^{2/3})^{\mathrm{Wick}} - (\lambda^{2/3})^w = r^w$$

with

$$r(x,y,\xi,\eta) = r(Z) = \int_0^1 \int_{\mathbb{R}^{12}} (1-\theta)(\lambda^{2/3})''(Z+\theta\tilde{Z})\tilde{Z}^2 e^{-2\pi|\tilde{Z}|^2} 2^6\, d\tilde{Z} d\theta.$$

Direct computation shows that if $|\alpha| + |\beta| + |\tilde{\alpha}| + |\tilde{\beta}| \geq 1$ then

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_y^{\tilde{\alpha}} \partial_\eta^{\tilde{\beta}} (\lambda^{2/3})\right| \leq C_{\alpha,\beta,\tilde{\alpha},\tilde{\beta}}\big(\,\langle\partial_x V(x)\rangle^{2/3} + \langle\xi\rangle^{2/3} + \langle y\rangle + \langle\eta\rangle\,\big).$$

As a result we have , with $Z = (x,y,\xi,\eta)$ and $\tilde{Z} = (\tilde{x},\tilde{y},\tilde{\xi},\tilde{\eta})$,

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_y^{\tilde{\alpha}} \partial_\eta^{\tilde{\beta}} r(Z)\right| \leq C_{\alpha,\beta,\tilde{\alpha},\tilde{\beta}}\big(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4\big), \quad (75)$$

where $\mathcal{L}_j$ is given by

$$\begin{aligned} \mathcal{L}_1 &= \int_0^1 \int_{\mathbb{R}^{12}} \langle\partial_x V(x+\theta\tilde{x})\rangle^{2/3} \left|\tilde{Z}\right|^2 e^{-2\pi|\tilde{Z}|^2} 2^6\, d\tilde{Z} d\theta, \\ \mathcal{L}_2 &= \int_0^1 \int_{\mathbb{R}^{12}} \langle\xi+\theta\tilde{\xi}\rangle^{2/3} \left|\tilde{Z}\right|^2 e^{-2\pi|\tilde{Z}|^2} 2^6\, d\tilde{Z} d\theta, \\ \mathcal{L}_3 &= \int_0^1 \int_{\mathbb{R}^{12}} \langle y+\theta\tilde{y}\rangle \left|\tilde{Z}\right|^2 e^{-2\pi|\tilde{Z}|^2} 2^6\, d\tilde{Z} d\theta, \\ \mathcal{L}_4 &= \int_0^1 \int_{\mathbb{R}^{12}} \langle\eta+\theta\tilde{\eta}\rangle \left|\tilde{Z}\right|^2 e^{-2\pi|\tilde{Z}|^2} 2^6\, d\tilde{Z} d\theta. \end{aligned}$$

As for the term $\mathcal{L}_1$, we use the condition (9) to compute

$$\begin{aligned}\mathcal{L}_1 &\lesssim \int_0^1 \int_{\mathbb{R}^{12}} \langle x+\theta\tilde{x}\rangle^{2M/3} \left|\tilde{Z}\right|^2 e^{-2\pi\left|\tilde{Z}\right|^2} 2^6\, d\tilde{Z} d\theta \\ &\lesssim \int_{\mathbb{R}^{12}} \langle x\rangle^{2M/3} \langle \tilde{x}\rangle^{2M/3} \left|\tilde{Z}\right|^2 e^{-2\pi\left|\tilde{Z}\right|^2} 2^6\, d\tilde{Z} \\ &\lesssim \langle x\rangle^{2M/3} \lesssim \langle \partial_x V(x)\rangle^{2/3}.\end{aligned}$$

Using the inequality

$$\langle \xi+\theta\tilde{\xi}\rangle^{2/3} \lesssim \langle \xi\rangle^{2/3} \langle \tilde{\xi}\rangle^{2/3},$$

we have

$$\mathcal{L}_2 \lesssim \langle \xi\rangle^{2/3} \int_{\mathbb{R}^{12}} \langle \tilde{\xi}\rangle^{2/3} \left|\tilde{Z}\right|^2 e^{-2\pi\left|\tilde{Z}\right|^2} 2^6\, d\tilde{Z} \lesssim \langle \xi\rangle^{2/3}.$$

Similarly

$$\mathcal{L}_3+\mathcal{L}_4 \lesssim \langle y\rangle + \langle \eta\rangle.$$

It follows from (75) and the above estimates that

$$r^w \big(\big(\langle \partial_x V(x)\rangle^{2/3} + \langle \xi\rangle^{2/3} + \langle y\rangle + \langle \eta\rangle \big)^{-1} \big)^w \in \mathrm{Op}\big(S(1,\ |dx|^2+|dy|^2+|d\xi|^2+|d\eta|^2)\big),$$

and thus

$$\begin{aligned}\left\|r^w u\right\|_{L^2} &\lesssim \left\|\big(\langle \partial_x V(x)\rangle^{2/3}\big)^w u\right\|_{L^2} + \left\|\big(\langle \xi\rangle^{2/3}\big)^w u\right\|_{L^2} + \left\|\langle y\rangle^w u\right\|_{L^2} + \left\|\langle \eta\rangle^w u\right\|_{L^2} \\ &\lesssim \left\|\langle \partial_x V(x)\rangle^{2/3} u\right\|_{L^2} + \left\|\langle D_x\rangle^{2/3} u\right\|_{L^2} + \left\|\langle y\rangle u\right\|_{L^2} + \left\|\langle D_y\rangle u\right\|_{L^2} \\ &\lesssim \left\|Pu\right\|_{L^2} + \left\|u\right\|_{L^2},\end{aligned}$$

the last inequality using (16), (62) and (6).

Step c) Supposing V satisfies the assumption (9), we show

$$\left\| \Big(\partial_x V(x) - \big(\partial_x V(x)\big)^{\mathrm{Wick(x)}}\Big) \cdot D_y u\right\|_{L^2} \lesssim \left\|Pu\right\|_{L^2} + \left\|u\right\|_{L^2}. \tag{76}$$

In fact, for each $1\le j\le 3$, one has , by (14)

$$\begin{aligned}&\partial_{x_j} V(x) - \big(\partial_{x_j} V(x)\big)^{\mathrm{Wick(x)}} \\ &\quad= \Big(\big(\partial_{x_j} V(x)\big)^{w(x)} - \big(\partial_{x_j} V(x)\big)^{\mathrm{Wick(x)}}\Big) + \Big(\partial_{x_j} V(x) - \big(\partial_{x_j} V(x)\big)^{w(x)}\Big) \\ &\quad= r_j^w + \tilde{r}_j^w\end{aligned}$$

with $\tilde{r}_j \in S(\langle \partial_x V(x)\rangle^{1/3},\ |dx|^2+|d\xi|^2)$ due to Theorem 2.3.18 in [16], and

$$r_j(x,\xi) = r(X) = \int_0^1 \int_{\mathbb{R}^6} (1-\theta)\big(\partial_{x_j} V\big)''(x+\theta\tilde{x})\tilde{x}^2 e^{-2\pi\left|\tilde{X}\right|^2} 2^3\, d\tilde{X} d\theta.$$

Applying (9) we have, for any $\alpha,\beta\in\mathbb{Z}_+^3$,

$$\begin{aligned}\left|\partial_x^\alpha \partial_\xi^\beta r_j(x,\xi)\right| &\lesssim \int_0^1 \int_{\mathbb{R}^6} \langle x+\theta\tilde{x}\rangle^{M/3} |\tilde{x}|^2 e^{-2\pi\left|\tilde{X}\right|^2} 2^3\, d\tilde{X} d\theta \\ &\lesssim \langle x\rangle^{M/3} \int_{\mathbb{R}^6} \langle \tilde{x}\rangle^{M/3} |\tilde{x}|^2 e^{-2\pi\left|\tilde{X}\right|^2} 2^3\, d\tilde{X} \\ &\lesssim \langle \partial_x V(x)\rangle^{1/3}.\end{aligned}$$

It then follows that

$$\tilde{r}_j^w\big(\langle\partial_x V(x)\rangle^{-1/3}\big)^w,\ r_j^w\big(\langle\partial_x V(x)\rangle^{-1/3}\big)^w \in \mathrm{Op}\big(S(1,\ |dx|^2+|d\xi|^2)\big),$$

and thus

$$\begin{aligned}\big\|r_j^w D_{y_j} u\big\|_{L^2} + \big\|\tilde{r}_j^w D_{y_j} u\big\|_{L^2} &\lesssim \big\|\langle\partial_x V(x)\rangle^{1/3}\langle D_y\rangle u\big\|_{L^2}\\ &\lesssim \big\|\langle\partial_x V(x)\rangle^{2/3} u\big\|_{L^2} + \big\|\langle D_x\rangle^2 u\big\|_{L^2}\\ &\lesssim \|Pu\|_{L^2} + \|u\|_{L^2},\end{aligned}$$

the last inequality using (16). The estimate (76) follows.

Step d) Combining (73), (74) and (76), we get the desired estimate (10), completing the proof of Theorem 1.1. □

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