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To cite this version:

HAL Id: hal-00492141
https://hal.archives-ouvertes.fr/hal-00492141v2
Submitted on 24 May 2011

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The Energy-Momentum Tensor on Spin\(^c\) Manifolds

Roger Nakad

March 7, 2011

Institut Élie Cartan, Université Henri Poincaré, Nancy I, B.P 239
54506 Vandoeuvre-Lès-Nancy Cedex, France.

nakad@iecn.u-nancy.fr

Abstract

On Spin\(^c\) manifolds, we study the Energy-Momentum tensor associated with a spinor field. First, we give a spinorial Gauss type formula for oriented hypersurfaces of a Spin\(^c\) manifold. Using the notion of generalized cylinders, we derive the variational formula for the Dirac operator under metric deformation and point out that the Energy-Momentum tensor appears naturally as the second fundamental form of an isometric immersion. Finally, we show that generalized Spin\(^c\) Killing spinors for Codazzi Energy-Momentum tensor are restrictions of parallel spinors.

Keywords: Spin\(^c\) structures; Spin\(^c\) Gauss formula; metric variation formula for the Dirac operator; Energy-Momentum tensor; generalized cylinder; generalized Killing spinors.

1 Introduction

In [14], O. Hijazi proved that on a compact Riemannian spin manifold \((M^n, g)\) any eigenvalue \(\lambda\) of the Dirac operator to which is attached an eigenspinor \(\psi\) satisfies

\[
\lambda^2 \geq \inf_M \left( \frac{1}{4} \text{Scal}_M + |\ell^\psi|^2 \right),
\]

(1)
where $\text{Scal}^M$ is the scalar curvature of the manifold $M$ and $\ell^\psi$ is the field of symmetric endomorphisms associated with the field of quadratic forms $T^\psi$ called the Energy-Momentum tensor. It is defined on the complement set of zeroes of the eigenspinor $\psi$, for any vector $X \in \Gamma(TM)$ by

$$T^\psi(X) = \text{Re} \langle X \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2} \rangle.$$ 

Here $\nabla$ denotes the Levi-Civita connection on the spinor bundle of $M$ and “$\cdot$” the Clifford multiplication. The limiting case of (1) is characterized by the existence of a spinor field $\psi$ satisfying for all $X \in \Gamma(TM)$,

$$\nabla_X \psi = -\ell^\psi(X) \cdot \psi.$$ 

(2)

For Spin$^c$ structures, the complex line bundle $L^M$ is endowed with an arbitrary connection and hence an arbitrary curvature $i\Omega^M$ which is an imaginary 2-form on the manifold. In terms of the Energy-Momentum tensor the author proved in [25] that on a compact Riemannian Spin$^c$ manifold any eigenvalue $\lambda$ of the Dirac operator to which is attached an eigenspinor $\psi$ satisfies

$$\lambda^2 \geq \inf_M \left( \frac{1}{4} \text{Scal}^M - \frac{c_n}{4} |\Omega^M| + |\ell^\psi|^2 \right),$$ 

(3)

where $c_n = 2[\frac{n}{2}]^2$. The limiting case of (3) is characterized by the existence of a spinor field $\psi$ satisfying for every $X \in \Gamma(TM)$,

$$\begin{cases}
\nabla^\Sigma_X \psi = -\ell^\psi(X) \cdot \psi, \\
\Omega^M \cdot \psi = i\frac{c_n}{2} |\Omega^M| \psi.
\end{cases}$$ 

(4)

Here $\nabla^\Sigma_X$ denotes the Levi-Civita connection on the Spin$^c$ spinor bundle and “$\cdot$” the Spin$^c$ Clifford multiplication. In [25], the author showed also that the sphere with a special Spin$^c$ structure is a limiting manifold for (3).

Studying the Energy-Momentum tensor on a Riemannian or semi-Riemannian spin manifolds has been done by many authors, since it is related to several geometric constructions (see [12], [2], [24] and [6] for results in this topic). In this paper we study the Energy-Momentum tensor on Riemannian and semi-Riemannian Spin$^c$ manifolds. First, we prove that the Energy-Momentum tensor appears in the study of the variations of the spectrum of the Dirac operator:

**Proposition 1.1** Let $(M^n, g)$ be a Spin$^c$ Riemannian manifold and $g_t = g + tk$ a smooth 1-parameter family of metrics. For any spinor field $\psi \in \Gamma(\Sigma M)$, we have

$$\frac{d}{dt} \bigg|_{t=0} (D^M \tau^t_0 \psi, \tau^t_0 \psi)_{g_t} = -\frac{1}{2} \int_M <k, T_\psi> dv_g,$$ 

(5)
where $(\), (\cdot,\cdot) = \int_M \text{Re} \langle (\cdot,\cdot) \rangle dv_g$, the Dirac operator $D^M_t$ is the Dirac operator associated with $M_t = (M, g_t)$ and $\tau_t^0 \psi$ is the image of $\psi$ under the isometry $\tau_t^0$ between the spinor bundles of $(M, g)$ and $(M, g_t)$. Here $T_\psi$ is defined by $T_\psi = |\psi|^2 T^\psi$ and $T^\psi$ is the symmetric bilinear form associated with the Energy-Momentum tensor, i.e. it is given for every $X, Y \in \Gamma(TM)$ by $T^\psi(X, Y) = \frac{1}{2} \text{Re} \left( X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2} \right)$. This was proven in [4] by J. P. Bourguignon and P. Gauduchon for spin manifolds.

Using this, we extend to Spin$^c$ manifolds a result by Th. Friedrich and E. C. Kim in [8] on spin manifolds:

**Theorem 1.2** Let $M$ be a Spin$^c$ Riemannian manifold. A pair $(g_0, \psi_0)$ is a critical point of the Lagrange functional

$$W(g, \psi) = \int_U \left( \text{Scal}^M_g + \varepsilon \lambda |\psi|^2_g - \varepsilon \text{Re} <D^g \psi, \psi>_g \right) dv_g,$$

$(\lambda, \varepsilon \in \mathbb{R})$ for all open subsets $U$ of $M$ if and only if $(g_0, \psi_0)$ is a solution of the following system

$$\begin{cases} 
D^g \psi = \lambda \psi, \\
\text{ric}^M_g - \frac{\text{Scal}^M_g}{2} g = \varepsilon \frac{1}{2} T^\psi,
\end{cases}$$

where $\text{ric}^M_g$ denotes the Ricci curvature of $M$ considered as a symmetric bilinear form.

Now, we interprete the Energy-Momentum tensor as the second fundamental form of a hypersurface. In fact, we prove the following:

**Proposition 1.3** Let $M^n \hookrightarrow (Z, g)$ be any compact oriented hypersurface isometrically immersed in an oriented Riemannian Spin$^c$ manifold $(Z, g)$, of mean curvature $H$ and Weingarten map $W$. Assume that $Z$ admits a parallel spinor field $\psi$, then the Energy-Momentum tensor associated with $\varphi = :\psi|_M$ satisfies

$$2\ell^\varphi = -W.$$ 

Moreover, if the mean curvature $H$ is constant, the hypersurface $M$ satisfies the equality case in (3) if and only if

$$\text{Scal}^Z - 2 \text{ric}^Z(\nu, \nu) - c_n |\Omega^M| = 0. \quad (6)$$

This was proven by Morel in [24] for a compact oriented hypersurface of a spin manifold carrying parallel spinor but in this case the hypersurface $M$ is directly a limiting manifold for (1) without the condition (6).
Finally, we study generalized Killing spinors on Spin$^c$ manifolds. They are characterized by the identity, for any tangent vector field $X$ on $M$,

$$\nabla^\Sigma_M X \psi = \frac{1}{2} F(X) \cdot \psi,$$

where $F$ is a given symmetric endomorphism on the tangent bundle. It is straightforward to see that

$$2T^\psi(X,Y) = - \langle F(X), Y \rangle.$$

These spinors are closely related to the so-called $T$–Killing spinors studied by Friedrich and Kim in [9] on spin manifolds. It is natural to ask whether the tensor $F$ can be realized as the Weingarten tensor of some isometric embedding of $M$ in a manifold $\mathbb{Z}^{n+1}$ carrying parallel spinors. Morel studied this problem in the case of spin manifolds where the tensor $F$ is parallel and in [2], the authors studied the problem in the case of semi-Riemannian spin manifolds where the tensor $F$ is a Codazzi-Mainardi tensor. We establish the corresponding result for semi-Riemannian Spin$^c$ manifolds:

**Theorem 1.4** Let $(M^n, g)$ be a semi-Riemannian Spin$^c$ manifold carrying a generalized Spin$^c$ Killing spinor $\varphi$ with a Codazzi-Mainardi tensor $F$. Then the generalized cylinder $Z := I \times M$ with the metric $dt^2 + g_t$, where $g_t(X,Y) = g((Id - tF)^2 X,Y)$, equipped with the Spin$^c$ structure arising from the given one on $M$ has a parallel spinor whose restriction to $M$ is just $\varphi$.

A characterisation of limiting 3-dimensional manifolds for (3), having generalized Spin$^c$ Killing spinors with Codazzi tensor is then given.

The paper is organised as follows: In Section 2, we collect basic material on spinors and the Dirac operator on semi-Riemannian Spin$^c$ manifolds. In Section 3, we study hypersurfaces of Spin$^c$ manifolds. We derive a spinorial Gauss formula after identifying the restriction of the Spin$^c$ spinor bundle of the ambient manifold with the Spin$^c$ spinor bundle of the hypersurface. In Section 4, we define the generalized cylinder of a Spin$^c$ manifold $M$ and we collect formulas relating the curvature of a generalized cylinder to geometric data on $M$. In section 5, we compare the Dirac operators for two different semi-Riemannian metrics, then one first has to identify the spinor bundles using parallel transport. In the last section, we interpret the Energy-Momentum tensor as the second fundamental form of a hypersurface and we study generalized Spin$^c$ Killing spinors. The author would like to thank Oussama Hijazi for his support and encouragements.
2 The Dirac operator on semi-Riemannian Spin\(^c\) manifolds

In this section, we collect some algebraic and geometric preliminaries concerning the Dirac operator on semi-Riemannian Spin\(^c\) manifolds. Details can be found in \cite{3} and \cite{2}. Let \(r + s = n\) and consider on \(\mathbb{R}^n\) the nondegenerate symmetric bilinear form of signature \((r, s)\) given by

\[
\langle v, w \rangle := \sum_{j=1}^r v_j w_j - \sum_{j=r+1}^n v_j w_j,
\]

for any \(v, w \in \mathbb{R}^n\). We denote by \(\text{Cl}_{r, s}\) the real Clifford algebra corresponding to \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\), this is the unitary algebra generated by \(\mathbb{R}^n\) subject to the relations

\[
e_j \cdot e_k + e_k \cdot e_j = \begin{cases} 
-2\delta_{jk} & \text{if } j \leq r, \\
2\delta_{jk} & \text{if } j > r,
\end{cases}
\]

where \((e_j)_{1 \leq j \leq n}\) is an orthonormal basis of \(\mathbb{R}^n\) of signature \((r, s)\), i.e., \(\langle e_j, e_k \rangle = \varepsilon_j \delta_{jk}\) and \(\varepsilon_j = \pm 1\). The complex Clifford algebra \(\text{Cl}_{r, s}^c\) is the complexification of \(\text{Cl}_{r, s}\) and it decomposes into even and odd elements \(\text{Cl}_{r, s} = \text{Cl}_{r, s}^0 \oplus \text{Cl}_{r, s}^1\). The real spin group is defined by

\[
\text{Spin}(r, s) := \{ v_1 \cdot \ldots \cdot v_{2k} \in \text{Cl}_{r, s} | v_j \in \mathbb{R}^n \text{ such that } \langle v_j, v_j \rangle = \pm 1 \}.
\]

The spin group \(\text{Spin}(r, s)\) is the double cover of \(\text{SO}(r, s)\), in fact the following sequence is exact

\[
1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \text{Spin}(r, s) \xrightarrow{\xi} \text{SO}(r, s) \longrightarrow 1,
\]

where \(\xi = \text{Ad}_{|_{\text{Spin}(r, s)}}\) and \(\text{Ad}\) is defined by

\[
\text{Ad} : \text{Cl}_{r, s}^c \longrightarrow \text{End}(\mathbb{R}^n)
\]

\[
w \mapsto \text{Ad}_w : v \mapsto \text{Ad}_w(v) = w \cdot v \cdot w^{-1}.
\]

Here \(\text{Cl}_{r, s}^c\) denotes the group of units of \(\text{Cl}_{r, s}\). Since \(\mathbb{S}^1 \cap \text{Spin}(r, s) = \{ \pm 1 \}\), we define the complex spin group by

\[
\text{Spin}^c(r, s) = \text{Spin}(r, s) \times_{\mathbb{Z}/2} \mathbb{S}^1.
\]

The complex spin group is the double cover of \(\text{SO}(r, s) \times \mathbb{S}^1\), this yields to the exact sequence

\[
1 \longrightarrow \mathbb{Z}/2 \longrightarrow \text{Spin}^c(r, s) \xrightarrow{\xi^c} \text{SO}(r, s) \times \mathbb{S}^1 \longrightarrow 1,
\]

where \(\xi^c = (\xi, \text{Id}^2)\). When \(n = 2m\) is even, \(\text{Cl}_{r, s}\) has a unique irreducible complex representation \(\chi_{2m}\) of complex dimension \(2^m\), \(\chi_{2m} : \text{Cl}_{r, s} \longrightarrow \text{End}(\Sigma_{r, s})\). If \(n = 5\)
2m + 1 is odd, $\mathbb{C}l_{r,s}$ has two inequivalent irreducible representations both of complex dimension $2^m$, $\chi_{2m+1}^j : \mathbb{C}l_{r,s} \rightarrow \text{End}(\Sigma^j_{r,s})$, for $j = 0$ or 1, where $\Sigma^j_{r,s} = \{ \sigma \in \Sigma_{r,s}, \; \chi^j_{2m+1}(\omega_{r,s})\sigma = (-1)^j \sigma \}$ and $\omega_{r,s}$ is the complex volume element

$$\omega_{r,s} = \begin{cases} 
    i^{m-s} e_1 \cdot \ldots \cdot e_n & \text{if } n = 2m, \\
    i^{m-1+s} e_1 \cdot \ldots \cdot e_n & \text{if } n = 2m + 1.
\end{cases}$$

We define the complex spinorial representation $\rho_n$ by the restriction of an irreducible representation of $\mathbb{C}l_{r,s}$ to $\text{Spin}^c(r, s)$:

$$\rho_n := \begin{cases} 
    \chi_{2m|\text{Spin}^c(r, s)} & \text{if } n = 2m, \\
    \chi_{2m+1|\text{Spin}^c(r, s)} & \text{if } n = 2m + 1.
\end{cases}$$

When $n = 2m$ is even, $\rho_n$ decomposes into two inequivalent irreducible representations $\rho^+_n$ and $\rho^-_n$, i.e., $\rho_n = \rho^+_n + \rho^-_n : \text{Spin}^c(r, s) \rightarrow \text{Aut}(\Sigma_{r,s})$. The space $\Sigma_{r,s}$ decomposes into $\Sigma_{r,s} = \Sigma^+_r \oplus \Sigma^-_r$, where $\omega_{r,s}$ acts on $\Sigma^+_r$ as the identity and minus the identity on $\Sigma^-_r$. If $n = r + s$ is odd and when restricted to $\text{Spin}^c(r, s)$, the two representations $\chi_{2m+1|\text{Spin}^c(r, s)}^0$ and $\chi_{2m+1|\text{Spin}^c(r, s)}^1$ are equivalent and we simply choose $\Sigma_{r,s} := \Sigma^+_r$. The complex spinor bundle $\Sigma_{r,s}$ carries a Hemitian symmetric bilinear $\text{Spin}^c(r, s)$-invariant form $\langle \cdot, \cdot \rangle$, such that

$$\langle v \cdot \sigma_1, \sigma_2 \rangle = (-1)^{s+1} \langle \sigma_1, v \cdot \sigma_2 \rangle$$

for all $\sigma_1, \sigma_2 \in \Sigma_{r,s}$ and $v \in \mathbb{R}^n$.

Now, we give the following isomorphism $\alpha$, which is of particular importance for the identification of the $\text{Spin}^c$ bundles in the context of immersions of hypersurfaces:

$$\alpha : \mathbb{C}l_{r,s} \rightarrow \mathbb{C}l_{r+1,s}^0$$

$$e_j \rightarrow \nu \cdot e_j,$$

(8)

where we look at an embedding of $\mathbb{R}^n$ onto $\mathbb{R}^{n+1}$ such that $(\mathbb{R}^n)^\perp$ is spacelike and spanned by a spacelike unit vector $\nu$.

Let $N^n$ be an oriented semi-Riemannian manifold of signature $(r, s)$ and let $P_{\text{SO}}N$ be the $\text{SO}(r, s)$-principal bundle of positively space and time oriented orthonormal tangent frames. A complex $\text{Spin}^c$ structure on $N$ is a $\text{Spin}^c(r, s)$-principal bundle $P_{\text{Spin}}N$ over $N$, an $\mathbb{S}^1$-principal bundle $P_{\mathbb{S}^1}N$ over $N$ together with a twofold covering map $\Theta : P_{\text{Spin}}N \rightarrow P_{\text{SO}}N \times_N P_{\mathbb{S}^1}N$ such that

$$\Theta(ua) = \Theta(u)\xi^c(a),$$

for every $u \in P_{\text{Spin}}N$ and $a \in \text{Spin}^c(r, s)$, i.e., $N$ has a $\text{Spin}^c$ structure if and only if there exists an $\mathbb{S}^1$-principal bundle $P_{\mathbb{S}^1}N$ over $N$ such that the transition functions $g_{\alpha\beta} \times l_{\alpha\beta} : U_{\alpha} \cap U_{\beta} \rightarrow \text{SO}(r, s) \times \mathbb{S}^1$ of the
$\text{SO}(r, s) \times S^1$-principal bundle $P_{\text{SO}(N)} \times_N P_{S^1}$ admit lifts to $\text{Spin}^c(r, s)$ denoted by $	ilde{g}_{\alpha\beta} \times \tilde{I}_{\alpha\beta} : U_{\alpha} \cap U_{\beta} \rightarrow \text{Spin}^c(r, s)$, such that $\xi^c \circ (\tilde{g}_{\alpha\beta} \times \tilde{I}_{\alpha\beta}) = g_{\alpha\beta} \times I_{\alpha\beta}$. This, anyhow, is equivalent to the second Stiefel-Whitney class $w_2(N)$ being equal, modulo 2, to the first Chern class $c_1(L^N)$ of the complex line bundle $L^N$. It is the complex line bundle associated with the $S^1$-principal fibre bundle via the standard representation of the unit circle.

Let $\Sigma N := P_{\text{Spin}^c} \times_{\rho_0} \Sigma_{r,s}$ be the spinor bundle associated with the spinor representation. A section of $\Sigma N$ will be called a spinor field. Using the co-cycle condition of the transition functions of the two principal fibre bundles $P_{\text{Spin}^c} \times_{\rho_0} \Sigma_{r,s}$ and $P_{\text{SO}(N)} \times_{\rho_1} P_{S^1}$, we can prove that

$$\Sigma N = \Sigma' N \otimes (L^N)^{1/2},$$

where $\Sigma' N$ is the locally defined spin bundle and $(L^N)^{1/2}$ is locally defined too but $\Sigma N$ is globally defined. The tangent bundle $TN = P_{\text{SO}(N)} \times_{\rho_0} \mathbb{R}^n$ where $\rho_0$ stands for the standard matrix representation of $\text{SO}(r, s)$ on $\mathbb{R}^n$, can be seen as the associated vector bundle $TN \simeq P_{\text{Spin}^c} \times_{\rho_1}\Sigma_{r,s} \mathbb{R}^n$ where $\rho_1$ is the first projection. One defines the Clifford multiplication at every point $p \in N$:

$$T_p N \otimes \Sigma_p N \rightarrow \Sigma_p N$$

$$[b, v] \otimes [b, \sigma] \rightarrow [b, v] \cdot [b, \sigma] := [b, v \cdot \sigma = \chi_n(v)\sigma],$$

where $b \in P_{\text{Spin}^c}, v \in \mathbb{R}^n, \sigma \in \Sigma_{r,s}$ and $\chi_n = \chi_{2m}$ if $n$ is even and $\chi_n = \chi_{2m+1}$ if $n$ is odd. The Clifford multiplication can be extended to differential forms. Clifford multiplication inherits the relations of the Clifford algebra, i.e., for $X, Y \in T_p N$ and $\varphi \in \Sigma_p N$ we have $X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi = -2 \langle X, Y \rangle \varphi$. In even dimensions the spinor bundle splits into $\Sigma N = \Sigma^+ N \oplus \Sigma^- N$, where $\Sigma^\pm N = P_{\text{Spin}^c} \times_{\rho_0} \Sigma^\pm_{r,s}$. Clifford multiplication by a non-vanishing tangent vector interchanges $\Sigma^+ N$ and $\Sigma^- N$. The $\text{Spin}^c(r, s)$-invariant nondegenerate symmetric sesquilinear form $\langle , \rangle$ on $\Sigma_{r,s}$ and $\Sigma^\pm_{r,s}$ induces inner products on $\Sigma N$ and $\Sigma^\pm N$ which we again denote by $\langle , \rangle$ and it satisfies

$$\langle X \cdot \psi, \varphi \rangle = (-1)^{s+1} \langle \psi, X \cdot \varphi \rangle,$$

for every $X \in \Gamma(TN)$ and $\psi, \varphi \in \Gamma(\Sigma N)$. Additionally, given a connection 1-form $A^N$ on $P_{S^1} N$, $A^N : T(P_{S^1} N) \rightarrow i\mathbb{R}$ and the connection 1-form $\omega^N$ on $P_{\text{SO}(N)}$ for the Levi-Civita connection $\nabla^N$, we can define the connection

$$\omega^N \times A^N : T(P_{\text{SO}(N)} \times_N P_{S^1}) \rightarrow \mathfrak{s}_n \oplus i\mathbb{R} = \mathfrak{spin}^c$$

on the principal fibre bundle $P_{\text{SO}(N)} \times_N P_{S^1}$ and hence a covariant derivative $\nabla^\Sigma N$ on $\Sigma N$ [7] given locally by

$$\nabla^\Sigma N_{\epsilon_k} \varphi = \left[ b \times s, e_k(\sigma) + \frac{1}{4} \sum_{j=1}^n \epsilon_j e_j \cdot \nabla_{e_k} e_j \cdot \sigma + \frac{1}{2} A^N(s_\epsilon(e_k))\sigma \right]$$

$$= e_k(\varphi) + \frac{1}{4} \sum_{j=1}^n \epsilon_j e_j \cdot \nabla_{e_k} e_j \cdot \varphi + \frac{1}{2} A^N(s_\epsilon(e_k))\varphi,$$
where $\varphi = \widetilde{b \times s, \sigma}$ is a locally defined spinor field, $b = (e_1, \ldots, e_n)$ is a local space and time oriented orthonormal tangent frame, $s : U \rightarrow P_{\mathfrak{g}_1} N$ is a local section of $P_{\mathfrak{g}_1} N$ and $\widetilde{b \times s}$ is the lift of the local section $b \times s : U \rightarrow P_{SO} N \times_N P_{\mathfrak{g}_1} N$ to the 2-fold covering $\Theta : P_{Spin^c} N \rightarrow P_{SO} N \times_N P_{\mathfrak{g}_1} N$. The curvature of $A^N$ is an imaginary valued 2-form denoted by $F_{AN} = dA^N$, i.e., $F_{AN} = i\Omega^N$, where $\Omega^N$ is a real valued 2-form on $P_{\mathfrak{g}_1} N$. We know that $\Omega^N$ can be viewed as a real valued 2-form on $N$. In this case $i\Omega^N$ is the curvature form of the associated line bundle $L^N$. The curvature tensor $R^N_{\Sigma^N}$ of $\nabla^{\Sigma^N}$ is given by

$$R^N_{\Sigma^N}(X,Y) \varphi = \frac{1}{4} \sum_{j,k=1}^n \varepsilon_j \varepsilon_k \left\langle R^N_{\Sigma^N}(X,Y) e_j, e_k \right\rangle e_j \cdot e_k \cdot \varphi + \frac{i}{2} \Omega^N(X,Y) \varphi, \quad (10)$$

where $R^N$ is the curvature tensor of the Levi-Civita connection $\nabla^N$. In the Spin$^c$ case, the Ricci identity translates, for every $X \in \Gamma(TN)$, to

$$\sum_{k=1}^n \varepsilon_k e_k \cdot R^N_{\Sigma^N}(e_k, X) \varphi = \frac{1}{2} \text{Ric}^N(X) \cdot \varphi - \frac{i}{2} (X \downarrow \Omega^N) \cdot \varphi. \quad (11)$$

Here $\text{Ric}^N$ denotes the Ricci curvature considered as a field of endomorphism on $TN$. The Ricci curvature considered as a symmetric bilinear form will be written $\text{ric}^N(Y,Z) = \left\langle \text{Ric}^N(Y), Z \right\rangle$. The Dirac operator maps spinor fields to spinor fields and is locally defined by

$$D^N \varphi = i^s \sum_{j=1}^n \varepsilon_j e_j \cdot \nabla^{\Sigma^N}_{e_j} \varphi,$$

for every spinor field $\varphi$. The Dirac operator is an elliptic operator, formally selfadjoint, i.e. if $\psi$ or $\varphi$ has compact support, then \( \int_N \left\langle D^N \varphi, \psi \right\rangle dv_g = \int_N \left\langle \varphi, D^N \psi \right\rangle dv_g \).

3 Semi-Riemannian Spin$^c$ hypersurfaces and the Gauss formula

In this section, we study Spin$^c$ structures of hypersurfaces, such as the restriction of a Spin$^c$ bundle of an ambient semi-Riemannian manifold and the complex spinorial Gauss formula.

Let $\mathcal{Z}$ be an oriented $(n + 1)$-dimensional semi-Riemannian Spin$^c$ manifold and $M \subset \mathcal{Z}$ a semi-Riemannian hypersurface with trivial spacelike normal bundle. This means that there is a vector field $\nu$ on $\mathcal{Z}$ along $M$ satisfying $\langle \nu, \nu \rangle = +1$ and $\langle \nu, TM \rangle = 0$. Hence if the signature of $M$ is $(r, s)$, then the signature of $\mathcal{Z}$ is $(r + 1, s)$.
Proposition 3.1 The hypersurface $M$ inherits a $\text{Spin}^c$ structure from that on $Z$, and we have
\[
\begin{align*}
\Sigma Z|_M &\simeq \Sigma M \quad \text{if } n \text{ is even,} \\
\Sigma^+ Z|_M &\simeq \Sigma M \quad \text{if } n \text{ is odd.}
\end{align*}
\]
Moreover Clifford multiplication by a vector field $X$, tangent to $M$, is given by
\[
X \bullet \varphi = (\nu \cdot X \cdot \psi)|_M,
\]
where $\psi \in \Gamma(\Sigma Z)$ (or $\psi \in \Gamma(\Sigma^+ Z)$ if $n$ is odd), $\varphi$ is the restriction of $\psi$ to $M$, “$\cdot$” is the Clifford multiplication on $Z$, and “$\bullet$” that on $M$.

Proof: The bundle of space and time oriented orthonormal frames of $M$ can be embedded into the bundle of space and time oriented orthonormal frames of $Z$ restricted to $M$, by
\[
\Phi : P_{\text{SO}} M \longrightarrow P_{\text{SO}} Z|_M
\]
where $\xi$ is the Clifford multiplication on $Z$, and $\xi^c$ that on $M$.

\[
\begin{array}{c}
\text{Spin}^c(r, s) \quad \leftarrow \quad \text{Spin}^c(r + 1, s) \\
\downarrow \quad \downarrow \\
\text{SO}(r, s) \times S^1 \quad \leftarrow \quad \text{SO}(r + 1, s) \times S^1
\end{array}
\]

The isomorphism $\alpha$, defined in (8) yields the following commutative diagram:
\[
\begin{array}{c}
P_{\text{Spin}^c} M \quad \longrightarrow \quad P_{\text{Spin}^c} Z|_M \\
\downarrow \quad \downarrow \\
P_{\text{SO}} M \times_M P_{S^1} Z|_M \longrightarrow P_{\text{SO}} Z|_M \times_M P_{S^1} Z|_M
\end{array}
\]
The $\text{Spin}^c(r, s)$-principal bundle $(P_{\text{Spin}^c} M, \pi, M)$ and the $S^1$-principal bundle $(P_{S^1} M =: P_{S^1} Z|_M, \pi, M)$ define a $\text{Spin}^c$ structure on $M$. Let $\Sigma Z$ be the spinor bundle on $Z$,
\[
\Sigma Z = P_{\text{Spin}^c} Z \times_{\rho_{n+1}} \Sigma_{r+1, s},
\]
where $\rho_{n+1}$ stands for the spinorial representation of $\text{Spin}^c(r + 1, s)$. Moreover, for any spinor $\psi = [b \times s, \sigma] \in \Sigma Z$ we can always assume that $pr_1 \circ \Theta(b \times s) = b$ is a local section of $P_{\text{SO}} Z$ with $\nu$ for first basis vector where $pr_1$ is the projection into $P_{\text{SO}} Z$. Then we have
\[
\psi|_M = [\widetilde{b} \times s|_{U \cap M}, \sigma|_{U \cap M}],
\]
where the equivalence class is reduced to elements of $\text{Spin}^c(r,s)$. It follows that one can realise the restriction to $M$ of the spinor bundle $\Sigma Z$ as

$$
\Sigma Z|_M = P_{\text{Spin}^c} M \times_{\rho_{n+1} \circ \alpha} \Sigma r+1,s.
$$

If $n = 2m$ is even, it is easy to check that $\chi^0_{2m+1} \circ \alpha = \chi^0_{2m+1} |_{\text{Cl}}$. Hence $\chi^0_{2m+1} \circ \alpha$ is an irreducible representation of $\text{Cl}_{r,s}$ of dimension $2^m$, and finally $\chi^0_{2m+1} \circ \alpha \cong \chi_{2m}$. We conclude that

$$
\rho_{2m+1} \circ \alpha \cong \rho_{2m}, \quad \Sigma Z|_M \cong \Sigma M.
$$

If $n = 2m + 1$ is odd, we know that $\chi^0_{2m+1}$ is the unique irreducible representation of $\text{Cl}_{r,s}$ of dimension $2^m$ for which the action of the complex volume form is the identity. Since $n+1 = 2m+2$ is even, $\Sigma Z$ decomposes into positive and negative parts, $\Sigma^\pm Z = P_{\text{Spin}^c} Z \times_{\rho_{2m+1}^\pm} \Sigma^{\pm r+1,s}$. It is easy to show that $\chi_{2m+2} \circ \alpha = \chi_{2m+2} |_{\text{Cl}}$, but $\chi_{2m+2} \circ \alpha$ can be written as the direct sum of two irreducible inequivalent representations, as $\chi_{2m+2} \cong \chi_{2m+1} \oplus \chi_{2m+1}$. Hence, we have

$$
\chi_{2m+2} \circ \alpha = (\chi_{2m+2} \circ \alpha)^+ \oplus (\chi_{2m+1} \circ \alpha)^-,
$$

where $(\chi_{2m+2} \circ \alpha)^\pm (\omega_{r,s}) = \pm \text{Id}_{\text{Cl}_{r,s}}$. The representation $\chi_{2m+1}^0$ being the unique representation of $\text{Cl}_{r,s}$ of dimension $2^m$ for which the action of the volume form is the identity, we get $(\chi_{2m+2} \circ \alpha)^+ \cong \chi_{2m+1}^0$. Finally,

$$
\rho_{2m+2}^+ \circ \alpha \cong \rho_{2m+1} \quad \text{and} \quad \Sigma^+ Z|_M \cong \Sigma M.
$$

Now, Equation (12) follows directly from the above identification.

**Remarks 3.2**

1. *The algebraic remarks in the previous section show that if $n$ is odd we can also get $\Sigma^+ Z|_M \cong \Sigma M$, where the Clifford multiplication by a vector field tangent to $M$ is given by $X \circ \varphi = -(\nu \cdot X \cdot \psi)|_M$.*

2. *The connection 1-form defined on the restricted $\mathbb{S}^1$-principal bundle $(P_{\mathbb{S}^1} M \cong \mathbb{S}^1, \pi, M)$, is given by

$$
A^M = A^2|_M : T(P_{\mathbb{S}^1} M) = T(P_{\mathbb{S}^1} Z)|_M \longrightarrow i\mathbb{R}.
$$

Then the curvature 2-form $i\Omega^M$ on the $\mathbb{S}^1$-principal bundle $P_{\mathbb{S}^1} M$ is given by $i\Omega^M = i\Omega^Z|_M$, which can be viewed as an imaginary 2-form on $M$ and hence as the curvature form of the line bundle $L^M$, the restriction of the line bundle $L^Z$ to $M$.*

3. *For every $\psi \in \Gamma(\Sigma Z) (\psi \in \Gamma(\Sigma^+ Z)$ if $n$ is odd), the real 2-forms $\Omega^M$ and $\Omega^Z$ are related by the following formulas:

$$
|\Omega^Z|^2 = |\Omega^M|^2 + |\nu \cdot \Omega^Z|^2,
$$

(14)
\[(\Omega^Z \cdot \psi)|_M = \Omega^M \cdot \varphi + (\nu \cdot \Omega^Z) \cdot \varphi.\]  \hfill (15)

In fact, we can write
\[
\Omega^Z = \sum_{i=1}^{n} \Omega^Z(\nu, e_i) \nu \wedge e_i + \sum_{i<j} \Omega^Z(e_i, e_j) e_i \wedge e_j = - (\nu \cdot \Omega^Z) \wedge \nu + \Omega^M,
\]
which is (14). When restricting the Clifford multiplication of \(\Omega^Z\) by \(\psi\) to the hypersurface \(M\) we obtain
\[
(\Omega^Z \cdot \psi)|_M = (\nu \cdot (\nu \cdot \Omega^Z) \cdot \psi)|_M + (\Omega^M \cdot \psi)|_M = (\nu \cdot \Omega^Z) \cdot \varphi + \Omega^M \cdot \varphi.
\]  \hfill (16)

**Proposition 3.3 (The spinorial Gauss formula)** We denote by \(\nabla^{\Sigma Z}\) the spinorial Levi-Civita connection on \(\Sigma Z\) and by \(\nabla^{\Sigma M}\) that on \(\Sigma M\). For all \(X \in \Gamma(TM)\) and for every spinor field \(\psi \in \Gamma(\Sigma Z)\), then
\[
(\nabla^{\Sigma Z}_X \psi)|_M = \nabla^{\Sigma M}_X \varphi - \frac{1}{2} W(X) \cdot \varphi,
\]  \hfill (17)

where \(W\) denotes the Weingarten map with respect to \(\nu\) and \(\varphi = \psi|_M\). Moreover, let \(D^Z\) and \(D^M\) be the Dirac operators on \(Z\) and \(M\). Denoting by the same symbol any spinor and it’s restriction to \(M\), we have
\[
\nu \cdot D^Z \varphi = \tilde{D} \varphi + \frac{i^* n}{2} H \varphi - i^* \nabla^{\Sigma Z}_\nu \varphi,
\]  \hfill (18)

where \(H = \frac{1}{n} \text{tr}(W)\) denotes the mean curvature and \(\tilde{D} = D^M \oplus (-D^M)\) if \(n\) is even and \(\tilde{D} = D^M\) if \(n\) is odd.

**Proof:** The Riemannian Gauss formula is given, for every vector fields \(X\) and \(Y\) on \(M\), by
\[
\nabla^Z_X Y = \nabla^M_X Y + \langle W(X), Y \rangle \nu.
\]  \hfill (19)

Let \((e_1, e_2, \ldots, e_n)\) a local space and time oriented orthonormal frame of \(M\), such that \(b = (e_0 = \nu, e_1, e_2, \ldots, e_n)\) is that of \(Z\). We consider \(\psi\) a local section of \(\Sigma Z\), \(\psi = [b \times s, \sigma]\) where \(s\) is a local section of \(P_{\Sigma Z}\). Using (9), (19) and the fact that \(X(\psi)|_M = X(\varphi)\) for \(X \in \Gamma(TM)\), we compute for \(j = 1, \ldots, n\)
\[
\big(\nabla^{\Sigma Z}_{e_j} \psi\big)|_M = e_j(\varphi) + \frac{1}{4} \sum_{k=0}^{n} \varepsilon_k (e_k \cdot \nabla^{\Sigma Z}_{e_j} e_k \cdot \psi)|_M + \frac{1}{2} A^Z(s_*(e_j)) \varphi
\]
\[
= e_j(\varphi) + \frac{1}{4} \sum_{k=1}^{n} \varepsilon_k (e_k \cdot \nabla^{\Sigma Z}_{e_j} e_k \cdot \psi)|_M + \frac{1}{4} (\nu \cdot \nabla^{\Sigma Z}_{e_j} \nu \cdot \psi)|_M + \frac{1}{2} A^Z(s_*(e_j)) \varphi
\]
\[
= \nabla^{\Sigma M}_{e_j} \varphi + \frac{1}{4} \sum_{k=1}^{n} \varepsilon_k <W(e_j), e_k > (e_k \cdot \nu \cdot \psi)|_M - \frac{1}{4} (\nu \cdot W(e_j) \cdot \psi)|_M
\]
\[
= \nabla^{\Sigma M}_{e_j} \varphi - \frac{1}{2} (\nu \cdot W(e_j) \cdot \psi)|_M
\]
\[
= \nabla^{\Sigma M}_{e_j} \varphi - \frac{1}{2} W(e_j) \cdot \varphi.
\]
Moreover \((D^Z \psi)|_M = i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla_{\varepsilon_j}^{\Sigma} \psi)|_M + i^s (\nu \cdot \nabla^{\Sigma} \psi)|_M\), and by (17),
\[
i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla_{\varepsilon_j}^{\Sigma \psi})|_M = i^s \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla_{\varepsilon_j}^{\Sigma \psi}) - i^s \frac{1}{2} \sum_{j=1}^n \varepsilon_j (e_j \cdot \nu \cdot W(e_j) \cdot \psi)|_M
\]
\[
= -i^s \nu \cdot \sum_{j=1}^n \varepsilon_j (e_j \cdot \nabla_{\varepsilon_j}^{\Sigma \psi}) + i^s \frac{1}{2} \sum_{j=1}^n \varepsilon_j (\nu \cdot e_j \cdot W(e_j) \cdot \psi)|_M
\]
\[
= -\nu \cdot \tilde{D}_{\psi} - \frac{i^s}{2} \text{tr}(W)(\nu \cdot \psi)|_M.
\]

**Proposition 3.4** Let \(Z\) be an \((n+1)\)-dimensional semi-Riemannian Spin\(^c\) manifold. Assume that \(Z\) carries a semi-Riemannian foliation by hypersurfaces with trivial spacelike normal bundle, i.e., the leaves \(M\) are semi-Riemannian hypersurfaces and there exists a vector field \(\nu\) on \(Z\) perpendicular to the leaves such that \(\langle \nu, \nu \rangle = 1\) and \(\nabla^Z \nu = 0\). Then the commutator of the leafwise Dirac operator and the normal derivative is given by
\[
i^{-s}[\nabla^Z_{\nu}, \tilde{D}] \varphi = D^W \varphi - \frac{n}{2} \nu \cdot \text{grad}^M (H) \cdot \varphi + \frac{1}{2} \nu \cdot \text{div}^M (W) \cdot \varphi + \frac{i}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi.
\]
Here \(\text{grad}^M\) denotes the leafwise gradient, \(\text{div}^M (W) = \sum_{i=1}^n \varepsilon_i (\nabla_{\varepsilon_i}^M W)(e_i)\) denotes the leafwise divergence of the endomorphism field \(W\) and \(D^W \varphi = \sum_{i=1}^n \varepsilon_i \nu \cdot e_i \cdot \nabla^M_{\varepsilon_i} \varphi\).

**Proof:** We choose a local oriented orthonormal tangent frame \((e_1, \ldots, e_n)\) for the leaves and we may assume for simplicity that \(\nabla^Z_{\nu} e_j = 0\). Now, we compute
\[
i^{-s}[\nabla^Z_{\nu}, \tilde{D}] \varphi = \sum_{j=1}^n \varepsilon_j \left(\nabla^Z_{\nu} (\nu \cdot e_j \cdot \nabla^M_{\varepsilon_j} \varphi) - \nu \cdot e_j \cdot \nabla^M_{\varepsilon_j} \nabla^Z_{\nu} \varphi\right)
\]
\[
= \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left[\nabla^Z_{\nu} (\nabla^M_{\varepsilon_j} + \frac{1}{2} \nu \cdot W(e_j)) - \nabla^M_{\varepsilon_j} \nabla^Z_{\nu} \varphi\right]
\]
\[
= \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left[\nabla^Z_{\nu} (\nabla^M_{\varepsilon_j} + \frac{1}{2} \nu \cdot W(e_j)) - \nabla^M_{\varepsilon_j} \nabla^Z_{\nu} \varphi\right]
\]
\[
= -\frac{i}{2} \nu \cdot \text{Ric}^Z (\nu) \cdot \varphi + \frac{i}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi
\]
\[
+ \sum_{j=1}^n \varepsilon_j \nu \cdot e_j \cdot \left(\nabla^Z_{\varepsilon_j} W(e_j) + \frac{1}{2} \nu \cdot (\nabla^Z_{\varepsilon_j} W)(e_j)\right) \varphi
\]

12
\[
\begin{align*}
&= \frac{1}{2} \nu \cdot \text{Ric}^Z(\nu) \cdot \varphi + i \frac{1}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi \\
&\quad + \sum_{j=1}^{n} \varepsilon_j \nu \cdot e_j \cdot \left( \nabla_{W(e_j)}^{\Sigma M} - \frac{1}{2} \nu \cdot W^2(e_j) + \frac{1}{2} \nu \cdot (\nabla^Z_W)(e_j) \right) \varphi \\
&= -\frac{1}{2} \nu \cdot \text{Ric}^Z(\nu) \cdot \varphi + i \frac{1}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi + \mathcal{D}^W \varphi \\
&\quad + \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j \cdot \left( -W^2(e_j) + (\nabla^Z_W)(e_j) \right) \varphi.
\end{align*}
\]

The Riccati equation for the Weingarten map \((\nabla^2_W X)(X) = R^Z(X, \nu) \nu + W^2(X)\) yields
\[
i^{-\kappa}[\nabla^Z_{\nu}, \tilde{D}] \varphi = -\frac{1}{2} \nu \cdot \text{Ric}^Z(\nu) \cdot \varphi + i \frac{1}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi + \mathcal{D}^W \varphi \\
+ \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j \cdot (R^Z(e_j, \nu) \nu) \cdot \varphi \\
= -\frac{1}{2} \nu \cdot \text{Ric}^Z(\nu) \cdot \varphi + i \frac{1}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi + \mathcal{D}^W \varphi + \frac{1}{2} \text{Ric}^Z(\nu, \nu) \varphi \\
= \mathcal{D}^W \varphi - \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j \text{ric}^Z(\nu, e_j) \nu \cdot e_j \cdot \varphi + i \frac{1}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi. \tag{20}
\]

The Codazzi-Mainardi equation for \(X, Y, V \in TM\) is given by \(\langle \nabla^M_Y W(X), V \rangle = \langle (\nabla^M_X W)(Y), V \rangle - \langle (\nabla^M_Y W)(X), V \rangle \). Thus,
\[
\text{ric}^Z(\nu, X) = \sum_{j=1}^{n} \varepsilon_j \langle R^Z(X, e_j) e_j, \nu \rangle \\
= \sum_{j=1}^{n} \varepsilon_j \left( \langle (\nabla^M_X W)(e_j), e_j \rangle - \langle (\nabla^M_Y W)(X), e_j \rangle \right) \\
= \text{tr}(\nabla^M_X W) - \langle \text{div}^M(W), X \rangle.
\]

Plugging this into (20) we get
\[
i^{-\kappa}[\nabla^Z_{\nu}, \tilde{D}] \varphi = \mathcal{D}^W \varphi - \frac{1}{2} \sum_{j=1}^{n} \varepsilon_j \left( \text{tr}(\nabla^M_{e_j} W) - \langle \text{div}^M(W), e_j \rangle \right) \nu \cdot e_j \cdot \varphi \\
+ i \frac{1}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi \\
= \mathcal{D}^W \varphi - \frac{n}{2} \nu \cdot \text{grad}^M(H) \cdot \varphi + \frac{1}{2} \nu \cdot \text{div}^M(W) \cdot \varphi + i \frac{1}{2} \nu \cdot (\nu \cdot \Omega^Z) \cdot \varphi.
\]
4 The generalized cylinder on semi-Riemannian Spin$^c$ manifolds

Let $M$ be an $n$-dimensional smooth manifold and $g_t$ a smooth 1-parameter family of semi-Riemannian metrics on $M$, $t \in I$ where $I \subset \mathbb{R}$ is an interval. We define the generalized cylinder by

$$Z := I \times M,$$

with semi-Riemannian metric $g_Z := \langle \cdot, \cdot \rangle = dt^2 + g_t$. The generalized cylinder is an $(n+1)$-dimensional semi-Riemannian manifold of signature $(r+1,s)$ if the signature of $g_t$ is $(r,s)$.

**Proposition 4.1** There is a 1-1-correspondence between the Spin$^c$ structures on $M$ and that on $Z$.

**Proof:** As explained in Section 3, Spin$^c$ structures on $Z$ can be restricted to Spin$^c$ structures on $M$. Conversely, given a Spin$^c$ structure on $M$ it can be pulled back to $I \times M$ via the projection $pr_2 : I \times M \rightarrow M$ yields a Spin$^c$ structure on $Z$. In fact, the pull back of the Spin$^c(r,s)$-principal bundle $P_{\text{Spin}^c} M$ on $M$ gives rise to a Spin$^c(r,s)$-principal bundle on $Z$ denoted by $P_{\text{Spin}^c} Z$

$$
\begin{array}{ccc}
P_{\text{Spin}^c} Z & \longrightarrow & P_{\text{Spin}^c} M \\
\downarrow \pi & & \downarrow \pi \\
Z = I \times M & \longrightarrow & M
\end{array}
$$

Enlarging the structure group via the embedding Spin$^c(r,s) \hookrightarrow$ Spin$^c(r+1,s)$, which covers the standard embedding

$$\text{SO}(r,s) \times S^1 \hookrightarrow \text{SO}(r+1,s) \times S^1$$

$$(a,z) \mapsto \left( \begin{array}{cc} 1 & 0 \\ 0 & a \end{array} \right), z,$$

gives a Spin$^c(r+1,s)$-principal fibre bundle on $Z$, denoted also by $P_{\text{Spin}^c} Z$. The pull back of the line bundle $L^M$ on $M$ defining the Spin$^c$ structure on $M$, gives a line bundle $L^Z$ on $Z$ such that the following diagram commutes

$$
\begin{array}{ccc}
L^Z & = & pr_2^*(L^M) \\
\downarrow \pi & & \downarrow \pi \\
Z = I \times M & \longrightarrow & M
\end{array}
$$

The line bundle $L^Z$ on $Z$ and the Spin$^c(r+1,s)$-principal fibre bundle $P_{\text{Spin}^c} Z$ on $Z$ yields the Spin$^c$ structure on $Z$ which restricts to the given Spin$^c$ structure on $M$. 

Remark 4.2 If $M$ is a Spin$^c$ Riemannian manifold and if we denote by $i\Omega^M$ the imaginary valued curvature on the line bundle $L^M$, we know that there exists a unique curvature 2-form, denoted by $i\Omega^Z$, on the line bundle $L^Z = pr_2^*(L^M)$, defined by $i\Omega^Z = pr_2^*(i\Omega^M)$. Thus we have

\[ \Omega^Z(X,Y) = \Omega^M(X,Y) \] and \[ \Omega^Z(\nu,Y) = 0 \] for any $X,Y \in \Gamma(TM)$.

Proposition 4.3 \[2\] On a generalized cylinder $Z = I \times M$ with semi-Riemannian metric $g^Z = \langle \cdot, \cdot \rangle = dt^2 + g_t$ we define, in every $p \in M$ and $X,Y \in T_p M$, the first and second derivatives of $g_t$ by

\[ \dot{g}_t(X,Y) := \frac{d}{dt}(g_t(X,Y)) \] and \[ \ddot{g}_t(X,Y) := \frac{d^2}{dt^2}(g_t(X,Y)). \]

Hence the following formulas hold:

\[
\langle W(X), Y \rangle = -\frac{1}{2} \dot{g}_t(X,Y), \quad (21)
\]

\[
\langle R^Z(U,V)X,Y \rangle = \langle R^M(U,V)X,Y \rangle \quad (22)
\]

\[
\ge \frac{1}{4} \left( \langle \nabla^M Y \dot{g}_t(U,V) \rangle - \langle \nabla^M X \dot{g}_t(V,U) \rangle \right), \quad (23)
\]

\[
\langle R^Z(X,Y)U,\nu \rangle = -\frac{1}{2} \left( \ddot{g}_t(X,Y) + \dot{g}_t(W(X),Y) \right), \quad (24)
\]

where $X,Y,U,V \in T_p M$, $p \in M$.

5 The variation formula for the Dirac operator on Spin$^c$ manifolds

First we give some facts about parallel transport on Spin$^c$ manifolds along a curve $c$. We consider a Riemannian Spin$^c$ manifold $N$, we know that there exists a unique correspondence which associates to a spinor field $\psi(t) = \psi(c(t))$ along a curve $c: I \rightarrow N$ another spinor field $\frac{D}{dt}\psi$ along $c$, called the covariant derivative of $\psi$ along $c$, such that

\[
\frac{D}{dt}(\psi + \varphi) = \frac{D}{dt}\psi + \frac{D}{dt}\varphi, \quad \text{for any } \psi \text{ and } \varphi \text{ along the curve } c,
\]

\[
\frac{D}{dt}(f\psi) = f\frac{D}{dt}\psi + \left( \frac{d}{dt}f \right) \psi, \quad \text{where } f \text{ is a differentiable function on } I,
\]

\[
\nabla_{\dot{c}(t)}^N \psi = \frac{D}{dt}\varphi, \quad \text{where } \varphi(t) = \psi(c(t)).
\]
A spinor field \( \psi \) along a curve \( c \) is called parallel when \( \frac{D}{dt} \psi(t) = 0 \) for all \( t \in I \). Now, if \( \psi_0 \) is a spinor at the point \( c(t_0) \), \( t_0 \in I, (\psi_0 \in \Sigma(c(t_0))N) \) then there exists a unique parallel spinor \( \varphi \) along \( c \), such that \( \psi_0 = \varphi(t_0) \). The linear isometry \( \tau_{t_0}^{t_1} \) defined by

\[
\tau_{t_0}^{t_1} : \Sigma_{c(t_0)} N \longrightarrow \Sigma_{c(t_1)} N
\]

\[
\psi_0 \longrightarrow \varphi(t_1),
\]

is called the parallel transport along the curve \( c \) from \( c(t_0) \) to \( c(t_1) \). The basic property of the parallel transport on a \( \text{Spin}^c \) manifold is the following: Let \( \psi \) be a spinor field on a Riemannian \( \text{Spin}^c \) manifold \( N \), \( X \in \Gamma(TN) \), \( p \in N \) and \( c : I \longrightarrow N \) an integral curve through \( p \), i.e., \( c(t_0) = p \) and \( \frac{d}{dt} c(t) = X(c(t)) \), we have

\[
(\nabla_X \Sigma N \psi)_p = \frac{d}{dt} \left( \tau_{t_0}^{t_1}(\psi(0)) \right)_{t=t_0}.
\]

(25)

Now, we consider \( g_t \) a smooth 1-parameter family of semi-Riemannian metrics on a \( \text{Spin}^c \) manifold \( M \) and the generalized cylinder \( Z = I \times M \) with semi-Riemannian metric \( g^Z = \langle \cdot , \cdot \rangle = dt^2 + g_t \). For \( t \in I \) we denote by \( M_t \) the manifold \( (M_t, g_t) \). Let us write “\( \cdot \)” for the Clifford multiplication on \( Z \) and “\( \bullet_t \)” for that on \( M_t \). Recall from Section 4 that \( \text{Spin}^c \) structures on \( M \) and \( Z \) are in 1-1-correspondence and \( \Sigma Z|_{M_t} = \Sigma M_t \) as hermitian vector bundles if \( n = r + s \) is even and \( \Sigma^+ Z|_{M_t} = \Sigma M_t \) if \( n \) is odd.

For a given \( x \in M \) and \( t_0, t_1 \in I \), parallel transport \( \tau_{t_0}^{t_1} \) on the generalized cylinder \( Z \) along the curve \( c : I \rightarrow I \times M, t \rightarrow (t, x) \) is given by

\[
\tau_{t_0}^{t_1} : \Sigma_{c(t_0)} Z \simeq \Sigma_x M_{t_0} \longrightarrow \Sigma_{c(t_1)} Z \simeq \Sigma_x M_{t_1}.
\]

This isomorphism satisfies

\[
\tau_{t_0}^{t_1}(X \bullet_{t_0} \varphi) = (\zeta_{t_0}^{t_1} \cdot X) \bullet_{t_1}(\tau_{t_0}^{t_1} \varphi),
\]

\[
<\tau_{t_0}^{t_1} \psi, \tau_{t_0}^{t_1} \varphi> = <\psi, \varphi>,
\]

where \( \zeta_{t_0}^{t_1} : T_{(t_0,x)} Z \simeq T_x M_{t_0} \longrightarrow T_{(t_1,x)} Z \simeq T_x M_{t_1} \) is the parallel transport on \( Z \) along the same curve \( c \), \( X \in T_x M_{t_0} \) and \( \psi, \varphi \in \Sigma_x M_{t_0} \).

**Theorem 5.1** On a \( \text{Spin}^c \) manifold \( M \), let \( g_t \) be a smooth 1-parameter family of semi-Riemannian metrics. Denote by \( D^M \) the Dirac operator of \( M_t \), and \( \mathcal{D}^g = \sum_{i,j=1}^n \varepsilon_i \varepsilon_j \hat{g}_0 (e_i, e_j) e_i \bullet_t \nabla^{\Sigma M_t} \). Then for any smooth spinor field \( \psi \) on \( M_{t_0} \) we have

\[
\frac{d}{dt} \bigg|_{t=t_0} \tau_{t_0}^{t_1} D^M \psi = -\frac{1}{2} \mathcal{D}^g \psi + \frac{1}{4} \text{grad}^M (\text{tr}_{g_{t_0}} (\hat{g}_{t_0})) \bullet_{t_0} \psi - \frac{1}{4} \text{div}^M (\hat{g}_{t_0}) \bullet_{t_0} \psi.
\]

**Proof:** The vector field \( \nu := \frac{\partial}{\partial t} \) is spacelike of unit length and orthogonal to the hypersurfaces \( M_t := \{ t \} \times M \). Denote by \( W_t \) the Weingarten map of \( M_t \) with respect to
ν and by \( H_t \) the mean curvature. If \( X \) is a local coordinate field on \( M \), then \( \langle X, \nu \rangle = 0 \) and \([X, \nu] = 0\). Thus

\[
0 = d_\nu \langle X, \nu \rangle = \langle \nabla^Z_\nu X, \nu \rangle + \langle X, \nabla^Z_\nu \nu \rangle = \langle \nabla^Z_\nu \nu, \nu \rangle + \langle X, \nabla^Z_\nu \nu \rangle = -\langle W_t(X), \nu \rangle + \langle X, \nabla^Z_\nu \nu \rangle = \langle X, \nabla^Z_\nu \nu \rangle
\]

and differentiating \( \langle \nu, \nu \rangle = 1 \) yields \( \langle \nu, \nabla^Z_\nu \nu \rangle = 0 \). Hence \( \nabla^Z_\nu \nu = 0 \), i.e., for \( x \in M \) the curves \( t \mapsto (t, x) \) are geodesics parametrized by arclength. So the assumptions of Proposition 3.4 are satisfied for the foliation \((M_t)_{t \in I}\). By Remark 4.2, the commutator formula of Proposition 3.4 gives for a section \( \varphi \) of \( \Sigma M_t \) (or \( \Sigma^+ M_t \) if \( n \) is odd)

\[
i^{-s}[\nabla^Z_\nu, D^{M_t}] \varphi = \Omega^{W_t} \varphi - \frac{n}{2} \text{grad}^{M_t} (H_t) \cdot_t \varphi + \frac{1}{2} \text{div}^{M_t} (W_t) \cdot_t \varphi. \tag{26}
\]

From Proposition 4.3 we deduce

\[
\text{div}^{M_t} (W_t) = -\frac{1}{2} \text{div}^{M_t} (\dot{g}_t) \quad \text{and} \quad \Omega^{W_t} = -\frac{1}{2} \Omega^{\dot{g}_t}.
\]

Thus (26) can be rewritten as

\[
i^{-s}[\nabla^Z_\nu, D^{M_t}] \varphi = -\frac{1}{2} \text{D}^{\dot{g}_t} \varphi + \frac{1}{4} \text{grad}^{M_t} (\text{tr}_{\dot{g}_t} (\dot{g}_t)) \cdot_t \varphi - \frac{1}{4} \text{div}^{M_t} (\dot{g}_t) \cdot_t \varphi. \tag{27}
\]

Now if \( \varphi \) is parallel along the curves \( t \mapsto (t, x) \), i.e., it is of the form \( \varphi(t, x) = \tau_0^t \psi(t_0, x) \) for some spinor field \( \psi \) on \( M_{t_0} \), then using (25) at \( t = t_0 \), the left hand side of (27) could be written as

\[
i^{-s} \frac{d}{dt} \bigg|_{t=t_0} \tau_0^t D^{M_t} \tau_0^t \psi = i^{-s} \frac{d}{dt} \bigg|_{t=t_0} \tau_0^t D^{M_t} \tau_0^t \psi = \tau_0^t D^{M_t} \tau_0^t \psi,
\]

which gives the variation formula for the Dirac operator.

**Corollary 5.2** Let \((M^n, g)\) be a Spin\(^c\) Riemannian manifold, if we consider the family of metrics defined by \( g_t = g + tk \), where \( k \) is a symmetric \((0, 2)\)-tensor, we have

\[
\frac{d}{dt} \bigg|_{t=0} \tau_0^t D^{M_t} \tau_0^t \psi = -\frac{1}{2} \text{D}^k \psi + \frac{1}{4} \text{grad}^M (\text{tr}_g (k)) \cdot \psi - \frac{1}{4} \text{div}^M (k) \cdot \psi, \tag{29}
\]

where \( \cdot = \cdot_{t_0=0} \) is the Clifford multiplication on \( M \).

This formula has been proved in [4], Theorem 21 for spin Riemannian manifolds and in [2] for spin semi-Riemannian manifolds.
6 Energy-Momentum tensor on Spin\(^c\) manifolds

In this section we study the Energy-Momentum tensor on Spin\(^c\) Riemannian manifolds from a geometric point of view. We begin by giving the proofs of Proposition 1.1, Theorem 1.2 and Proposition 1.3.

**Proof of Proposition 1.1** : Using Equation (29) we calculate

\[
\frac{d}{dt}\bigg|_{t=0} (\tau^0 \tau_t^0 D^M, \tau^0_0 \psi, \psi) = \frac{d}{dt}\bigg|_{t=0} (D^M \tau^0_0 \psi, \tau^0_0 \psi) = -\frac{1}{2} \sum_{i,j} k(e_i, e_j)(e_i \cdot \nabla^M_{e_j} \psi, \psi) = -\frac{1}{2} \int_M <k, T_{\psi} > dv_g.
\]

**Proof of Theorem 1.2** : The Proof of this Theorem will be omitted since it is similar to the one given by Friedrich and Kim in [8] for spin manifolds.

**Proof of Proposition 1.3** : Let \( \psi \) be any parallel spinor field on \( Z \). Then Equation (17) yields

\[
\nabla^M_X \varphi = \frac{1}{2} W(X) \cdot \varphi.
\]

Let \((e_1, \ldots, e_n)\) be a positively oriented local orthonormal basis of \( TM \). For \( j = 1, \ldots, n \) we have

\[
\nabla^M_{e_j} \varphi = \frac{1}{2} \sum_{k=1}^n W_{jk} e_k \cdot \varphi.
\]

Taking Clifford multiplication by \( e_i \) and the scalar product with \( \varphi \), we get

\[
\text{Re} \langle e_i \cdot \nabla^M_{e_j} \varphi, \varphi \rangle = \frac{1}{2} \sum_{k=1}^n W_{jk} \text{Re} \langle e_i \cdot e_k \cdot \varphi, \varphi \rangle = -W_{ij} |\varphi|^2.
\]

Therefore, \( 2\ell^2 = -W \). Using Equation (18) it is easy to see that \( \varphi \) is an eigenspinor associated with the eigenvalue \( -\frac{n}{2} H \) of \( \tilde{D} \). Since \( \text{Scal}^Z = \text{Scal}^M + 2 \text{ric}^Z(\nu, \nu) - n^2 H^2 + |W|^2 \) we get

\[
\frac{1}{4} (\text{Scal}^M - c_n |\Omega^M|) + |T^\varphi|^2 = \frac{1}{4} (\text{Scal}^Z - 2 \text{ric}^Z(\nu, \nu) - c_n |\Omega^M|) + n^2 H^2 / 4,
\]

hence \( M \) satisfies the equality case in (3) if and only if (6) holds.
Corollary 6.1 Under the same conditions as Proposition 1.3, if \( n = 2 \) or \( 3 \), the hypersurface \( M \) satisfies the equality case in (3) if \( \text{Ric}^Z(\nu) = 0 \) and \( \text{Scal}^Z \geq 0 \).

Proof: Since \( Z \) has a parallel spinor, we have (see [7])

\[
|\text{Ric}^Z(\nu)| = |\nu \cdot \Omega^Z|, \tag{31}
\]

\[
i(Y \cdot \Omega^Z) \cdot \psi = \text{Ric}^Z(Y) \cdot \psi \quad \text{for every } Y \in \Gamma(\Sigma Z). \tag{32}
\]

For \( Y = e_j \) in Equation (32) then taking Clifford multiplication by \( e_j \) and summing from \( j = 1, \ldots, n+1 \), we get

\[
i \sum_{j=1}^{n+1} e_j \cdot (e_j \cdot \Omega^Z) \cdot \psi = \sum_{j=1}^{n+1} e_j \cdot \text{Ric}^Z(e_j) \cdot \psi = -\text{Scal}^Z \psi.
\]

But 2 \( \Omega^Z \cdot \psi = \sum_{j=1}^{n+1} e_j \cdot (e_j \cdot \Omega^Z) \cdot \psi \), hence we deduce that \( \Omega^Z \cdot \psi = i\frac{\text{Scal}^Z}{2} \psi \). By (31) and (15) we obtain \( \Omega^M \cdot \varphi = i\frac{\text{Scal}^M}{2} \varphi \). Since \( n = 2 \) or \( 3 \) we have \( |\Omega^M| = \frac{\text{Scal}^Z}{2} \) and Equation (6) is satisfied.

Corollary 6.2 Under the same conditions as Proposition 1.3, if the restriction of the complex line bundle \( L^Z \) is flat, i.e., \( L^M \) is a flat complex line bundle \((\Omega^M = 0)\), the hypersurface \( M \) is a limiting manifold for (3).

Proof: Since \( \Omega^M = 0 \), Equation (15) yields \( i\frac{\text{Scal}^Z}{2} \varphi = \Omega^Z \cdot \psi|_M = (\nu \cdot \Omega^Z) \cdot \varphi \). But,

\[
i(\nu \cdot \Omega^Z) \cdot \varphi = i(\nu \cdot (\nu \cdot \Omega^Z) \cdot \psi)|_M = (\nu \cdot \text{Ric}^Z(\nu) \cdot \psi)|_M
\]

\[
= -\text{ric}^Z(\nu,\nu) \varphi + \sum_{j=1}^{n} \text{ric}^Z(\nu, e_j) e_j \cdot \varphi. \tag{33}
\]

Taking the real part of the scalar product of Equation (33) with \( \varphi \) yields \( \frac{\text{Scal}^Z}{2} = \text{ric}^Z(\nu,\nu) \), hence Equation (6) is satisfied.

Now, let \( M \) be a Spin\(^c\) Riemannian manifold having a generalized Killing spinor field \( \varphi \) with a symmetric endomorphism \( F \) on the tangent bundle \( TM \). As mentioned in the introduction, it is straightforward to see that \( 2T^\varphi(X,Y) = -\langle F(X),Y \rangle \). We will study these generalized Killing spinors when the tensor \( F \) is a Codazzi-Mainardi tensor, i.e., \( F \) satisfies

\[
(\nabla^M_X F)(Y) = (\nabla^M_Y F)(X) \quad \text{for } X,Y \in \Gamma(TM). \tag{34}
\]

For this, we give the following lemma whose proof will be omitted since it is similar to Lemma 7.3 in [2].
Lemma 6.3 [2] Let $g_t$ be a smooth 1-parameter family of semi-Riemannian metrics on a Spin$^c$ manifold $(M^n, g = g_0)$ and let $F$ be a field of symmetric endomorphisms of $TM$. We consider the metric $g_Z = (\cdot, \cdot) = dt^2 + g_t$ on $Z$ such that $g_t(X,Y) = g((1d - tF)^2X,Y)$ for all vector fields $X,Y$ on $M$. We have $\langle R_Z(U,\nu)\nu, V \rangle = 0$ for all vector fields $U,V$ tangent to $M$ and if $F$ satisfies the Codazzi-Mainardi equation then $\langle R_Z(U,V)W, \nu \rangle = 0$ for all $U,V$ and $W$ on $Z$.

Proof of Theorem 1.4: We define $\psi_{(0,x)} := \varphi_x$ via the identification $\Sigma_xM \cong \Sigma_{(0,x)}Z$ (resp. $\Sigma^+_{(0,x)}Z$ for $n$ odd) and $\psi_{(t,x)} = \frac{1}{t} \psi_{(0,x)}$. By Equation (21), the endomorphism $F$ is the Weingarten tensor of the immersion of $\{0\} \times M$ in $Z$ and hence by construction we have for all $X \in \Gamma(TM)$

$$\nabla_X^{\Sigma_Z} \psi_{\{0\} \times M} = 0 \quad \text{and} \quad \nabla_{\nu}^{\Sigma_Z} \psi = 0.$$  \hspace{1cm} (35)

Since the tensor $F$ satisfies the Codazzi-Mainardi equation, Lemma 6.3 yields $g_Z(R^Z(U,V)W,\nu) = 0$ for all $U,V$ and $W \in \Gamma(Z)$ and $g_Z(R^Z(X,\nu)\nu, Y) = 0$ for all $X$ and $Y$ tangent to $M$. Hence $R^Z(\nu, X) = 0$ for all $X \in \Gamma(TM)$. Let $X$ be a fixed arbitrary tangent vector field on $M$. Using (10) and (35) we get

$$\nabla_{\nu}^{\Sigma_Z} \psi = R^Z(\nu, X)\psi = \frac{1}{2} R^Z(X,\nu) \cdot \psi + i \frac{1}{2} \Omega^Z(X,\nu)\psi = 0.$$

Thus showing that the spinor field $\nabla_{\nu}^{\Sigma_Z} \psi$ is parallel along the geodesics $\mathbb{R} \times \{x\}$. Now (35) shows that this spinor vanishes for $t = 0$, hence it is zero everywhere on $Z$. Since $X$ is arbitrary, this shows that $\psi$ is parallel on $Z$.

Corollary 6.4 Let $(M^3, g)$ be a compact, oriented Riemannian manifold and $\varphi$ an eigenspinor associated with the first eigenvalue $\lambda_1$ of the Dirac operator such that the Energy-Momentum tensor associated with $\varphi$ is a Codazzi tensor. $M$ is a limiting manifold for (3) if and only if the generalized cylinder $Z^4$, equipped with the Spin$^c$ structure arising from the given one on $M$, is Kähler of positive scalar curvature and the immersion of $M$ in $Z$ has constant mean curvature $H$.

Proof: First, we should point out that every 3-dimensional compact, oriented, smooth manifold has a Spin$^c$ structure. Now, if $M^3$ is a limiting manifold for (3), by Theorem 1.4, the generalized cylinder has a parallel spinor whose restriction to $M$ is $\varphi$. Since $Z$ is a 4-dimensional Spin$^c$ manifold having parallel spinor, $Z$ is Kähler [1]. Moreover, using (15), we have

$$\Omega^M \cdot \varphi = \frac{i}{2} \text{Scal}^Z \varphi = \frac{\mathcal{C}_n}{2} |\Omega^M| \varphi,$$

so $\text{Scal}^Z \geq 0$. Finally $H = \frac{1}{n} \text{tr}(W) = \frac{1}{2} \text{tr}(-2T^c) = -\frac{2}{n} \lambda_1$, which is a constant. Now if the generalized cylinder is Kähler and $M$ is a compact hypersurface of constant mean curvature $H$, thus $M$ is compact hypersurface immersed in a Spin$^c$ manifold having parallel spinor with constant mean curvature. Since $\text{Scal}^Z \geq 0$ and $\nu, \Omega^Z = \text{Ric}^Z(\nu) = 0$, Corollary 6.1 gives the result.

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References


