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Option pricing with Levy process using Mellin Transform

Jules SADEFO KAMDEM ¹

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Abstract

In this paper, we use Mellin transform to get the expression for the free boundary and price of an American finite-lived option, when the underlying is governed by the Levy process. We have also derived the free boundary and price of an American perpetual put as the limit of the preceded finite-lived option. We then show how to compute the price of an American option on a basket of stocks using Mellin transform of several variables.

Key Words: American put option, Basket put option, Levy process, Mellin transform, Convolution, Free boundary, PIDE.

1 Introduction

The valuation of American options under Levy process driven models is a quite hard task and no general analytical solutions are available. For the perpetual case, i.e. with infinite time horizon, one can use the Wiener-Hopf factorization theory. For the valuation of finite time horizon American options one can follow two numerical methods such as numerically solving PIDE or Monte Carlo methods adapted for optimal stopping problems.

In this paper, following [10], we choose the jump-diffusion approach with constant coefficients and we compute the price of an option on a basket of stocks using Mellin transforms in several variables. We then find that verification for a single stock using Mellin transform a closed form formula for the American option, when the underlying asset is governed by the Levy process. More precisely, by using the Mellin transform, we solve the PIDE that is associated to the model.

There are several papers in the literature dealing with the valuation of jump-diffusion processes. Even though, there are both models analytical formulas for the solutions of the PIDE, either as an infinite sum or in terms of an integral; the last one used the Fourier transform (see Carr and al.[5], Bayarchenko and al. [3]). But most of the authors used numerical methods (see for example Matache and al. [9], where the value of American options using Merton's model is found implicitly by the penalty method). Here we intend to price an option (European and American) on a stock that is governed by Levy process.

1.1 Solve the PIDE using Mellin transform

In this section, we use Mellin transforms to derive the following equation

$$\frac{\partial V}{\partial t}(t, S) + rS \frac{\partial V}{\partial S}(t, S) + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2}(t, S) - rV(t, S) + \int_{\mathbb{R}} \left[V(t, Se^y) - V(t, S) - S(e^y - 1) \frac{\partial V}{\partial S}(t, S) \right] \nu(dy) = f(t, S), \quad (1)$$

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on $[0, T[\times]0, \infty[$ with terminal conditions $\forall S > 0, \quad V(T, S) = H(S)$.

Assume that there exist a solution $V(S, t)$ of equation (1) such that regarded as a function of a variable S , it is Mellin transformable as well as $\frac{\partial V}{\partial S}, \frac{\partial^2 V}{\partial S^2}, \frac{\partial V}{\partial t}$. Note that a function $f(s)$ defined on the positive real line is Mellin transformable if the function $f(s)s^{k-1} \in L^1([0, \infty[)$ for some $k > 0$.

Let $\hat{V}(t, \eta)$ denote the Mellin transform of $V(t, S)$ which is defined by the relation

$$\hat{V}(t, \eta) = \int_0^\infty V(t, S) S^{\eta-1} dS \quad (2)$$

where η is a complex variable such that $Re(\eta) < k$. The largest open strip $] \eta_l, \eta_u[$ in which the integral converges is called the fundamental strip.

Conversely, the inverse Mellin transform of $\hat{V}(t, \eta)$ is given by

$$V(t, S) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{V}(t, \eta) S^{-\eta} d\eta, \quad k < c. \quad (3)$$

Since the Mellin Transform is linear, if we assume that $\lim_{S \rightarrow 0} S^{k-1} V(t, S) = 0$, $\lim_{S \rightarrow 0} S^{k-2} \frac{\partial V}{\partial S}(t, S) = 0$, then the Mellin transform of the (1) yield

$$\frac{d\hat{V}}{dt} + \left[\frac{\sigma^2}{2}(\eta^2 + \eta) - r\eta - r + \int_{\mathbb{R}} [e^{\eta y} - 1 - (e^y - 1)\eta] \nu(dy) \right] \hat{V} = \hat{f}(\eta, t). \quad (4)$$

The homogeneous equation ($\hat{f} = 0$), has the general solution

$$\hat{V}(t, \eta) = A(\eta) \exp \left(- \left[\frac{\sigma^2}{2}(\eta^2 + \eta) - r\eta - r + \int_{\mathbb{R}} [e^{\eta y} - 1 - (e^y - 1)\eta] \nu(dy) \right] t \right), \quad (5)$$

where $A(\eta) = \hat{H}(\eta) = \hat{V}(T, \eta)$. When taking into account the final time condition $\hat{H}(\eta) = \hat{V}(T, \eta)$, we have the following general solution of the equation (4):

$$\hat{V}(t, \eta) = \hat{H}(\eta) \exp(q(\eta, r, \sigma)(T - t)) + \int_t^T \hat{f}(\eta, s) \exp(q(\eta, r, \sigma)(t - s)) ds$$

where

$$q(\eta, r, \sigma) = \frac{\sigma^2}{2}(\eta^2 + \eta) - r\eta - r + q_\eta. \quad (6)$$

Conversely,

$$V(t, S) = v(t, S) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} S^{-\eta} \left[\int_t^T \hat{f}(\eta, s) \exp(q(\eta, r, \sigma)(t - s)) ds \right] d\eta, \quad (7)$$

where

$$v(t, S) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{H}(\eta) \exp(q(\eta, r, \sigma)(T - t)) S^{-\eta} d\eta,$$

is the price of an European option ($\hat{f} \equiv 0$.)

1.2 Examples of Levy measure

Example 1.1

$$\begin{cases} \nu(dy) = \sum_{\epsilon=\pm} \epsilon c_\epsilon \lambda_\epsilon e^{\lambda_\epsilon y} \mathbf{1}_{A_{-\epsilon}} \\ \psi(\xi) = \frac{\sigma^2}{2} \xi^2 - ib\xi + \sum_{\epsilon=\pm} \frac{ic_\epsilon \xi}{\lambda_\epsilon + i\xi}, \end{cases}$$

where $A_- = -\infty, 0, A_+ = 0, \infty, \lambda_+ > 0 > \lambda_-, c_\epsilon > 0$ for $\epsilon = \pm$ and $\lambda_- < -1$.

Since $\nu(\{0\}) = 0$ and if $\int_{\mathbb{R}} \nu(dy) = 1$, we have

$$\begin{aligned} q(\eta, r, \sigma) &= \frac{\sigma^2}{2}(\eta^2 + \eta) - r\eta - r + \int_{\mathbb{R}} [e^{\eta y} - 1 - (e^y - 1)\eta] d\nu(y) \\ &= \frac{\sigma^2}{2}(\eta^2 + \eta) - r\eta - r + \eta - 1 + \sum_{\epsilon=\pm} c_\epsilon \lambda_\epsilon \frac{(1-\eta)(\lambda_\epsilon + 1 + \eta)}{(\lambda_\epsilon + \eta)(1 + \lambda_\epsilon)}. \end{aligned} \quad (8)$$

2 Case of American Put option with Lévy process

In this section, we derive the integral equations to determine the free boundary and the price of an American put option on one underlying stock govern by Levy process, by using Mellin Transform.

The free boundary is given by the critical stock price, on one side of which it is optimal to hold the option (this side is called the continuation region) and on the other side of which to exercise it (exercise region).

For an American put option, at each time t , there is a value of the stock price $S^* = S^*(t)$ such that for $0 < S < S^*$, it is optimal to exercise the option (and the option price is given by the pay-off) and for $S^* < S < \infty$, it is optimal to hold the option.

In the case of American option, $P = P(S, t)$ satisfies the non homogeneous equation 1 with the conditions $0 < t < T, 0 < S < \infty$, and

$$f(S, t) = \begin{cases} -rE, & \text{if } 0 < S < S^*(t) \\ 0, & \text{if } S > S^*(t) \end{cases}$$

and the final time condition is $P(S, T) = H(S) = (E - S)^+$. Both the price of the option $P(S, t)$ and the boundary conditions at $S^*(t)$ are unknown functions to be found. The smooth pasting conditions determine the free boundary:

$$P(S^*, T) = E - S^* \quad \text{and} \quad \frac{\partial P}{\partial S} \Big|_{S=S^*} = -1. \quad (9)$$

The Mellin transform of f is

$$\hat{f}(\eta, t) = - \int_0^{S^*(t)} rE S^{\eta-1} dS = -\frac{rE}{\eta} (S^*(t))^\eta. \quad (10)$$

By replacing in the general solution of the equation (1), we have that

$$\hat{P}(t, \eta) = - \int_t^T \frac{rE}{\eta} (S^*(s))^\eta \exp(q(\eta, r, \sigma)(s-t)) ds + \check{D}(\eta, t, T)$$

where $\hat{H}(\eta) = \frac{E^{-\eta+2}}{(2-\eta)(1-\eta)}$ and $\check{D}(\eta, t, T) = \hat{H}(\eta) \exp(q(\eta, r, \sigma)(T-t))$.

When using the Mellin-inversion, we then have that

$$\begin{aligned} P(t, S) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} S^{-\eta} \left[-\frac{rE}{\eta} \int_t^T (S^*(s))^\eta e^{q(\eta, r, \sigma)(s-t)} ds + \check{D}(\eta, t, T) \right] d\eta \\ &= p(t, S) - \frac{rE}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{\eta} \left[\int_t^T \left(\frac{S^*(s)}{S} \right)^\eta e^{q(\eta, r, \sigma)(s-t)} ds \right] d\eta \end{aligned} \quad (11)$$

When substituting $S = S^*(t)$ in (11) and using the relation (9), we obtain the following integral equation for the free boundary:

$$E - S^*(t) = \frac{E}{2\pi i} \left[- \int_{c-i\infty}^{c+i\infty} \frac{r}{\eta} \left[\int_t^T \left(\frac{S^*(s)}{S^*(t)} \right)^\eta e^{q(\eta, r, \sigma)(s-t)} ds \right] d\eta + \int_{c-i\infty}^{c+i\infty} \left(\frac{E}{S^*(t)} \right)^\eta \frac{\exp(q(\eta, r, \sigma)(T-t))}{\eta(1+\eta)} d\eta \right] \quad (12)$$

We therefore introduces the following theorem:

Theorem 2.1 *If the price of a stock price $S_t = \exp(rt + L_t)$ is govern by the exponential Levy process under some good conditions, the price of an American put on this stock with the exercise price E and the maturity date T is given by*

$$P(S, t) = \frac{E}{2\pi i} \left[- \int_{c-i\infty}^{c+i\infty} \frac{r}{\eta} \left[\int_t^T \left(\frac{S^*(s)}{S} \right)^\eta e^{q(\eta, r, \sigma)(s-t)} ds \right] d\eta + \int_{c-i\infty}^{c+i\infty} \left(\frac{E}{S} \right)^\eta \frac{\exp(q(\eta, r, \sigma)(T-t))}{\eta(1+\eta)} d\eta \right] \quad (13)$$

where $\text{Re}(\eta) > 0$, $\text{Re}(q(\eta, r, \sigma)) < 0$ and the free boundary $S^*(t)$ is the solution of the following integral equation

$$E - S^*(t) = \frac{E}{2\pi i} \left[- \int_{c-i\infty}^{c+i\infty} \frac{r}{\eta} \left[\int_t^T \left(\frac{S^*(s)}{S^*(t)} \right)^\eta e^{q(\eta, r, \sigma)(s-t)} ds \right] d\eta + \int_{c-i\infty}^{c+i\infty} \left(\frac{E}{S^*(t)} \right)^\eta \frac{\exp(q(\eta, r, \sigma)(T-t))}{\eta(1+\eta)} d\eta \right] \quad (14)$$

If we change the time variable t to $\tau = T - t$, we have

Corollary 2.2 *The free boundary $S^*(t) = b(T - t)$ is the solution of the following integral equation*

$$E - b(\tau) = \frac{E}{2\pi i} \left[\int_0^\tau \left[\int_{c-i\infty}^{c+i\infty} \frac{r}{\eta} \left(\frac{b(u)}{b(\tau)} \right)^\eta e^{q(\eta, r, \sigma)u} d\eta \right] du + \int_{c-i\infty}^{c+i\infty} \left(\frac{E}{b(\tau)} \right)^\eta \frac{\exp(q(\eta, r, \sigma)\tau)}{\eta(1+\eta)} d\eta \right] \quad (15)$$

where we introduces the function b such that $b(\tau) = S^*(T - \tau)$ for all $0 \leq \tau \leq T$.

In the preceded section, one would applied the Cauchy's residue theorem even thought such application will depend to the analyticity and the zeros of $q(\eta, r, \sigma)$.

2.1 Application to Kou's Model

In the case where we consider the Levy measure given by Kou(2002), we have:

$$q(\eta, r, \sigma) = \frac{\sigma^2}{2}(\eta^2 + \eta) - r\eta - r + \eta - 1 + \sum_{\epsilon=\pm} c_\epsilon \lambda_\epsilon \frac{(1-\eta)(\lambda_\epsilon + 1 + \eta)}{(\lambda_\epsilon + \eta)(1 + \lambda_\epsilon)}. \quad (16)$$

is the fraction of polynomial function in \mathbb{C} . If we write $q(\eta, r, \sigma) = q_1(\eta)/q_2(\eta)$, we have that $d(q_2) = 2$ and the $d(q_2) = 4$. It is clear that $q(\eta, r, \sigma)$ is a meromorphic function in $\mathbb{C} - \{-\lambda_-, -\lambda_+\}$, this means that $g(\eta) = (\eta + \lambda_+)(\eta + \lambda_-)q(\eta, r, \sigma)$ is an analytic function in $D = \{\eta/q(\eta, r, \sigma) > 0, \eta = c + i y \text{ with } y \in \mathbb{R} \text{ \& } 0 < c < k\}$.

2.2 Estimation of integrals

In this section, we indicate how to estimate an integral of complex function.

If we pose $\eta = c + iy$ and $q(\eta, r, \sigma) = q_1(y) + i q_2(y)$ with $q_2(y) = (1 + 2c - k_1)y + \int_{\mathbb{R}} [e^{cz} \sin(yz) - (e^z - 1)y] \nu(dz)$, $q_1(y) = \frac{c + c^2 - ck_1 - y^2}{k_1} + \int_{\mathbb{R}} [e^{cz} \cos(yz) - 1 - (e^z - 1)c] \nu(dz)$ and $k_1 = 2r/\sigma^2$, we have:

$$\begin{aligned} \frac{E}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{E}{b(\tau)} \right)^\eta \frac{e^{q(\eta, r, \sigma)\tau}}{\eta(1+\eta)} d\eta &= \frac{E}{2\pi} \left(\frac{E}{b(\tau)} \right)^c \int_{-\infty}^{\infty} \frac{\bar{\eta}(\bar{\eta} + 1)}{|\eta(1+\eta)|^2} e^{q_1(y)\tau + i(\tau q_2(y) - y \ln(\frac{E}{b(\tau)}))} dy \\ &= \frac{E}{2\pi} \left(\frac{E}{b(\tau)} \right)^c \int_{-\infty}^{\infty} (a_1(y) + i a_2(y)) [\cos(\beta(y)) + i \sin(\beta(y))] e^{q_1(y)\tau} dy \\ &= \frac{E}{2\pi} \left(\frac{E}{b(\tau)} \right)^c \int_{-\infty}^{\infty} [a_1(y) \cos(\beta(y)) - a_2(y) \sin(\beta(y))] e^{q_1(y)\tau} dy \quad (17) \end{aligned}$$

$$\begin{aligned} &= \frac{2E}{2\pi} \left(\frac{E}{b(\tau)} \right)^c \int_0^{\infty} [a_1(y) \cos(\beta(y)) - a_2(y) \sin(\beta(y))] e^{q_1(y)\tau} dy \quad (18) \\ &= I_1 + I_2, \quad (19) \end{aligned}$$

where $\beta(y) = \tau q_2(y) - y \ln \left(\frac{E}{b(\tau)} \right)$, $a_1(y) = \frac{(c+c^2-y^2)}{(c+c^2-y^2)^2+y^2(2c+1)^2}$ and $a_2(y) = \frac{(2c+1)y}{(c+c^2-y^2)^2+y^2(2c+1)^2}$. Also since $b_1(y) = \operatorname{Re}(\frac{\bar{\eta}}{|\eta|^2}) = c/c^2 + y^2$ is an even function and $b_2(y) = \operatorname{Im}(\frac{\bar{\eta}}{|\eta|^2}) = y/(c^2 + y^2)$ is an odd function and because $\gamma(y) = \tau q_2(y) - y \ln \left(\frac{b(u)}{b(\tau)} \right)$ is an odd function, we have

$$\begin{aligned} \frac{E}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{r}{\eta} \left(\frac{b(u)}{b(\tau)} \right)^\eta e^{q(\eta, r, \sigma)u} d\eta &= \frac{E}{2\pi} \left(\frac{b(u)}{b(\tau)} \right)^c \int_{-\infty}^{\infty} (b_1(y) + i b_2(y)) [\cos(\gamma(y)) + i \sin(\gamma(y))] e^{q_1(y)u} dy \\ &= \frac{E}{2\pi} \left(\frac{b(u)}{b(\tau)} \right)^c \int_{-\infty}^{\infty} [b_1(y)\cos(\gamma(y)) - b_2(y)\sin(\gamma(y))] e^{q_1(y)u} dy \quad (20) \end{aligned}$$

$$= \left(\frac{b(u)}{b(\tau)} \right)^c \frac{E}{\pi} \int_0^{\infty} [b_1(y)\cos(\gamma(y)) - b_2(y)\sin(\gamma(y))] e^{q_1(y)u} dy \quad (21)$$

$$= \left(\frac{b(u)}{b(\tau)} \right)^c \frac{E}{\pi} [Q_1(u) - Q_2(u)]. \quad (22)$$

We would use the N-point gaussian-Hermite quadrature scheme to evaluated respectively $Q_1(u)$ and $Q_2(u)$. But before this, since $\operatorname{Re}(c) > 0$ and $y \in \mathbb{R}$ by respectively replace the cosine and sine transforms in $Q_1(u)$ and $Q_2(u)$, we have

$$\begin{aligned} Q_1(u) &= \int_0^{\infty} c \frac{\cos(\gamma(y))}{c^2 + y^2} e^{q_1(y)u} dy \\ &= \int_0^{\infty} \cos(\gamma(y)) \left[\int_0^{\infty} e^{-cz} \cos(yz) dz \right] e^{q_1(y)u} dy \quad (23) \end{aligned}$$

$$= \int_0^{\infty} \left[\int_0^{\infty} e^{-cz} \cos(yz) \cos(\gamma(y)) dz \right] e^{q_1(y)u} dy \quad (24)$$

$$= \frac{1}{2} \int_0^{\infty} e^{q_1(y)u} \left[\int_0^{\infty} e^{-cz} [\cos(yz + \gamma(y)) + \cos(yz - \gamma(y))] dz \right] e^{q_1(y)u} dy, \quad (25)$$

Also

$$\begin{aligned} Q_2(u) &= \int_0^{\infty} \frac{y \cos(\gamma(y))}{c^2 + y^2} e^{q_1(y)u} dy \\ &= \int_0^{\infty} \cos(\gamma(y)) \left[\int_0^{\infty} e^{-cz} \sin(yz) dz \right] e^{q_1(y)u} dy \quad (26) \end{aligned}$$

$$= \int_0^{\infty} \left[\int_0^{\infty} e^{-cz} \sin(yz) \cos(\gamma(y)) dz \right] e^{q_1(y)u} dy \quad (27)$$

$$= \frac{1}{2} \int_0^{\infty} e^{q_1(y)u} \left[\int_0^{\infty} e^{-cz} [\sin(yz + \gamma(y)) + \sin(yz - \gamma(y))] dz \right] e^{q_1(y)u} dy. \quad (28)$$

We then have

$$\frac{E}{2\pi i} \int_0^{\tau} \left[\int_{c-i\infty}^{c+i\infty} \frac{r}{\eta} \left(\frac{b(u)}{b(\tau)} \right)^\eta e^{q(\eta, r, \sigma)u} d\eta \right] du = \frac{E}{\pi} \int_0^{\tau} \left(\frac{b(u)}{b(\tau)} \right)^c \left[\int_0^{\infty} \frac{c-y}{c^2 + y^2} \cos(\gamma(y)) e^{q_1(y)u} dy \right] du,$$

Remark 2.3 It's straightforward to see that the evaluation of the preceded integrals will depend to the choice of the Levy measure ν .

3 Free boundary and price for the perpetual put option

Since in section 1, we can derived the finite-lived American put by using inverse Mellin transform, we use the smooth pasting condition to derive for the free boundary $S_\infty^* = S_\infty^*(t)$ of the perpetual put, which is a constant, for all t . We then use the value S_∞^* to derive an expression for the perpetual price put $P_\infty(S, t)$.

When we consider the put option price $P(S, t)$ as given in (2.1) and $T \rightarrow \infty$, the smooth pasting condition is

$$\left. \frac{\partial P}{\partial S} \right|_{S=S_\infty^*} = -1 \quad (29)$$

$$= \frac{E}{2\pi i S_\infty^*} \int_{c-i\infty}^{c+i\infty} \frac{1}{q(\eta, r, \sigma)} d\eta. \quad (30)$$

If $\eta = c + i y$, because $q_2(y)$ is odd and $q_1(y)$ is even, therefore the free boundary S_∞^* is:

$$S_\infty^* = -\frac{E}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{q(\eta, r, \sigma)} d\eta \quad (31)$$

$$\begin{aligned} &= \frac{E}{2\pi} \int_{-\infty}^{\infty} \frac{-q_1(y) + i q_2(y)}{(q_1(y))^2 + (q_2(y))^2} dy \\ &= -\frac{E}{\pi} \int_0^{\infty} \frac{(q_1(y))}{(q_1(y))^2 + (q_2(y))^2} dy \end{aligned} \quad (32)$$

with $Re(q(\eta, r, \sigma)) < 0$.

While S_∞^* is known, since the price of the perpetual European option is zero because it can never be exercised, following the theorem (2.1) and taking $T \rightarrow \infty$ in (13), we obtain the following price of the perpetual put for $S_\infty^* < S(t)$:

$$P_\infty(S, t) = \frac{r E}{2\pi i} \left[-\int_{c-i\infty}^{c+i\infty} \frac{1}{\eta} \left[\int_t^\infty \left(\frac{S_\infty^*(t)}{S} \right)^\eta e^{q(\eta, r, \sigma)(s-t)} ds \right] d\eta \right] \quad (33)$$

$$= \frac{r E}{2\pi i} \left[\int_{c-i\infty}^{c+i\infty} \frac{1}{\eta q(\eta, r, \sigma)} \left(\frac{S_\infty^*}{S} \right)^\eta d\eta \right] \quad (34)$$

$$= \frac{r E}{2\pi} \left(\frac{S_\infty^*}{S} \right)^c \int_{-\infty}^{\infty} \frac{\cos(y \ln(S/S_\infty^*)) - i \sin(y \ln(S/S_\infty^*))}{(c+iy)(q(c+iy, r, \sigma))} dy \quad (35)$$

$$= \frac{r E}{\pi} \left(\frac{S_\infty^*}{S} \right)^c [J_1 - J_2]. \quad (36)$$

where $Re(q(\eta, r, \sigma)) < 0$, $\eta = c + i y$,

$$J_1 = \int_0^\infty \frac{(c q_1(y) - y q_2(y)) \cos(y \ln(S/S_\infty^*))}{(c^2 + y^2)((q_1(y))^2 + (q_2(y))^2)} dy, \quad (37)$$

and

$$J_2 = \int_0^\infty \frac{(c q_2(y) + y q_1(y)) \sin(y \ln(S/S_\infty^*))}{(c^2 + y^2)((q_1(y))^2 + (q_2(y))^2)} dy. \quad (38)$$

Remark 3.1 To estimate (31) and (34), one can applied the Cauchy residue Theorem. But this application will depend to the choice of the measure ν , since we need to know the roots and the poles of $q(\eta, r, \sigma)$.

3.1 Perpetual option with Kou's Model

4 Generalization to an American basket option

In this section, following the preceded sections, we use Mellin transform with n variables to derive an integral equation for the price of an American put $P(S_1, \dots, S_n, t)$ on a basket with n stocks. We will then simply showed how to treated the simple case where $n = 2$.

In the multidimensional case, the early exercise feature of the American option gives rise to a free boundary problem. For an option on a basket of n stocks, the free boundary is determined by a euro-value $SS^*(t)$ for each t . The put is held for $SS = \sum_{i=1}^n S_i > SS^*(t)$ and in this region the price is determined as the solution of the n -dimensional PIDE. For $SS \leq SS^*(t)$, the put is exercised and $P(S_1, \dots, S_n, t)$ is given by the payoff function $\max(K - SS, 0)$. One will extended the domain of PIDE by setting $P(S_1, \dots, S_n, t) = E - SS$ for $SS \leq SS^*(t)$. The price of the put can the be determined by the following nonhomogeneous PIDE:

5 Conclusion

A A PIDE for the Option Pricing Function

Since the option price is

$$V(t) = f(X_t, t) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [w(e^{X_t + L_{T-t}})] \quad (39)$$

with $f(x, t) := e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [w(e^{x + L_{T-t}})]$, we have the following theorem:

Theorem A.1 Assume that the function $f(x, t)$ defined in (39) is of class $C^{(2;1)}(\mathbb{R}, \mathbb{R}^+)$, that is, it is twice continuously differentiable in the variable x and once continuously differentiable in the variable t . Assume further that the law of L_t has support \mathbb{R} . Then $f(x, t)$ satisfies the following partial integro-differential equation (PIDE):

$$\frac{\partial f}{\partial t}(x, t) + b \frac{\partial f}{\partial x}(x, t) + \frac{c}{2} \frac{\partial^2 f}{\partial x^2}(x, t) - r f(x, t) + \int_{\mathbb{R}} \left[f(x + y, t) - f(x, t) - y \frac{\partial f}{\partial x}(x, t) \right] \nu(dy) = 0, \quad (40)$$

where $w(e^x) = f(x, T)$, for $x \in \mathbb{R}$ and $t \in]0, T[$.

Here (b, c, ν) is the Levy parameters under the risk-neutral probability measure \mathbb{Q} . r is the short-term interest-rate.

Proof. Indications....

By using Ito Lemma to approximate $dV(t)$, we have

$$d(e^{-rt} V(t)) = -re^{-rt} V(t) dt + e^{-rt} dV(t) \quad (41)$$

$$\begin{aligned} &= e^{-rt} \left[-rV(t) dt + \frac{\partial f}{\partial t}(X_{t-}, t) dt + \frac{\partial f}{\partial X}(X_{t-}, t) dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial X^2}(X_{t-}, t) d\langle X^c, X^c \rangle_t \right] \\ &\quad + e^{-rt} \int_{\mathbb{R}} \left[f(X_{t-} + y, t) - f(X_{t-}, t) - y \frac{\partial f}{\partial X}(X_{t-}, t) \right] \mu^{(X)}(dy, dt), \end{aligned} \quad (42)$$

where $\mu^{(X)}$ is the Levy measure associated with X_t . Because

$$X_t = \ln(S_0) + rT + L_t,$$

the jumps of X and those of the Levy process L coincide, and so do the jump measures. Furthermore, the stochastic differentials of X and $\langle X^c, X^c \rangle$ coincide with the corresponding differentials for the Levy process L . Hence we get

$$\begin{aligned} d(e^{-rt}V(t)) &= -re^{-rt}V(t)dt + e^{-rt}dV(t) \\ &= e^{-rt} \left[-rV(t)dt + \frac{\partial f}{\partial t}(X_{t-}, t)dt + \frac{\partial f}{\partial X}(X_{t-}, t)dL_t + \frac{1}{2} \frac{\partial^2 f}{\partial X^2}(X_{t-}, t) d\langle X^c, X^c \rangle_t \right] \\ &\quad + e^{-rt} \int_{\mathbb{R}} \left[f(X_{t-} + y, t) - f(X_{t-}, t) - y \frac{\partial f}{\partial X}(X_{t-}, t) \right] \mu^L(dy, dt). \end{aligned} \quad (43)$$

Since $V(t) = f(X_{t-}, t)$, the right-hand side of (43) can be written as the sum of a local martingale and a predictable process of bounded variation, whose differential is given by

$$\begin{aligned} &e^{-rt} \left[-rf(X_{t-}, t)dt + \frac{\partial f}{\partial t}(X_{t-}, t)dt + \frac{\partial f}{\partial X}(X_{t-}, t)dL_t + \frac{1}{2} \frac{\partial^2 f}{\partial X^2}(X_{t-}, t) d\langle X^c, X^c \rangle_t \right] \\ &+ e^{-rt} \int_{\mathbb{R}} \left[f(X_{t-} + y, t) - f(X_{t-}, t) - y \frac{\partial f}{\partial X}(X_{t-}, t) \right] \nu^{(L)}(dy)dt. \end{aligned} \quad (44)$$

By the argument above, this process vanishes identically. By continuity, this means that for all values x from the support of the risk neutral measure $\mathbb{Q}^{X_{t-}}$, (that is, by assumption, for all $x \in \mathbb{R}$), we have the equation (40).

QED

References

- [1] J. Almendral, C. W. Oosterlee(2005): Numerical valuation of options with jumps in the underlying, *Applied Numerical Mathematics* **53**(2005) 1-18.
- [2] O. E. Barndorff (1998): Process normal inverse Gaussian type, *Finance stochastic 2* (1998) 41-68.
- [3] S. I. Bayarchenko, S. Z. Levendorskii (2002): Non-gaussian Merton-Black-Scholes Theory, Advanced Series on Statistical Science & Applied probability, **vol. 9**, World scientific River Edge, NJ.
- [4] J. Bertoin (1996): Levy processes, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge 1996.
- [5] P. P Carr, D. B. Madan (1999): Option valuation using the fourier transform, *J. Comput. Finance* **2** 61-73
- [6] E. Eberlein (2001): Application of generalized hyperbolic Levy motions to finance, in: *O. Barndorff-Nielsen, T. Mikoson, S. Resnick (Eds.), Lévy Processes-Theory and applications, Birkhauser, Boston, MA, 2001, pp. 319-336*
- [7] P. Flageolet, X. Gourdon, P. Dumas (1998): Mellin transforms and asymptotics: Harmonics sums, INRIA Project.
- [8] J. Hull, A. White (1987): The pricing of options with stochastic volatilities, *J. Finance* **42**, 281-300.
- [9] A. M. Matache, T. vonPetersdorff, C. Schwab (2002): Fast deterministic pricing on Lévy diving assets, *Working paper, ETH Zurich*.
- [10] R. Panini and R. P. Srivastav (2004): Option Pricing with Mellin Transforms, *Mathematical and Computer Modelling*, **40**, (2004) 43-56.
- [11] P. Willmott, J. Dewynne and S. Howison (1995): *The Mathematics of financial derivatives: A Student Introduction*, Cambridge University Press, Cambridge, U.K.