Noncommutative symmetric functions IV : Quantum linear groups and Hecke algebras at q = 0

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Abstract

We present representation theoretical interpretations of quasi-symmetric functions and noncommutative symmetric functions in terms of quantum linear groups and Hecke algebras at $q = 0$. We obtain in this way a noncommutative realization of quasi-symmetric functions analogous to the plactic symmetric functions of Lascoux and Schützenberger. The generic case leads to a notion of quantum Schur function.

1 Introduction

This paper, which is intended as a sequel to [9, 21, 6], is devoted to the representation theoretical interpretation of noncommutative symmetric functions and quasi-symmetric functions. These objects, which are two different generalizations of ordinary symmetric functions [10, 9], build up two Hopf algebras dual to each other, and have been shown to provide a Frobenius type theory for Hecke algebras of type A at $q = 0$, playing the same rôle as the classical correspondence between symmetric functions and characters of symmetric groups [7] (which extends to the case of the generic Hecke algebra).

In the classical case, the interpretation of symmetric functions in terms of representations of symmetric groups is equivalent, via Schur-Weyl duality, to the fact that Schur functions are the characters of the irreducible polynomial representations of general linear groups. Equivalently, instead of working with polynomial representations of $GL(n)$, one can use comodules over the Hopf algebra of polynomial functions over $GL(n)$ [11]. This Hopf algebra is known to admit interesting q-deformations (quantized function algebras; see [8] for instance) to which Schur-Weyl duality can be extended for generic values of q, the symmetric group being replaced by the Hecke algebra.

The standard version of the quantum linear group is not defined for $q = 0$. The theory of crystal bases [16], which allows to "take the limit $q \to 0$" in certain modules by working with renormalized operators modulo a lattice, describes the combinatorial aspects of the generic case, and provides illuminating interpretations of classical constructions such as the Robinson-Schensted correspondence, the Littlewood-Richardson rule and the plactic monoid [3, 17, 24, 26].

However, another version exists [4] which plays an equivalent rôle for generic values of q, but in which one can specialize q to 0. This specialization is quite different of what is obtained with crystal bases, and leads to an new interpretation of quasi-symmetric functions and noncommutative symmetric functions analogous to the interpretation of ordinary symmetric functions as polynomial characters of $GL(n)$. Moreover, this interpretation allows to give a realization of quasi-symmetric functions similar to the plactic interpretation of symmetric functions (see

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dinary Young tableaux one has to use skew tableaux of ribbon shape, and dual objects called quasi-ribbons, for which Schensted type algorithms can be constructed. In fact, most aspects of the classical theory can be adapted to this highly degenerate case. As this is an example of a non-semisimple case for which everything can be worked out explicitely, one can expect that this treatment could serve as a guide for understanding the more complicated degeneracies at roots of unity.

This paper is structured as follows. We first recall the basic definitions concerning noncommutative symmetric functions and quasisymmetric functions (Section 2) and review the Frobenius correspondence for the generic Hecke algebras (Section 3). Next we introduce the Dipper-Donkin version of the quantized function algebra of the space of $n \times n$ matrices (Section 4). We describe some interesting subspaces (Sections 4.5 and 4.6), and prove that the $q = 0$ specialization of the diagonal subalgebra is a quotient of the plactic algebra, which we call the hypoplactic algebra (Section 4.7). Next, we review the representation theory of the 0-Hecke algebra and its interpretation in terms of quasi-symmetric functions and noncommutative symmetric functions, providing the details which were omitted in [7]. In Section 6, we introduce a notion of noncommutative character for $A_q(n)$-comodules, and prove that these characters live in the diagonal subalgebra. For generic q, the characters of irreducible comodules are quantum analogues of Schur functions. For $q = 0$, we show that hypoplactic analogues of the fundamental quasi-symmetric functions F_I (quasi-ribbons) can be obtained as the characters of irreducible $A_0(n)$ comodules, and give a similar construction for the ribbon Schur functions. These constructions lead to degenerate versions of the Robinson-Schensted correspondence, which are discussed in Section 7.

2 Noncommutative symmetric functions and quasi-symmetric functions

2.1 Noncommutative symmetric functions

The algebra of *noncommutative symmetric functions* [9] is the free associative algebra $\mathbf{Sym} = \mathbb{Q}\langle S_1, S_2, \ldots\rangle$ generated by an infinite sequence of noncommutative indeterminates S_k, called the *complete* symmetric functions. One defines $S^I = S_{i_1} S_{i_2} \ldots S_{i_r}$ for any composition $I = (i_1, i_2, \ldots, i_r) \in (\mathbb{N}^*)^r$. The family (S^I) is a linear basis of $\mathbf{Sym}$. Although it is convenient to define $\mathbf{Sym}$ as an abstract algebra, a useful realisation can be obtained by taking an infinite alphabet $A = \{a_1, a_2, \ldots\}$ and defining its complete homogeneous symmetric functions by

$$\overrightarrow{\prod_{i\geq 1}} (1 - ta_i)^{-1} = \sum_{n\geq 0} t^n S_n(A) \tag{1}$$

Although these elements are not symmetric for the usual action of permutations on the free algebra, they are invariant under the Lascoux-Schützenberger action of the symmetric group [25], which can now be interpreted as a particular case of Kashiwara's action of the Weyl group on the $U_q(\mathfrak{sl}_n)$-crystal graph of the tensor algebra [24].

The set of all compositions of a given integer n is equipped with the *reverse refinement order*, denoted $\preceq$. For instance, the compositions J of 4 such that $J \preceq (1, 2, 1)$ are exactly $(1, 2, 1)$, $(3, 1)$, $(1, 3)$ and (4). The *ribbon Schur functions* (R_I) can then be defined by

$$S^I = \sum_{J\preceq I} R_J \qquad \text{or} \qquad R_I = \sum_{J\preceq I} (-1)^{\ell(I)-\ell(J)} S^J ,$$

where $\ell(I)$ denotes the *length* of I. The family (R_I) is another homogenous basis of $\mathbf{Sym}$.

ric function f obtained by applying to F the algebra morphism which maps S_n to the complete homogeneous function h_n (our notations for commutative symmetric functions will be those of [28]). The ribbon Schur function R_I is then mapped to the corresponding ordinary ribbon Schur function, which will be denoted by r_I.

Ordinary symmetric functions are endowed with an extra product $*$, called the internal product, which corresponds to the multiplication of central functions on the symmetric group. A noncommutative analog of this product can be defined, the character ring of $\mathfrak{S}_n$ being replaced by its descent algebra [35] (see also below) .

Recall that i is said to be a *descent* of $\sigma \in \mathfrak{S}_n$ if $\sigma(i) > \sigma(i+1)$. The set $\mathrm{Des}\,(\sigma)$ of these integers is called the *descent set* of σ. If $I = (i_1, \ldots, i_r)$ is a composition of n, one associates with it the subset $D(I) = \{ d_1, \ldots, d_{r-1} \}$ of $[1, n-1]$ defined by $d_k = i_1 + \ldots + i_k$ for $k \in [1, r-1]$. Let D_I be the sum in $\mathbb{Z}[\mathfrak{S}_n]$ of all permutations with descent set $D(I)$. As shown by Solomon [35], the D_I form a basis of a subalgebra of $\mathbb{Z}[\mathfrak{S}_n]$ called the *descent algebra* of $\mathfrak{S}_n$ and denoted by Σ_n. One can define an isomorphism of graded vector spaces

$$\alpha : \ \mathbf{Sym} = \bigoplus_{n \geq 0} \mathbf{Sym}_n \ \longrightarrow \ \Sigma = \bigoplus_{n \geq 0} \Sigma_n$$

by setting $\alpha(R_I) = D_I$. Observe that $\alpha(S^I)$ is then equal to $D_{\subseteq I}$, i.e. to the sum of all permutations of $\mathfrak{S}_n$ whose descent set is contained in $D(I)$.

2.2 Quasi-symmetric functions

As proved in [29] (see also [9]), the algebra of noncommutative symmetric functions is in natural duality with the algebra of quasi-symmetric functions, introduced by Gessel in [10]. Let $X = \{ x_1, x_2, \ldots, x_n \ldots \}$ be a totally ordered set of commutative indeterminates. An element $f \in \mathbb{C}[X]$ is said to be a *quasi-symmetric function* if for each composition $K = (k_1, \ldots, k_m)$ all the monomials $x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_m}^{k_m}$ with $i_1 < i_2 < \ldots < i_m$ have the same coefficient in f. The quasi-symmetric functions form a subalgebra $QSym$ of $\mathbb{C}[X]$.

One associates with a composition $I = (i_1, \ldots, i_m)$ the *quasi-monomial function*

$$M_I = \sum_{j_1 < \ldots < j_m} x_{j_1}^{i_1} \ldots x_{j_m}^{i_m} \ .$$

The family of quasi-monomial functions is clearly a basis of $QSym$. Another important basis of $QSym$ is formed by *quasi-ribbon functions* which are defined by

$$F_I = \sum_{I \preceq J} M_J \ ,$$

e.g. $F_{122} = M_{122} + M_{1112} + M_{1211} + M_{11111}$. The pairing $\langle \cdot, \cdot \rangle$ between **Sym** and $QSym$ [29] is then defined by $\langle S^I, M_J \rangle = \delta_{IJ}$ or equivalently by $\langle R_I, F_J \rangle = \delta_{IJ}$. This duality is essentially equivalent to the noncommutative Cauchy identity

$$\prod_{i \geq 1}^{\longrightarrow} \left(\prod_{j \geq 1}^{\longrightarrow} (1 - x_i a_j)^{-1} \right) = \sum_I F_I(X) \, R_I(A) \ , \qquad (2)$$

and can also be interpreted as the canonical duality between Grothendieck groups asociated to 0-Hecke algebras [7] (see Section 5).

3.1 Hecke algebras

The Hecke algebra $H_N(q)$ of type A_{N-1} is the $\mathbb{C}(q)$-algebra generated by $N-1$ elements $(T_i)_{i=1,N-1}$ with relations

$$\left\{\begin{array}{ccll} T_i^2 & = & (q-1)\,T_i + q & \text{for } i \in [1, N-1]\ , \\ T_i\,T_{i+1}\,T_i & = & T_{i+1}\,T_i\,T_{i+1} & \text{for } i \in [1, N-2]\ , \\ T_i\,T_j & = & T_j\,T_i & \text{for } |i-j| > 1\ . \end{array}\right.$$

The Hecke algebra $H_N(q)$ is a deformation of the $\mathbb{C}$-algebra of the symmetric group $\mathfrak{S}_N$ (obtained for $q=1$). For generic complex values of q, it is isomorphic to $\mathbb{C}[\mathfrak{S}_N]$ (and hence semi-simple) except when $q=0$ or when q is a root of unity. The first relation is often replaced by

$$T_i^2 = (q - q^{-1})\,T_i + 1 \tag{3}$$

which is invariant under the substitution $q \longrightarrow -q^{-1}$ and is more convenient for working with Kazhdan-Lusztig polynomials and canonical bases. However the convention adopted here, i.e.

$$T_i^2 = (q-1)\,T_i + q\ , \tag{4}$$

is the natural one when q is interpreted as the cardinality of a finite field and $H_N(q)$ as the endomorphism algebra of the permutation representation of $GL_N(\mathbb{F}_q)$ on the set on complete flags [14]. Moreover one can specialize $q=0$ in relation (4). In the modular representation theory of $GL_N(\mathbb{F}_q)$, the Hecke algebra corresponding to this specialization occurs when q is a power of the characteristic of the ground field. For this reason, among others, it is interesting to consider the 0-Hecke algebra $H_N(0)$ which is the $\mathbb{C}$-algebra obtained by specialization of the generic Hecke algebra $H_N(q)$ at $q=0$. This algebra is therefore presented by

$$\left\{\begin{array}{ccll} T_i^2 & = & -T_i & \text{for } i \in [1, N-1]\ , \\ T_i\,T_{i+1}\,T_i & = & T_{i+1}\,T_i\,T_{i+1} & \text{for } i \in [1, N-2]\ , \\ T_i\,T_j & = & T_j\,T_i & \text{for } |i-j| > 1\ . \end{array}\right.$$

The representation theory of $H_N(0)$ was investigated by Norton who obtained a fairly complete picture [31]. Important specific features of the type A are described by Carter in [1]. The 0-Hecke algebra can also be realized as an algebra of operators acting on the equivariant Grothendieck ring of the flag manifold [22].

3.2 The Frobenius correspondence

We will see that the 0-Hecke algebra is the right object for giving a representation theoretical interpretation of noncommutative symmetric functions and of quasi-symmetric functions. To emphasize the parallel with the well-known correspondence between representations of the symmetric group and symmetric functions, we first recall the main points of the classical theory.

Let *Sym* be the ring of symmetric functions and let

$$R[\mathfrak{S}] = \bigoplus_{N\geq 0} R[\mathfrak{S}_N]$$

be the ring of equivalence classes of finitely generated $\mathbb{C}[\mathfrak{S}_N]$-modules (with sum and product corresponding to direct sum and induction product). We know from the work of Frobenius that

map $\mathcal{F} : R[\mathfrak{S}] \longrightarrow Sym$ which sends the class of a Specht module V_λ to the Schur function s_λ. The first point is that $\mathcal{F}$ is a ring homomorphism. That is,

$$\mathcal{F}([U \otimes V] \uparrow_{\mathfrak{S}_N \times \mathfrak{S}_M}^{\mathfrak{S}_{N+M}}) = \mathcal{F}([U])\,\mathcal{F}([V])$$

for a $\mathfrak{S}_N$-module U and a $\mathfrak{S}_M$-module V. The second one is the character formula, which can be stated as follows: for any finite dimensional $\mathfrak{S}_N$-module V, the value of the chararacter of V on a permutation of the conjugacy class labelled by the partition μ is equal to the scalar product

$$\chi(\mu) = \langle \mathcal{F}(V)\,, p_\mu \rangle$$

where p_μ is the product of power sums $p_{\mu_1} \cdots p_{\mu_r}$.

This theory can be extended to the Hecke algebra $H_N(q)$ when q is neither 0 nor a root of unity. The characteristic map is independent of q, and still maps the q-Specht module $V_\lambda(q)$ to the Schur function s_λ. The induction formula remains valid and the character formula has to be modified as follows (see [2, 18, 19, 32, 36, 37]). Define for a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ of N the element

$$w_\mu = (\sigma_1 \ \ldots \ \sigma_{\mu_1 - 1})\,(\sigma_{\mu_1+1} \ \ldots \ \sigma_{\mu_1+\mu_2-1}) \ \ldots \ (\sigma_{\mu_1+\ldots+\mu_{r-1}+1} \ \ldots \ \sigma_{N-1})$$

(where σ_i is the elementary transposition $(i \ i{+}1)$). The character formula for $H_N(q)$ gives the value χ_μ^λ on T_{w_μ} of the character of the irreducible q-Specht module $V_\lambda(q)$. It reads

$$\chi_\mu^\lambda = \mathrm{tr}_{\,V_\lambda(q)}\,(T_{w_\mu}) = \langle \mathcal{F}(V_\lambda(q)), C_\mu(q) \rangle = \langle s_\lambda, C_\mu(q) \rangle$$

where $C_\mu(q) = (q-1)^{l(\mu)}\, h^\mu((q-1)X)$ (in λ-ring notation, $h^\mu((q-1)X)$ denotes the image of the homogeneous symmetric function $h^\mu(X)$ under the ring homomorphism $p_k \mapsto (q^k - 1)\, p_k$).

4 The quantum coordinate ring $A_q(n)$

4.1 Tensor representations of $H_N(q)$

Let $E = \{e_1, \ldots, e_n\}$ be a finite set and let

$$V = \bigoplus_{i=1}^{n} \mathbb{C}(q)\, e_i$$

be the $\mathbb{C}(q)$-vector space with basis (e_i). For $\mathbf{v} = e_{k_1} \otimes \ldots \otimes e_{k_N} \in V^{\otimes N}$ and $i \in [1, N{-}1]$, we define $\mathbf{v}^{\sigma_i}$ by setting

$$\mathbf{v}^{\sigma_i} = e_{k_1} \otimes \ldots e_{k_{i-1}} \otimes e_{k_{i+1}} \otimes e_{k_i} \otimes e_{k_{i+2}} \otimes \ldots \otimes e_{k_N} \ .$$

Following [15, 4, 5], one defines a right action of $H_N(q)$ on $V^{\otimes N}$ by

$$\begin{cases} \mathbf{v} \cdot T_i &= \quad \mathbf{v}^{\sigma_i} & \text{if } k_i < k_{i+1} \ , \\ \mathbf{v} \cdot T_i &= \quad q\,\mathbf{v} & \text{if } k_i = k_{i+1} \ , \\ \mathbf{v} \cdot T_i &= \quad q\,\mathbf{v}^{\sigma_i} + (q-1)\,\mathbf{v} & \text{if } k_i > k_{i+1} \ . \end{cases}$$

This is a variant of Jimbo's action [15] itself defined by

$$\begin{cases} \mathbf{v} \cdot T_i &= \quad q^{1/2}\,\mathbf{v}^{\sigma_i} & \text{if } k_i < k_{i+1} \ , \\ \mathbf{v} \cdot T_i &= \quad q\,\mathbf{v} & \text{if } k_i = k_{i+1} \ , \\ \mathbf{v} \cdot T_i &= \quad q^{1/2}\,\mathbf{v}^{\sigma_i} + (q-1)\,\mathbf{v} & \text{if } k_i > k_{i+1} \ . \end{cases}$$

on $(V^*)^{\otimes N}$ is given by

$$\begin{cases} \mathbf{v}^* \cdot T_i &= q\,(\mathbf{v}^*)^{\sigma_i} & \text{if } k_i < k_{i+1}\ , \\ \mathbf{v}^* \cdot T_i &= q\,\mathbf{v}^* & \text{if } k_i = k_{i+1}\ , \\ \mathbf{v}^* \cdot T_i &= (\mathbf{v}^*)^{\sigma_i} + (q-1)\,\mathbf{v}^* & \text{if } k_i > k_{i+1}\ . \end{cases}$$

Example 4.1 Let $V = \mathbb{C}(q)\,e_1 \oplus \mathbb{C}(q)\,e_2$. The matrices describing the right action of T_1 on $V \otimes V$ and on $V^* \otimes V^*$ in the canonical bases of these spaces are

$$\check{R} = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 1 & q-1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \quad , \quad \check{R}^* = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & q & q-1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} .$$

We also need the left actions of $H_N(q)$ on $V^{\otimes N}$ and $(V^*)^{\otimes N}$ defined by

$$\begin{cases} T_i \cdot \mathbf{v} &= -q\,\mathbf{v} \cdot T_i^{-1} &= -\mathbf{v} \cdot T_i + (q-1)\,\mathbf{v}\ , \\ T_i \cdot \mathbf{v}^* &= -q\,\mathbf{v}^* \cdot T_i^{-1} &= -\mathbf{v}^* \cdot T_i + (q-1)\,\mathbf{v}^*\ . \end{cases}$$

Equivalently, for $\mathbf{v} = e_{k_1} \otimes \ldots \otimes e_{k_N} \in (V)^{\otimes N}$ and $\mathbf{v}^* = e^*_{k_1} \otimes \ldots \otimes e^*_{k_N} \in (V^*)^{\otimes N}$,

$$\begin{cases} T_i \cdot \mathbf{v} = -\mathbf{v}^{\sigma_i} + (q-1)\,\mathbf{v}\ , & T_i \cdot \mathbf{v}^* = -q\,(\mathbf{v}^*)^{\sigma_i} + (q-1)\,\mathbf{v}^* & \text{if } k_i < k_{i+1}\ , \\ T_i \cdot \mathbf{v} = -\mathbf{v}\ , & T_i \cdot \mathbf{v}^* = -\mathbf{v}^* & \text{if } k_i = k_{i+1}\ , \\ T_i \cdot \mathbf{v} = -q\,\mathbf{v}^{\sigma_i}\ , & T_i \cdot \mathbf{v}^* = -(\mathbf{v}^*)^{\sigma_i} & \text{if } k_i > k_{i+1}\ . \end{cases}$$

4.2 The Hopf algebra $A_q(n)$

The quantum group $A_q(n)$ is the $\mathbb{C}(q)$-algebra generated by the n^2 elements $(x_{ij})_{1\le i,j\le n}$ subject to the defining relations

$$\begin{cases} x_{jk}\,x_{il} &= q\,x_{il}\,x_{jk} & \text{for } i<j,\ k \le l\ , \\ x_{ik}\,x_{il} &= x_{il}\,x_{ik} & \text{for every } i,k,l\ , \\ x_{jl}\,x_{ik} - x_{ik}\,x_{jl} &= (q-1)\,x_{il}\,x_{jk} & \text{for } i<j,\ k<l\ . \end{cases}$$

This algebra is a quantization of the Hopf algebra of polynomial functions on the variety of $n \times n$ matrices introduced by Dipper and Donkin in [4]. It is not isomorphic to the classical quantization of Faddeev-Reshetikin-Takhtadzhyan [8], and although for generic values of q both versions play essentially the same rôle, an essential difference is that the Dipper-Donkin algebra is defined for $q = 0$.

$A_q(n)$ is a Hopf algebra with comultiplication Δ defined by

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}\ .$$

Moreover one can define a left coaction δ of $A_q(n)$ on $V^{\otimes N}$ by

$$\delta(e_i) = \sum_{j=1}^{n} x_{ij} \otimes e_j$$

and the following property shows that $A_q(n)$ is related to the Hecke algebras in a similar way as GL_n and the symmetric groups.

$H_N(q)$ on $V^{\otimes N}$. That is, the following diagram is commutative

$$
\begin{tikzcd}
V^{\otimes N} \arrow[r, "\delta^{\otimes N}"] \arrow[d, "h"'] & A_q(n) \otimes V^{\otimes N} \arrow[d, "Id \otimes h"] \\
V^{\otimes N} \arrow[r, "\delta^{\otimes N}"'] & A_q(n) \otimes V^{\otimes N}
\end{tikzcd}
$$

for every element $h \in H_N(q)$ considered as an endomorphism of $V^{\otimes N}$.

This property still holds for $q = 0$. Thus, for any $h \in H_N(0)$, $V^{\otimes N}\, h$ will be a sub-$A_0(n)$-comodule of $V^{\otimes N}$. This is this property which will allow us to define a plactic-like realization of quasi-symmetric functions. For later reference, note that the defining relations of $A_0(n)$ are

$$
\left\{
\begin{array}{lcll}
x_{jk}\, x_{il} & = & 0 & \text{for } i < j,\ k \le l \ , \\
x_{ik}\, x_{il} & = & x_{il}\, x_{ik} & \text{for every } i, k, l \ , \\
x_{jl}\, x_{ik} & = & x_{ik}\, x_{jl} - x_{il}\, x_{jk} & \text{for } i < j,\ k < l \ .
\end{array}
\right.
\tag{5}
$$

4.3 Some notations for the elements of $A_q(n)$

Each generator x_{ij} of $A_q(n)$ will be identified with a two row array and with an element of $V \otimes V^*$ modulo certain relations as described below:

$$
x_{ij} = \begin{bmatrix} i \\ j \end{bmatrix} = e_i \otimes e_j^* \ .
$$

For $\mathbf{i} = (i_1, \ldots, i_r)$, $\mathbf{j} = (j_1, \ldots, j_r) \in [1, n]^r$, let $e_{\mathbf{i}} = e_{i_1} \otimes \ldots \otimes e_{i_r}$ and $e_{\mathbf{j}}^* = e_{j_1}^* \otimes \ldots \otimes e_{j_r}^*$. One can then identify the monomial $x_{\mathbf{i}\,\mathbf{j}} = x_{i_1 j_1} \, \ldots \, x_{i_r j_r}$ of $A_q(n)$ with the two row array

$$
\begin{bmatrix} i_1 & i_2 & \ldots & i_r \\ j_1 & j_2 & \ldots & j_r \end{bmatrix} ,
$$

itself regarded as the class of the tensor $e_{\mathbf{i}} \otimes e_{\mathbf{j}}^* \in T^r(V \otimes V^*)$ modulo the relations

$$
\left\{
\begin{array}{lcll}
e_{\mathbf{i}} \otimes e_{\mathbf{j}}^* & \equiv & e_{\mathbf{i}}^{\sigma_r} \otimes (e_{\mathbf{j}}^* \cdot T_r) & \text{for each } r \text{ such that } i_r > i_{r+1} \ , \\
e_{\mathbf{i}} \otimes e_{\mathbf{j}}^* & \equiv & e_{\mathbf{i}} \otimes (e_{\mathbf{j}}^*)^{\sigma_r} & \text{for each } r \text{ such that } i_r = i_{r+1} \ .
\end{array}
\right.
\tag{6}
$$

These relations are equivalent to

$$
e_{\mathbf{i}} \otimes e_{\mathbf{j}}^* \ \equiv \ -T_r \cdot e_{\mathbf{i}} \otimes (e_{\mathbf{j}}^*)^{\sigma_r} \qquad \text{for each } r \text{ such that } j_r \le j_{r+1} \ . \tag{7}
$$

4.4 Linear bases of $A_q(n)$

For every $\mathbf{i} = (i_1, \ldots, i_r) \in [1, n]^r$, let $I(\mathbf{i}) \in \mathbb{N}^n$ be defined by

$$
I(\mathbf{i})_p = \mathrm{Card}\, \{\, i_k,\ k \in [1, r]\, ,\ i_k = p \,\}
$$

for $p \in [1, n]$. For $I, J \in \mathbb{N}^n$, set

$$
A_q(I, J) = \sum_{I(\mathbf{i}) = I,\, I(\mathbf{j}) = J} \mathbb{C}(q)\, x_{\mathbf{i}\mathbf{j}} \ .
$$

A monomial basis compatible with this grading is constructed in [4]. The basis vectors, which are labelled by matrices $M = (m_{ij})_{1\le i,j\le n} \in \mathcal{M}_n(\mathbb{N})$ are

$$x_M = (x_{11}^{m_{11}}\, x_{12}^{m_{12}} \,\dots\, x_{1n}^{m_{1n}}) \;\dots\; (x_{n1}^{m_{n1}}\, x_{n2}^{m_{n2}} \,\dots\, x_{nn}^{m_{nn}}) \;\in A_q(n)\ .$$

It will be useful to introduce another monomial basis (x^M) of $A_q(n)$, labelled by the same matrices, and defined by

$$x^M = (x_{1n}^{m_{1n}}\, x_{2n}^{m_{2n}} \,\dots\, x_{nn}^{m_{nn}}) \;\dots\; (x_{11}^{m_{11}}\, x_{21}^{m_{21}} \,\dots\, x_{n1}^{m_{n1}}) \;\in A_q(n)\ .$$

Proposition 4.3 *For any $q \in \mathbb{C}$, the family $(x^M)_{M\in\mathcal{M}_n(\mathbb{N})}$ is a homogeneous linear basis of $A_q(n)$.*

Proof — It is clearly sufficient to prove that each basis element x_M can be expressed in terms of the x^N. Using the array and tensor notations, such an element can be represented by

$$x_M = \begin{bmatrix} \dots & i_1 & i_2 & \dots \\ \dots & j_1 & j_2 & \dots \end{bmatrix} = e_{\mathbf{i}} \otimes e_{\mathbf{j}}^* \ ,$$

where j_1 is the maximal element of the second row of this array and where $i_1 \le i_2$. The maximality of j_1 and relation (7) imply

$$x_M = (-1)^{\ell(\sigma)} T_\sigma e_{\mathbf{i}} \otimes e_{\mathbf{j}} = \begin{pmatrix} (-1)^{l(\sigma)}\, T_\sigma \\ Id \end{pmatrix} \cdot \begin{bmatrix} i_1 & \dots & i_2 & \dots \\ j_1 & \dots & j_2 & \dots \end{bmatrix}$$

for some permutation σ. By induction on the length of x_M, there exists some other permutation τ such that

$$x_M = \begin{pmatrix} (-1)^{l(\tau)}\, T_\tau \\ Id \end{pmatrix} \cdot \begin{bmatrix} i_1 & \dots & i_r & \dots & k_1 & \dots & k_s \\ j_1 & \dots & j_1 & \dots & 1 & \dots & 1 \end{bmatrix} .$$

Going back to the definition of the left action of $H_N(q)$ and to relations (6), this implies that x_M is a $\mathbb{Z}[q]$-linear combination of elements of the form

$$\begin{bmatrix} i_1 & \dots & i_r & \dots & k_1 & \dots & k_s \\ n & \dots & n & \dots & 1 & \dots & 1 \end{bmatrix} ,$$

from which the conclusion follows immediately. $\square$

4.5 The standard subspace of $A_q(n)$

The restrictions to the standard component of $A_q(n)$ of the transition matrices between the two bases (x_M) and (x^M) have an interesting description.

Definition 4.4 *The* standard subspace *$S_q(n)$ of $A_q(n)$ is*

$$S_q(n) = A_q(1^n, 1^n) = \bigoplus_{\sigma\in\mathfrak{S}_n} \mathbb{C}(q)\, x_\sigma$$

where $x_\sigma = x_{1\sigma(1)}\, x_{2\sigma(2)}\, \dots\, x_{n\sigma(n)}$ for $\sigma \in \mathfrak{S}_n$.

The following result is an immediate consequence of Proposition 4.3.

$$S_q(n) = \bigoplus_{\sigma\in\mathfrak{S}_n} \mathbb{C}(q)\, x^{\sigma}$$

where $x^{\sigma} = x_{\sigma(n)n}\ \dots\ x_{\sigma(2)2}\, x_{\sigma(1)1}$.

The elements of the transition matrices between the two bases $(x_{\sigma})_{\sigma\in\mathfrak{S}_n}$ and $(x^{\sigma})_{\sigma\in\mathfrak{S}_n}$ of $S_q(n)$ are R-polynomials. Recall that the family $(R_{\tau,\sigma}(q))_{\sigma,\tau\in\mathfrak{S}_n}$ of *R-polynomials* is defined by

$$(T_{\sigma^{-1}})^{-1} = \varepsilon(\sigma)\, q^{-l(\sigma)} \sum_{\tau\leq\sigma} \varepsilon(\tau)\, R_{\tau,\sigma}(q)\, T_{\tau}\ \in H_n(q)$$

for $\sigma\in\mathfrak{S}_n$ (cf. [13]). The R-polynomial $R_{\tau,\sigma}(q)$ is in $\mathbb{Z}[q]$, has degree $l(\sigma)-l(\tau)$ and its constant term is $\varepsilon(\sigma\tau)$.

Proposition 4.6 *The bases* (x^{σ}) *and* (x_{σ}) *are related by*

$$x^{\sigma} = \sum_{\tau\leq\sigma} R_{\tau,\sigma}(q)\ x_{\omega\tau} \quad \textit{and} \quad x_{\sigma} = \sum_{\tau\leq\sigma} R_{\tau,\sigma}(-q)\ x^{\omega\tau}$$

where $\omega = (n\ n{-}1\ \dots\ 1)$ *and where* $\leq$ *is the Bruhat order.*

Proof — In the notation of Section 4.3, we can write

$$\begin{aligned}
x^{\sigma} &= e_{\sigma}\otimes e_{\omega}^{*} = e_{12\dots n}^{\sigma}\otimes e_{\omega}^{*} \equiv e_{12\dots n}\otimes e_{\omega}^{*}\cdot T_{\sigma^{-1}} \\
&\equiv e_{12\dots n}\otimes(-q)^{l(\sigma)}\, T_{\sigma^{-1}}^{-1}\cdot e_{\omega}^{*} \equiv e_{12\dots n}\otimes\Big(\sum_{\tau\leq\sigma}\varepsilon(\tau)\, R_{\tau,\sigma}(q)\, T_{\tau}\Big)\cdot e_{\omega}^{*} \\
&\equiv e_{12\dots n}\otimes\Big(\sum_{\tau\leq\sigma} R_{\tau,\sigma}(q)\, e_{\omega\tau}^{*}\Big) = \sum_{\tau\leq\sigma} R_{\tau,\sigma}(q)\, x_{\omega\tau}\ .
\end{aligned}$$

The second relation can be proved in the same way. □

Corollary 4.7 *In* $A_0(n)$*, one has*

$$x^{\sigma} = \sum_{\tau\leq\sigma} \varepsilon(\sigma\,\tau)\ x_{\omega\tau} \quad \textit{and} \quad x_{\sigma} = \sum_{\tau\leq\sigma} \varepsilon(\sigma\tau)\ x_{\omega\tau}\ .$$

4.6 Decomposition of left and right standard subspaces at $q = 0$

In the array notation, the standard subspace is spanned by arrays whose both rows are permutations. If one requires one row to be a fixed permutation σ, one obtains the left and right subcomodules of $A_q(n)$ which are independent of σ for generic q, but not for $q = 0$.

Definition 4.8 *The* left *and* right standard subspaces *of* $A_q(n)$*, respectively denoted by* $L_q(n)$ *and* $R_q(n)$*, are defined by*

$$L_q(n) = \bigoplus_{J=(j_1,\dots,j_n)\in\mathbb{N}^n} \mathbb{C}(q)\, x_{1,j_1}\, x_{2,j_2}\ \dots\ x_{n,j_n}\ ,$$

$$R_q(n) = \bigoplus_{I=(i_1,\dots,i_r)\in\mathbb{N}^r} \mathbb{C}(q)\, x_{i_n,n}\ \dots\ x_{i_2,2}\, x_{i_1,1}\ .$$

$$L_q(n;\sigma) = \sum_{J=(j_1,\dots,j_n)\in\mathbb{N}^n} \mathbb{C}(q)\, x_{\sigma(1),j_1}\, x_{\sigma(2),j_2} \,\cdots\, x_{\sigma(n),j_n} \ ,$$

$$R_q(n;\sigma) = \sum_{I=(i_1,\dots,i_r)\in\mathbb{N}^r} \mathbb{C}(q)\, x_{i_n,\sigma(n)} \,\cdots\, x_{i_2,\sigma(2)}\, x_{i_1,\sigma(1)} \ .$$

For generic q, all the left (resp. right) subspaces are the same.

Proposition 4.9 *If $q \in \mathbb{C}$ is nonzero and not a root of unity,*

$$L_q(n;\sigma) = L_q(n) \quad \textit{and} \quad R_q(n;\sigma) = R_q(n) \ .$$

Proof — Using the tensor notation of Section 4.3, we can write

$$e_\sigma \otimes e_J^* = e_{12\dots n}^\sigma \otimes e_J^* \equiv e_{12\dots n} \otimes e_J^* \cdot T_{\sigma^{-1}} \equiv (e_{12\dots n} \otimes e_J^*) \cdot (Id \otimes T_{\sigma^{-1}})$$

for every $J \in \mathbb{N}^n$, so that $L_q(n;\sigma) \subset L_q(n)$. Moreover the asumptions on q imply that $T_{\sigma^{-1}}$ is invertible. The previous relation can therefore be read as

$$e_{12\dots n} \otimes e_J^* \equiv (e_\sigma \otimes e_J^*) \cdot (Id \otimes (T_{\sigma^{-1}})^{-1}) \ ,$$

from which we get that $L_q(n) \subset L_q(n;\sigma)$. The second equality is obtained in the same way. $\square$

When $q = 0$, the subspaces $L_0(n;\sigma)$ and $R_0(n;\sigma)$ are not equal to $L_0(n)$ and $R_0(n)$. However, the proof of Proposition 4.9 shows that $L_0(n;\sigma) \subset L_0(n)$ and $R_0(n;\sigma) \subset R_0(n)$. The subspaces $L_0(n;\sigma)$ (resp. $R_0(n;\sigma)$ are right (resp. left) sub-$A_0(n)$-comodules of $L_0(n)$ (resp. $R_0(n)$), of which they form a filtration with respect to the weak order on the symmetric group. To prove this, let us introduce some notations. We associate to an integer vector $I = (i_1,\dots,i_n)$ of $\mathbb{N}^n$ the two sets

$$\mathrm{Inv}\,(I) = \{\, (k,l),\ 1 \le k < l \le n-1,\ i_k > i_l \,\} \ ,$$

$$\mathrm{Pos}\,(I) = \{\, (k,l),\ 1 \le k < l \le n-1,\ i_k < i_l \,\} \ .$$

Proposition 4.10 *For every permutation $\sigma \in \mathfrak{S}_n$, one has*

$$L_0(n;\sigma) = \bigoplus_{\substack{J=(j_1,\dots,j_n)\\ \mathrm{Inv}\,(\sigma)\subset \mathrm{Inv}\,(J)}} \mathbb{C}\, x_{\sigma(1),j_1} \,\cdots\, x_{\sigma(n),j_n} \ ,$$

$$R_0(n;\sigma) = \bigoplus_{\substack{I=(i_1,\dots,i_n)\\ \mathrm{Pos}\,(I)\subset \mathrm{Pos}\,(\sigma)}} \mathbb{C}\, x_{i_n,\sigma(n)} \,\cdots\, x_{i_1,\sigma(1)} \ .$$

Proof — We only show the first identity, the second one being proved in the same way.

Lemma 4.11 *Let $l \ge 1$ be such that $\sigma(i) > \sigma(i+l)$ and $j_i \le j_{i+l}$. Then, in $A_0(n)$*

$$x_{\sigma(i),j_i} \,\cdots\, x_{\sigma(i+l),j_{i+l}} = 0 \ .$$

result holds for $l-1$. Two cases are to be considered.

1) $\sigma(i+l-1) > \sigma(i+l)$. If $j_{i+l-1} \le j_{i+l}$, one clearly has

$$x_{\sigma(i),j_i} \ldots x_{\sigma(i+l),j_{i+l}} = x_{\sigma(i),j_i} \ldots x_{\sigma(i+l-1),j_{i+l-1}} x_{\sigma(i+l),j_{i+l}} = 0 \ .$$

On the other hand, if $j_{i+l-1} > j_{i+l}$, we can write

$$x_{\sigma(i),j_i} \cdots x_{\sigma(i+l),j_{i+l}} = x_{\sigma(i),j_i} \cdots x_{\sigma(i+l),j_{i+l}} x_{\sigma(i+l-1),j_{i+l-1}} - x_{\sigma(i),j_i} \cdots x_{\sigma(i+l),j_{i+l-1}} x_{\sigma(i+l-1),j_{i+l}} \cdot$$

We have here $j_i \le j_{i+l-1}$ so that the right hand side is zero, as required.

2) $\sigma(i+l-1) < \sigma(i+l)$. Thus $\sigma(i) > \sigma(i+l-1)$ and we just have to check the case $j_i > j_{i+l-1}$. Then, $j_{i+l-1} < j_{i+l}$ so that

$$x_{\sigma(i),j_i} \cdots x_{\sigma(i+l),j_{i+l}} = x_{\sigma(i),j_i} \cdots x_{\sigma(i+l),j_{i+l}} x_{\sigma(i+l-1),j_{i+l-1}} + x_{\sigma(i),j_i} \cdots x_{\sigma(i+l-1),j_{i+l}} x_{\sigma(i+l),j_{i+l-1}} \, ,$$

which is indeed zero by induction. □

It follows from the lemma that

$$L_0(n;\sigma) \ = \sum_{\substack{J=(j_1,\ldots,j_n) \\ \mathrm{Inv}\,(\sigma)\subset \mathrm{Inv}\,(J)}} \mathbb{C}\, x_{\sigma(1),j_1} \ldots x_{\sigma(n),j_n} \ ,$$

and it remains to prove that the sum is direct. Let $J=(j_1,\ldots,j_n) \in \mathbb{N}^n$ such that $\mathrm{Inv}\,(\sigma) \subset \mathrm{Inv}\,(J)$. Using the same argument as in the proof of Proposition 4.6 we can write

$$x_{\sigma(1),j_1} \ldots x_{\sigma(n),j_n} = e_\sigma \otimes e_J^* = e_{12\ldots n}^\sigma \otimes e_J^* \equiv e_{12\ldots n} \otimes (e_J^* \cdot T_{\sigma^{-1}}) = \sum_{\tau \le \sigma} \varepsilon(\sigma\tau)\, e_{12\ldots n} \otimes e_{J\cdot\tau}$$

where $\le$ is the Bruhat order on $\mathfrak{S}_n$ and $J \cdot \tau = (j_{\tau(1)}, \ldots, j_{\tau(n)})$. This last formula clearly shows that the family $(x_{\sigma(1),j_1} \ldots x_{\sigma(n),j_n})_{\mathrm{Inv}\,(\sigma)\subset \mathrm{Inv}\,(J)}$ is free. □

We can now prove that the "left cells" $L_0(n;\sigma)$ form a filtration of the right comodule $L_0(n)$ with respect to the weak order.

Proposition 4.12 *Let $\sigma \in \mathfrak{S}_n$ and let $i \in [1, n-1]$ such that $\sigma(i) > \sigma(i+1)$. Then $L_0(n;\sigma)$ is strictly included into $L_0(n;\sigma\,\sigma_i)$.*

Proof — The inclusion $L_0(n;\sigma) \subset L_0(n;\sigma\,\sigma_i)$ is immediate. Thus it suffices to show that this inclusion is strict. One can easily construct an element $\mathbf{x} \in L_0(n;\sigma\,\sigma_i)$ of the form $\mathbf{x} = \ldots x_{\sigma(i+1),k}\, x_{\sigma(i),k} \ldots$ Using the formalism of Section 4.3, one checks that $(T_i \otimes Id)\cdot \mathbf{x} = -\mathbf{x} \neq 0$. On the other hand, $(T_i \otimes Id) \cdot L_0(n;\sigma) = 0$. Thus $\mathbf{x} \notin L_0(n;\sigma)$. □

4.7 The diagonal subalgebra and the quantum pseudoplactic algebra

Definition 4.13 *The* quantum diagonal algebra $\Delta_q(n)$ *is the subalgebra of $A_q(n)$ generated by $x_{11}, \ldots, x_{nn}$.*

The character theory of $A_q(n)$-comodules described in Section 6 will show that the noncommutative algebra $\Delta_q(n)$ contains a subalgebra isomorphic to the algebra of ordinary symmetric polynomials, exactly as in the case of the plactic algebra.

$PPl_q(A)$ is the quotient of $\mathbb{C}(q)\langle A\rangle$ by the relations

$$\begin{cases} q\,a\,a\,b-(q+1)\,a\,b\,a+b\,a\,a &= 0 \qquad \textit{for } a<b\ , \\ q\,a\,b\,b-(q+1)\,b\,a\,a+b\,b\,a &= 0 \qquad \textit{for } a<b\ , \\ c\,a\,b-a\,c\,b-b\,c\,a+b\,a\,c &= 0 \qquad \textit{for } a<b<c\ . \end{cases}$$

The third relation is the Lie relation $[\,[a,c],b]=0$ where $[x,y]$ is the usual commutator $x\,y-y\,x$. For $q=1$, the first two relations become $[a,[a,b]]=[b,[b,a]]=0$ and $PPl_1(A)$ is the universal enveloping algebra of the Lie algebra defined by these relations.

It should be noted that the classical plactic algebra is not obtained by any specialization of $PPl_q(A)$. The motivation for the introduction of $PPl_q(A)$ comes from the following conjecture.

Conjecture 4.15 *Let $A=\{a_1,\ldots,a_n\}$ be a totally ordered alphabet of cardinality n. For generic q, the mapping $\varphi: a_i\to x_{ii}$ induces an isomorphism between $PPl_q(A)$ and the diagonal algebra $\Delta_q(n)$.*

Our conjecture is stated for generic values of q, i.e. when q is considered as a free variable, or avoiding a discrete set in $\mathbb{C}$. It is clearly not true for arbitrary complex values of q. For example, for $q=1$, the diagonal algebra $\Delta_1(n)$ is an algebra of commutative polynomials. The diagonal algebra at $q=0$ is also particularly interesting, and its structure will be investigated in the forthcoming section.

4.8 The hypoplactic algebra

Let again A be a totally ordered alphabet. We recall that the *plactic algebra* on A is the $\mathbb{C}$-algebra $Pl(A)$, quotient of $\mathbb{C}\langle A\rangle$ by the relations

$$\begin{cases} aba=baa \ ,\quad bba=bab \qquad \text{for } a<b\ , \\ acb=cab \ ,\quad bca=bac \qquad \text{for } a<b<c\ . \end{cases}$$

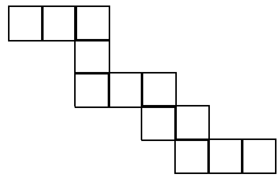
These relations, which were obtained by Knuth [20], generate the equivalence relation identifying two words which have the same P-symbol under the Robinson-Schensted correspondence. Though Schensted had shown that the construction of the P-symbol is an associative operation on words, the monoid structure on the set of tableaux has been mostly studied by Lascoux and Schützenberger [25] under the name 'plactic monoid'. These authors showed, for example, that the Littlewood-Richardson rule is essentially equivalent to the fact that plactic Schur functions, defined as sums of all tableaux with a given shape, are the basis of a commutative subalgebra of the plactic algebra. This point of view is now explained by Kashiwara's theory of crystal bases [16, 17], which also leads to the definition of plactic algebras associated to all classical simple Lie algebras [24, 27]. Other interpretations of the Robinson-Schensted correspondence and of the plactic relations can be found in [3, 26].

Kashiwara's crystallization process describes the generic situation modulo a certain lattice, but does not amount to put $q=0$ in the defining relations of quantum groups, which is generally impossible due to the symmetric rôles played by q and q^{-1}. The specialization $q=0$ in $A_q(n)$ or in $H_N(q)$ leads to a different combinatorics, and describes a truly degenerate case, rather than combinatorial aspects of the generic situation. In particular, the specialization of the diagonal algebra is a remarkable quotient of the plactic algebra that we shall now introduce.

by the quartic relations

$$\begin{cases} baba = abab \quad , \quad baca = abac & \text{for } a < b < c \ , \\ cacb = acbc \quad , \quad cbab = bacb & \text{for } a < b < c \ , \\ badc = dbca \quad , \quad acbd = cdab & \text{for } a < b < c < d \ . \end{cases}$$

The combinatorial objects playing the rôle of Young tableaux are the so-called ribbons and quasi-ribbons. We recall first that a *ribbon diagram* is a skew Young diagram containing no 2×2 block of boxes. A ribbon diagram with n boxes is naturally encoded by a composition $I = (i_1, \ldots, i_r)$ of n, called the *shape* of the diagram, whose parts are the lengths of its rows (starting from the top). For instance, the following skew diagram is a ribbon diagram of shape $(3, 1, 3, 2, 3)$.

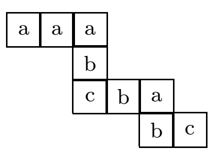


Let I be a composition. A *quasi-ribbon tableau* of shape I is then obtained by filling a ribbon diagram r of shape I by letters of A in such a way that

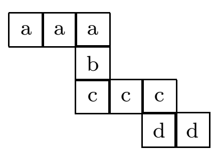
- each row of r is nondecreasing from left to right;
- each column of r is strictly increasing from *top to bottom*.

A word is said to be a *quasi-ribbon word* of shape I if it can be obtained by reading from *bottom to top* and from left to right the columns of a quasi-ribbon diagram of shape I. Observe that this convention allows to read the shape of a quasi-ribbon word on the word itself.

Example 4.17 The word $u = aacbabbac$ is not a quasi-ribbon word since the planar representation of u obtained by writing its decreasing factors as columns is not a quasi-ribbon tableau, as one can see on the picture. On the other hand, the word $v = aacbacdcd$ is a quasi-ribbon word of shape $(3, 1, 3, 2)$. The quasi-ribbon tableau corresponding to v is also given below.



aacbabbac



aacbacdcd

The central result of this section is the following.

Theorem 4.18 *The classes of the quasi-ribbon words form a linear basis of the hypoplactic algebra $HPl(A)$.*

Proof — We first prove that every word w of A^* is equivalent to some quasi-ribbon word with respect to the hypoplactic congruence $\equiv$. It is sufficient to prove this for standard words (i.e. permutations), since the hypoplactic congruence is compatible with standardization. The standardized $std(w)$ of a word w is the permutation obtained by the following process. Reading w from left to right, label $1, 2, \ldots$ the successive occurences of the smallest letter a of w, then

labelled letters a_i regarded as elements of the alphabet $A \times \mathbb{N}$ endowed with the lexicographic order. Then replace each labelled letter by the integers $1, 2, \ldots$ according to its rank in the lexicographic order, as in the following example:

$$w = ababca \longrightarrow a_1 b_1 a_2 b_2 c_1 a_3 \longrightarrow std(w) = 142563 \ .$$

This standardization process, due to Schensted [34], is compatible with the plactic relations [25]. One can also check that it is compatible with the quartic hypoplactic relations (used in connection with the usual plactic relations). The standardization of the first hypoplactic relation $baba = abab$ leads for instance to $b_1 a_1 b_2 a_2 = a_1 b_1 a_2 b_2$ which is a consequence of a plactic relation $(bac = bca)$ and of the last hypoplactic relation $(cdab = acbd)$:

$$b_1 a_1 b_2 a_2 = b_1 b_2 a_1 a_2 = a_1 b_1 a_2 b_2 \ .$$

The other verifications are done in the same way. This implies therefore that $u \equiv v$ iff $std(u) \equiv std(v)$.

Thus, if we assume that the theorem holds for standard words, $std(w)$ is equivalent to some standard quasi-ribbon word r. Compatibility with the hypoplactic congruence imply that $w \equiv r'$ where r' is the word obtained from r by replacing each integer $i \in [1, n]$ by the letter of A occupying the i-th position in $std(w)^{-1}$. But in a standard word, the hypoplactic relations preserve the relative order of all pairs $(i, i+1)$. It follows that the image in r' of a column of the ribbon diagram of r is still a strictly decreasing sequence of letters, so that r' is still a quasi-ribbon word of the same shape as r. Hence w is equivalent to a quasi-ribbon word.

Example 4.19 Let again $w = ababca$. Then we have $std(w) = 142563$ and

$$142\underline{563} \equiv 1\underline{4253}6 \equiv 124356$$

(the places where a rewriting rule has been applied are underlined). Hence $std(w)$ is equivalent to the standard quasi-ribbon word $r = 124356$ of shape $(3, 3)$. The compatibility of the standardization process with $\equiv$ implies that $w \equiv r' = aababc$. The quasi-ribbon representation of r' is

$$\begin{array}{ccccc} \boxed{\mathrm{a}} & \boxed{\mathrm{a}} & \boxed{\mathrm{a}} & & \\ & & \boxed{\mathrm{b}} & \boxed{\mathrm{b}} & \boxed{\mathrm{c}} \end{array}$$

We now turn back to the standard case. We have to prove that every permutation of $\mathfrak{S}_n$ (considered as a word over $[1, n]$) is equivalent to some (standard) quasi-ribbon word over $[1, n]$. The proof proceeds by induction on n. Suppose that the result is true up to some $n \geq 1$, and let $w = u\, a$ be a permutation of $\mathfrak{S}_{n+1}$ where $|u| = n$ and $a \in [1, n+1]$. Applying the induction hypothesis to u, we can write $w \equiv r\, a$ where r is a standard quasi-ribbon word over $[1, n{+}1] - \{a\}$. Decompose r as $r = c_1 \ldots c_l$ where c_i is the word obtained by reading from bottom to top the i-th column of the quasi-ribbon tableau associated with r. Thus $w \equiv c_1 \ldots c_l\, a$. Since r is a quasi-ribbon word, the first column c_1 has to be of one of the following two types:

1. $c_1 = j\ j{-}1\ \ldots\ 1$ for some $j \in [1, n+1]$. In this case, the conclusion follows by applying the induction hypothesis to $c_2 \ldots c_l\, a$.

2. $c_1 = j\ j{-}1\ \ldots\ i{+}1\ i{-}1\ \ldots\ 1$ for some $j \in [1, n{+}1]$. In this case, the induction hypothesis allows us to write $c_2 \ldots c_l\, i \equiv d_2 \ldots d_m$ where $d_2 \ldots d_m$ is the column decomposition of some standard quasi-ribbon word. Since i is the minimal letter of this last word, we must have $d_2 = d_2'\, i$, and the conclusion is implied by the following lemma.

$$(j\ \ldots\ i{+}1\ i{-}1\ \ldots\ 1)\ (n\ \ldots\ j{+}1\ i)\ \equiv\ (i{-}1\ \ldots\ 1)\ (j\ \ldots i)\ (n\ \ldots\ j{+}1)\ ,$$

where $x \ldots y$ denotes the concatenation of the elements of the interval $[x, y]$, which is the empty word for $x > y$.

Proof of the lemma – We argue by induction on n. For $n = 3$, the two possible situations covered by the lemma correspond exactly to the two standard plactic relations written as

$$\begin{array}{cc} 3 & \\ 1 & 2 \end{array} \equiv \begin{array}{cc} 1 & 3 \\ & 2 \end{array}\ , \qquad \begin{array}{cc} 2 & 3 \\ & 1 \end{array} \equiv \begin{array}{cc} 2 & \\ 1 & 3 \end{array}\ .$$

The standard quartic hypoplactic relations are also special instances of the lemma:

$$\begin{array}{cc} 4 & \\ 2 & 3 \\ & 1 \end{array} \equiv \begin{array}{cc} 2 & \\ 1 & 4 \\ & 3 \end{array}\ , \qquad \begin{array}{ccc} 3 & 4 & \\ & 1 & 2 \end{array} \equiv \begin{array}{ccc} 1 & 3 & \\ & 2 & 4 \end{array}\ .$$

Let now $n \geq 4$ and suppose that the lemma holds up to order $n - 1$. Suppose first that $i = 1$. If $j + 1 = n$, the formula is obtained by application of a single plactic relation. So, we can suppose that $j + 1 < n$. In this case, the result follows by successively applying a plactic relation, the induction hypothesis and a hypoplactic relation as shown below

$$\begin{array}{rlcl} & (j\ \ldots\ 2)\,(n\ n{-}1\ \ldots\ j{+}1\ 1) & \equiv & (j\ \ldots\ 3)\,(n\ 2)\,(n{-}1\ \ldots\ j{+}1\ 1) \\ \equiv & (j\ \ldots\ 3)\,(n\ 2\ 1)\,(n{-}1\ \ldots\ j{+}1) & \equiv & (j\ \ldots\ 2\ 1)\,(n\ n{-}1\ \ldots\ j{+}1\ 1)\ . \end{array}$$

Consider now the case $i = 2$. Suppose first that $j = 3$. For $n = 4$, the result to be proved is exactly one of the standard hypoplactic relations. Thus we can assume that $n > 4$. Using successively the fact that $(3\ 1)\,(n \ldots 4\ 2)$ is plactically equivalent to $(n\ \ldots\ 3\ 1)\,(4\ 2)$, a hypoplactic relation and then a plactic relation, we obtain

$$(3\ 1)\,(n\ \ldots\ 4\ 2) \equiv (n\ \ldots\ 3\ 1)\,(4\ 2) \equiv (n\ \ldots\ 1)\,(3\ 2)\,(4) \equiv (n\ \ldots\ 3\ 1)\,(2)\,(4)\ .$$

Applying twice the induction hypothesis, we can rewrite the right hand side as

$$(n\ \ldots\ 3\ 1)\,(2)\,(4) \equiv (1)\,(n\ \ldots\ 3\ 2)\,(4) \equiv (1)\,(3\ 2)\,(n\ \ldots\ 4)\ .$$

If $j > 3$, we reach the desired conclusion by first applying the induction hypothesis and then the fact that $j\ \ldots\ 1\ 3$ is plactically equivalent to $1\ j \ldots\ 3$ as described below

$$(j\ \ldots\ 3\ 1)\,(n\ \ldots\ j{+}1\ 2) \equiv (j\ \ldots\ 1)\,(3\ 2)\,(n\ \ldots\ j{+}1) \equiv (1)\,(j\ \ldots\ 3\ 2)\,(n\ \ldots\ j{+}1)\ .$$

The general case $i \geq 3$ follows then by iterated applications of the induction hypothesis as described below

$$\begin{array}{rlcl} & (j\ \ldots\ i{+}1\ i{-}1\ \ldots\ 2\ 1)\,(n\ \ldots\ j{+}1\ i) & \equiv & (j\ \ldots\ i{+}1\ i{-}1\ \ldots\ 2)\,(n\ \ldots\ j{+}1\ i\ 1) \\ \equiv & (i{-}1\ \ldots\ 2)\,(j\ \ldots\ i)\,(n\ \ldots\ j{+}1\ 1) & \equiv & (i{-}1\ \ldots\ 2)\,(j\ \ldots\ i\ 1)\,(n\ \ldots\ j{+}1) \\ \equiv & (i{-}1\ \ldots\ 2\ 1)\,(j\ \ldots\ i)\,(n\ \ldots\ j{+}1)\ . & & \end{array}$$

$\square$

At this point, we have shown that every word of A^* is equivalent to some quasi-ribbon word. To conclude the proof of the theorem, it remains to show that the hypoplactic classes of quasi-ribbon words are linearly independent. Again, by the standardization argument, it suffices to

suppose that $A = [1,n]$ for some $n \geq 1$. The point is now that the hypoplactic relations are satisfied by the generators of the 0-diagonal algebra $\Delta_0(n)$. Hence one can define a morphism φ from $HPl(A)$ onto $\Delta_0(n)$ by $\varphi(i) = x_{ii}$ for every $i \in A = [1,n]$.

Let $w \in \mathfrak{S}_n$ be a standard quasi-ribbon word of length n over $[1,n]$. By definition, there exists a strictly increasing sequence $1 = k_1 < k_2 < \ldots < k_{l+1} = n+1$ such that $w = c_1 \ \ldots \ c_l$ with $c_i = k_{i+1}-1 \ \ldots \ k_i$ for $i \in [1,l]$. Consider the Young subgroup $\mathfrak{S}_w = \mathfrak{S}_{[k_1,k_2-1]} \times \ldots \times \mathfrak{S}_{[k_l,k_{l+1}-1]}$ of $\mathfrak{S}_n$. Applying Corollary 4.7 to each strictly decreasing word c_i, one obtains that

$$\varphi(w) = \sum_{\sigma \in \mathfrak{S}_w} \varepsilon(\sigma) \begin{bmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{bmatrix}$$

in the notation of Section 4.3. Observe that w is the unique permutation of maximal length occuring in the sum. This property implies immediately that $\varphi(\mathcal{B}_n)$ is free in $\Delta_0(n)$. It follows that $\mathcal{B}_n$ is itself free in $HPl(A)$ as desired. $\square$

Let $C(\sigma)$ be the unique composition I such that $D(\sigma) = D(I)$. By the *evaluation* $\mathrm{ev}\,(w)$ of a word $w \in A^*$, we mean the vector $\mathrm{ev}\,(w) = (|w|_a)_{a \in A} \in \mathbb{N}^A$ whose entries are just the different numbers of occurences of each letter $a \in A$ in w.

As an interesting consequence of the proof of Theorem 4.18 and of Lemma 4.20, we obtain:

Proposition 4.21 *Let w be a word over a totally ordered alphabet A, let λ be its evaluation and let $\sigma = std(w)$. The unique quasi-ribbon word to which w is equivalent with respect to the hypoplactic congruence is the unique quasi-ribbon word of evaluation λ and of shape $C(\sigma^{-1})$.*

Example 4.22 Consider again $w = ababca$. Then $\lambda = (3,2,1)$ and $std(w) = \sigma = 142563$. Hence $\sigma^{-1} = 136245$ and $C(\sigma^{-1}) = (3,3)$. The unique quasi-ribbon word of evaluation $(3,2,1)$ and of shape $(3,3)$ is $aababc$. Thus $w \equiv aababc$ as already seen in Example 4.19.

The importance of the hypoplactic algebra comes from the following isomorphism, wich follows directly from the previous considerations.

Theorem 4.23 *The ring homomorphism defined by $\varphi : a_i \longrightarrow x_{ii}$ is an isomorphism between the hypoplactic algebra $HPl(A)$ and the crystal limit $\Delta_0(n)$ of the quantum diagonal algebra.*

We have already seen that the quantum diagonal algebra $\Delta_q(n)$ is not isomorphic to the quantum plactic algebra when $q \in \{0,1\}$. We conjecture that these two degenerate cases are the only exceptions.

5 Characteristics of $H_N(0)$-modules

5.1 Grothendieck rings associated with finitely generated $H_N(0)$-modules

Let $G_0(H_N(q))$ be the Grothendieck group of the category of finitely generated $H_N(q)$-modules and let $K_0(H_N(q))$ be the Grothendieck group of equivalence classes of finitely generated projective $H_N(q)$-modules. When q is not 0 and not a root of unity, the Hecke algebra $H_N(q)$ is semi-simple and these two groups coincide. Moreover their direct sum for all $n \geq 0$ endowed with the induction product is isomorphic to the ring *Sym* of (commutative) symmetric functions.

When $q = 0$, $H_N(0)$ is not semi-simple. In particular, indecomposable $H_N(0)$-modules are not necessarily irreducible, and the Grothendieck rings

$$\mathcal{G} = \bigoplus_{N \geq 0} G_0(H_N(0)) \qquad \text{and} \qquad \mathcal{K} = \bigoplus_{N \geq 0} K_0(H_N(0))$$

of quasi-symmetric functions and **Sym** of noncommutative symmetric functions. The duality between **Sym** and $QSym$ (cf. Section 2.2) can therefore be traced back to a general fact in representation theory.

5.2 Simple $H_N(0)$-modules

There are 2^{N-1} simple $H_N(0)$-modules, all of dimension 1 [1, 31]. To see this, it is sufficient to observe that $(T_iT_{i+1} - T_{i+1}T_i)^2 = 0$. Thus, all the commutators $[T_i, T_j]$ are in the radical of $H_N(0)$. But the quotient of $H_N(0)$ by the ideal generated by these elements is the *commutative* algebra generated by $N-1$ elements $t_1, \ldots, t_{N-1}$ subject to $t_i^2 = -t_i$. It is easy to check that this algebra has no nilpotent elements, so that it is $H_N(0)/\mathrm{rad}\,(H_N(0))$. The irreducible representations are thus obtained by sending a set of generators to -1 and its complement to 0. We shall however label these representations by compositions rather than by subsets. Let I be a composition of N and let $D(I)$ the associated subset of $[1, N-1]$. The irreducible representation φ_I of $H_N(0)$ is then defined by

$$\varphi_I(T_i) = \begin{cases} -1 & \text{if } i \in D(I)\ , \\ 0 & \text{if } i \notin D(I)\ . \end{cases}$$

The associated $H_N(0)$-module will be denoted by $\mathbf{C}_I$. These modules (when I runs over all compositions of N) form a complete system of simple $H_N(0)$-modules.

The simple modules can also be realized as minimal left ideals of $H_N(0)$. To describe the generators, we associate with a composition I of N two permutations $\alpha(I)$ and $\omega(I)$ of $\mathfrak{S}_N$ defined by

- $\alpha(I)$ is the permutation obtained by filling the columns of the skew Young diagram of ribbon shape I from bottom to top and from left to right with the numbers $1, 2, \ldots, N$, i.e. the standard quasi-ribbon word of shape I;
- $\omega(I)$ is the permutation obtained by filling the rows of the skew Young diagram of ribbon shape I from left to right and from bottom to top with the numbers $1, 2, \ldots, N$.

Example 5.1 Consider the composition $I = 22113$ of 9. The fillings of the ribbon diagram of shape I corresponding to $\alpha(I)$ and $\omega(I)$ are

$$\begin{array}{ccccc} 1 & 3 & & & \\ & 2 & 7 & & \\ & & 6 & & \\ & & 5 & & \\ & & 4 & 8 & 9 \end{array} \qquad\qquad \begin{array}{ccccc} 8 & 9 & & & \\ & 6 & 7 & & \\ & & 5 & & \\ & & 4 & & \\ & & 1 & 2 & 3 \end{array}$$

$\alpha(22113)$ $\qquad$ $\omega(22113)$

Thus $\alpha(22113) = 132765489$ and $\omega(22113) = 896754123$.

Recall that the *permutohedron* of order N is the Hasse diagram of the weak order on $\mathfrak{S}_N$, that is, the graph whose vertices are the elements of $\mathfrak{S}_N$ and where an edge labelled $i \in [1, N-1]$ between σ and τ means that $\tau = \sigma_i\,\sigma$.

Lemma 5.2 *Let I be a composition of N. The descent class $D_I = \{\,\sigma \in \mathfrak{S}_N,\ D(\sigma) = D(I)\,\}$ is the interval $[\alpha(I), \omega(I)]$ for the weak order on $\mathfrak{S}_N$.*

$$\begin{cases} \square_i^2 = \square_i & \text{for } i \in [1, N-1] \ , \\ \square_i \, \square_j = \square_j \, \square_i & \text{for } |i-j| > 1 \ , \\ \square_i \, \square_{i+1} \, \square_i = \square_{i+1} \, \square_i \, \square_{i+1} & \text{for } i \in [1, N-2] \ . \end{cases}$$

In particular, the morphism defined by $T_i \longrightarrow -\square_i$ is an involution of $H_N(0)$. As the $\square_i$ satisfy the braid relations, one can associate to each permutation $\sigma \in \mathfrak{S}_N$ the element $\square_\sigma$ of $H_N(0)$ defined by $\square_\sigma = \square_{i_1} \dots \square_{i_r}$ where $\sigma_{i_1} \dots \sigma_{i_r}$ is an arbitrary reduced decomposition of σ.

For a composition $I = (i_1, \dots, i_r)$ we denote by $\overline{I} = (i_r, \dots, i_1)$ its mirror image and by $I^\sim$ its conjugate composition, i.e. the composition obtained by writing from right to left the lengths of the columns of the ribbon diagram of I. For instance, $\overline{(3,2,1)} = (1,2,3)$ and $(3,2,1)^\sim = (2,2,1,1)$.

Proposition 5.3 *The simple $H_N(0)$ module $\mathbf{C}_I$ is isomorphic to the minimal left ideal $H_N(0)\,\eta_I$ of $H_N(0)$ where $\eta_I = T_{\omega(\overline{I})}\,\square_{\alpha(I^\sim)}$.*

Proof — Observe first that $\omega(\overline{I})^{-1} = \omega(I)$ and $\alpha(I) = \omega(I^\sim)\,\omega_N$ (where ω_N is the maximal permutation of $\mathfrak{S}_N$). It follows that $\mathrm{Des}\,(\omega(\overline{I})^{-1}) = D(I)$ and $\mathrm{Des}\,(\alpha(I^\sim)^{-1}) = [1, n{-}1] - D(\overline{I})$. Thus, taking into account the fact that $T_i\,\square_i = 0$, one checks that η_I is different from 0 and that

$$T_i\,T_{\omega(\overline{I})}\,\square_{\alpha(I^\sim)} = \begin{cases} -\,T_{\omega(\overline{I})}\,\square_{\alpha(I^\sim)} & \text{if } i \in D(I) \ , \\ 0 & \text{if } i \notin D(I) \ . \end{cases} \qquad \square$$

5.3 Indecomposable projective $H_N(0)$-modules

The indecomposable projective $H_N(0)$-modules have also been classified by Norton (cf. [1, 31]). One associates with a composition I of N the unique indecomposable projective $H_N(0)$-module $\mathbf{M}_I$ such that $\mathbf{M}_I/\mathrm{rad}\,(\mathbf{M}_I) \simeq \mathbf{C}_I$. This module can be realized as the left ideal

$$\mathbf{M}_I = H_N(0)\,\nu_I$$

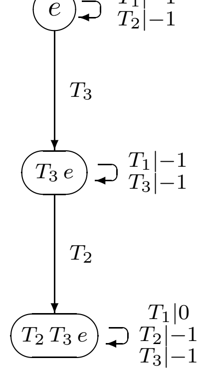
where $\nu_I = T_{\alpha(I)}\,\square_{\alpha(\overline{I}^\sim)}$. Since $\alpha(I^\sim)^{-1} = \omega_n\,\omega(\overline{I})$, one gets $D(\alpha(\overline{I}^\sim)^{-1}) = [1, n{-}1] - D(I)$. It follows that the generator ν_I of $\mathbf{M}_I$ is different from 0 and that a basis of $\mathbf{M}_I$ is given by

$$\{\ T_\sigma\,\square_{\alpha(\overline{I}^\sim)},\ \mathrm{Des}\,(\sigma) = D(I)\ \} = \{\ T_\sigma\,\square_{\alpha(\overline{I}^\sim)},\ \sigma \in [\alpha(I), \omega(I)]\ \} \ ,$$

according to Lemma 5.2. Hence the dimension of $\mathbf{M}_I$ is equal to the cardinality of the descent class D_I. Also, every interval of the form $[\alpha(I), \omega(I)]$ in the permutohedron can be interpreted as the "graph" of some indecomposable projective $H_N(0)$-module (cf. Example 5.4 below). The family $(\mathbf{M}_I)_{|I|=N}$ forms a complete system of projective indecomposable $H_N(0)$-modules, and

$$H_n(0) = \bigoplus_{|I|=N} \mathbf{M}_I \ . \qquad (8)$$

Example 5.4 Let $I = (1,1,2)$. Then $I^\sim = (1,3)$, $\overline{I} = (2,1,1)$, $\overline{I}^\sim = (3,1)$, $\alpha(I) = 3214$ and $\alpha(\overline{I}^\sim) = 1243$. Hence $\nu_{112} = T_2\,T_1\,T_2\,\square_3$. The module $\mathbf{M}_{112}$ can be described by the following automaton. An arrow labelled T_i going from f to g means that $T_i \cdot f = g$, and a loop on the vertex f labelled $T_i|\epsilon$ (with $\epsilon = 0$ or $\epsilon = -1$) means that $T_i \cdot f = \epsilon\, f$.



This is also the graph of the interval $[3214, 4312] = D_{112}$ in the permutohedron of $\mathfrak{S}_4$. The (-1)-loops correspond to the descents of the inverse permutation.

5.4 A Frobenius type characteristic for finite dimensional $H_N(0)$-modules

Let M be a finite dimensional $H_N(0)$-module and consider a composition series for M, i.e. a decreasing sequence $M_1 = M \supset M_2 \supset \ldots \supset M_k \supset M_{k+1} = \mathbb{C}$ of submodules where the successive quotients M_i/M_{i+1} are simple. Therefore each M_i/M_{i+1} is isomorphic to some $\mathbf{C}_{I_i}$, and the Jordan-Hölder theorem ensures that the quasi-symmetric function

$$\mathcal{F}(M) = \sum_{i=1}^{k} F_{I_i}$$

is independent of the choice of the composition series. This quasi-symmetric function is called the *characteristic* of M. Its properties are quite similar to those of the usual Frobenius characteristic of a $\mathfrak{S}_N$-module [7]. However, the characteristic $\mathcal{F}(M)$ of a $H_N(0)$-module M does not specify it up to isomorphism.

The character formula for $H_N(0)$-modules can be stated in a form similar to the Frobenius character formula. For a composition I, we denote by $C_I(q)$ the noncommutative symmetric function $C_I(q) = (q-1)^{l(I)}\, S^I((q-1)A)$, in the noncommutative λ-ring notation introduced in [21]. Let also w_J be the permutation of the Young subgroup $\mathfrak{S}_J$ defined by

$$w_J = (\sigma_1 \, \ldots \, \sigma_{j_1-1})\, (\sigma_{j_1+1} \, \ldots \, \sigma_{j_1+j_2-1}) \, \ldots \, (\, \sigma_{j_1+\ldots+j_{r-1}+1} \, \ldots \, \sigma_{n-1}) \ .$$

The character of a module M is then determined by its values $\chi_M(T_{w_J}) = \mathrm{tr}_{\,M}(T_{w_J})$ on the special elements T_{w_J}.

Proposition 5.5 [7] (Character formula) *The character of M is given by*

$$\chi_M(T_{w_J}) = \langle \mathcal{F}(M)\, , \, C_J(0) \rangle$$

where $\langle\ ,\ \rangle$ is the pairing between QSym and **Sym**.

One can refine $\mathcal{F}$ into a graded version of the characteristic, at least when M is a cyclic module i.e. when $M = H_N(0)\, e$. In this case, the length filtration

$$H_N(0)^{(k)} = \bigoplus_{l(\sigma) \geq k} \mathbb{C}\, T_\sigma$$

suggests to introduce the *graded characteristic* $\mathcal{F}_q(M)$ of M defined by

$$\mathcal{F}_q(M) = \sum_{k\geq 0} q^k\, \mathcal{F}(M^{(k)}/M^{(k+1)}) \ .$$

The ordinary characteristic $\mathcal{F}(M)$ is then the specialization of $\mathcal{F}_q(M)$ at $q = 1$.

The graded characteristic is in particular defined for the modules induced by tensor products of simple 1-dimensional modules

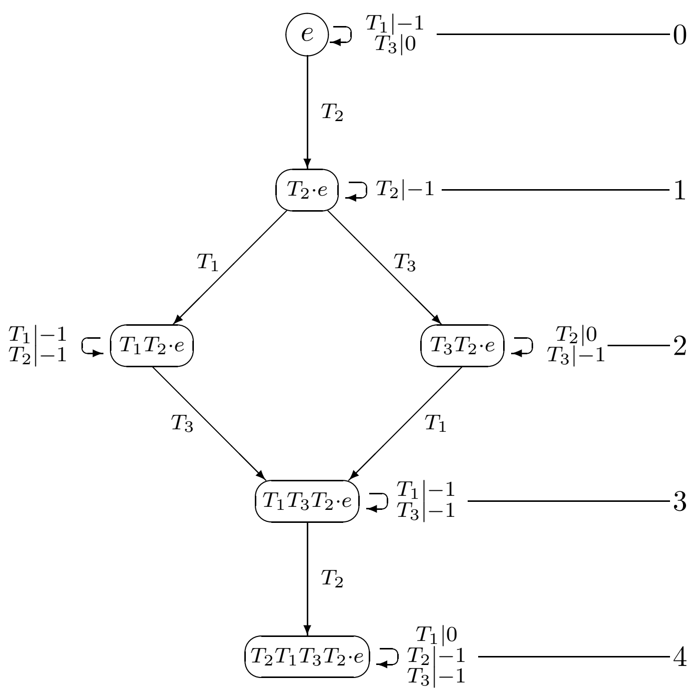
$$M_{I_1,\ldots,I_r} = \mathbf{C}_{I_1} \otimes \cdots \otimes \mathbf{C}_{I_r} \uparrow_{H_{n_1}(0)\otimes\cdots\otimes H_{n_r}(0)}^{H_{n_1+\cdots+n_r}(0)} \ ,$$

the characteristic of which being equal to the product $F_{I_1} \cdots F_{I_r}$. The induction formula can be stated in terms of the graded characteristic, which leads to a q-analogue of the algebra of quasi-symmetric functions. This q-analogue is defined in terms of the q-shuffle product [6]. Let A be an alphabet and let q be an indeterminate commuting with A. The q-shuffle is the bilinear operation of $\mathbb{N}[q]\langle A\rangle$ denoted by $\odot_q$ and recursively defined on words by the relations

$$\begin{cases} 1 \odot_q u = u \odot_q 1 = u \ , \\ (a\,u) \odot_q (b\,v) = a\,(u \odot_q b\,v) + q^{|au|}\, b\,(a\,u \odot_q v) \ , \end{cases}$$

where 1 is the empty word, $u, v \in A^*$ and $a, b \in A$. One can show that $\odot_q$ is associative (cf. [6]).

Example 5.6 Let $M_{(11),(2)}$ denote the $H_4(0)$-module obtained by inducing to $H_4(0)$ the $H_2(0)\otimes H_2(0)$-module $\mathbf{C}_{11} \otimes \mathbf{C}_2$, identifying $H_2(0) \otimes H_2(0)$ with the subalgebra of $H_4(0)$ generated by T_1 and T_3. This $H_4(0)$-module is generated by a single element e on which T_1 and T_3 act by $T_1 \cdot e = -e$ and by $T_3 \cdot e = 0$. The following automaton gives a complete description of this module. The states (vertices) correspond to the images of e under the action of some element of $H_4(0)$, which form a linear basis of $M_{(11),(2)}$.



The automaton is graded by the distance $d(f)$ of a state f to the initial state e as indicated on the picture. This grading is precisely the one described by $\mathcal{F}_q$. That is, if we associate with each state

we find

$$\mathcal{F}_q(M_{(11,2)}) = \sum_f q^{d(f)}\, F_{I(f)} = F_{13} + q\, F_{22} + q^2\,(F_{112} + F_{31}) + q^3\, F_{121} + q^4\, F_{211}\ .$$

This equality can also be read on the q-shuffle of 21 and 34:

$$21 \odot_q 34 = 2134 + q\, 2314 + q^2\, 2341 + q^2\, 3214 + q^3\, 3241 + q^4\, 3421\ .$$

One obtains the graded characteristic by replacing each permutation σ in this expansion by the quasi-symmetric function $F_{C(\sigma)}$.

This example illustrates the general fact that the graded characteristic of an induced module as above is always given by the q-shuffle. As it is an associative operation, one obtains in this way a q-deformation of the ring of quasi-symmetric functions.

Proposition 5.7 [7, 6] *Let I and J be compositions of N and M. Let also $\sigma \in \mathfrak{S}_{[1,N]}$ and $\tau \in \mathfrak{S}_{[N+1,N+M]}$ be such that* $\mathrm{Des}\,(\sigma) = D(I)$ *and* $\mathrm{Des}\,(\tau) = D(J)$. *The $H_{N+M}(0)$-module obtained by inducing to $H_{N+M}(0)$ the $H_N(0)\otimes H_M(0)$-module $\mathbf{C}_I \otimes \mathbf{C}_J$ (identifying $H_N(0)\otimes H_M(0)$ to the subalgebra of $H_{N+M}(0)$ generated by $T_1, \ldots, T_{N-1}, T_{N+1}, \ldots, T_{N+M-1}$) is cyclic, and its graded characteristic is given by*

$$\mathcal{F}_q(\mathbf{C}_I \otimes \mathbf{C}_J \uparrow_{H_N(0)\otimes H_M(0)}^{H_{N+M}(0)}) = \sum_{\nu\in\mathfrak{S}_{N+M}} q^{d(\nu)}\, F_{C(\nu)}$$

where $C(\nu)$ denotes the composition associated with the descent set of ν and where

$$\sigma \odot_q \tau = \sum_{\nu\in\mathfrak{S}_{N+M}} q^{d(\nu)}\, \nu\ .$$

For $q = 1$, we obtain the following result [7].

Corollary 5.8 *The characteristic $\mathcal{F}$ is an isomorphism between $\mathcal{G}$ and the $\mathbb{Z}$-algebra of quasi-symmetric functions.*

5.5 A noncommutative characteristic for finite dimensional projective $H_N(0)$-modules

Let M be a finite dimensional projective $H_N(0)$-module. Hence M is isomorphic to a direct sum of indecomposable projective modules

$$M = \bigoplus_{i=1}^{m} \mathbf{M}_{I_i}\ .$$

The *noncommutative Frobenius characteristic* of M is the noncommutative symmetric function $\mathcal{R}(M)$ defined by

$$\mathcal{R}(M) = \sum_{i=1}^{m} R_{I_i}\ .$$

The characteristic $\mathcal{R}(M)$ does characterize every finite dimensional projective $H_N(0)$-module M up to isomorphism, and is therefore stronger than $\mathcal{F}$. The following proposition, which is a reformulation of Carter's expression of the Cartan invariants of $H_N(0)$ shows in particular how to compute $\mathcal{F}(M)$ from $\mathcal{R}(M)$.

acteristic $\mathcal{F}(M)$ of M is a symmetric function which is the commutative image of $\mathcal{R}(M)$.

Proof — It suffices to prove the result when $M = \mathbf{M}_I$. In this case,

$$\mathcal{F}(\mathbf{M}_I) = \sum_{J \vDash N} c_{IJ}\, F_J \ ,$$

where the Cartan invariant c_{IJ} is equal to the number of permutations σ of $\mathfrak{S}_N$ such that $D(\sigma) = I$ and $D(\sigma^{-1}) = J$ (see [1]). On the other hand, by a formula of Gessel [10], we have $c_{IJ} = (r_I, r_J) = \langle r_I\, , \, R_J \rangle$, where $(\cdot,\cdot)$ denotes the usual scalar product of commutative symmetric functions (see [28]) and where r_I is the commutative image of the ribbon Schur function R_I. Using the fact that the quasi-ribbons F_I and noncommutative ribbon Schur functions R_J are dual bases, it follows that $\mathcal{F}(\mathbf{M}_I) = r_I$. $\square$

The induction from a tensor product of projective modules is described by the product of noncommutative symmetric functions.

Proposition 5.10 *Let $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_s)$ be compositions of N and M. Then,*

$$\mathcal{R}(\mathbf{M}_I \otimes \mathbf{M}_J \uparrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)}) = R_I\, R_J = R_{I \cdot J} + R_{I \triangleright J} \ , \tag{9}$$

where we set $I \cdot J = (i_1, \ldots, i_r, j_1, \ldots, j_s)$ and $I \triangleright J = (i_1, \ldots, i_{r-1}, i_r + j_1, j_2, \ldots, j_s)$.

Proof — The formula for the product of two noncommutative ribbon Schur functions is proved in [9], and we just have to show that

$$\mathbf{M}_I \otimes \mathbf{M}_J \uparrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)} \simeq \ \mathbf{M}_{I \cdot J} \oplus \mathbf{M}_{I \triangleright J} \ .$$

Using the duality between simple modules and indecomposable projective modules, we obtain

$$\mathbf{M}_I \otimes \mathbf{M}_J \uparrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)} \simeq \bigoplus_{K \vDash N+M} \dim \mathrm{Hom}_{\,H_{N+M}(0)} \left(\mathbf{M}_I \otimes \mathbf{M}_J \uparrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)} , \, \mathbf{M}_K \right) \ \mathbf{M}_K \ .$$

By Frobenius reciprocity, we have

$$\mathbf{M}_I \otimes \mathbf{M}_J \uparrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)} \simeq \bigoplus_{K \vDash N+M} \dim \mathrm{Hom}_{\,H_N(0) \otimes H_M(0)} \left(\mathbf{M}_I \otimes \mathbf{M}_J \, , \, \mathbf{C}_K \downarrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)} \right) \mathbf{M}_K \ .$$

Observe now that the description of the family $(\mathbf{M}_I)$ given in Section 5.3 implies that

$$\dim \mathrm{Hom}_{\,H_N(0)} \, (\mathbf{M}_J, \mathbf{C}_I) = \begin{cases} 1 & \text{if } I = J \ , \\ 0 & \text{if } I \neq J \ , \end{cases}$$

so that

$$\dim \mathrm{Hom}_{\,H_N(0) \otimes H_M(0)} \left(\mathbf{M}_I \otimes \mathbf{M}_J, \mathbf{C}_K \downarrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)} \right)$$

is equal to 0 if $D(K) \cap [1, N] \neq D(I)$ or $D(K) \cap [N{+}1, N{+}M] \neq N{+}D(J)$ and equal to 1 when $D(K) \cap [1, N] = D(I)$ and $D(K) \cap [N{+}1, N{+}M] = N{+}D(J)$, i.e. when $K = I \cdot J$ or $K = I \triangleleft J$ as desired. $\square$

Thus, we have the following interpretation of the algebra of noncommutative symmetric functions.

Corollary 5.11 *The characteristic map $\mathcal{R}$ is an isomorphism between the Grothendieck ring $\mathcal{K}$ and the $\mathbb{Z}$-algebra of noncommutative functions.*

6.1 The character of an $A_q(n)$-comodule

Let M be a finite dimensional $A_q(n)$-comodule with coaction δ. Let $(m_i)_{i=1,m}$ be a basis of M. There exists elements $(a(i,j))_{1\le i,j\le m}$ of $A_q(n)$ such that

$$\delta(m_i) = \sum_{j=1}^{m} a(i,j) \otimes m_j$$

for $i \in [1,m]$. The element

$$\sum_{i=1}^{m} a(i,i)$$

of $A_q(n)$ is independent of the choice of the basis (m_i). It will be denoted by $\chi(M)$ and called the *character* of the $A_q(n)$-comodule M.

Proposition 6.1 *Let M, N, M', M'' be $A_q(n)$-comodules.*

1. *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence, $\chi(M) = \chi(M') + \chi(M'')$;*
2. *$\chi(M \otimes N) = \chi(M)\,\chi(N)$;*
3. *if $M \simeq N$, then $\chi(M) = \chi(N)$.*

It happens that for generic values of q, the character of an $A_q(n)$-comodule is always an element of the quantum diagonal algebra.

Theorem 6.2 *Let q be an indeterminate and let M be an $A_q(n)$-comodule. Then the character $\chi(M)$ belongs to the diagonal algebra $\Delta_q(n)$.*

Proof — The basic observation is the following lemma.

Lemma 6.3 *The quantum determinant of $A_q(n)$ can be expressed by means of q-commutators as follows:*

$$\begin{vmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix}_q \stackrel{\text{def}}{=} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\, x_{1\sigma(1)} \dots x_{n\sigma(n)}$$

$$= \frac{1}{(1-q)^{n-1}}\, [x_{nn}, [\dots, [x_{22}, x_{11}]_q \dots]_q]_q$$

where $[P,Q]_q = P\,Q - q\,Q\,P$.

Proof of the lemma – Observe first that the lemma is equivalent to the identity

$$x_{nn} \begin{vmatrix} x_{11} & \dots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-1,1} & \dots & x_{n-1,n-1} \end{vmatrix} - q \begin{vmatrix} x_{11} & \dots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-1,1} & \dots & x_{n-1,n-1} \end{vmatrix} x_{nn} = (1-q) \begin{vmatrix} x_{11} & \dots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{vmatrix} .$$

Using the tensor notation of Section 4.3, this can be rewritten as

$$(1-q)\Big(\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\, e_{12\dots n} \otimes e_\sigma^*\Big) + q\Big(\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(n)=n}} \varepsilon(\sigma)\, e_{12\dots n} \otimes e_\sigma^*\Big) = \sum_{\sigma\in\mathfrak{S}_{n-1}} \varepsilon(\sigma)\, e_{n12\dots n-1} \otimes e_{n\sigma}^* \,,$$

$$(-1)^{n-1}\Big(\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(1)=n}} \varepsilon(\sigma)\, e^*_\sigma\Big)\cdot T_1\, T_2\, \dots\, T_{n-1} = (1-q)\Big(\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\, e^*_\sigma\Big) + q\Big(\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(n)=n}} \varepsilon(\sigma)\, e^*_\sigma\Big)\,.$$

This last formula is now easily proved by induction on n. □

As a consequence of Lemma 6.3, we obtain that the character of the r-th exterior power $\Lambda^r_q(V)$ of V (cf. [4]) is equal to

$$\chi(\Lambda^r_q(V)) = \Lambda_r(q;\Delta) \overset{\text{def}}{=} \frac{1}{(1-q)^{n-1}}\Big(\sum_{1\le i_1<\dots<i_r\le n} [x_{i_r i_r}, [\dots, [x_{i_2 i_2}, x_{i_1 i_1}]_q \dots]_q]_q\Big)$$

where $\Delta = \{\, x_{11}, \dots, x_{nn}\,\}$. Let now $\lambda = (1^{l_1}, \dots, n^{l_n})$ be a partition of n. It follows then from Proposition 6.1 that the character of the comodule

$$M_{\lambda,q} = V^{\otimes l_1} \otimes \Lambda^2_q(V)^{\otimes l_2} \otimes \dots \otimes \Lambda^n_q(V)^{\otimes l_n}$$

is also in the diagonal algebra $\Delta_q(n)$.

On the other hand, it has been shown by Dipper and Donkin [4] that one can associate with every partition λ of n an irreducible $A_q(n)$-comodule $L_{\lambda,q}$ and that the family $(L_{\lambda,q})_{\lambda\vdash n}$ forms a complete system of irreducible $A_q(n)$-comodules. They also proved that for an appropriate ordering $<$ on partitions of n, the products of exterior powers decompose as

$$M_{\lambda,q} \simeq L_{\lambda,q} \;\oplus\; \bigoplus_{\mu<\lambda} a_\mu\;\, L_{\mu,q}\;.$$

Applying Proposition 6.1, we see that the matrix giving the decomposition of $(\chi(M_{\lambda,q}))_{\lambda\vdash n}$ on $(\chi(L_{\lambda,q}))_{\lambda\vdash n}$ is unitriangular. It follows that the character $\chi(L_{\lambda,q})$ is a linear combination of elements of the family $(\chi(M_{\lambda,q}))_{\lambda\vdash n}$. Hence $\chi(L_{\lambda,q}) \in \Delta_q(n)$. □

Note 6.4 The commutative image of $\chi(M)$ is the formal commutative character introduced in [4]. But the formal character of $L_{\lambda,q}$ is the Schur function S_λ. Thus the characters (in our sense) of the irreducible $A_q(n)$-comodules are quantum analogues of Schur functions.

Note 6.5 Let $Char_{\mathbb{Z}}(q;n)$ denote the $\mathbb{Z}$-lattice of $\Delta_n(q)$ spanned by characters of $A_q(n)$-comodules. The proof of Theorem 6.2 shows that $Char_{\mathbb{Z}}(q;n)$ is the subring of $\Delta_n(q)$ generated by the n quantum elementary functions $\Lambda_r(q;\Delta)$ with $1\le r\le n$. Moreover since the composition series of the two $A_n(q)$-comodules $\Lambda^r_q(V)\otimes\Lambda^s_q(V)$ and $\Lambda^s_q(V)\otimes\Lambda^r_q(V)$ are the same, these quantum elementary functions commute. It follows that $Char_{\mathbb{Z}}(q;n)$ is a commutative $\mathbb{Z}$-algebra isomorphic to the algebra of symmetric functions in n variables.

Note 6.6 Although Theorem 6.2 has been stated for generic values of q, it is not difficult to see that it holds for $q\in\mathbb{C}-\{\,0,1\,\}$. In the usual commutative case (i.e. $q=1$), the result becomes false. On the other hand we conjecture that it still holds for $q=0$ (cf. Conjecture 6.13).

6.2 A family of irreducible $A_0(n)$-comodules

Let I be a composition of N. The element $\eta_I = T_{\omega(\overline{I})}\,\square_{\alpha(I^\sim)}$ of $H_N(0)$ generates the one-dimensional left $H_N(0)$-module $\mathbf{C}_I$. One can use it to construct the $A_0(n)$-comodule

$$\mathbf{D}_I = V^{\otimes N}\cdot\eta_I\;.$$

Let A be a noncommutative totally ordered alphabet and let I be a composition. We denote by $F_I(A)$ the sum of all quasi-ribbon words of shape I. According to a result of Gessel [10], the commutative image of $F_I(A)$ is the quasi-symmetric function F_I.

$$\begin{array}{|c|c|}\hline a & a \\ \hline \multicolumn{1}{c|}{} & b \\ \cline{2-2}\end{array} \qquad \begin{array}{|c|c|}\hline a & a \\ \hline \multicolumn{1}{c|}{} & c \\ \cline{2-2}\end{array} \qquad \begin{array}{|c|c|}\hline a & b \\ \hline \multicolumn{1}{c|}{} & c \\ \cline{2-2}\end{array} \qquad \begin{array}{|c|c|}\hline b & b \\ \hline \multicolumn{1}{c|}{} & c \\ \cline{2-2}\end{array}$$

Thus $F_{21}(a,b,c) = aba + aca + acb + bcb$. The commutative image of $F_{21}(a,b,c)$ is clearly equal to $M_{21} + M_{111} = F_{21}$, as desired.

Proposition 6.8 *Let I be a composition of N. Then $\chi(\mathbf{D}_I) = F_I(x_{11}, \ldots, x_{nn})$.*

Proof — Let $QR(I)$ be the set of all quasi-ribbon words of shape I. We associate with every word $w = a_{k_1} \ldots a_{k_n}$ of A^* the tensor $\mathbf{w} = a_{k_1} \otimes \ldots \otimes a_{k_n}$ of $V^{\otimes n}$.

Lemma 6.9 *The family $(\mathbf{w} \cdot \eta_I)_{w \in QR(I)}$ is a linear basis of the $A_0(n)$-comodule $\mathbf{D}_I$.*

Proof of the lemma – Define the natural reading $n(r)$ of a quasi-ribbon tableau r of shape I as the word obtained by reading r from left to right. If w is the quasi-ribbon word associated with r, we also denote by $n(w)$ the natural reading of r. For example, $n(ababdc) = aabbcd$ is the natural reading of the quasi-ribbon tableau

$$\begin{array}{|c|c|c|c|}\cline{1-2} a & a & \multicolumn{2}{c}{} \\ \hline \multicolumn{1}{c|}{} & b & b & c \\ \cline{2-4} \multicolumn{3}{c|}{} & d \\ \cline{4-4}\end{array} \quad .$$

Let now $i \in D(I)$. By definition of η_I, one has $T_i\,\eta_I = -\eta_I$. Hence, we get

$$\mathbf{v} \cdot \eta_I = -\,(\mathbf{v} \cdot T_i) \cdot \eta_I = \begin{cases} 0 & \text{if } k_i = k_{i+1} \ , \\ -\,\mathbf{v}^{\sigma_i} \cdot \eta_I & \text{if } k_i < k_{i+1} \ , \end{cases}$$

for every $\mathbf{v} = a_{k_1} \otimes \ldots \otimes a_{k_N} \in V^{\otimes N}$. In particular,

$$\mathbf{v} \cdot \eta_I = -\,\mathbf{v}^{\sigma_i} \cdot \eta_I \tag{10}$$

when $k_i \neq k_{i+1}$. Suppose now that $i \notin D(I)$. Then $T_i\,\eta_I = 0$. Thus we can write

$$\mathbf{v} \cdot \eta_I = (\mathbf{v}^{\sigma_i} \cdot T_i) \cdot \eta_I = \mathbf{v}^{\sigma_i} \cdot T_i\,\eta_I = 0$$

for every $\mathbf{v} = a_{k_1} \otimes \ldots \otimes a_{k_N} \in V^{\otimes N}$ such that $k_i > k_{i+1}$. It follows that the family of all tensors of the form $(a_{k_1} \otimes \ldots \otimes a_{k_N}) \cdot \eta_I$ with $k_i \leq k_{i+1}$ when $i \notin D(I)$ and $k_i < k_{i+1}$ when $i \in D(I)$ spans the comodule $\mathbf{D}_I$. In other words, we get a generating family of $\mathbf{D}_I$ by taking the set $\mathcal{R}$ formed of all $\mathbf{w} \cdot \eta_I$ where w runs over the natural readings of all quasi-ribbon tableaux of shape I. Moreover it is easy to see that these elements are not zero.

Now, there is at most one increasing word of a given evaluation which can be the natural reading of some quasi-ribbon tableau of shape I. It follows that $\mathcal{R}$ is a linear basis of $\mathbf{D}_I$. Finally, formula (10) shows that $\mathbf{w} \cdot \eta_I = \pm\, \mathbf{n(w)} \cdot \eta_I$ for a quasi-ribbon word w of shape I. $\square$

We are now in position to compute $\chi(\mathbf{D}_I)$. In the notation of Section 4.3,

$$\delta(\mathbf{w}) = \sum_{|u|=|w|} (w \otimes u^*) \otimes \mathbf{u}$$

for every $w \in A^*$. Hence, according to Proposition 4.2,

$$\delta(\mathbf{w} \cdot \eta_I) = \sum_{|u|=|w|} (w \otimes u^*) \otimes \mathbf{u} \cdot \eta_I \ ,$$

$$\chi(D_I) = \sum_{w \in QR(I)} \left(\sum_{\substack{|u|=|w| \\ \mathbf{u}\cdot\eta_I = \mathbf{w}\cdot\eta_I}} w \otimes u^* \right) . \tag{11}$$

Let now w be a quasi-ribbon word of shape I. Let also $u = a_{k_1} \dots a_{k_N}$ be a word distinct from w such that $\mathbf{w} \cdot \eta_I = \mathbf{u} \cdot \eta_I$. Let $r(u)$ be the ribbon diagram of shape I obtained by filling the boxes from left to right by the letters of u. Let then $r'(u)$ be the quasi-ribbon tableau of shape I obtained from $r(u)$ by sorting all columns in increasing order from top to bottom. Let us finally denote by $v(u)$ the word obtained by reading from left to right the letters of $r'(u)$. Using again the arguments of the proof of Lemma 6.9, we see that $\mathbf{v(u)} \cdot \eta_I = 0$ if $v(u)$ is not the natural reading of a quasi-ribbon tableau of shape I. It follows that the alphabets of all columns of $r(u)$ and $r(w)$ must coincide. Since $u \neq w$, there must exist integers $i < j$ and $k < l$ such that

$$w \otimes u^* = \begin{bmatrix} \dots & l & k & \dots \\ \dots & i & j & \dots \end{bmatrix}$$

which is therefore equal to 0. Hence, we have

$$\sum_{\substack{|u|=|w| \\ \mathbf{u}\cdot\eta_I = \mathbf{w}\cdot\eta_I}} w \otimes u^* = w \otimes w^* \ .$$

Going back to formula (11), we see that

$$\chi(\mathbf{D}_I) = \sum_{w \in QR(I)} w \otimes w^* = F_I(x_{11}, \dots, x_{nn}) \ . \qquad \square$$

Note 6.10 The same argument as in Note 6.5 shows that the noncommutative quasi-ribbon functions $F_I(A)$ span a commutative subalgebra of the hypoplactic algebra $HPl(A)$, isomorphic to the algebra of quasi-symmetric functions over a commutative alphabet of the same cardinality as A. This property can in fact also be proved in a purely combinatorial way.

Example 6.11 Let $n = 3$, $N = 4$ and $I = (3,1)$. Then $\eta_{31} = T_3\, T_2\, T_1\, (1+T_2)\,(1+T_3)\,(1+T_2)$ and $\mathbf{D}_{31} = V^{\otimes 4} \cdot \eta_{31}$. By computing the images under η_{31} of the canonical basis vectors of $V^{\otimes 4}$, one gets

$$\mathbf{D}_{31} = \mathbb{C}\, a_1a_2a_3a_2 \cdot \eta_{31} \oplus \mathbb{C}\, a_2a_2a_3a_2 \cdot \eta_{31} \oplus \mathbb{C}\, a_1a_1a_3a_2 \cdot \eta_{31} \oplus \mathbb{C}\, a_1a_1a_3a_1 \cdot \eta_{31} \oplus \mathbb{C}\, a_1a_1a_2a_1 \cdot \eta_{31} \ .$$

Thus,

$$\begin{aligned} \chi(\mathbf{D}_{31}) &= x_{22}x_{11}x_{11}x_{11} + +x_{33}x_{11}x_{11}x_{11} + x_{33}x_{22}x_{22}x_{22} + x_{33}x_{11}x_{11}x_{22} + x_{33}x_{11}x_{22}x_{22} \\ &= x_{11}x_{11}x_{22}x_{11} + +x_{11}x_{11}x_{33}x_{11} + x_{22}x_{22}x_{33}x_{22} + x_{11}x_{11}x_{33}x_{22} + x_{11}x_{22}x_{33}x_{22} \ . \end{aligned}$$

This last expression is exactly the sum of the quasi-ribbons words associated with the five quasi-ribbon tableaux

$$\begin{array}{ccc} 1 & 1 & 1 \\ & & 2 \end{array} \qquad \begin{array}{ccc} 1 & 1 & 1 \\ & & 3 \end{array} \qquad \begin{array}{ccc} 1 & 1 & 2 \\ & & 3 \end{array} \qquad \begin{array}{ccc} 1 & 2 & 2 \\ & & 3 \end{array} \qquad \begin{array}{ccc} 2 & 2 & 2 \\ & & 3 \end{array}$$

Hence $\chi(\mathbf{D}_{31}) = F_{31}(x_{11}, x_{22}, x_{33})$ as desired.

Proposition 6.12 *The $\mathbf{D}_I$ are irreducible, pairwise non-isomorphic $A_0(n)$-comodules.*

to prove that these comodules are irreducible. Let I be a composition of N and let M be a nonzero subcomodule of $\mathbf{D}_I$. According to Lemma 6.9, there exists a family R of quasi-ribbon words of shape I and a family $(m_w)_{w\in R}$ of nonzero complex numbers such that

$$m = \sum_{w\in R} m_w \, \mathbf{w} \cdot \eta_I \ \in M \ .$$

Using the tensor formalism of Section 4.3, it follows that

$$\delta(m) = \sum_{|u|=N} \Big(\sum_{w\in R} m_w \, w \otimes u^* \Big) \otimes \mathbf{u} \cdot \eta_I \ \in A_0(n) \otimes M \ .$$

As there is at most one quasi-ribbon word of shape I and of a given evaluation, we deduce by homogeneity with respect to the first component of $A_0(n)$ that

$$\delta(w) = \sum_{|u|=N} (w \otimes u^*) \otimes \mathbf{u} \cdot \eta_I \ \in A_0(n) \otimes M$$

for every $w \in R$. Let now w be an arbitrary quasi-ribbon word of R and let $u = a_{k_1} \dots u_{k_N}$ be a word of length N. Note that $w \otimes u^* = 0$ if $k_i \leq k_{i+1}$ when $i \in D(I)$. On the other hand, $\mathbf{u} \cdot \eta_I = 0$ if $k_i > k_{i+1}$ and $i \notin D(I)$ according to the proof of Lemma 6.9. Hence

$$\delta(w) = \sum_{u\in QR(I)} (w \otimes u^*) \otimes \mathbf{u} \cdot \eta_I \ .$$

The monomials $w \otimes u^*$, where u runs over all quasi-ribbon words of shape I, are nonzero and pairwise distincts elements of $A_0(n)$. Since $\delta(w) \in A_0(n) \otimes M$, all the tensors $\mathbf{u} \cdot \eta_I$ are M. According to Lemma 6.9, this shows that $M = \mathbf{D}_I$. □

Conjecture 6.13 *The family $(\mathbf{D}_I)_I$ (where I runs through all compositions) is a complete system of irreducible $A_0(n)$-comodules.*

Note 6.14 Conjecture 6.13 would imply that the character of every $A_0(n)$-comodule is an element of $\Delta_0(n)$, showing therefore that Theorem 6.2 is still valid when $q = 0$. Moreover according to Note 6.10, it would also imply that the character ring $Char_{\mathbb{Z}}(0; n)$ is isomorphic to the $\mathbb{Z}$-algebra of quasi-symmetric functions over an n-letter alphabet. □

6.3 Another family of $A_0(n)$-comodules

Let I be a composition of N. The element $\nu_I = T_{\alpha(I)} \square_{\alpha(\bar{I}^{\sim})}$ of the 0-Hecke algebra generates the indecomposable projective left $H_n(0)$-module $\mathbf{M}_I$. One can also use to construct the $A_0(n)$-comodule $\mathbf{N}_I$ defined by

$$\mathbf{N}_I = V^{\otimes N} \cdot \nu_I \ .$$

A word will be said to be of *ribbon shape* I (where I is a composition) if it can be obtained by reading from left to right and from top to bottom the columns of a skew Young tableau of ribbon shape I. We denote by $R_I(A)$ the sum of all words of A^* of ribbon shape I.

Proposition 6.15 *Let I be a composition of N. Then $\chi(\mathbf{N}_I) = R_I(x_{11}, \dots, x_{nn})$.*

Proof — We use the same notation as in the proof of Proposition 6.8. Let also $R(I)$ be the set of all words of ribbon shape I.

Proof of the lemma – Note that $T_i\,\nu_I = -\nu_I$ for $i \in D(I)$. It follows that

$$\mathbf{v}\cdot\nu_I = -(\mathbf{v}\cdot T_i)\,\nu_I = \begin{cases} 0 & \text{if } k_i = k_{i+1}\ , \\ -\mathbf{v}^{\sigma_i}\cdot\nu_I & \text{if } k_i < k_{i+1}\ , \\ \mathbf{v}\cdot\nu_I & \text{if } k_i > k_{i+1}\ , \end{cases}$$

for $i \in D(I)$ and $\mathbf{v} = a_{k_1}\otimes\ldots\otimes a_{k_N} \in V^{\otimes N}$. Hence we can rewrite (up to a sign) every $\mathbf{v}\cdot\nu_I$ in such a way that $k_i > k_{i+1}$ for $i \in D(I)$. The structure of the right action of $H_N(0)$ on $V^{\otimes N}$ implies that such an element is equal to $\mathbf{w}\cdot\Box_{\alpha(\overline{I}^\sim)}$ where we still have $w = a_{k_1}\,\ldots\,a_{k_N}$ with $k_i > k_{i+1}$ for $i \in D(I)$. Let now $i \notin D(I)$. Then $\Box_{\alpha(\overline{I}^\sim)} = \Box_i\,\Box_{\alpha(\overline{I}^\sim)}$. Hence

$$\mathbf{w}\cdot\Box_{\alpha(\overline{I}^\sim)} = \mathbf{w}\cdot\Box_i\,\Box_{\alpha(\overline{I}^\sim)} = \mathbf{w}^{\sigma_i}\cdot T_i\,\Box_i\,\Box_{\alpha(\overline{I}^\sim)} = 0$$

when $k_i > k_{i+1}$. Every $\mathbf{v}\cdot\nu_I$ can therefore be rewritten as $\pm\mathbf{w}\cdot\Box_{\alpha(\overline{I}^\sim)}$ where $w \in R(I)$. In other words, the family $(\mathbf{w}\cdot\Box_{\alpha(\overline{I}^\sim)})_{w\in R(I)}$ spans $\mathbf{N}_I$.

Now, it follows from (8) that

$$V^{\otimes N} = \sum_{I\vdash N}\mathbf{N}_I\ . \tag{12}$$

Since any word of A^N has a unique ribbon shape, we deduce that

$$\sum_{I\vdash N}|R(\overline{I})| = \dim\,V^{\otimes N} \le \sum_{I\vdash N}\dim\,\mathbf{N}_I \le \sum_{I\vdash N}|R(\overline{I})|$$

from which we get that dim $\mathbf{N}_I$ is equal to the number of words of ribbon shape I. $\Box$

This argument also shows that decomposition (12) is in fact a direct sum. Arguing as in the proof of Proposition 6.8, we see that

$$\delta(w\cdot\Box_{\alpha(I^\sim)}) = \sum_{u\in R(\overline{I})}(w\otimes u^*)\,\mathbf{u}\cdot\Box_{\alpha(I^\sim)}$$

for $w \in R(I)$, whence the theorem. $\Box$

Example 6.17 Let $n = 3$, $N = 4$ and $I = (1,1,2)$. Then $\nu_{112} = T_1\,T_2\,T_1\,(1+T_3)$ and $\mathbf{N}_{112} = V^{\otimes 4}\cdot\nu_{112}$. By computing the action of ν_{211} on the standard basis of $V^{\otimes 4}$, one gets

$$\mathbf{N}_{112} = \mathbb{C}\,a_3\,a_2\,a_1\,a_1\cdot\nu_{112}\ \oplus\ \mathbb{C}\,a_3\,a_2\,a_1\,a_2\cdot\nu_{112}\ \oplus\ \mathbb{C}\,a_3\,a_2\,a_1\,a_3\cdot\nu_{112}\ .$$

Then,

$$\chi(\mathbf{N}_{112}) = x_{33}\,x_{22}\,x_{11}\,x_{11} + x_{33}\,x_{22}\,x_{11}\,x_{22} + x_{33}\,x_{22}\,x_{11}\,x_{33}\ .$$

This expression is the sum of the ribbon words associated with the 3 ribbon tableaux

$$\begin{array}{|c|c}\hline 3 & \\ \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \qquad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

and $\chi(\mathbf{N}_{112}) = R_{112}(x_{11}, x_{22}, x_{33})$ as desired. $\Box$

Note 6.18 Using the same kind of argument as in Section 6.2, one can prove that $\mathbf{N}_I$ is an indecomposable $A_n(q)$-comodule. $\Box$

In the classical case (corresponding to $q = 1$), the Robinson-Schensted correspondence is the combinatorial counterpart of the decomposition of $V^{\otimes N}$ into $GL_n(\mathbb{C}) \times \mathfrak{S}_N$-bimodules. On the other hand, for $q = 0$, there are two natural ways of decomposing $V^{\otimes N}$ into $A_0(n) \times H_N(0)$-bicomodules. This leads to two different Robinson-Schensted type correspondences, involving here ribbon and quasi-ribbon diagrams.

7.1 A first Robinson-Schensted type correspondence

The first combinatorial algorithm corresponds to the decomposition

$$V^{\otimes N} = \bigoplus_{I \vDash N} \mathbf{N}_I \tag{13}$$

(cf. the proof of Proposition 6.15). Recall that any right $H_N(0)$-submodule of $V^{\otimes N}$ can also be regarded as a left module, the action being given by $\mathbf{v}T_i = -\Box_i\,\mathbf{v}$. It follows then from Lemma 6.16 that $\mathbf{N}_I$ is a left $H_N(0)$-module whose all composition factors are equal to $\mathbf{C}_{\overline{I}^{\sim}}$. This observation gives us a basis of $V^{\otimes N}$ indexed by pairs (r, qr) where r is a word of ribbon shape I and where qr is the (unique) standard quasi-ribbon word of shape $\overline{I}^{\sim}$. The corresponding Robinson-Schensted map is therefore trivial. It just associates to a word w its ribbon diagram. It can clearly be recursively defined by an insertion process as follows.

Let r be the ribbon diagram of w, let x be the letter which is in the last box of r and let $a \in A$. The ribbon diagram of wa is then obtained from r by glueing a at the end of the last row of r if $x \le a$ or under the last box of the last row of r if $a < x$. For exemple, with $w = baccb$, we have

$$\emptyset \longrightarrow \boxed{\mathrm{b}} \longrightarrow \begin{matrix}\boxed{\mathrm{b}}\\ \boxed{\mathrm{a}}\end{matrix} \longrightarrow \begin{matrix}\boxed{\mathrm{b}} & \\ \boxed{\mathrm{a}} & \boxed{\mathrm{c}}\end{matrix} \longrightarrow \begin{matrix}\boxed{\mathrm{b}} & & \\ \boxed{\mathrm{a}} & \boxed{\mathrm{c}} & \boxed{\mathrm{c}}\end{matrix} \longrightarrow \begin{matrix}\boxed{\mathrm{b}} & & \\ \boxed{\mathrm{a}} & \boxed{\mathrm{c}} & \boxed{\mathrm{c}} \\ & & \boxed{\mathrm{b}}\end{matrix}$$

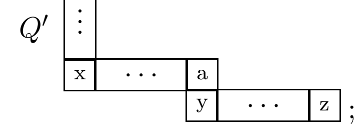
This construction is clearly bijective (the standard quasi-ribbon does not bring here any supplementary information).

7.2 A second Robinson-Schensted type correspondence

The second Robinson-Schensted type algorithm is related to the composition factors of $V^{\otimes N}$. Using Lemmas 6.9 and 6.16, one can see that these compositions factors are exactly the comodules $\mathbf{D}_I$ each of them occuring $|QR(I)|$ times. But $\mathbf{D}_I$ considered as a left $H_N(0)$-module is isomorphic to $\mathbf{M}_I$. It follows that there exists a basis of $V^{\otimes N}$ indexed by pairs (Q, R) where Q is a quasi-ribbon word of shape I and where R is a standard ribbon word of the same shape. The corresponding Robinson-Schensted type algorithm which associates to each word $w \in A^*$ the pair (Q, R) is described below.

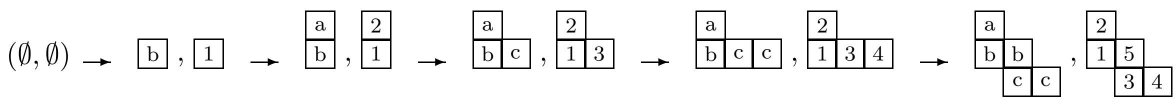
Let Q be a quasi-ribbon diagram and let $a \in A$. Let Q' be the diagram obtained from Q by deleting its last row and let x (resp. z) be the first (resp. last) letter of the last row of Q. The result $\mathcal{Q}$ of the insertion of a in Q is defined by the following rules:

- if $z \le a$, $\mathcal{Q}$ is obtained by adding a box containing a at the end of the last row of Q
- if $x \le a < z$, let y be the first letter of the last row of Q which is strictly greater than a. The quasi-ribbon diagram $\mathcal{Q}$ is then



- if $a < x$, $\mathcal{Q}$ is obtained by inserting a in Q' and glueing under the quasi-ribbon obtained in this way the last row of Q.

Let $w = a_1 \ldots a_n$ be a word of length n. The pair (Q, R) associated with w can be defined as follows. The quasi-ribbon diagram Q is obtained by inserting the letters of w (from left to right), starting from an empty diagram. The standard ribbon diagram R is iteratively constructed by putting at each step $i \in [1, n]$ of the algorithm the number i in the box that contains at this moment in Q the letter a_i inserted at this step. Let us illustrate again this correspondence on $w = baccb$.



The correspondence $w \to (Q, R)$ is clearly a bijection. In fact, the quasi-ribbon diagram Q associated with w is of shape $C(\sigma^{-1})$ where $\sigma = std(w)$. Going back to Proposition 4.21, this gives the following property, which is the quasi-ribbon version of Knuth's theorem [20].

Proposition 7.1 *Let $u, v \in A^*$. Then, u and v correspond to the same quasi-ribbon Q under the second algorithm iff $u \equiv v$ with respect to the hypoplactic congruence.*

In other words, the hypoplactic relations play, for quasi-ribbons, the same rôle as the plactic relations for Young tableaux.

References

[1] R.W. Carter, *Representation theory of the* 0*-Hecke algebra*, J. Algebra **15** (1986), 89–103.

[2] I.V. Cherednik, *An analogue of the character formula for Hecke algebras*, Funct. Anal. Appl. **21** (1987), 172–174.

[3] E. Date, M. Jimbo and T. Miwa, *Representations of $U_q(gl_n)$ at $q = 0$ and the Robinson-Schensted correspondence*, in "Physics and mathematics of strings, Memorial volume of V. Knizhnik", L. Brink, D. Friedan, A.M. Polyakov eds., World Scientific, 1990.

[4] R. Dipper and S. Donkin, *Quantum GL_n*, Proc. London Math. Soc. **63** (1991), 165–211.

[5] G. Duchamp, D. Krob, A. Lascoux, B. Leclerc, T. Scharf and J.-Y. Thibon, *Euler-Poincaré characteristic and polynomial representations of Iwahori-Hecke algebras*, Publ. RIMS, Kyoto Univ., **31** (1995), 179–201.

[6] G. Duchamp, A. Klyachko, D. Krob and J.-Y. Thibon, *Noncommutative symmetric functions III: Deformations of Cauchy and convolution algebras*, LITP preprint 96/08, Paris, 1996.

fonctions symétriques non-commutatives, et algèbres de Hecke à $q = 0$, C.R. Acad. Sci. Paris **322** (1996), 107–112.

[8] L.D. Faddeev, N.Y. Reshetikin and L.A. Takhtadzhyan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. **1** (1990), 193–225.

[9] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh and J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. in Math. **112** (1995), 218–348.

[10] I. Gessel, *Multipartite P-partitions and inner product of skew Schur functions*, Contemp. Math. **34** (1984), 289–301.

[11] J. A. Green, *Polynomial representations of GL_n*, Springer Lecture Notes in Math. **830**, 1980.

[12] N. Hoefsmit, *Representations of Hecke algebras of finite groups with BN-pairs of classical types*, Thesis, University of British Columbia, 1974.

[13] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1990.

[14] N. Iwahori, *On the structure of the Hecke ring of a Chevalley group over a finite field*, J. Fac. Sci. Univ. Tokyo Sect. I **10** (1964), 215–236.

[15] M. Jimbo, *A q-analogue of $U(gl(N+1))$, Hecke algebra and the Yang-Baxter equation*, Lett. Math. Phys. **11** (1986) 247–252.

[16] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465-516.

[17] M. Kashiwara and T. Nakashima, *Crystal graphs for representations of the q-analogue of classical Lie algebras*, J. Algebra **165** (1994), 295-345.

[18] S.V. Kerov and A.M. Vershik, *Characters and realizations of representations of an infinite dimensional Hecke algebra, and knot invariants*, Soviet Math. Dokl. **38** (1989) 134–137.

[19] R.C. King and B.G. Wybourne, *Representations and trace of the Hecke algebra $H_n(q)$ of type A_{n-1}*, J. Math. Phys. **33** (1992), 4–14.

[20] D.E. Knuth, *Permutations, matrices and generalized Young tableaux*, Pacific J. Math. **34** (1970), 709–727.

[21] D. Krob, B. Leclerc and J.-Y. Thibon, *Noncommutative symmetric functions II: Transformations of alphabets*, LITP preprint 95-07, 1995.

[22] A. Lascoux, *Anneau de Grothendieck de la variété de drapeaux*, in *The Grothendieck Festschrift*, P. Cartier et al. eds, Birkhäuser, 1990, 1-34.

[23] A. Lascoux, B. Leclerc and J.-Y. Thibon, *Une conjecture pour le calcul des matrices de décomposition des algèbres de Hecke de type A aux racines de l'unité*, C.R. Acad. Sci. Paris, Ser. A, **321**, (1995), 511-516.

[24] A. Lascoux, B. Leclerc and J.-Y. Thibon, *Crystal graphs and q-analogues of weight multiplicities for the root system A_n*, Lett. Math. Phys., **35**, (1995), 359–374.

109 (1981), 129–156.

[26] B. LECLERC and J.-Y. THIBON, *The Robinson-Schensted correspondence, crystal bases, and the quantum straightening at* $q = 0$, The Electronic J. of Combinatorics **3** (1996), # 11.

[27] P. LITTELMAN, *A plactic algebra for each semisimple Lie algebra*, preprint, 1994.

[28] I.G. MACDONALD, *Symmetric functions and Hall polynomials*, Oxford, 1979; 2nd ed. 1995.

[29] C. MALVENUTO and C. REUTENAUER, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra **177** (1995), 967-982.

[30] C. MALVENUTO and C. REUTENAUER, *Plethysm and conjugation of quasi-symmetric functions*, Preprint, 1995.

[31] P.N. NORTON, 0-*Hecke algebras*, J. Austral. Math. Soc. Ser. A **27** (1979), 337–357.

[32] A. RAM, *A Frobenius formula for the characters of the Hecke algebras*, Invent. Math. **106** (1991), 461–488.

[33] C. REUTENAUER, *Free Lie algebras*, Oxford, 1993.

[34] C. SCHENSTED, *Longest increasing and decreasing subsequences*, Canad. J. Math. **13** (1961), 179-191.

[35] L. SOLOMON, *A Mackey formula in the group ring of a Coxeter group*, J. Algebra **41** (1976), 255–268.

[36] A.J. STARKEY, *Characters of the generic Hecke algebra of a system of BN-pairs*, Thesis, University of Warwick, 1975.

[37] K. UENO and Y. SHIBUKAWA, *Character table of Hecke algebra of type* A_{N+1} *and representations of the quantum group* $U_q(gl_{n+1})$, in *Infinite Analysis* Part B, A. TSUCHIYA, T. EGUCHI and M. JIMBO eds., World Scientific, Singapore, 1992, 977-984.