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Abstract: The problem under consideration is the controllability of a wide class of convolution Volterra systems, namely the class of "diffusive" systems, for which there exists an input-output state realization whose state evolves in the so-called diffusive representation space. We first show that this universal state variable is approximately controllable, and then deduce that such Volterra systems always possess suitable controllability properties, stated and proved. Then, we show how to solve the optimal null control problem in an LQ sense. A numerical example finally highlights these results.

Keywords: controllability, operators, infinite dimensional system, integral equations, diffusive representation, state realization, optimal control

1. INTRODUCTION

There exists a large literature considering the control problem of infinite dimensional systems, and now different techniques are available to compute suitable control laws for linear or nonlinear partial differential equations. See e.g. Tucsnak and Weiss (2009) for a recent textbook on the control of linear operators, or Coron (2007) for techniques adapted to nonlinear dynamical equations. The class of systems under consideration here are described by a pseudo-differential operator of diffusive type. These models appear in various applications such as in acoustics (see e.g., Fellah et al. (2001); Polack (1991)), in combustion (see e.g. Rouzaud (2003)), in electrical engineering (see e.g. Bidan et al. (2001)), in biology (see e.g. Topaz and Bertozzi (2004)), etc. The operators under consideration in this paper belong more precisely to the general class of "diffusive operators" introduced in Montseny (2005). Such operators can be realized by means of input-output dissipative infinite dimensional equations, as very early considered for example in Kirkwood and Fuoss (1941); Rouse Jr (1953); Macdonald and Brachman (1956), or more recently in Montseny et al. (1993); Staffans (1994); Montseny (1998), etc. Various aspects have been studied in the literature for this class of systems. A lot of results can be found in Montseny (2005); we can also mention other works about the identification (Casenave and Montseny (2009)), the numerical simulation (Montseny (2004)), the inversion (Casenave (2009)), or the dissipativity (Matignon and Prieur (2005)) of such systems.
The aim of this paper is to prove the controllability or more precisely the approximate controllability of diffusive operators. In the sequel, we consider the Volterra input-output model of the form:

$$x(t) = \int_0^t h(t-s)\, u(s)\, ds,\ \forall t \geqslant 0, \tag{1}$$

with $h \in L^1_{\text{loc}}(\mathbb{R}^+_t)$, $x \in C^0(\mathbb{R}^+)$. Note that in this case, we necessarily have: $x(0) = \int_0^0 h(t-s)\, u(s)\, ds = 0$. Model (1) can be rewritten under the symbolic form:

$$x = H(\partial_t)u, \tag{2}$$

where $H(p)$ is the symbol (or transfer function, non necessarily rational) of the operator $H(\partial_t)$.
Note that model (2) can represent a wide variety of dynamical systems. Let us give the following examples:

- models of the form $K(\partial_t)x = \lambda x + u,\ \lambda \in \mathbb{R}$, are a particular case of (2) with $H(\partial_t) = (K(\partial_t) - \lambda I)^{-1}$. If $K(\partial_t) = \partial_t$, we get a classical differential model of the form $\partial_t x = \lambda x + u,\ x(0) = 0$.
- SISO models of the form:
$$\begin{cases} \dot{X} = AX + Bu,\ X(0) = 0 \\ x = CX \end{cases} \tag{3}$$
with $X(t) \in \mathbb{R}^n$, can also be rewritten under the form (2), with $H(p) = C(pI - A)^{-1}B$.

In this paper, we study the controllability of systems of the form (2) with $H(p)$ non rational. Such systems do not admit any state representation in $\mathbb{R}^n$. So the notion of controllability, well-defined in the case of systems with finite dimensional state representations, has to be reformulated. For that aim, we use the so-called diffusive representation (Montseny (2005)), which enables to realize the operator $H(\partial_t)$ by means of a suitable (infinite-dimensional) diffusive state equation with input u, from which we define an approximate controllability in a state space well adapted to the state realization. We then establish that all systems of the form (1), admitting such a diffusive realization, are approximately controllable.

This paper is organized as follows. Some preliminaries and the problem statement are given in Section 2. The main results are given in Section 3. The numerical simulation and the effective computation of the control are considered in Section 4. In particular an example is introduced and considered in this section. Section 5 contains some concluding remarks and point out some further research lines. Finally, the Appendix A collects the proofs of the main results. It necessitates to develop precisely the framework associated with the diffusive representation.

2. PRELIMINARIES

2.1 Problem statement

Let us consider the following Cauchy problem, on which will be based the state realization of (1):

$$\left\{\begin{array}{ll} \partial_t\psi(t,\xi) &= \gamma(\xi)\,\psi(t,\xi)+u(t),\ t>0,\ \xi\in\mathbb{R},\\ \psi(0,\xi) &= \psi_0(\xi), \end{array}\right. \tag{4}$$

where $\gamma \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R};\mathbb{C})$ defines an infinite simple arc in $\mathbb{C}^- + a$, $a \in \mathbb{R}$, closed at ∞. The problem is supposed to be well posed in the space $C^0(0,T;\Psi)$ of measurable functions with values in a suitable topological state space Ψ. For all $\psi_0 \in \Psi$ and $u \in L^2(0,T)$, (4) admits a unique solution, given by:

$$\psi(t,\xi) = e^{\gamma(\xi)t}\psi_0(\xi) + \int_0^t e^{\gamma(\xi)s}\,u(t-s)\,ds,\ t\in[0,T]. \tag{5}$$

This solution will be denoted $\psi(t;\psi_0,u)$ in the sequel.
Let $\mathcal{S}_T : L^2(0,T) \longrightarrow \Psi$ be the operator defined by:

$$\mathcal{S}_T(u) = \int_0^T e^{\gamma(\xi)\,s}\,u(T-s)\,ds. \tag{6}$$

We introduce the set:

$$\mathcal{R}_T = \left\{\psi(T;\psi_0,u);\ u\in L^2(0,T)\right\};$$

$\mathcal{R}_T$ is called the *reachable set* of system (4) at time T.
We now consider the following definitions:

Definition 1. System (4) is said to be:

- *controllable (in Ψ) on* $[0;T]$, if $\mathcal{R}_T$ is equal to Ψ;
- *approximately controllable (in Ψ) on* $[0;T]$, if $\mathcal{R}_T$ is densely embedded in Ψ, that is: $\overline{\mathcal{R}_T}^{\Psi} = \Psi$;
- *approximately controllable (in Ψ)* if it is approximately controllable on $[0,T]$ for any $T > 0$.

For all $\psi_0 \in \Psi$, note that $\mathcal{R}_T = \mathcal{S}_T(L^2(0,T)) + e^{\gamma(.)T}\psi_0$. Thus the set $\mathcal{R}_T$ is dense in Ψ if and only if $\mathcal{S}_T(L^2(0,T))$ is dense in Ψ. More precisely, we have (see e.g. Tucsnak and Weiss (2009)):

Proposition 2. Consider a topological vector space Ψ such that ψ_0, $e^{\gamma(.)T}\psi_0 \in \Psi$. The system (4) is approximately controllable in Ψ if and only if $\mathcal{S}_T(L^2(0,T))$ is dense in Ψ for any $T > 0$.

The problem under consideration in this paper is the approximate controllability of (4) and then of (2).

2.2 Notation and introduction of the control space

Let us first introduce the space:

$$\mathcal{D}_\infty = \left\{\phi\in C^\infty(\mathbb{R}),\ \forall n\in\mathbb{N},\ \sqrt{1+(.)^2}\,\partial_\xi^n\phi\in L^\infty(\mathbb{R})\right\};$$

it is classically a Fréchet space with topology defined by the countable set of norms: $\|\phi\|_n = \|\sqrt{1+(.)^2}\,\partial_\xi^n\phi\|_{L^\infty}$.

By denoting $L^2_c(\mathbb{R}^+)$ the space of functions of $L^2(\mathbb{R}^+)$ with compact support and $\mathcal{L}_\gamma$ the operator:

$$\mathcal{L}_\gamma : u\in L^2(\mathbb{R}^+) \longrightarrow \mathcal{L}_\gamma u := \int_0^{+\infty} e^{\gamma\,s}\,u(s)\,ds, \tag{7}$$

thanks to the property $u \in L^2(0,T) \Leftrightarrow u(T-\,.) \in L^2(0,T)$, we have:

$$\bigcup_{T>0} \mathcal{S}_T(L^2(0,T)) = \mathcal{L}_\gamma(L^2_c(\mathbb{R}^+)). \tag{8}$$

Let us introduce the following space:

Definition 3. $\mathbf{\Delta}_\gamma$ is the completion of $\mathcal{L}_\gamma(L^2_c(\mathbb{R}^+))$ in $\mathcal{D}_\infty$.

We denote Ω^-_γ and Ω^+_γ the two open domains delimited by γ such that $\Omega^+_\gamma \supset (a,+\infty)$. Assume there exists $\alpha_\gamma \in (\frac{\pi}{2},\pi)$ such that:

$$e^{\mathrm{i}[-\alpha_\gamma,\,\alpha_\gamma]}\mathbb{R}_+ + a \ \subset\ \Omega^+_\gamma. \tag{9}$$

Given γ_n a sequence of regular functions such that $\gamma_n(\mathbb{R}) \subset \Omega^+_{\gamma_{n+1}}$ and $\gamma_n \overset{W^{1,\infty}_{\mathrm{loc}}}{\to} \gamma$, the topological vector space Δ_γ is defined as the inductive limit associated with an inductive system $(\mathbf{\Delta}_{\gamma_n},\phi_n)$ where ϕ_n are topological isomorphisms such that $\phi_n(\Delta_{\gamma_n}) \hookrightarrow \phi_{n+1}(\Delta_{\gamma_{n+1}})$. The topological vector space $\Delta_\gamma := \varinjlim \phi_n(\Delta_{\gamma_n})$ is complete and locally convex, with $\phi_n(\mathbf{\Delta}_{\gamma_n}) \hookrightarrow \Delta_\gamma$ continuous and dense. It can be shown (see Montseny (2005)) that for any $u \in C^0(0,T)$, (4) is well-posed in $C^0(0,T;\Delta_\gamma)$.

3. CONTROLLABILITY RESULTS

3.1 Approximate controllability results

We are now in position to state the main results about the controllability of (4), the proofs of which are given in Appendix A.3:

Theorem 4. System (4) is approximately controllable in Δ_γ.

Corollary 5. System (4) is approximately controllable in any topological space Ψ such that $\Delta_\gamma \hookrightarrow \Psi$ with dense embedding.

In particular, from the dense embedding: $\Delta_\gamma \hookrightarrow L^2_\gamma := \overline{\Delta_\gamma}^{L^2(\mathbb{R})}$, (4) is approximately controllable in the Hilbert subspace L^2_γ.

Remark 6. The sector condition (9) appears as the cornerstone in the proof of Theorem 4: it is indeed at the origin of the analyticity of $\mathcal{L}^*_\gamma\mu$ on which the result is based; it also expresses the diffusive nature of system (4). Due to $\mathcal{L}_\gamma(L^2_c(\mathbb{R})) \subsetneq \Delta_\gamma$, system (4) is *not exactly controllable*; this is a consequence of the analyticity in $\mathbb{C}$ of $p \mapsto \int_0^{+\infty} e^{p\,s}\,u(s)\,ds$ when the support, supp u, of the function u is compact.

3.2 Controllability of the Volterra problem (2)

Consider an operator $H(\partial_t)$ admitting a so-called γ-symbol [1] $\mu \in \Delta'_\gamma$, that is, such that model (2) admits the input-output state representation (see Appendix A):

$$\left\{\begin{array}{l} \partial_t\psi = \gamma\,\psi + u,\quad \psi(0,\xi)=0\\ x = \langle\mu,\psi\rangle_{\Delta'_\gamma,\Delta_\gamma}. \end{array}\right. \tag{10}$$

[1] As usual, Δ'_γ designates the topological dual of Δ_γ.

Definition 7. The system $u \mapsto H(\partial_t)u$ is said (approximately) controllable if there exists $\mu \in \Delta'_\gamma$ such that for any $u \in L^2_{\rm loc}(\mathbb{R}^+)$:

- (10) is approximately controllable,
- for any (ψ, x) solution of (10), we have: $H(\partial_t)u = x$.

Then, from Theorem 4, we get:

Corollary 8. If $H(\partial_t)$ admits a γ-symbol, then the system (2) is approximately controllable.

Example 9. The "fractional" differential model of the form:

$$\partial_t^\alpha x = \lambda x + u, \; x(0^+) = 0, \text{ with } 0 < \alpha < 1, \tag{11}$$

is approximately controllable. Indeed, (11) is equivalently written: $x = (\partial_t^\alpha - \lambda)^{-1} u$ and $(\partial_t^\alpha - \lambda)^{-1}$ admits a γ-symbol as soon as $\lambda \in \overline{\Omega_\gamma^-}$ (see Appendix A). More generally, for any $\alpha > 0$, the system $x = \partial_t^{-\alpha}(\lambda x + u)$ is approximately controllable.

Remark 10. As in the finite dimensional case, it is interesting to introduce, for the controllability of (2), the notion of minimal state realization. Indeed, in the state realization (10), which is of the same type as (3) but in infinite dimension, we note that only the $\xi \in \operatorname{supp} \mu$ are involved in the synthesis of x. Assuming that $\operatorname{supp} \mu$ is a Lebesgue non-negligible set, we then introduce the seminorm in L^2 defined by:

$$p_\mu(\psi) = \sqrt{\int_{\operatorname{supp}\mu} |\psi|^2 \, d\xi},$$

and consider the quotient Hilbert space $\Delta_{\gamma,\mu} := \Delta_\gamma / \ker p_\mu$. From the above, we deduce that system (2) is (a fortiori) approximately controllable in $\Delta_{\gamma,\mu}$. Moreover, if $\mu \sim f \in L^1_{\rm loc}(\mathbb{R})$, the following ($\gamma$-)realization is minimal:

$$\begin{cases} \partial_t \psi = \gamma\, \psi + u, \quad \psi(0, \xi) = 0, \quad \xi \in \operatorname{supp}\mu \\ x = \langle \mu, \psi \rangle_{\Delta'_{\gamma,\mu}, \Delta_{\gamma,\mu}} = \displaystyle\int_{\operatorname{supp}\mu} f\, \psi \, d\xi. \end{cases} \tag{12}$$

3.3 About null control of (2)

We now state some results relating to approximative controllability to zero of (2). We suppose that the operator $H(\partial_t)$ admits a γ-symbol $\mu \in \Delta'_\gamma$. First we state an approximate controllability result in finite time.

Lemma 11. Let (x, u) be a solution of (2) on $[0, t_0]$; then, $\forall \varepsilon > 0$, $\forall T > t_0$, $\exists \tilde{u} \in L^2(0, T)$ such that $\tilde{u}|_{[0,t_0]} = u$ and, with $(\tilde{u}, \tilde{x})$ solution of (2) on $[0, T]$, $|\tilde{x}(T)| \leqslant \varepsilon$.

A stronger result of approximate controllability can in fact be stated, which ensures that the value of x after the control time T can also be controlled. This is the second main result (Lemma 11 and Theorem 12 are both proved in Appendix A.3).

Theorem 12. Let (x, u) be a solution of (2) on $[0, t_0]$; then $\forall \varepsilon > 0$, $\forall T > t_0$, $\forall T' > T$, $\exists \tilde{u} \in L^2(0, T')$ such that $\tilde{u}|_{[0,t_0]} = u$ and, with $(\tilde{u}, \tilde{x})$ solution of (2) on $[0, T']$:

$$|\tilde{x}(t)| \leq \varepsilon \;\; \forall t \in [T, T'].$$

4. OPTIMAL CONTROL

4.1 Problem formulation and analysis

Consider the system (2) and its γ-realization (10). Suppose that a control $u \in L^2(0, t_0)$ has been applied to the system: the state ψ reached at time t_0 is denoted ψ_0. By change of time variable $t := t - t_0$, the state equation now becomes:

$$\partial_t \psi = \gamma\, \psi + u, \quad \psi(0, .) = \psi_0. \tag{13}$$

Here we consider the problem of approximate null controllability, that is the approximate controllability to zero as considered e.g. in Crépeau and Prieur (2008). It consists in finding a control $u \in L^2(0, T)$ such that:

$$\psi(T, .) = e^{\gamma\, T} \psi_0 + \mathcal{S}_T u \simeq 0 \Leftrightarrow \mathcal{S}_T u \simeq -e^{\gamma\, T} \psi_0. \tag{14}$$

From Theorem 11, we would then have: $\psi(t, .) \simeq 0$ on $[T, T']$ (and so, thanks to the continuity of $\psi \mapsto \langle \mu, \psi \rangle$: $x(t) \simeq 0$ on $[T, T']$).

Because $S_T(L^2(0, T))$ is only densely embedded in Δ_γ, the null control problem in $L^2(0, T)$ is ill posed: it has in general no solution (i.e. $e^{\gamma t} \psi_0$ is not in $S_T(L^2(0, T))$). For the same reason, the set $S_T(L^2(0, T))$ is in general not closed in usual Hilbert spaces containing Δ_γ and orthogonal projection on $S_T(L^2(0, T))$ cannot be defined. This leads to consider the following weakened null control problem in the Hilbert space L^2_γ, with $\varepsilon > 0$:

$$\min_{u \in L^2(0,T)} \left\{ \| e^{\gamma\, T} \psi_0 + \mathcal{S}_T u \|^2_{L^2} + \varepsilon \| u \|^2_{L^2(0,T)} \right\}. \tag{15}$$

This problem is compatible with the property of approximate controllability; it can indeed be shown that this problem is well posed [2], that is there exists a unique solution u, given by:

$$u^{\rm opt}_\varepsilon = -(\mathcal{S}_T^* \mathcal{S}_T + \varepsilon I)^{-1} \mathcal{S}_T^* \, (e^{\gamma\, T} \psi_0).$$

4.2 Discrete formulation

Consider a mesh $\{t_k\}_{k=0:K}$ of the time variable t, such that $t_K = T$. The control u is computed as:

$$u = \sum_{k=0}^{K-1} u_k \, \mathbf{1}_{(t_k, t_k+1]}. \tag{16}$$

We have: $(\mathcal{S}_T u)(\xi_l) = \sum_k \int_{t_k}^{t_{k+1}} e^{\gamma(\xi_l)\,(T-s)} \, ds \, u_k = \sum_k \mathbf{S}_l^k \, u_k$, with:

$$\mathbf{S}_l^k = \int_{t_k}^{t_{k+1}} e^{\gamma(\xi_l)\,(T-s)} \, ds = \frac{e^{\gamma(\xi_l)\,(T-t_{k+1})} - e^{\gamma(\xi_l)\,(T-t_k)}}{-\gamma(\xi_l)}.$$

Under numerical approximation, problem (15) is then written:

$$\min_{u_K \in \mathbb{R}^K} \left\{ \| e^{\gamma\, T} \psi_0 + \mathbf{S}\, u_K \|^2_{\mathbb{R}^L} + \varepsilon \| u_K \|^2_{\mathbb{R}^K} \right\}, \tag{17}$$

with $u_K := [u_k]_{k=1:K}$, $e^{\gamma T} \psi_0 = \left[e^{\gamma(\xi_l) T} \psi_0(\xi_l) \right]_{l=1:L}$ and $\mathbf{S}$ the matrix of elements $\mathbf{S}_l^k$. The solution is given by:

$$u^{\rm opt}_{K,\varepsilon} = -(\mathbf{S}^* \mathbf{S} + \varepsilon \mathbf{I})^{-1} \mathbf{S}^* \, (e^{\gamma\, T} \psi_0). \tag{18}$$

4.3 Numerical example

Let $H(\partial_t)$ be the non rational operator with symbol:

$$H(p) = \frac{\ln(p)}{p + 200}. \tag{19}$$

H is holomorphic in $\mathbb{C} \backslash \mathbb{R}^-$ and $H(p) \to 0$ when $p \to \infty$ in $\mathbb{C} \backslash \mathbb{R}^-$; then $H(\partial_t)$ admits a diffusive state realization with γ defined by $\gamma(\xi) = -|\xi|$. The γ-symbol of $H(\partial_t)$ identifies with the distribution μ given by:

$$\mu = \mathrm{pv} \frac{\mathbf{1}_{\mathbb{R}^+}}{|\xi| - 200} + \ln(200)\, \delta_{200}.$$

[2] The image of the operator: $u \in L^2(0, T) \mapsto (u, \mathcal{S}_T u) \in L^2(0, T) \times L^2(\mathbb{R}_\xi)$ is closed for the graph norm.

For the finite dimensional approximate state realization[3] of $H(\partial_t)$, we consider a mesh $\{\xi_l\}_{l=1:L}$ of $L = 70$ discretization points geometrically spaced between $\xi_1 = 10^{-3}$ and $\xi_{70} = 10^4$. From $t = 0$ to $t = t_0 = 5$, we apply the control $u = \sin(\frac{\pi}{2}(\frac{t}{t_0})^2)$; we get:

$$\psi_0(\xi) = \left((\partial_t - \gamma(\xi))^{-1} \sin(\frac{\pi}{2}(\frac{\cdot}{t_0})^2) \right)_{|t=t_0}. \qquad (20)$$

Then, from $t = t_0$ to $t = T' = 25$, we apply the control:

$$u = u^{\text{opt}}_{K,\varepsilon} \mathbf{1}_{[t_0,T]}, \qquad (21)$$

where $T = 15$ and $u^{\text{opt}}_{K,\varepsilon}$ is obtained by (18) with $\varepsilon = 10^{-4}$ and $K = 10^4 + 1$ (i.e. $\Delta t = 10^{-3}$).
Figure 1 gives the time-evolution of the control u, of the output x and of the state ψ. It can be checked that the input u succeeds to control the state from its initial value to the final value 0 within time T and that the evolution after T confirms the result of Theorem 12.

5. CONCLUSION

In this paper, the controllability problem for a wide class of Volterra (scalar) systems has been studied. By considering the diffusive representation approach, the result is that any diffusive Volterra system $x = H(\partial_t)u$ is approximately controllable. This result can be trivially extended to general diffusive non linear and/or non t-invariant Volterra systems of the form $x = H(t, \partial_t) f(t, x, v)$ with f an invertible function[4], simply by replacing $\mu(\xi)$ by $\mu(t, \xi)$. In that sense, this controllability result can be viewed as a consistent extension of controllability of scalar differential systems $\partial_t x = f(t, x, v)$, $x(0) = 0$. It seems now to be interesting to tackle the controllability problem for *vector* systems of the form $x = H(t, \partial_t)u$ with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$.

Appendix A. DIFFUSIVE REPRESENTATION AND PROOF OF THE MAIN RESULTS

A complete statement of diffusive representation can be found in Montseny (2005); a shortened one is presented in Casenave and Montseny (2010).

A.1 Basic principle of diffusive representation

We consider a causal convolution operator defined, on any continuous function $u : \mathbb{R}^+ \to \mathbb{R}$, by:

$$u \mapsto \left(t \mapsto \int_0^t h(t-s)\, u(s)\, ds \right). \qquad (A.1)$$

We denote H the Laplace transform of h and $H(\partial_t)$ the convolution operator defined by (A.1).

Let $u^t(s) = \mathbf{1}_{(-\infty,t]}(s)\, u(s)$ be the restriction of u to its past and $u_t(s) = u^t(t-s)$ the so-called "history" of u. From causality of $K(\partial_t)$, we deduce:

$$[H(\partial_t)(u - u^t)](t) = 0 \text{ for all } t; \qquad (A.2)$$

then, we have for any continuous function u:

$$[H(\partial_t)\, u](t) = \left[\mathcal{L}^{-1}(H\, \mathcal{L}u)\right](t) = \left[\mathcal{L}^{-1}\left(H\, \mathcal{L}u^t\right)\right](t), \quad (A.3)$$

[3] Details about the numerical approximation of μ and ψ can be found in Montseny (2005); Casenave and Montseny (2010).
[4] in the sense that there exists g such that $v := g(t, x, u)$ with $u := f(t, x, v)$.

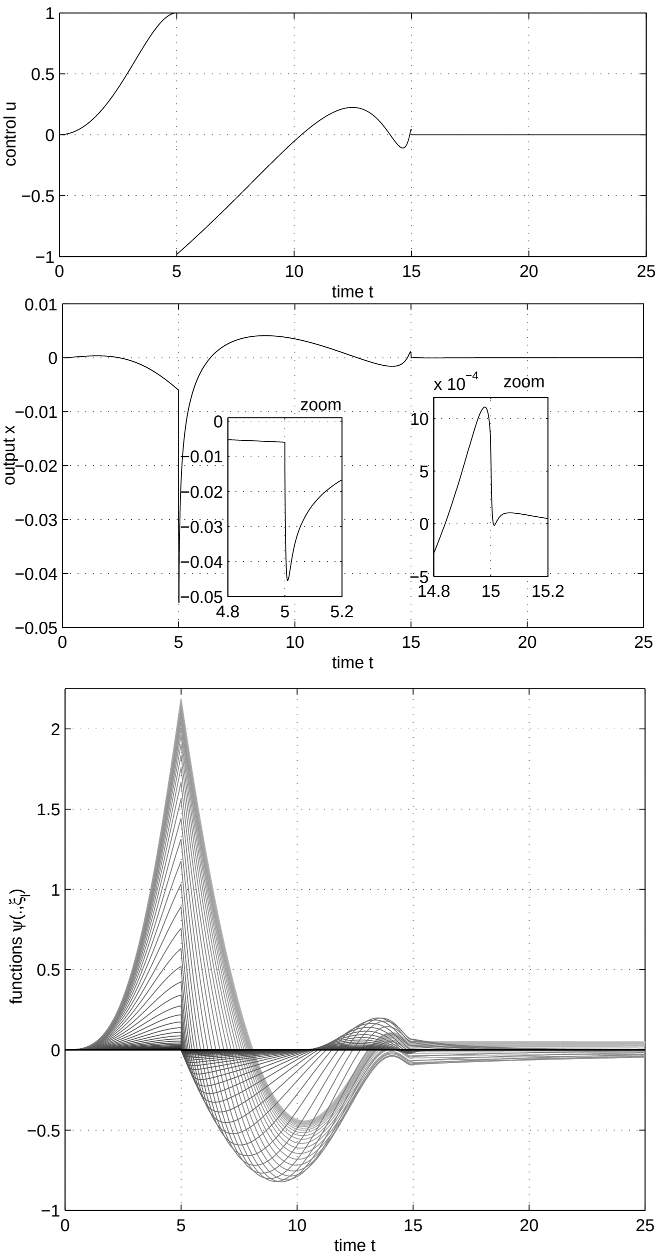

Fig. 1. Numerical results. Time-evolution of the control u (top), of the output x (middle) and of the state ψ (down) for t in $[0, 25]$

where $\mathcal{L}$ and $\mathcal{L}^{-1}$ are the Laplace and the inverse Laplace transforms defined by $(\mathcal{L}f)(p) = \int_0^\infty e^{-pt} f(t) dt$ and by $\left(\mathcal{L}^{-1}F\right)(t) = \frac{1}{2i\pi}\int_{b-i\infty}^{b+i\infty} e^{pt} F(p) dp$.

We define $\Psi_u(t,p) := e^{p\,t}\,(\mathcal{L}u^t)(p) = (\mathcal{L}u_t)(-p)$; by computing $\partial_t \mathcal{L}u_t$, Laplace inversion and use of (A.3):

Lemma 13. Ψ_u is solution of the differential equation:

$$\partial_t \Psi(t,p) = p\, \Psi(t,p) + u, \quad t > 0, \quad \Psi(0,p) = 0, \qquad (A.4)$$

and:

$$[H(\partial_t)\, u](t) = \frac{1}{2i\pi} \int_{b-i\infty}^{b+i\infty} H(p)\, \Psi_u(t,p)\, dp, \forall b \geqslant 0. \quad (A.5)$$

Now, let's consider γ, Ω_γ^+, Ω_γ^- as defined in Section 2, with γ regular. By use of standard techniques (Cauchy theorem, Jordan lemma), it can be shown:

Lemma 14. For γ such that H is holomorphic in $\overline{\Omega_\gamma^+}$, if $H(p) \to 0$ when $p \to \infty$ in Ω_γ^+, then:

$$[H(\partial_t)\, u]\,(t) = \frac{1}{2i\pi} \int_\gamma H(p)\, \Psi_u(t,p)\, dp. \tag{A.6}$$

Under assumptions of Lemma 14, we have (we use the notation $\langle \mu, \psi \rangle = \int \mu\, \psi\, d\xi$):

Proposition 15. Denoting $\mu = \frac{\gamma'}{2i\pi} H \circ \gamma$ and $\psi(t,.) = \Psi_u(t,.) \circ \gamma$, we have, for all $t \geq 0$,

$$[H(\partial_t)\, u]\,(t) = \langle \mu, \psi(t,.) \rangle\,, \tag{A.7}$$

where ψ is the solution on $(t,\xi) \in \mathbb{R}^{*+} \times \mathbb{R}$ of:

$$\partial_t \psi(t,\xi) = \gamma(\xi)\, \psi(t,\xi) + u(t),\ \psi(0,\xi) = 0. \tag{A.8}$$

Definition 16. The function μ defined in Proposition 15 is called *γ-symbol* of operator $H(\partial_t)$. The function ψ solution of (A.8) is called the *γ-representation* of u.

Proposition 17. The impulse response $h = \mathcal{L}^{-1} H$ of operator $H(\partial_t)$ is given by:

$$h(t) = \langle \mu, e^{\gamma\, s} \rangle = \mathcal{L}_\gamma^* \mu; \tag{A.9}$$

furthermore, h is holomorphic in[5] $\mathbb{R}^{+*}$.

Proof. (A.9) is obtained by setting u as the Dirac measure in (A.8). Analyticity of h is a consequence of the sector condition (9) (which makes equation (A.8) of diffusive nature). □

A.2 General topological framework

The results of Proposition 15 can be extended to a wide class of operators, provided that the associated γ-symbols are extended to suitable distribution spaces we will introduce in the sequel (so, the expression $\langle \mu, \psi \rangle$ will refer to a topological duality product).

Let consider the spaces $\mathcal{D}_\infty$ and $\boldsymbol{\Delta}_\gamma$ as defined in Section 2.2, and first note that we have the continuous and dense embeddings:

$$\mathcal{D} \hookrightarrow \mathcal{D}_\infty \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{D}'_\infty \hookrightarrow \mathcal{D}'. \tag{A.10}$$

We can show:

Proposition 18. $\mathcal{S}_T(L^2(0,T)) \subset \mathcal{D}_\infty$.

Proof. $\forall u \in L^2(0,T)$, $|\mathcal{S}_T\, u| \leqslant \int_0^T \left| e^{\gamma(\xi)(T-s)}\, u(s) \right| ds$

$$\leqslant \|u\|_{L^2} \sqrt{\int_0^T e^{2\mathrm{Re}(\gamma(\xi))(T-s)} ds} = \|u\|_{L^2} \sqrt{\frac{e^{2\mathrm{Re}(\gamma(\xi))T} - 1}{2\mathrm{Re}(\gamma(\xi))}}.$$

Under the hypotheses made on γ, we can show from simple analysis that there exists $c > 0$ such that, for all $\xi \geq 0$:

$$\sqrt{\frac{e^{2\mathrm{Re}(\gamma(\xi))T} - 1}{2\mathrm{Re}(\gamma(\xi))}} \leqslant \frac{c}{\sqrt{1+\xi^2}}.$$

The extension to $\partial_\xi^n(\mathcal{S}_T\, u)$ is then obtained by induction. □

Proposition 19. $\boldsymbol{\Delta}_\gamma$ is a strict subspace of $\mathcal{D}_\infty$ (Montseny (2005)).

[5] That is: there exists an open domain $D \subset \mathbb{C}$ such that $\mathbb{R}^{+*} \subset D$ and h admits an analytical continuation on D.

So, with $\mathcal{L}_\gamma^*$ the adjoint of[6] $\mathcal{L}_\gamma^*$, we have $\ker(\mathcal{L}_\gamma^*) \neq \{0\}$ and the dual $\boldsymbol{\Delta}'_\gamma$ is the quotient space:

$$\boldsymbol{\Delta}'_\gamma = \mathcal{D}'_\infty / \ker(\mathcal{L}_\gamma^*).$$

We have the following continuous and dense embeddings:

$$\boldsymbol{\Delta}_\gamma \hookrightarrow L^2_\gamma,\quad L^{2\prime}_\gamma \hookrightarrow \boldsymbol{\Delta}'_\gamma.$$

Remark 20. Thanks to (A.7), if $\gamma(0) = 0$, then the Dirac distribution δ is a γ-symbol of the operator $u \mapsto \int_0^t u(s)\, ds$, denoted ∂_t^{-1}.

Now suppose that $\gamma \in W^{1,\infty}_{\mathrm{loc}}$ (so, γ can be non regular), and consider the space Δ_γ defined in Section 2.2. The so-defined space Δ_γ has the following properties (Montseny (2005)):

- $\Delta_\gamma = \bigcup_n \phi_n(\boldsymbol{\Delta}_{\gamma_n})$ and Δ_γ is independent of the choice of the sequence γ_n.
- $\mathcal{L}_\gamma(L^2_c(\mathbb{R})) \hookrightarrow \phi_n(\boldsymbol{\Delta}_{\gamma_n})\ \forall n$; so, $\forall u \in L^2_c(\mathbb{R})$, $\psi_u = \mathcal{L}_\gamma u_t \in \phi_n(\boldsymbol{\Delta}_{\gamma_n})$.
- The dual Δ'_γ of Δ_γ is a complete, locally convex topological vector space; we have $\Delta'_\gamma = \bigcap_n [\phi_n(\boldsymbol{\Delta}_{\gamma_n})]'$.
- With ψ_u defined by (4) and $\mu \in \Delta'_\gamma$, the symbol H of the convolution operator defined by $u \mapsto \langle \mu, \psi_u \rangle_{\Delta'_\gamma, \Delta_\gamma}$ is holomorphic in Ω_γ^+ and $H(p) \to 0$ when $p \to \infty$ in Ω_γ^+.
- The impulse response h of operator $u \mapsto \langle \mu, \psi_u \rangle_{\Delta'_\gamma, \Delta_\gamma}$ is $h(t) = \langle \mu, e^{\gamma\, t} \rangle_{\Delta'_\gamma, \Delta_\gamma} = \mathcal{L}_\gamma^* \mu$; it is holomorphic in $\mathbb{R}^{+*}$.
- When μ can be identified with a locally integrable function (yet denoted μ), then $\langle \mu, \psi \rangle_{\Delta'_\gamma, \Delta_\gamma}$ can be expressed by the integral (in the Lebesgue sense):
$$\langle \mu, \psi \rangle_{\Delta'_\gamma, \Delta_\gamma} = \int \mu\, \psi\, d\xi.$$
- If γ is regular, we have the following continuous and dense embeddings: $\Delta_\gamma \hookrightarrow \boldsymbol{\Delta}_\gamma \hookrightarrow \boldsymbol{\Delta}'_\gamma \hookrightarrow \Delta'_\gamma$.
- $\forall \mu \in \Delta'_\gamma$, $\forall \psi \in \mathcal{L}_\gamma(L^2_c(\mathbb{R}))$, $\forall n \in \mathbb{N}$, $\exists!(\mu_n, \psi_n) \in \boldsymbol{\Delta}'_{\gamma_n} \times \boldsymbol{\Delta}_{\gamma_n}$ with $\frac{\mu_n}{\sqrt{1+\xi^2}} \in L^1(\mathbb{R})$ such that:
$$\langle \mu, \psi \rangle_{\Delta'_\gamma, \Delta_\gamma} = \langle \mu_n, \psi_n \rangle_{\boldsymbol{\Delta}'_{\gamma_n}, \boldsymbol{\Delta}_{\gamma_n}} = \int \mu_n\, \psi_n\, d\xi.$$

A.3 Proof of the main results

Proof of Theorem 4. Thanks to Proposition 2, we have to prove that $\mathcal{S}_T(L^2(0,T))$ is dense in Δ_γ for any $T > 0$. Let us consider the (continuous) extension of operator $\mathcal{L}_\gamma$ to $L^2(\mathbb{R}^+)$; we have:

$$\mathcal{L}_\gamma : L^2(\mathbb{R}^+) \to \mathcal{D}_\infty \text{ and } \mathcal{S}_T : L^2(0,T) \to \mathcal{D}_\infty,$$

and then, by identifying L^2 spaces with their duals:

$$\mathcal{L}_\gamma^* : \mathcal{D}'_\infty \to L^2(\mathbb{R}^+) \text{ and } \mathcal{S}_T^* : \mathcal{D}'_\infty \to L^2(0,T).$$

From Fubini theorem:

$$\langle \mu, \mathcal{S}_T u \rangle = \langle \mu, \int_0^T e^{\gamma\, s}\, u\, ds \rangle = \int_0^T u\, \langle \mu, e^{\gamma\, s} \rangle\, ds$$

$$= \int_0^T u\, \mathcal{S}_T^* \mu\, ds \Rightarrow \mathcal{S}_T^* \mu = \langle \mu, e^{\gamma\, s} \rangle_{\mathcal{D}'_\infty, \mathcal{D}_\infty} \in L^2(0,T);$$

similarly: $\mathcal{L}_\gamma^* \mu = \langle \mu, e^{\gamma\, s} \rangle_{\mathcal{D}'_\infty, \mathcal{D}_\infty} \in L^2(\mathbb{R}^+)$.

We can deduce: $\forall \mu \in \mathcal{D}'_\infty$, $\mathcal{S}_T^* \mu = \mathcal{L}_\gamma^* \mu|_{[0,T]}$. Then, due

[6] Defined by $\langle \mu, \mathcal{L}_\gamma u \rangle_{\mathcal{D}'_\infty, \mathcal{D}_\infty} = \int_0^{+\infty} (\mathcal{L}_\gamma^* \mu)\, u\, ds\ \forall \mu \in \mathcal{D}'_\infty$, $\forall u \in L^2(\mathbb{R}^+)$.

to Proposition 17, $\mathcal{L}_\gamma^*\mu$ is holomorphic on $\mathbb{R}^{+*}$. It follows that:

$$\mathcal{L}_\gamma^*\mu = 0_{L^2(\mathbb{R}^+)} \Leftrightarrow \mathcal{S}_T^*\mu = 0_{L^2(0,T)}$$

and so, $\ker \mathcal{L}_\gamma^* = \ker \mathcal{S}_T^*$. Consequently, $\overline{\mathrm{Im}\mathcal{S}_T}^{\Delta_\gamma} = \overline{\mathrm{Im}\mathcal{L}_\gamma}^{\Delta_\gamma} = \Delta_\gamma$, i.e. $\mathrm{Im}\mathcal{S}_T$ is dense in Δ_γ. □

Proof of Proposition 11. Let $\mathbf{x}$ be the trajectory defined on $[0,T]$ by:

$$\forall t \in [0,T],\ \mathbf{x}(t) = H(\partial_t)(u\,\mathbf{1}_{[0,t_0]}) = \langle \mu, \mathcal{S}_T(u\,\mathbf{1}_{[0,t_0]})\rangle.$$

We obviously have: $\mathbf{x}|_{[0,t_0]} = x$. Consider the open set $\mathcal{O}_1 = (-\mathbf{x}(T)-\varepsilon, -\mathbf{x}(T)+\varepsilon)$ of $\mathbb{R}$. As the operator $\phi \in \Delta_\gamma \mapsto \langle \mu, \phi\rangle_{\Delta_\gamma,\Delta'_\gamma} \in \mathbb{R}$ is continuous, the inverse image of $\mathcal{O}_1$ is an open set of Δ_γ, denoted $\mathcal{O}_2$. As $\mathrm{Im}(\mathcal{S}_{T-t_0})$ is dense in Δ_γ, then, there exists $v \in L^2(0, T-t_0)$ such that $\mathcal{S}_{T-t_0}v \in \mathcal{O}_2$, and we have

$$\langle \mu, \mathcal{S}_{T-t_0}v\rangle \in \mathcal{O}_1. \tag{A.11}$$

By simple computations, we can show that $\mathcal{S}_{T-t_0}v = \mathcal{S}_T\left(v(.-t_0)\mathbf{1}_{[t_0,T]}\right)$; so, with $\tilde{u}$ defined by:

$$\tilde{u} = u\,\mathbf{1}_{[0,t_0]} + v(.-t_0)\mathbf{1}_{[t_0,T]},$$

we have:

$$\begin{aligned}\tilde{x}(T) &= \langle \mu, \mathcal{S}_T\tilde{u}\rangle = \langle \mu, \mathcal{S}_T(u\,\mathbf{1}_{[0,t_0]})\rangle + \langle \mu, \mathcal{S}_T(v(.-t_0)\mathbf{1}_{[t_0,T]})\rangle \\ &= \mathbf{x}(T) + \langle \mu, \mathcal{S}_{T-t_0}v\rangle.\end{aligned}$$

From (A.11), we then deduce that $\tilde{x}(T) \in [-\varepsilon, \varepsilon]$. □

Proof of Theorem 12. From the definition of Δ_γ, $\mu \in \Delta'_\gamma \Rightarrow \mu \in \phi_n(\boldsymbol{\Delta}_{\gamma_n})'\ \forall n$; furthermore, if ψ is solution of (13), then $\psi(t,.) \in \phi_n(\boldsymbol{\Delta}_{\gamma_n})\ \forall n, t$ (Montseny (2005)). We then have: $\langle \mu, \psi\rangle_{\Delta'_\gamma,\Delta_\gamma} = \langle \mu_n, \psi_n\rangle_{\boldsymbol{\Delta}'_{\gamma_n},\boldsymbol{\Delta}_{\gamma_n}}$ with μ_n and ψ_n regular functions; hence, from properties of μ_n and ψ_n:

$$\begin{aligned}|x(t)| &= |\langle \mu, \psi(t,.)\rangle_{\Delta'_\gamma,\Delta_\gamma}| = |\int \mu_n(\xi)\,\psi_n(t,\xi)\,d\xi| \\ &= |\langle \sqrt{1+\xi^2}\,\psi_n(t,.), \tfrac{\mu_n}{\sqrt{1+\xi^2}}\rangle_{L^\infty,L^1}| \\ &\leqslant \|\sqrt{1+\xi^2}\,\psi_n(t,.)\|_{L^\infty}.\|\tfrac{\mu_n}{\sqrt{1+\xi^2}}\|_{L^1} \\ &\leqslant \|\sqrt{1+\xi^2}\,\psi_n(T,\xi)\,e^{\gamma_n(\xi)(t-T)}\|_{L^\infty}.\|\tfrac{\mu_n}{\sqrt{1+\xi^2}}\|_{L^1} \\ &\leqslant \sup_{\xi,t\in[0,T]}|e^{\gamma_n(\xi)(t-T)}|.\|\tfrac{\mu_n}{\sqrt{1+\xi^2}}\|_{L^1}.\sup_\xi|\sqrt{1+\xi^2}\,\psi_n(T,\xi)| \\ &= K\sup_\xi|\sqrt{1+\xi^2}\,\psi_n(T,\xi)|.\end{aligned}$$

Consider now an open ball $B(0,\eta) \subset L^\infty(\mathbb{R})$; then $V_n = \left(\frac{1}{\sqrt{1+\xi^2}}B(0,\eta)\right) \cap \boldsymbol{\Delta}_{\gamma_n}$ is an open set in the Fréchet space $\boldsymbol{\Delta}_{\gamma_n}$ and $V = \phi_n(V_n)$ is an open set in[7] Δ_γ. So, let $u \in L^2(0,T)$ such that $\psi(T,.) = \mathcal{S}_T u \in V$; it follows: $|x(t)| \leqslant K\sup_\xi|\sqrt{1+\xi^2}\,\psi_n(T,\xi)| \leqslant K\eta \leqslant \varepsilon$ if $\eta \leqslant \dfrac{\varepsilon}{K}$. □

[7] W is an open set in Δ_γ iff $\exists n$ s.t. W is an open set in $\phi_n(\boldsymbol{\Delta}_{\gamma_n})$.

REFERENCES

Bidan, P., Lebey, T., Montseny, G., and Saint-Michel, J. (2001). Transient voltage distribution in inverter fed motor windings: Experimental study and modeling. *IEEE Trans. Power Electronics*, 16, 92–100.

Casenave, C. (2009). *Représentation diffusive et inversion opératorielle pour l'analyse et la résolution de problèmes dynamiques non locaux.* Ph.D. thesis, Université Paul Sabatier, Toulouse.

Casenave, C. and Montseny, G. (2009). Optimal identification of delay-diffusive operators and application to the acoustic impedance of absorbent materials. In *Topics in Time Delay Systems: Analysis, Algorithms and Control*, volume 388, 315–328. Springer Verlag, Lecture Notes in Control and Information Science (LNCIS).

Casenave, C. and Montseny, G. (2010). Introduction to diffusive representation. In *4th IFAC Symposium on System, Structure and Control.* Ancona, Italy.

Coron, J.M. (2007). *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI.

Crépeau, E. and Prieur, C. (2008). Approximate controllability of a reaction-diffusion system. *Systems Control Lett.*, 57(12), 1048–1057.

Fellah, Z., Depollier, C., and Fellah, M. (2001). Direct and inverse scattering problem in porous material having a rigid frame by fractional calculus based method. *J. Sound and Vibration*, 244, 3659–3666.

Kirkwood, J. and Fuoss, R. (1941). Anomalous dispersion and dielectric loss in polar polymers. *The Journal of Chemical Physics*, 9, 329–340.

Macdonald, J. and Brachman, M. (1956). Linear-system integral transform relations. *Reviews of modern physics*, 28(4), 393–422.

Matignon, D. and Prieur, C. (2005). Asymptotic stability of linear conservative systems when coupled with diffusive systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 11(3), 487–507.

Montseny, G. (1998). Diffusive representation of pseudo-differential time-operators. *ESAIM: proceedings*, 5, 159–175.

Montseny, G. (2004). Simple approach to approximation and dynamical realization of pseudodifferential time operators such as fractional ones. *IEEE Transactions on Circuits and Systems II*, 51(11), 613–618.

Montseny, G. (2005). *Représentation diffusive.* Hermès-science, Paris.

Montseny, G., Audounet, J., and Mbodje, B. (1993). Optimal models of fractional integrators and application to systems with fading memory. In *Proceedings of the International Conference on Systems, Man and Cybernetics conference*, volume 5, 65–70. IEEE, Le Touquet (France).

Polack, J.D. (1991). Time domain solution of Kirchhoff's equation for sound propagation in viscothermal gases: a diffusion process. *J. Acoustique*, 4, 47–67.

Rouse Jr, P. (1953). A theory of the linear viscoelastic properties of dilute solutions of coiling polymers. *The Journal of Chemical Physics*, 21, 1272–1280.

Rouzaud, H. (2003). Long-time dynamics of an integro-differential equation describing the evolution of a spherical flame. *Revista matemática complutense*, 16(1), 207.

Staffans, O. (1994). Well-posedness and stabilizability of a viscoelastic equation in energy space. *Transactions of the American Mathematical Society*, 345(2), 527–575.

Topaz, C.M. and Bertozzi, A.L. (2004). Swarming patterns in a two-dimensional kinematic model for biological groups. *SIAM Journal On Applied Mathematics*, 65(1), 152–174.

Tucsnak, M. and Weiss, G. (2009). *Observation and Control for Operator Semigroups.* Birkhäuser, Basel.