ON THE DYNAMIC OF MONOTONE GRAPH, DENDRITE AND DENDROID MAPS

ISSAM NAGHMOUCHI

Abstract. We show that, for monotone graph map \( f \), all the \( \omega \)-limit sets are finite whenever \( f \) has periodic point and for monotone dendrite map, any infinite \( \omega \)-limit set does not contain periodic points. As a consequence, monotone graph and dendrite maps have no Li-Yorke pairs. However, we built a homeomorphism on a dendroid with a scrambled set having nonempty interior.

1. Introduction

Let \( X \) be a compact metric space with metric \( d \) and \( f : X \to X \) a continuous map. Denote by \( f^n \) the \( n \)-th iterate of \( f \); that is, \( f^0 = \text{Identity and } f^n = f \circ f^{n-1} \) if \( n \geq 1 \). For any \( x \in X \) the subset \( O_f(x) = \{ f^n(x) : n \in \mathbb{N} \} \) is called the orbit of \( x \) (under \( f \)). A point \( x \in X \) is called periodic of prime period \( n \in \mathbb{N}^* \) if \( f^n(x) = x \) and \( f^i(x) \neq x \) for \( 1 \leq i \leq n - 1 \). We define the \( \omega \)-limit set of a point \( x \) to be the set

\[
\omega_f(x) = \{ y \in X : \exists n_i \in \mathbb{N}, n_i \to \infty, \lim_{i \to \infty} d(f^{n_i}(x), y) = 0 \}
\]

\[= \bigcap_{n \in \mathbb{N}} \{ f^k(x) : k \geq n \}.
\]

A point \( x \in X \) is said to be recurrent for \( f \) if \( x \in \omega_f(x) \). We will denote by \( \text{Fix}(f) \), \( \text{Per}(f) \) and \( \text{R}(f) \) the set of fixed points, periodic points and recurrent points, respectively. Then we have the following inclusion relation \( \text{Fix}(f) \subset \text{Per}(f) \subset \text{R}(f) \). A subset \( A \) of \( X \) is called \( f \)-invariant if \( f(A) \subset A \). The set \( \omega_f(x) \) is a non-empty, closed and strongly invariant set, i.e. \( f(\omega_f(x)) = \omega_f(x) \).

A pair \((x, y) \in X \times X \) is called proximal if \( \lim \inf_{n, \to \infty} d(f^n(x), f^n(y)) = 0 \), it is called distal if \( \lim \inf_{n, \to \infty} d(f^n(x), f^n(y)) > 0 \). If \( \lim \sup_{n, \to \infty} d(f^n(x), f^n(y)) = 0 \), \((x, y)\) is called asymptotic. We say that \( x \) is asymptotic to a periodic point \( z \) if \( (x, z) \) is asymptotic. A pair \((x, y)\) is called a Li-Yorke pair (of \( f \)) if it is proximal but not asymptotic that is:

\[
\lim \inf_{n, \to \infty} d(f^n(x), f^n(y)) = 0
\]

and

\[
\lim \sup_{n, \to \infty} d(f^n(x), f^n(y)) > 0.
\]

This work was supported by the research unit 99UR/15-15
2000 Mathematics Subject Classification. 37E25, 37B40.

Key words and phrases. graph map, graph, dendrite, dendroid, Li-Yorke pair, \( \omega \)-limit set.
A subset $S$ of $X$ containing at least two points is called a *Li-Yorke scrambled set* (of $f$) if for any $x, y \in S$ with $x \neq y$, $(x, y)$ is a Li-Yorke pair. We say that $f$ is *Li-Yorke chaotic* if there exists an uncountable scrambled set of $f$, [17]. The topological entropy of $f$ will be denoted $h(f)$. It is proved in [5] that if $h(f) > 0$ then $f$ is Li-Yorke chaotic.

In this paper, we consider the case of $X$ being either a finite graph $G$, a dendrite $D$ or a dendroid $Y$. A continuum is a compact connected metric space. A topological space is arcwise connected if any two of its points can be joined by an arc. We use the terminology from Nadler [19].

By a *graph* $G$, we mean a continuum which can be written as the union of finitely many arcs such that any two of them are either disjoint or intersect only in one or both of their endpoints. Each of these arcs is called an *edge* of the graph. An arc is any space homeomorphic to the closed interval $[0, 1]$. A point $v \in G$ is called a *vertex* if it admits a neighborhood $U$ in $G$ homeomorphic to the set $\{ z \in \mathbb{C} : z^r \in [0, 1] \}$ with the natural topology for some $r \geq 3$, with the homeomorphism mapping $v$ to $0$. If $r = 1$, then we call $v$ an *endpoint* of $G$. Denote by $V(G)$ and $E(G)$ the sets of vertices and endpoints of $G$ respectively, then an edge is the closure of some connected component of $G \setminus V(G)$, it is homeomorphic to $[0, 1]$. A subset of $G$ homeomorphic to the circle is called a *cycle*. A subgraph of $G$ is a subset of $G$ which is a graph itself. Every sub-continuum of a graph is a graph ([19], Corollary 9.10.1), circles and trees are particular kinds of subgraphs.

For any $x, y \in G$, define the distance $d(x, y)$ to be the minimal length of arcs $[x, y]$ in $G$ whose endpoints are $x$ and $y$. We write $(x, y) = [x, y] \setminus \{x, y\}$.

By a *dendrite* $D$, we mean a locally connected continuum which contains no cycle. Every sub-continuum of a dendrite is a dendrite ([19], Theorem 10.10) and every connected subset of $D$ is arcwise connected ([19], Proposition 10.9).

By a *dendroid* $Y$, we mean an arcwise connected topological space $Y$, hereditarily unicoherent continuum; that is that each closed connected subsets $F$ of $Y$ is unicoherent: this means that whenever $A$ and $B$ are closed, connected subsets of $F$ such that $F = A \cup B$, then $A \cap B$ is connected. We notice that a dendrite is a locally connected dendroid. Since any dendroid $Y$ is unicoherent and arcwise connected, any two distinct points $x, y \in Y$ can be joined by a unique arc with endpoints $x$ and $y$.

A continuous map from a circle, graph (resp. dendrite; resp. dendroid) into itself is called a circle map, graph map (resp. dendrite map; resp. dendroid map). Every dendroid $Y$ (and in particular, every dendrite) has the fixed point property (see [19]); that is when $f$ is a dendroid map, then $\text{Fix}(f) \neq \emptyset$.

**Definition 1.1.** ([15]) *Let $X$, $Y$ be two topological spaces. A map $f : X \to Y$ is said to be monotone if for any connected subset $C$ of $Y$, $f^{-1}(C)$ is connected.*
We note that:

- Definition 1.1 is equivalent to that the preimage of any point by \( f \) is connected when \( f \) is a closed map (see [15], Theorem 9, p. 131). In particular, if \( X \) is compact and \( f : X \to X \) is a continuous map, then \( f \) is monotone if the preimage of any point of \( X \) is connected.
- For continuous interval map \( f : I \to \mathbb{R} \), \( f \) is monotone if and only if \( f \) is increasing or decreasing.

In recent years, subjects such as topological structure of minimal sets, depth of the center, topological entropy, Li-Yorke pairs, size of scrambled sets, etc..., of finite graph maps and dendrites maps have been studied by many authors (see [7], [18], [13], [8], [9], [4], [2], [3], [1]).

Effremova and Makhrova studied in [8] monotone map of a class of dendrite \( D \) and they showed that for any \( x \in D \), \( \omega_f(x) \) is a periodic orbit, moreover they also built a dendrite homeomorphism having an infinite \( \omega \)-limit set. Hric and Malek proved in [12] that any \( \omega \)-limit set of a graph map is either a periodic orbit, or an infinite compact nowhere dense set, or a finite union of subgraphs (which form a periodic orbit). In this paper, we studied monotone graph and dendrite map \( f \). We prove in particular, when \( f \) is a monotone graph map with periodic point, any \( \omega \)-limit set is a periodic orbit. Moreover, we built a homeomorphism on a dendroid having a scrambled with nonempty interior.

Our main results are the following:

**Theorem A.** Let \( f : G \to G \) be a monotone graph map and onto. If \( G \) is different of a cycle then, for any \( x \in G \), \( \omega_f(x) \) is a periodic orbit.

**Theorem B.** Let \( f \) be a monotone graph map. Then, the following statements are equivalent:

- i) \( f \) has a periodic point.
- ii) For any \( x \in G \), \( \omega_f(x) \) is a periodic orbit.

**Theorem C.** Let \( f : D \to D \) be a monotone dendrite map. Then, any infinite \( \omega \)-limit set contains no periodic points.

**Corollary 1.2.** Let \( X \) be a finite graph or a dendrite and \( f : X \to X \) a monotone continuous map. Then, \( f \) has no Li-Yorke pair.

As a consequence, we obtain by [5], the following corollary:

**Corollary 1.3.** Let \( X \) be a finite graph or a dendrite and \( f : X \to X \) a monotone continuous map. Then \( h(f) = 0 \).
Theorem D. There exists a homeomorphism on a dendroid having scrambled set with nonempty interior.

This paper is organized as follows: In Section 2, we give some preliminaries results which are useful for the rest of the paper. In Section 3, we show that for monotone circle map, any \( \omega \)-limit set is a periodic orbit whenever \( f \) has a periodic point. In Section 4, we extend this result to the case of monotone graph map by proving that the \( \omega \)-limit sets of an onto monotone graph map \( f : G \to G \) are periodic orbits if \( G \) is not a cycle. In result, monotone graph maps have no Li-Yorke pair. Section 5 deals with monotone dendrite maps, we prove that for such maps, if an \( \omega \)-limit set contains a periodic point then it is a periodic orbit and as a consequence, monotone dendrite maps have no Li-Yorke pair. Section 6 is devoted to the construction of a homeomorphism of a dendroid having a scrambled set with a nonempty interior.

2. Some results

Let \( \mathbb{N}^* \) denote the set of positive integers and \( \mathbb{N} = \mathbb{N}^* \cup \{0\} \).

Lemma 2.1. \([6] \) Let \( X \) be a compact metric space, \( f : X \to X \) a continuous map and \( x \in X \). If \( \omega_f(x) \) is finite then it is a periodic orbit.

Lemma 2.2. Let \( X \) be a compact space, \( f : X \to X \) a continuous map and \( x \in X \). If \( \omega_{f^m}(x) \) is finite for some \( m \in \mathbb{N}^* \) then \( \omega_f(x) \) is finite.

Proof. Suppose that \( \omega_{f^m}(x) \) is finite and let \( z \in \omega_f(x) \). Then there exists \( 0 \leq r < m \) such that \( z \in f^r(\omega_{f^m}(x)) \). Hence \( \omega_f(x) \subset \cup_{i=0}^{m-1} f^i(\omega_{f^m}(x)) \) and so \( \omega_f(x) \) is finite. \( \square \)

Lemma 2.3. Let \( X \) be a compact metric space and \( f : X \to X \) a continuous map. If \( x, y \in X \) such that \( \omega_f(x) \) and \( \omega_f(y) \) are finite then \( (x, y) \) is either asymptotic or distal. In particular, \( (x, y) \) is not a Li-Yorke pair.

Proof. By hypothesis, \( x \) (resp. \( y \)) is asymptotic to a periodic point \( p \) (resp. \( q \)); that is \( \lim_{n \to \infty} d(f^n(x), f^n(p)) = 0 \) and \( \lim_{n \to +\infty} d(f^n(x), f^n(q)) = 0 \). Hence, if \( p = q \), \( (x, y) \) is asymptotic and if \( p \neq q \), \( (x, y) \) is distal. \( \square \)

Lemma 2.4. \((6), p. 6\) If \( J \) is a compact subinterval of \( \mathbb{R} \) and \( f : J \to \mathbb{R} \) is a continuous map such that \( f(J) \supseteq J \), then \( f \) has a fixed point in \( J \).

Lemma 2.5. If \( J \) is a compact subinterval of \( \mathbb{R} \) and \( f : J \to J \) is a continuous monotone map, then for any \( x \in J \), \( \omega_f(x) \) is either a fixed point or an orbit of period 2. In particular, \( f \) has no Li-Yorke pair.
Proof. By hypothesis, $f$ is monotone then $f$ is either an increasing or decreasing map. If $f$ is increasing then for every $x \in J$ satisfying $f(x) \geq x$ (resp. $f(x) \leq x$), $(f^n(x))_{n \in \mathbb{N}}$ is an increasing (resp. decreasing) sequence, so $\omega_f(x)$ is a fixed point. If $f$ is decreasing then $f^2$ is increasing and hence (by the above), $\omega_{f^2}(x)$ is a fixed point for $f^2$. By uniform continuity of $f$, we have $\lim_{n \to +\infty} |f^n(x) - f^n(a)| = 0$. So, $\omega_f(x) = O_f(a) = \{x\}$ or $\{a, f(a)\}$. By Lemma 2.3, $f$ has no Li-Yorke pair.

Let $S^1 = [0, 1]_{0 < 1}$ be a circle endowed of the orientation: the counter clockwise sense induced via the natural projection $[0, 1] \to S^1$.

Corollary 2.6. If $f : S^1 \to S^1$ is a monotone circle map and not onto, then for any $x \in S^1$, $\omega_f(x)$ is either a fixed point or an orbit of period 2.

Proof. Since $f$ is continuous and not onto, $f(S^1) \subset S^1$ and there exist $a \neq b \in S^1$ such that $\cap_{n \in \mathbb{N}} f^n(S^1) = [a, b]$. Hence, $f([a, b]) = [a, b]$. Let $x \in S^1$. If for some $n \in \mathbb{N}$, $f^n(x) \in [a, b]$ then by Lemma 2.5, $\omega_f(x) = \omega_f(f^n(x))$ is either a fixed point or an orbit of period 2. Otherwise, $\omega_f(x) \subset \{a, b\}$ (since any $\omega$-limit set is included in $\cap_{n \in \mathbb{N}} f^n(S^1)$). This completes the proof.

Proposition 2.7. ([14]) If $f : S^1 \to S^1$ is a circle map with no periodic point, then it has no Li-Yorke pair.
Let \( f([a, b]) \supset [f(b), f(a)] \), a similar proof will be repeated to show that \( f([a, b]) = [f(b), f(a)] \). \( \square \)

**Lemma 3.2.** Let \( f : S^1 \to S^1 \) be an onto monotone circle map. If \( f \) has at least two periodic point then, for any \( x \in S^1 \), \( \omega_f(x) \) is a periodic orbit.

**Proof.** Let \( a, b \in S^1 \) be two periodic points of \( f \) and let \( n \) be such that \( f^n(a) = a \) and \( f^n(b) = b \). Take \( g = f^n \), then \( g \) is monotone continuous circle map and onto. By Lemma 3.1, we have either \( g([a, b]) = [a, b] \) or \( g((a, b]) = [b, a] \). If \( g([a, b]) = [a, b] \) (resp. \( [b, a] \)), then \( g([b, a]) = [b, a] \) (resp. \( [a, b] \)). In either cases, we get \( g^2([a, b]) = [a, b] \) and \( g^2([b, a]) = [b, a] \). Hence by Lemma 2.5, \( \omega_{g^2}(x) \) is finite for any \( x \in S^1 \) and so is \( \omega_f(x) \). \( \square \)

**Lemma 3.3.** Let \( f : S^1 \to S^1 \) be an onto monotone circle map. If \( f \) has only one periodic point \( a \), then \( a \in \text{Fix}(f) \) and for any \( x \in S^1 \), \( \omega_f(x) = \{a\} \).

**Proof.** Let \( I = f^{-1}(\{a\}) \). Since \( f \) is monotone and continuous, \( I \) is a closed connected subset containing \( a \). Let \( x \in S^1 \). If for some \( n \in \mathbb{N} \), \( f^n(x) \in I \), then it is clear that \( \omega_f(x) = \{a\} \) since \( f^k(x) = a \) for every integer \( k > n \). Now, assume that \( f^n(x) \notin I \) for all \( n \in \mathbb{N} \). We distinguish two cases:

- **Case 1.** \( f(x) \in [x, a] \).
  
  **Claim 1.** \( f^2(x) \in [f(x), a] \).
  
  Indeed, otherwise \( f^2(x) \notin [a, f(x)] \) and \( a \in [f(x), f^2(x)] \). By Lemma 3.1, \( f([x, f(x)]) \) is either \( [f(x), f^2(x)] \) or \( [f^2(x), f(x)] \). Since \( a \in [f(x), f^2(x)] \) and \( [x, f(x)] \cap I = \emptyset \), it follows that \( f([x, f(x)]) = [f^2(x), f(x)] \).
  
  - If \( f^2(x) \in [x, f(x)] \) (see Figure a), then \( f([x, f(x)]) = [f^2(x), f(x)] \subset [x, f(x)] \), so apply Lemma 2.5 to the interval \( J = [x, f(x)] \), we see that \( \omega_f(x) \) is a periodic orbit distinct of \( \{a\} \), a contradiction.
  
  - If \( f^2(x) \notin [x, f(x)] \) (see Figure b), then \( f([x, f(x)]) = [f^2(x), f(x)] \supset [x, f(x)] \) and by applying Lemma 2.4 to the interval \( J = [x, f(x)] \), then \( [x, f(x)] \) contains a fixed point, a contradiction since \( a \notin [x, f(x)] \).

- **Claim 2.** \( f([x, a]) = [f(x), a] \).
  
  By Lemma 3.1, \( f([x, a]) \) is either \( [f(x), a] \) or \( [a, f(x)] \) and we have \( f(x) \in
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Let \([x, a]\), then \(f^2(x) \in f([x, a])\). Since \(f^2(x) \notin [a, f(x)]\), it follows that \(f([x, a]) = [f(x), a]\).

We conclude that \(f([x, a]) = [f(x), a] \subset [x, a]\). By Lemma 2.5, \(\omega_f(x)\) is a periodic orbit and so \(\omega_f(x) = \{a\}\).

- Case 2. \(f(x) \in [a, x]\). By considering the clockwise sense of \(S^1\), the same proof works as in case 1. \(\square\)

**Proposition 3.4.** Let \(f : S^1 \to S^1\) be a monotone circle map. Then, the following statements are equivalent:

i) \(f\) has a periodic point.

ii) for any \(x \in S^1\), \(\omega_f(x)\) is a periodic orbit.

**Proof.** ii) \(\Rightarrow\) i): is obvious. i) \(\Rightarrow\) ii): Assume that \(f\) has a periodic point. If \(f\) is not onto, then by Corollary 2.6, any \(\omega\)-limit set is a periodic orbit. If \(f\) is onto then by Lemmas 3.2 and 3.3, any \(\omega\)-limit set is a periodic orbit. This completes the proof. \(\square\)

**Corollary 3.5.** Let \(f : S^1 \to S^1\) be a monotone circle map. Then \(f\) has no Li-Yorke pair.

**Proof.** If \(f\) has no periodic point then by Proposition 2.7, it has no Li-Yorke pair. If \(f\) has a periodic point then by Theorem A, \(\omega_f(x)\) is a periodic orbit for any \(x \in S^1\). Hence, for any \(x, y \in S^1\), the pair \((x, y)\) is not Li-Yorke, by Lemma 2.3. So \(f\) has no Li-Yorke pair. \(\square\)

Hric has proved in [11] that the topological sequence entropy (see [10]) of a circle map \(f\) is zero if and only if \(f\) has no Li-Yorke pair. As a consequence, we obtain by Corollary 3.5, the following:

**Corollary 3.6.** Every monotone circle map has zero sequence topological entropy.

4. Monotone Graph Map

In this section, \(G\) is a finite graph. Write by \(V = V(G) := \{v_1, v_2, \ldots, v_n\}\) and \(E = E(G) := \{e_1, e_2, \ldots, e_m\}\) the set of vertices and the set endpoints of \(G\) respectively. Denote by

- \(I = \{I_1, I_2, \ldots, I_r\}\) the set of edges in \(G\) having one endpoint in \(V\) and the other in \(E\),
- \(J = \{J_1, J_2, \ldots, J_s\}\) the set of edges in \(G\) with endpoints in \(V\),
- \(C = \{C_1, C_2, \ldots, C_t\}\) the set of cycles in \(G\) containing only one vertex.

Then \(G = (\bigcup_{1 \leq i \leq r} I_i) \cup (\bigcup_{1 \leq i \leq s} J_i) \cup (\bigcup_{1 \leq i \leq t} C_i)\) (See Figure 1).
Lemma 4.1. Let $G$ be a graph different from a cycle and $f : G \to G$ an onto monotone graph map. Then:

i) for every $v \in V$, $f^{-1}(\{v\})$ contains only one vertex. In result, $f(V) = V$.

ii) for every $e \in E$, $f^{-1}(\{e\})$ contains only one endpoint. In result, $f(E) = E$.

Proof. i) Let $v \in V$. Since $f$ is onto, continuous and monotone, $f^{-1}(\{v\})$ is a non empty closed connected subset of $G$, hence it is a subgraph of $G$.

Let us prove that $f^{-1}(\{v\})$ contains a vertex:

Otherwise, $f^{-1}(\{v\}) := [a, b]$ is an arc containing no vertex. Let $O$ be a closed connected neighborhood of $v$ so that $O \setminus \{v\}$ has $r$ connected components ($r \geq 3$). Then since $f$ is monotone and onto, $f^{-1}(O)$ is a non empty subgraph of $G$ and $f^{-1}(O) \setminus f^{-1}(\{v\})$ has $r$ connected components.

As $[a, b] \subset f^{-1}(O)$, so $f^{-1}(O) \setminus [a, b]$ has at most 2 connected components, a contradiction. Now, since the sets $f^{-1}(\{v\}) \cap V$, $v \in V$, are non empty and pairwise disjoint, so $f^{-1}(\{v\}) \cap V$ is a single point. As a consequence, we have $f(V) = V$.

ii) Let $e \in E$. Since $f$ is onto, continuous and monotone, $f^{-1}(\{e\})$ is a non empty closed connected subset of $G$, hence it is a subgraph of $G$.

Suppose that $f^{-1}(\{e\})$ contains no endpoint. Then by i), it also contains no vertex, hence $f^{-1}(\{e\}) = [a, b]$ is a closed arc. Let $O$ be an closed connected neighborhood of $e$ in $G$ containing no vertex. Then $f^{-1}(O)$ is a closed connected neighborhood of any point $x \in [a, b]$ and by i), it contains no vertex. As $f^{-1}(\{e\}) \subset f^{-1}(O)$, then $f^{-1}(O) \setminus [a, b]$ is not connected. However, $O \setminus \{e\}$ is connected and so is $f^{-1}(O \setminus \{e\}) = f^{-1}(O) \setminus f^{-1}(\{e\})$ since $f$ is monotone, a contradiction. Now, the sets $f^{-1}(\{e\}) \cap E$, $e \in E$, are nonempty and
pairwise disjoint, then \( f^{-1}(\{e\}) \cap E \) is a single point. As a consequence, we have \( f(E) = E \). \( \square \)

**Lemma 4.2.** If \( G \) is a graph different of a cycle and \( f : G \to G \) an onto monotone graph map, then:

i) for every \( I \in \mathcal{I} \), there is only one \( I' \in \mathcal{I} \) such that \( f(I') = I \).

ii) for every \( J \in \mathcal{J} \), there is only one \( J' \in \mathcal{J} \) such that \( f(J') = J \).

iii) for every \( C \in \mathcal{C} \), there is only one \( C' \in \mathcal{C} \) such that \( f(C') = C \).

**Proof.** i) Let \( I \in \mathcal{I} \), then by Lemma 4.1, \( f^{-1}(I) \) contains only one endpoint and only one vertex. Since \( f \) is monotone, \( f^{-1}(I) \) is connected, hence arcwise connected and so there is only one \( I' \in \mathcal{I} \) such that \( I' \subset f^{-1}(I) \) and then \( f(I') = I \).

ii) Let \( J \in \mathcal{J} \), then by Lemma 4.1, \( f^{-1}(J) \) contains only two vertices. Since \( f \) is monotone, \( f^{-1}(J) \) is connected (hence, arcwise connected) and so it contains some \( J' \in \mathcal{J} \) so \( f(J') = J \). Let \( \mathcal{J}(J) = \{ J' \in \mathcal{J} : f(J') = J \} \). If \( J \neq K \in \mathcal{J} \) and \( J' \in \mathcal{J}(J) \), \( K' \in \mathcal{K}(J) \) then \( f(J') = J \) and \( f(K') = K \), hence, \( J' \neq K' \). Thus, the sets \( \mathcal{J}(J) \), \( J \in \mathcal{J} \), are pairwise disjoint then they contain only one element.

iii) From i), ii), we have

\[
(4.1) \quad f(\bigcup_{K \in \mathcal{I} \cup \mathcal{J}} K) = \bigcup_{K \in \mathcal{I} \cup \mathcal{J}} K.
\]

Let \( C \in \mathcal{C} \) and let \( v \in C \cap V \). Then \( C \setminus \{v\} \subset G \setminus (\bigcup_{K \in \mathcal{I} \cup \mathcal{J}} K) \), by (4.1),

\[
(4.2) \quad f^{-1}(C \setminus \{v\}) \subset G \setminus (\bigcup_{K \in \mathcal{I} \cup \mathcal{J}} K) \subset \bigcup_{C \in \mathcal{C}} C.
\]

By Lemma 4.1, \( f^{-1}(C \setminus \{v\}) \) is a connected subset containing no vertices and no endpoints, so by (4.2), \( f^{-1}(C \setminus \{v\}) \subset C' \) for some \( C' \in \mathcal{C} \). Now, lets prove that \( f(C') = C \):

we have \( f(C') \supset C \setminus \{v\} \). Since \( f(C') \) is closed,

\[
(4.3) \quad f(C') \supset C.
\]

write \( \{v\} = C' \cap V \). We will prove that \( f(v') = v \): By Lemma 4.1, \( f^{-1}(\{v\}) \) contains a vertex and by (4.3), it also contains a point from \( C' \). Since \( f^{-1}(\{v\}) \) is arcwise connected, it contains an arc \( K \) joining this two points. As \( v' \in K \), hence, \( v' \in f^{-1}(\{v\}) \) and then \( f(v') = v \). Let \( y \in C' \) so that \( f(y) \in C \setminus \{v\} \). Let \( x \in C' \setminus \{v\} \) and let \([x, y] \subset C' \) be the arc so that \( v' \notin [x, y] \). Then, \( f([x, y]) \supset [f(x), f(y)] \). Suppose that \( f(x) \notin C \). Then, \( f([x, y]) \) contains \( v \), and therefore, there is \( w \in [x, y] \cap f^{-1}(\{v\}) \). Since \( f^{-1}(\{v\}) \) is connected, \( f^{-1}(\{v\}) \) contains \([w, v'] \) or \([v', w] \). As \( y \in [w, v'] \) (resp. \( x \in [v', w] \)), then \( f(x) = v \) or \( f(y) = v \), a contradiction. We conclude that \( f(x) \in C \) and therefore \( C \subset f(C') \). In result, for any \( C \in \mathcal{C} \), there is \( C' \in \mathcal{C} \) such that \( f(C') = C \). The sets, \( \mathcal{C}(C) := \{ C' \in \mathcal{C} : f(C') = C \} \), \( C \in \mathcal{C} \), are pairwise disjoint and non empty then they contain only one element. This ends the proof. \( \square \)
Lemma 4.3. Let $G$ be a graph different of a cycle and $f : G \rightarrow G$ an onto monotone graph map. Then there exist $N \in \mathbb{N}^*$ such that for any $I \in \mathcal{I}$, $J \in \mathcal{J}$ and $C \in \mathcal{C}$, we have, $f^N(I) = I$, $f^N(J) = J$ and $f^N(C) = C$.

Proof. i) By Lemma 4.2, there exist an $r$–permutation $\sigma_r$, an $s$–permutation $\sigma_s$ and an $t$–permutation $\sigma_t$ such that $f(I_i) = I_{\sigma_r(i)}$, $f(J_j) = J_{\sigma_s(j)}$ and $f(C_k) = C_{\sigma_t(k)}$, for every $(i, j, k) \in \{1, \ldots, r\} \times \{1, \ldots, s\} \times \{1, \ldots, t\}$. Taking $N = (rst)!$, the proof is complete. □

Proof of Theorem A. Let $x \in G$. If $x \in K$ where $K \in \mathcal{I}$ (resp. $\mathcal{J}$), then by Lemma 2.5, $\omega^N(x)$ is finite since $f^N$ is monotone. If $x \in C_k$ for some $1 \leq k \leq t$. We have $f^N|_{C_k}$ is a monotone map from $C_k$ to itself. Moreover, $f^N|_{C_k}$ has the vertex $v_k$ in $C_k$ as a fixed point by Lemma 4.3, i). Therefore, by Proposition 3.4, $\omega^N(x)$ is finite. Hence, for all $x \in G$, $\omega_f(x)$ is a periodic orbit by Lemma 2.1 and 2.2. The proof is complete. □

Proof of Theorem B. ii) $\Rightarrow$ i): is obvious. i) $\Rightarrow$ ii): Let $G_\infty := \cap_{n \in \mathbb{N}} f^n(G)$. Then $G_\infty$ is either a cycle or a graph different of the cycle and $f|_{G_\infty} : G_\infty \rightarrow G_\infty$ is an onto monotone graph map. Suppose $f$ has a periodic point $p$, then $p \in G_\infty$. If $G_\infty$ is a graph different of the cycle, then by Theorem A, $\omega_f(x)$ is a periodic orbit for any $x \in G_\infty$. If $G_\infty$ is a cycle, then $f|_{G_\infty} : G_\infty \rightarrow G_\infty$ is a monotone circle map with a periodic point $p$, hence, by Proposition 3.4, $\omega_f(x)$ is a periodic orbit for any $x \in G_\infty$. In either cases, $\omega_f(x)$ is a periodic orbit for any $x \in G_\infty$. Now, let $x \in G$. If for some $n \in \mathbb{N}$, $f^n(x) \in G_\infty$, then by above, $\omega_f(f^n(x))$ is a periodic orbit and so is $\omega_f(x)$ (since $\omega_f(x) = \omega_f(f^n(x))$). If $f^n(x) \notin G_\infty$ for all $n \in \mathbb{N}$, then $\omega_f(x)$ is included in the boundary of $G_\infty$ which is a finite set. Hence, $\omega_f(x)$ is finite and so it is a periodic orbit by Lemma 2.1. This completes the proof. □

Proof of Corollary 1.2 (the graph case): Assume that $X = G$ is a finite graph. If $f$ has a periodic point then by Theorem B, for all $x \in G$, $\omega_f(x)$ is finite, hence by Lemma 2.3, $f$ has no Li-Yorke pair. If $f$ has no periodic point, then $G_\infty = \cap_{n \in \mathbb{N}} f^n(G)$ is a cycle (otherwise, $G_\infty$ is a graph different of a cycle and $f|_{G_\infty} : G_\infty \rightarrow G_\infty$ is an onto monotone graph map then by Theorem A, $f|_{G_\infty}$ has a periodic point, a contradiction). Moreover, for any $x \in G$, there is $n \in \mathbb{N}$ such that $f^n(x) \in G_\infty$ (otherwise, for all $n \in \mathbb{N}$, $f^n(x) \notin G_\infty$ and hence $\omega_f(x)$ is included in the boundary of $G_\infty$ which is a finite set and then $\omega_f(x)$ is a periodic orbit, a contradiction). Take now $x, y \in G$. There is $n, m \in \mathbb{N}$ such that $f^n(x), f^m(y) \in G_\infty$. Since $G_\infty$ is invariant, one can choose $f^n(x), f^m(y) \in G_\infty$. By Corollary 3.5, the pair $(f^n(x), f^m(y))$ is not a Li-Yorke pair, hence, $(x, y)$ is not a Li-Yorke pair. □

5. The dendrite case

To prove Theorem C, we need the following Lemmas.
Lemma 5.1. ([18], Lemma 2.1) Let \((D, d)\) be a dendrite and \(f : D \rightarrow D\) a dendrite map. Then for every \(\varepsilon > 0\), there exists \(\delta = \delta(\varepsilon) > 0\) such that, for any \(x, y \in D\) with \(d(x, y) \leq \delta\), \(\text{diam}([x, y]) < \varepsilon\).

Here \(\text{diam}(A) := \sup_{x,y \in A} d(x, y)\) is the diameter of a subset \(A\) of \(D\).

Lemma 5.2. ([18], Lemma 2.3) Let \((C_i)_{i \in \mathbb{N}}\) be a sequence of connected subsets of a dendrite \((D, d)\). If \(C_i \cap C_j = \emptyset\) for all \(i \neq j\), then
\[
\lim_{n \to +\infty} \text{diam}(C_n) = 0.
\]

Lemma 5.3. Let \((D, d)\) be a dendrite and \(f : D \rightarrow D\) a monotone dendrite map. Then for any \(x, y \in D\), \(f([x, y]) = [f(x), f(y)]\).

Proof. Since \(f\) is continuous and monotone, we have \(f([x, y]) \supseteq [f(x), f(y)]\) and \(f^{-1}([f(x), f(y)]) \supseteq [x, y]\) respectively. Hence, \([f(x), f(y)] \supseteq f([x, y])\) and therefore, \(f([x, y]) = [f(x), f(y)]\). \(\square\)

Lemma 5.4. Let \((D, d)\) be a dendrite and \(f : D \rightarrow D\) a monotone dendrite map. If \((x, y)\) is a proximal pair then the set \([x, y]\) is synchronously proximal, that is
\[
\liminf_{n \to +\infty} \text{diam}(f^n([x, y])) = 0.
\]

Proof. Let \((k_i)_{i \in \mathbb{N}}\) be an infinite sequence of positive integers such that
\[
\lim_{i \to +\infty} d(f^{k_i}(x), f^{k_i}(y)) = 0.
\]
Take \(\varepsilon > 0\) and let \(\delta = \delta(\varepsilon) > 0\) be as in Lemma 5.1. Then, there exists \(i_0 \in \mathbb{N}\) such that for every \(i \geq i_0\), \(d(f^{k_i}(x), f^{k_i}(y)) \leq \delta\). By Lemmas 5.1 and 5.3, \(\text{diam}(f^{k_i}([x, y])) = \text{diam}([f^{k_i}(x), f^{k_i}(y)]) < \varepsilon\). Hence,
\[
\lim_{i \to +\infty} \text{diam}(f^{k_i}([x, y])) = 0.
\]
\(\square\)

Lemma 5.5. Let \((D, d)\) be a dendrite and \(f : D \rightarrow D\) a monotone dendrite map. Suppose that \(a \in \text{Fix}(f)\) and let \(x \in D\). If for some \(m \in \mathbb{N}\) and \(n \in \mathbb{N}^+\), \((a, f^m(x)] \cap (a, f^{m+n}(x)] \neq \emptyset\), then \((a, x] \cap (a, f^n(x)] \neq \emptyset\).

Proof. Suppose that \((a, f^m(x)] \cap (a, f^{m+n}(x)] \neq \emptyset\) and \((a, x] \cap (a, f^n(x]) = \emptyset\). Take \(z \in (a, f^m(x)] \cap (a, f^{m+n}(x)]\). Since \((a, f^m(x)] \subset f^m([a, x])\) (resp. \((a, f^{m+n}(x]) \subset f^m([a, f^n(x)])\)), there is \(y_1 \in [a, x]\) (resp. \(y_2 \in [a, f^n(x)]\)) such that \(f^m(y_1) = f^m(y_2) = z\), hence, \(a \in [y_1, y_2]\) and by Lemma 5.3, \(f^m([y_1, y_2]) = \{z\}\), thus \(a = f^m(a) = z\), a contradiction. \(\square\)

Lemma 5.6. Let \((D, d)\) be a dendrite and \(f : D \rightarrow D\) a monotone dendrite map. Let \(a \in \text{Fix}(f)\) and \(x \in D\) be such that \((a, f(x)] \cap (a, x] = (a, u)\) with \(u \in (a, x)\) and \(f(u) = a\) then the sets \((a, f^i(x)]\), \(i \geq 1\), are pairwise disjoint.
Proof. Since \( f \) is monotone and \( f(u) = f(a) = a \), we have \( f([a, u]) = \{a\} \). Then,

\[(a, f^2(x)) \cap (a, x) = \emptyset.\]  

Indeed, on the contrary, \((a, f^2(x)) \cap (a, f(x)) \neq \emptyset\), since \((a, f^2(x)) \cap (a, u) \neq \emptyset\) and \((a, u) \subset (a, f(x))\). Take \( y \in (a, f^2(x)) \cap (a, f(x)) \), there is \( z_1 \in [a, x] \) and \( z_2 \in [a, f(x)] \) such that \( f(z_1) = f(z_2) = y \), hence by Lemma 5.3, we have \( f([z_1, z_2]) = \{y\} \). Since \([z_1, z_2] \cap [a, u] \neq \emptyset\) and \( f([a, u]) = \{a\} \), one has \( y = a \), a contradiction. Suppose now that there is \( j > i \geq 1 \) such that \((a, f^i(x)) \cap (a, f^j(x)) \neq \emptyset\). If \( j - i = 1 \), then there are \( y \in (a, f^i(x)) \cap (a, f^{i+1}(x)) \), \( z_1 \in [a, x] \) and \( z_2 \in [a, f(x)] \) such that \( f^i(z_1) = f^i(z_2) = y \), by Lemma 5.3, \( f^i([z_1, z_2]) = \{y\} \). Again, since \([z_1, z_2] \cap [a, u] \neq \emptyset\), we get \( y = a \), a contradiction.

If \( j - i \geq 2 \), put \( k = j - i \). By Lemma 5.5, \((a, x) \cap (a, f^k(x)) \neq \emptyset\) and hence, \((a, f(x)) \cap (a, f^k(x)) \neq \emptyset\), since \((a, f^k(x)) \cap (a, u) \neq \emptyset\) and \((a, u) \subset (a, f(x))\). Again, by Lemma 5.5, \((a, x) \cap (a, f^{k-1}(x)) \neq \emptyset\). We continuous this process by induction on \( k \) to get that \((a, x) \cap (a, f^2(x)) \neq \emptyset\), which contradicts (5.1). \(\square\)

Lemma 5.7. Let \((D, d)\) be a dendrite and \( f : D \to D \) a monotone dendrite map. Let \( a \in \text{Fix}(f) \) and let \( x \in D \setminus \text{Fix}(f) \) such that \( \omega_f(x) \) contains \( a \). Then \( \omega_f(x) = \{a\} \).

Proof. We distinguish two cases:

- Case 1. the connected subsets \((a, f^n(x))_{n \in \mathbb{N}}\), are pairwise disjoint. By Lemma 5.2, we have \( \lim_{n \to \infty} \text{diam}([a, f^n(x)]) = 0 \) and therefore \( \omega_f(x) = \{a\} \).
- Case 2. the connected subsets \((a, f^n(x))_{n \in \mathbb{N}}\) are not pairwise disjoint. Then there exist \( n \in \mathbb{N}, m \in \mathbb{N}^* \) such that \([a, f^n(x)] \cap [a, f^{n+m}(x)] \neq \emptyset\). By Lemma 5.5, \([a, x] \cap [a, f^m(x)]\) is a non degenerate arc, so there exists \( u_1 \in (a, x] \cap (a, f^m(x)]\) such that \([a, x] \cap [a, f^m(x)] = [a, u_1]\). (See Figure 2)
As \( [a, f^m(x)] = f^m([a, x]) \), there exists \( u_0 \in [a, x] \) such that \( f^m(u_0) = u_1 \). Moreover, \( u_0 \neq a \) since \( u_1 \neq a \). Set \( u_k = f^{km}(u_0), \ k \in \mathbb{N} \). Let prove that,

\[
(\ast) \quad \text{for any } v \in (a, u_0], \ f^m(v) \in [a, v).
\]

Indeed, we have \( f^m(v) \in [a, u_1] \subset [a, x] \). If \( f^m(v) \notin [a, v] \), then \( f^m(v) \in [v, x] \), so by induction on \( k \), for every \( k \in \mathbb{N} \), \( [a, f^{km}(v)] \supset f^{km}(a, v) = [a, f^{km}(v)] \) holds. As \( [a, f^{km}(x)] = f^{km}([a, x]) \supset f^{km}([a, v]) = [a, f^{km}(v)] \), it follows that, for every \( k \in \mathbb{N} \),

\[
\text{diam}([a, f^{km}(x)]) \geq \text{diam}([a, f^{km}(v)]) \geq \text{diam}([a, v]).
\]

We deduce from Lemma 5.1 that \( \lim \inf_{k \to +\infty} d(a, f^{km}(x)) > 0 \) and so \( a \notin \omega_f(x) \). Then, \( a \notin \omega_f(x) \), since \( a \in \text{Fix}(f) \), a contradiction. From (\ast), we have in particular \( u_k \in [a, u_0] \) for every \( k \in \mathbb{N}^* \). Hence, \( [a, u_0] \) is \( f^m \)-invariant. Therefore, for any \( v \in [a, u_0] \), \( \omega_f(v) = \{a\} \) since \( f^m(v) \in [a, v] \).

In particular, \( \omega_f(u_0) = \{a\} \). Now, we distinguish two sub-cases.

- **Case 2.1.** There exists \( k \in \mathbb{N} \) such that \( f^{km}(x) \in [a, u_0] \).

As \( \omega_f(x) = \omega_f(f^{km}(x)) \), then \( \omega_f(x) = \{a\} \). By Lemma 2.2, \( \omega_f(x) \) is finite, hence \( \omega_f(x) = O_f(a) = \{a\} \).

- **Case 2.2.** For any \( k \in \mathbb{N}, \ f^{km}(x) \notin [a, u_0] \).

Let prove, in this case, that for every \( k \in \mathbb{N}^* \): \( [u_k, f^{km}(x)] \cap [a, u_0] = \{u_k\} \).

We proceed by induction on \( k \).

For \( k = 1 \), \( [u_1, f^m(x)] \cap [a, u_0] = \{u_1\} \). Now suppose that for some \( k \in \mathbb{N}^* \), \( [u_k, f^{km}(x)] \cap [a, u_0] = \{u_k\} \). (See Figure 3)
Assume that \([u_{k+1}, f^{(k+1)m}(x)] \cap [a, u_0] \supsetneq \{u_{k+1}\}\). There is \(z \in (u_{k+1}, f^{(k+1)m}(x)] \cap [a, u_1]\). Since \(f^m([a, u_0]) = [a, u_1]\) and \(f^m([u_k, f^{km}(x)]) = [u_{k+1}, f^{(k+1)m}(x)]\), there exist \(w \in [a, u_0]\) and \(v \in [u_k, f^{km}(x)]\) such that \(f^m(w) = f^m(v) = z\). As \(f\) is monotone, so is \(f^m\) and then \(f^{-m}(\{z\}) \supset [w, v]\). As \(u_k \in [w, v]\), so \(f^m(u_k) = z \neq u_{k+1}\), a contradiction.

\[\star\text{ Case 2.2.1.}\] If for any \(k \in N\), \(u_k \neq a\), then the sets \([u_k, f^{km}(x)], k \in N\) are pairwise disjoint.

Indeed, if there is \(z \in [u_k, f^{km}(x)] \cap [u_j, f^{jm}(x)]\) for some integers \(k, j\) with \(k \neq j\), then \([z, u_k] \cup [u_k, u_j] = [z, u_j] \subset [u_j, f^{jm}(x)]\). It follows that, \([u_j, f^{jm}(x)] \cap [a, u_0] \supset [u_k, u_j]\). As \(u_{k+1} \in [a, u_k]\) for every \(k \in N\), then \(u_k \neq u_j\), so \([u_j, f^{jm}(x)] \cap [a, u_0] \supset [u_k, u_j] \neq \{u_j\}\), a contradiction.

\[\star\text{ Case 2.2.2.}\] If for some \(k_0 \in N\), \(u_{k_0+1} = a\) and \(u_{k_0} \neq a\), then the sets \(\{a, f^{km}(x)\}, k \geq k_0\), are pairwise disjoint:

Indeed, we have \((a, f^{(k_0-1)m}(x)] \cap (a, f^{km}(x)] = (a, u_{k_0}]\) and \(f^m(u_{k_0}) = u_{k_0+1} = a\).
By applying Lemma 5.6 to the map $f^m$, we get that $\{a, f^{km}(x)\}$ are pairwise disjoint for $k \geq k_0$.

In either cases (case 2.2.1 or case 2.2.2), for sufficiently large $k \geq 1$, the sets $(u_k, f^{km}(x))$ are pairwise disjoint.

We conclude by Lemma 5.2, that $\lim_{k \to +\infty} \text{diam}(u_k, f^{km}(x)) = 0$, so the pair $(x, u_0)$ is asymptotic for $f^m$. Hence, $\omega_{f^m}(x) = \omega_{f^m}(u_0) = \{a\}$. By Lemma 2.2, $\omega_f(x) = \{a\}$. This completes the proof.

Proof of Theorem C. Let $x \in D$ so that $\omega_f(x)$ contains a periodic point $a$. Let $m$ be the period of $a$. Then $a \in \text{Fix}(f^m)$ and $\omega_{f^m}(x)$ contains $a$. By Lemma 5.7, $\omega_{f^m}(x) = \{a\}$ and therefore $\omega_f(x) = O_f(a)$ by Lemma 2.2. The proof is complete.

Proof of Corollary 1.2. (The dendrite case): Assume that $X = D$ is a dendrite. We will show that if $(x, y)$ is a proximal pair for $f$ then $(x, y)$ is an asymptotic pair for $f$.

If the sets $[f^n(x), f^n(y)]$, $(n \in \mathbb{N})$ are pairwise disjoint, then by Lemma 5.2, the pair $(x, y)$ is asymptotic for $f$. If there are $n, m \in \mathbb{N}^*$ such that $[f^n(x), f^n(y)] \cap [f^{m+n}(x), f^{m+n}(y)] \neq \emptyset$, then since $[f^{m+n}(x), f^{m+n}(y)] = f^m([f^n(x), f^n(y)])$ (Lemma 5.3), there is $z \in [f^n(x), f^n(y)]$ such that $f^m(z) \in [f^n(x), f^n(y)]$. By Lemma 5.4, the set $[x, y]$ is synchronously proximal, so is $[f^n(x), f^n(y)] = f^n([x, y])$ (since $f^n$ is uniformly continuous). It follows that the pair $(z, f^m(z))$ is proximal. Therefore, $\omega_f(z)$ contains a periodic point $b \in \text{Fix}(f^m)$. By Theorem C, $\omega_f(z) = O_f(b)$. As $[f^n(x), f^n(y)]$ is synchronously proximal and $z \in [f^n(x), f^n(y)]$, then $\omega_f(f^n(x)) \cap \omega_f(z) \neq \emptyset \neq \omega_f(f^n(y)) \cap \omega_f(z)$. Thus, $\omega_f(x) = \omega_f(f^n(x))$ and $\omega_f(y) = \omega_f(f^n(y))$ contain $O_f(b)$. By Theorem C, $\omega_f(x) = \omega_f(y) = O_f(b)$. As a consequence, the pair $(x, y)$ is asymptotic since it is proximal.
6. **Proof of Theorem D**

We give the construction of a homeomorphism of a dendroid \( Y \) having a scrambled set with nonempty interior and we show that no homeomorphism on \( Y \) can be *completely scrambled* (i.e. the whole space \( Y \) is a scrambled set).

- **Construction of the dendroid \( Y \).**
  Let \( f \) be an increasing homeomorphism on \([0, 1]\) such that \( f(x) > x \) for every \( x \in (0, 1) \). Then \( f(0) = 0 \) and \( f(1) = 1 \). Let \( a_0 \in (0, 1) \) be a real number. Write \( a_{n+1} = f(a_n), \, n \in \mathbb{Z} \) and define

\[
Y = (\cup_{n \in \mathbb{Z}} I_n) \cup ([0, 1] \times \{0\}) \cup (\{0\} \times [0, 1]) \cup (\{1\} \times [0, 1])
\]

where \( I_n = \{a_n\} \times [0, 1] \) (see Figure 4). We endow \( Y \) by the metric \( d \) where

\[
d((x, y), (x', y')) = |x - x'| + |y - y'|
\]
for any \((x, y), (x', y') \in Y\). It is clear that \((Y, d)\) is a dendroid but not a dendrite since it is not locally connected.

![Figure 5. The dendroid Y](image)

- **Construction of the homeomorphism \( F \).**
  For every \( b \in (0, 1) \) and \( \lambda \in (0, \frac{1}{b}) \), define:

\[
h_{\lambda,b}(x) = \begin{cases} 
\lambda x, & \text{if } x \in [0, b], \\
\frac{1}{b-1} ((\lambda b - 1)x + b(1 - \lambda)), & \text{if } x \in [b, 1]
\end{cases}
\]
Then $h_{\lambda,b} \geq id_{[0,1]}$ if $\lambda \geq 1$ (resp. $h_{\lambda,b} \leq id_{[0,1]}$ if $\lambda \leq 1$), (see Figure 5).
Choose $n_0 \in \mathbb{N}^*$ so that $1 + \frac{1}{n_0} < \frac{4}{3}$. Define for each $n \in \mathbb{N}$:

\[
\begin{aligned}
\lambda_{2n} &= 1 - \frac{1}{n_0+2n} \\
\lambda_{2n+1} &= 1 + \frac{1}{n_0+(2n+1)}.
\end{aligned}
\]

Now, define the sequences $(k_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ by induction as follows:

Write $b_0 = \frac{3}{4}$ and $k_0 = 0$. Since $\lambda_0 < 1$, we choose $k_1 \in \mathbb{N}$ such that $h^{k_1}_{\lambda_0,b_0}(b_0) < \frac{1}{2}$ and so define $b_1 = h^{k_1}_{\lambda_0,b_0}(b_0)$. Since $\lambda_1 > 1$, we choose $k_2 \in \mathbb{N}^*$ so that $h^{k_2}_{\lambda_1,b_0}(b_1) > b_0$ and $h^{k_2-1}_{\lambda_1,b_0}(b_1) \leq b_0$ and so define $b_2 = h^{k_2}_{\lambda_1,b_0}(b_1)$.

- If $n \in \mathbb{N}^*$ is an odd integer, there exists $k_{n+1} \in \mathbb{N}^*$ such that $h^{k_{n+1}}_{\lambda_n,b_0}(b_n) > b_0$ and $h^{k_{n+1}-1}_{\lambda_n,b_0}(b_n) \leq b_0$, so we let $b_{n+1} = h^{k_{n+1}}_{\lambda_n,b_0}(b_n)$.
- If $n \in \mathbb{N}^*$ is an even integer, there exists $k_{n+1} \in \mathbb{N}^*$ such that $h^{k_{n+1}}_{\lambda_n,b_0}(b_n) < \frac{1}{2^{n+1}}$, so we let $b_{n+1} = h^{k_{n+1}}_{\lambda_n,b_0}(b_n)$.

For each $n \in \mathbb{N}$, set $h_{2n} = h_{\lambda_{2n},b_{2n}}$ and $h_{2n+1} = h_{\lambda_{2n+1},b_0}$. Then for any $n \in \mathbb{N}$, we have the following properties:

\[
\begin{aligned}
(6.1) & \quad 0 < b_{2n+1} < \frac{1}{2^{2n+1}} \\
(6.2) & \quad b_{2n+2} = h^{k_{2n+2}}_{2n+1}(b_{2n+1}) > b_0 \\
(6.3) & \quad h^{k_{n+1}}_{n}(b_n) = \left(\prod_{i=0}^{n} \lambda^{k_{i+1}}_{i}\right)b_0
\end{aligned}
\]
By (6.2) and (6.3), we have

\[
\prod_{i=0}^{2n+1} \lambda_i^{k_{i+1}} > 1.
\]

For each \( n \in \mathbb{N} \), set \( \alpha_n = \sum_{i=0}^{n} k_i \). In particular, \( \alpha_0 = 0 \).

Let \( F : Y \to Y \) be the map defined as follows:

For any \( x \in [0, 1] \), \( n \in \mathbb{N} \) and \( \alpha_n \leq i < \alpha_{n+1} \):

\[
\begin{align*}
F(x, 0) &= (f(x), 0), \\
F(0, x) &= (0, x), \\
F(1, x) &= (1, x), \\
F(a_i, x) &= (a_{i+1}, h_n(x)), \\
F(a_{-(i+1)}, x) &= (a_{-i}, h_n^{-1}(x)).
\end{align*}
\]

Then for any \( x \in [0, b_0] \), we have

\[
\begin{align*}
F^1(a_0, x) &= (a_{k_1}, \lambda_0^{k_1} x), \\
F^{k_1+k_2}(a_0, x) &= (a_{k_1+k_2}, \lambda_0^{k_1} \lambda_1^{k_2} x), \\
F^{k_1+k_2+k_3}(a_0, x) &= (a_{k_1+k_2+k_3}, \lambda_0^{k_1} \lambda_1^{k_2} \lambda_2^{k_3} x), \\
& \vdots
\end{align*}
\]

\[
F^{\alpha_n}(a_0, x) = (a_{\alpha_n}, \prod_{i=0}^{n} \lambda_i^{k_{i+1}} x), \quad \text{for any } n \in \mathbb{N}.
\]

**Claim 1.** \( F \) is a homeomorphism on \( Y \):

For each \( n \in \mathbb{Z} \), we have \( F(I_n) = I_{n+1} \). Since \( F(\{0\} \times [0, 1]) = \{0\} \times [0, 1] \) and \( F(\{1\} \times [0, 1]) = \{1\} \times [0, 1] \), \( F(Y) = Y \). As the \( h_n \) are one to one, so does \( F \) and hence \( F : Y \to Y \) is bijective.

Now, to prove that \( F \) is a homeomorphism on \( Y \), it suffices to show that \( F \) is continuous on \( Y \). It is clear that \( F \) is continuous on \( X \setminus \{\{1\} \times [0, 1] \cup \{0\} \times [0, 1] \cup \{(a_n, 0)\}_{n \in \mathbb{Z}}\} \). For any \( y \in [0, 1] \), \( n \in \mathbb{Z} \), \( F(a_n, y) = (a_{n+1}, h_n(y)) \) for some \( i(n) \in \mathbb{N} \). Since \( f(a_n) = a_{n+1} \), \( F \) is continuous on each \( (a_n, 0) \). Take a point \( (1, y_0) \in \{1\} \times [0, 1] \). Take \( \varepsilon > 0 \) and choose \( N \in \mathbb{N}^+ \) so that for any integers \( n \geq \alpha_N \), \( m \geq N \), we have \( 1 - a_n \leq \frac{\varepsilon}{2} \) and \( d_{\infty}(h_m, id_{[0,1]}) \leq \frac{\varepsilon}{2} \) where \( d_{\infty}(h_m, id_{[0,1]}) := \sup_{x \in [0,1]} |h_m(x) - x| \). Denote by \( U := Y \cap ((a_{\alpha_N}, 1] \times (y_0 - \delta, y_0 + \delta)) \), it is an open neighborhood of \( (1, y_0) \) in \( Y \) where \( \delta < \frac{\varepsilon}{4} \) and \( (y_0 - \delta, y_0) \subseteq (0, 1) \). For \( (x, y) \in U \cap \{1\} \times [0, 1] \), we have \( x = 1 \), \( |y - y_0| < \delta \) and

\[
d(F(x, y), F(1, y_0)) = d((x, y), (1, y_0)) < \delta < \varepsilon.
\]
For \((x, y) \in U \cap \cup_{n \in \mathbb{Z}} I_n\), there exist \(n \geq \alpha_N \) and \(i(n) \geq N\) such that \((x, y) = (a_n, y)\) and \(F(x, y) = (a_{n+1}, h_{i(n)}(y))\). Hence,
\[
d(F(x, y), F(1, y_0)) = d((a_{n+1}, h_{i(n)}(y)), (1, y_0))
\]
\[
\leq d((a_{n+1}, h_{i(n)}(y)), (a_{n+1}, y)) + d((a_{n+1}, y), (1, y_0))
\]
\[
\leq |h_{i(n)}(y) - y| + |a_{n+1} - 1| + |y - y_0|
\]
\[
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \delta
\]
\[
\leq \varepsilon.
\]
Similarly, \(F\) is continuous at \((1, 0)\) by taking \(U = (a_N, 1) \times [0, \frac{\varepsilon}{4})\). By the same way, we prove that \(F\) is continuous on \(\{0\} \times [0, 1]\).

**Claim 2.** The subset \(\{a_0\} \times [0, b_0] \subset I_0\) is a scrambled set for both \(F\) and \(F^{-1}\).

Let \((a, y), (a, z) \in \{a_0\} \times [0, b]\) with \(z \neq y\). Then by (6.5):
\[
d(F^{\alpha_{2n+1}}(a_0, y), F^{\alpha_{2n+1}}(a, z)) = d((a_{\alpha_{2n+1}}, (\prod_{i=0}^{2n+1} \lambda_{i+1}^{k_i} y)), (a_{\alpha_{2n+1}}, (\prod_{i=0}^{2n+1} \lambda_{i+1}^{k_i} z))).
\]
By (4.4), we have
\[
d(F^{\alpha_{2n+1}}(a_0, y), F^{\alpha_{2n+1}}(a_0, z)) > |z - y|.
\]
Hence \(((a_0, y), (a_0, z))\) is not asymptotic pair and for every \(n \in \mathbb{N}\), we have
\[
F^{\alpha_{2n}}(\{a_0\} \times [0, b_0]) \subset \{a_{\alpha_{2n}}\} \times [0, b_{2n+1}] \subset \{a_{\alpha_{2n}}\} \times [0, \frac{1}{2^{2n+1}}].
\]
It follows that any pair of \(\{a_0\} \times [0, b_0]\) is proximal. We conclude that \(((a_0, y), (a_0, z))\) is a Li-Yorke pair for \(F\). By the same proof (replace \(F\) by \(F^{-1}\)), any proper pair in \(\{a_0\} \times [0, b_0]\) is a Li-Yorke pair for \(F^{-1}\).

**Remark 6.1.**

(i) The set of recurrent points of \(F\) is \((\{0\} \times [0, 1]) \cup (\{1\} \times [0, 1])\) where all are fixed points. In particular, the scrambled set \(\{a_0\} \times [0, b_0]\) has no recurrent point.

(ii) From the construction of the map \(F\), we deduce that for every \(y \in [0, 1]\), the pair \(((a_0, y), F(a_0, y))\) is asymptotic for both \(F\) and \(F^{-1}\), since the sequence \((h_n)_{n \in \mathbb{N}}\) converges uniformly to \(id_{[0,1]}\).

(iii) There is no completely scrambled homeomorphism on \(Y\). Indeed, suppose that there is a completely scrambled homeomorphism \(g : Y \to Y\). Then \(g^2(\{0\} \times [0, 1]) = \{0\} \times [0, 1]\) and \(g^2(\{1\} \times [0, 1]) = \{1\} \times [0, 1]\) since the only points \((x, y)\) having a non connected neighborhood are in \((\{0\} \times (0, 1]) \cup (\{1\} \times (0, 1])\). By Lemma 2.5, the sets \(\{0\} \times [0, 1]\) and \(\{1\} \times [0, 1]\) contain no Li-Yorke pairs for \(g^2\) and so does for \(g\).
Acknowledgements. I am greatly indebted to Professor Habib Marzougui for helpful discussions on the subject of the paper. I am also grateful to Professor Francisco Balibrea for valuable information.

REFERENCES


Issam Naghmouchi, University of 7th November at Carthage, Faculty of Science of Bizerte, Department of Mathematics, Jarzouna, 7021, Tunisia.

E-mail address: issam.nagh@gmail.com