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Magnetic bottles on geometrically finite hyperbolic surfaces

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Abstract

We consider a magnetic Laplacian \(-\Delta_A = (id + A)^*(id + A)\) on a hyperbolic surface \(M\), when the magnetic field \(dA\) is infinite at the boundary at infinity. We prove that the counting function of the eigenvalues has a particular asymptotic behavior when \(M\) has an infinite area.

1 Introduction

We consider a smooth, connected, complete and oriented Riemannian surface \((M, g)\) and a smooth, real one-form \(A\) on \(M\). We define the magnetic Laplacian

\[-\Delta_A = (i d + A)^*(i d + A) ,
( i d + A)u = i du + uA , \forall u \in C_0^\infty(M; \mathbb{C}) \ .\]

(1.1)

The magnetic field is the exact two-form \(\rho_B = dA\).

If \(dm\) is the Riemannian measure on \(M\), then

\[\rho_B = \tilde{b} \, dm , \quad \text{with} \quad \tilde{b} \in C^\infty(M; \mathbb{R}) .\]

(1.2)

The magnetic intensity is \(b = |\tilde{b}|\).

\(^1\) Keywords: spectral asymptotics, magnetic bottles, hyperbolic surface.
It is well known, (see [Shu]), that $-\Delta_A$ has a unique self-adjoint extension on $L^2(M)$, containing in its domain $C^\infty_0(M;\mathbb{C})$, the space of smooth and compactly supported functions.

When $b$ is infinite at the infinity, (with some additional assumption), the spectrum of $-\Delta_A$ is discrete, and we denote by $(\lambda_j)_j$ the increasing sequence of eigenvalues of $-\Delta_A$, (each eigenvalue is repeated according to its multiplicity). Let

$$N(\lambda) = \sum_{\lambda_j < \lambda} 1.$$  \hspace{1cm} (1.3)

We are interested by the hyperbolic surfaces $M$, when the curvature of $M$ is constant and negative.

In this case, when $M$ has finite area, the asymptotic behavior of $N(\lambda)$ seems to be the Weyl formula: $N(\lambda) \sim_{+\infty} \frac{\lambda}{4\pi} |M|$. S. Golénia and S. Moroianu in [Go-Mo] have such examples.

In the case of the Poincaré half-plane, $M = \mathbb{H}$, we prove in [Mo-Tr] that the Weyl formula is not valid: $\lim_{\lambda \to +\infty} \lambda^{-1} N(\lambda) = +\infty$.

For example when $b(z) = a_0^2 (x/y)^{2m_0} + a_1^2 y^{m_1} + a_2^2 / y^{m_2}$, $a_j > 0$ and $m_j \in \mathbb{N}^*$, then

$$N(\lambda) \sim_{+\infty} \lambda^{1+1/(2m_0)} \ln(\lambda) \alpha(m_0, m_1, m_2).$$

In this paper, we are interested by the hyperbolic surfaces with infinite area. When $M$ is a geometrically finite hyperbolic surface of infinite area and when the above example is arranged for this new situation, ($m_0$ is absent, $m_1$ appears in the cusps and $m_2$ in the funnels), we get

$$N(\lambda) \sim_{+\infty} \lambda^{1+1/m_2} \alpha(m_2);$$

the cusps do not contribute to the leading part of $N(\lambda)$.

## 2 Main result

We assume that $(M, g)$ is a smooth connected Riemannian manifold of dimension two, which is a geometrically finite hyperbolic surface of infinite area; (see [Per] or [Bor] for the definition and the related references). More precisely

$$M = \bigcup_{j=0}^{J_1} M_j \bigcup_{k=1}^{J_2} F_k;$$  \hspace{1cm} (2.1)
where the $M_j$ and the $F_k$ are open sets of $\mathbf{M}$, such that the closure of $M_0$ is compact, and if $J_1 > 0$, the other $M_j$ are cuspidal ends of $\mathbf{M}$, and the $F_k$ are funnel ends of $\mathbf{M}$.

This means that, for any $j$, $1 \leq j \leq J_1$, there exist strictly positive constants $a_j$ and $L_j$ such that $M_j$ is isometric to $\mathbb{S} \times ]a_j^2, +\infty[$, equipped with the metric

$$ds_j^2 = y^{-2}(L_j^2 \, d\theta^2 + dy^2) ; \quad (2.2)$$

($\mathbb{S} = \mathbb{S}^1$ is the unit circle.)

In the same way, for any $k$, $1 \leq k \leq J_2$, there exist strictly positive constants $a_k$ and $L_k$ such that $F_k$ is isometric to $\mathbb{S} \times ]a_k^2, +\infty[$, equipped with the metric

$$ds_k^2 = \tau_k^2 \cosh^2(t)d\theta^2 + dt^2 ; \quad (2.3)$$

moreover, for any two integers $j$, $k > 0$, we have $M_j \cap F_k = \emptyset$ and $M_j \cap M_k = F_j \cap F_k = \emptyset$ if $j \neq k$.

Let us choose some $z_0 \in M_0$ and let us define

$$d : \mathbf{M} \rightarrow \mathbb{R}^+ ; \quad d(z) = d_g(z, z_0) ; \quad (2.4)$$

d_g(. , . ) denotes the distance with respect to the metric $g$.

We assume the smooth one-form $A$ to be given such that the magnetic field $\tilde{b}$ satisfies

$$\lim_{d(z) \to \infty} b(z) = +\infty . \quad (2.5)$$

If $J_1 > 0$, there exists a constant $C_1 > 0$ such

$$|X\tilde{b}(z)| \leq C_1(b(z) + 1)e^{d(z)}|X|_g ; \quad (2.6)$$

$\forall z \in M_j$, $\forall X \in T_z\mathbf{M}$ and $\forall j = 1, \ldots J_1$.

There exists a constant $C_2 > 0$ such

$$|X\tilde{b}(z)| \leq C_2(b(z) + 1)|X|_g ; \quad (2.7)$$

$\forall z \in F_k$, $\forall X \in T_z\mathbf{M}$ and $\forall k = 1, \ldots J_2$.

For any self-adjoint operator $P$, and for any real $\lambda$, we will denote by $E_\lambda(P)$ its spectral projection, and when its trace is finite we will denote it by

$$N(\lambda; P) = Tr(E_\lambda(P)) .$$

$N(\lambda; P)$ is the number of eigenvalues of $P$, (counted with their multiplicity), which are in $]-\infty, \lambda[.$
Theorem 2.1 Under the above assumptions, $-\Delta_A$ has a compact resolvent and for any $\delta \in \left[\frac{1}{3}, \frac{2}{5}\right]$, there exists a constant $C > 0$ such that
\[
\frac{1}{2\pi} \int_M \left(1 - \frac{C}{(b(m) + 1)^{2/5}}\right) N(\lambda(1 - C\lambda^{-3\delta + 1}) - \frac{1}{4} b(m)) \, dm \\
\leq N(\lambda, -\Delta_A) \leq \frac{1}{2\pi} \int_M \left(1 + \frac{C}{(b(m) + 1)^{2/5}}\right) N(\lambda(1 + C\lambda^{-3\delta + 1}) - \frac{1}{4} b(m)) \, dm
\]
where
\[
N(\mu, b(m)) = b(m) \sum_{k=0}^{+\infty} \mu - (2k + 1)b(m) ]_+ \quad \text{if } b(m) > 0 ,
\]
and
\[
N(\mu, b(m)) = \mu/2 \quad \text{if } b(m) = 0 .
\]

$[\rho]_+^0$ is the Heaviside function:
\[
[\rho]_+^0 = \begin{cases} 1 , & \text{if } \rho > 0 \\ 0 , & \text{if } \rho \leq 0 . \end{cases}
\]

The Theorem remains true if we replace $\int_M$ by $\sum_{k=1}^{J_2} \int_{F_k}$, due to the fact that the other parts are bounded by $C\lambda$.

Corollary 2.2 Under the assumptions of Theorem 2.1 and if the function
\[
\omega(\mu) = \int_M [\mu - b(m)]^0 \, dm
\]
satisfies, $\exists C_1 > 0$ s.t. $\forall \mu > C_1 , \forall \tau \in ]0,1[, \omega ((1 + \tau) \mu) - \omega(\mu) \leq C_1 \tau \omega(\mu) ,
\]
then
\[
N(\lambda, -\Delta_A) \sim \frac{1}{2\pi} \int_M N(\lambda - \frac{1}{4}, b(m)) \, dm .
\]
For example this allows us to consider magnetic fields of the following type:

\[
F_k, \quad b(\theta, t) = p_k \left( \frac{1}{cosh(t)} \right),
\]

and on \( M_j, \quad j > 0, \quad b(\theta, y) = q_j(y), \)

where the \( p_k(s) \) and the \( q_j(s) \) are, for large \( s \), polynomial functions of order \( \geq 1 \). In this case, if \( d \) is the largest order of the \( p_k(s) \), then

\[
N(\lambda; -\Delta_A) \sim \alpha \lambda^{1+1/d},
\]

for some constant \( \alpha > 0 \), depending only on the funnels \( F_k \) where the order of \( p_k(s) \) is \( d \).

3 Estimate for Dirichlet operators

3.1 The main propositions

In this section, we consider some particular open set \( \Omega \) of \( M \) with smooth boundary. To \( \Omega \) and \( -\Delta_A \), we associate the Dirichlet operator \( -\Delta^\Omega \), and we estimate \( N(\lambda; -\Delta^\Omega) \).

**Proposition 3.1** Let \( \Omega \) an open set of \( M_0 \) with smooth boundary. Then there exists a constant \( C_\Omega > 0 \) s.t.

\[
\left| N(\lambda; -\Delta^\Omega) - \frac{|\Omega|}{4\pi} \lambda \right| \leq C_\Omega \sqrt{\lambda}; \quad \forall \lambda > 1.
\]

As \( \overline{\Omega} \) is compact, the above estimate is well known. See for example Theorem 29.3.3 in [Hor].

**Proposition 3.2** Let \( j > 0 \) and \( \Omega \) an open set of the cusp \( M_j \), isometric to \( S \times (a^2, +\infty] \), equipped with the metric

\[
ds^2 = y^{-2} \left( L^2 d\theta^2 + dy^2 \right); \quad (a \text{ and } L \text{ are strictly positive constants}).
\]

Then \( -\Delta^\Omega \) has a compact resolvent and

\[
N(\lambda; -\Delta^\Omega) \sim \frac{|\Omega|}{4\pi} \lambda; \quad \text{as } \lambda \to +\infty.
\]
We will prove it in the next subsection.

**Proposition 3.3** Let \( \Omega \) an open set of a funnel \( F_k \), isometric to \( S \times [a^2, +\infty[ \), equipped with the metric 

\[
ds^2 = L^2 \cosh^2(t) \, dt^2 + \, dt^2 ; \quad (a \text{ and } L \text{ are strictly positive constants}).
\]

Then \( -\Delta^\Omega_A \) has a compact resolvent and for any \( \delta \in \left[ \frac{1}{3}, \frac{2}{5} \right] \), there exists a constant \( C > 0 \) such that 

\[
\frac{1}{2\pi} \int_\Omega \left( 1 - \frac{C}{(b(m) + 1)^{(2-5\delta)/2}} \right) N(\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4}, b(m)) \, dm \\
\leq N(\lambda, -\Delta^\Omega_A) \leq \frac{1}{2\pi} \int_\Omega \left( 1 + \frac{C}{(b(m) + 1)^{(2-5\delta)/2}} \right) N(\lambda(1 + C\lambda^{-3\delta+1}) - \frac{1}{4}, b(m)) \, dm
\]

The proof comes easily following the ones in the Poincaré half-plane of \([\text{Mo-Tr}]\), using the method of \([\text{Col}]\), in the neighbourhood of the boundary at infinity. It corresponds to a context where the partitions of unity were fine, so they can be performed on \( S \times [a^2, +\infty[ \), (instead of \( \mathbb{R} \times ]-\infty,0[ \) ) .

### 3.2 Proof of Proposition 3.2

For simplicity we change the unit circle \( S = S_1 \) into the circle \( S_L \) , of radius \( L \) , so 

\[
\Omega = S_L \times [a^2, +\infty[ , \quad ds^2 = y^{-2}(dx^2 + dy^2) , \quad \text{and} \quad -\Delta^\Omega_A u(z) = y^2[(D_x - A_1)^2 u(z) + (D_y - A_2)^2 u(z)] ;
\]

moreover 

\[
d(z, z') = \text{arg} \cosh \frac{y^2 + y'^2 + d^2_{S_L}(x, x')}{2yy'}.
\]

We begin by proving the compactness of the resolvent of \( -\Delta^\Omega_A \).

**Lemma 3.4** There exists \( C_0 > 1 \) such that 

\[
\int_\Omega (b(z) - C_0)|u(z)|^2 \, dm \leq \int_\Omega -\Delta^\Omega_A u(z) \overline{u(z)} \, dm ; \quad \forall u \in C_0^\infty(\Omega).
\]
Proof. Let us denote the quadratic form
\[ q_\Omega^A(u) = \int_\Omega -\Delta_\Omega^A u(z)\overline{u(z)}\,dm \quad \forall \, u \in C^\infty_0(\Omega). \] (3.2)
Then
\[ q_\Omega^A(u) = \int_\Omega \left[ |(D_x - A_1)u|^2 + (D_y - A_2)u|^2 \right] \,dxdy, \]
and
\[ \left| \int_\Omega \overline{\tilde{b}(z)}|u(z)|^2\,dm \right| = \left| \int_\Omega [(D_x - A_1)u(z)(D_y - A_2)u(z) - (D_y - A_2)u(z)(D_x - A_1)u(z)]\,dxdy \right|. \]
Therefore we get that
\[ \left| \int_\Omega \overline{\tilde{b}(z)}|u(z)|^2\,dm \right| \leq q_\Omega^A(u). \]
As \( b(z) = |\tilde{b}(z)| \to +\infty \) at the infinity, the Lemma comes easily.

The Lemma 3.4 and the assumption (2.5) prove that \( -\Delta_\Omega^A \) has compact resolvent.

Later on, we will need that the assumptions (2.5) and (2.6) ensure that there exists \( C > 1 \) such that
\[ b(z)/C \leq b(z') \leq Cb(z), \quad \text{if} \quad |y - y'| \leq 1 \text{ and } y > C. \] (3.3)
This comes from the fact that \( d(z) \) is equivalent to \( \ln(y) \) for \( y(>1) \) large enough, so the assumption (2.6) ensures that \( |\partial_x b(z)| + |\partial_y b(z)| \leq C(|b(z)| + 1). \)

Lemma 3.5 There exists a constant \( C_0 > 1 \) such that, for any \( \lambda > 1 \) and for any \( K \subset \Omega \) isometric to \( I_1 \times I_2 \), endowed with the metric in (3.4), with
\[ I_1 = ]x_0 - \epsilon_1, x_0 + \epsilon_1[, \quad I_2 = ]y_0 - \epsilon_2, y_0 + \epsilon_2[, \]
\[ \epsilon_1 \in ]C_0^{-1}, 1[, \quad \epsilon_2 = \sqrt{y_0}/\sqrt{b(z_0)}, \quad (y_0 > C_0); \]
the following estimates hold:
\[ [\lambda(1 - \frac{1}{\sqrt{y_0}}) - C_0]\frac{|K|_y}{4\pi} \leq N(\lambda; -\Delta_K^A) \leq [\lambda(1 + \frac{1}{\sqrt{y_0}}) + C_0]\frac{|K|_y}{4\pi}. \] (3.4)

Proof. If \( b(z_0) > C\lambda \), then, according to the estimate of Lemma 3.4 with \( K \) instead of \( \Omega \), \( N(\lambda; -\Delta_K^A) = 0 \).
So we can assume that \( b(z_0) \leq C\lambda \).
We use that the spectrum of $-\Delta^K_A$ is gauge-invariant, so we can suppose that in $K$

$$A_2 = 0 \quad \text{and} \quad A_1(x, y) = -\int_{y_0}^y \frac{\tilde{b}(x, \rho)}{\rho^2} d\rho.$$ 

Then $|A_1(x, y)| \leq C \epsilon^2 \frac{b(z_0)}{y_0^2}$. From this estimate, we get that for any $\epsilon \in ]0, 1[$,

$$-(1-\epsilon)\Delta^K_0 - C \epsilon^2 \frac{b^2(z_0)}{\epsilon y_0^2} \leq -\Delta^K_A \leq -(1+\epsilon)\Delta^K_0 + C \epsilon^2 \frac{b^2(z_0)}{\epsilon y_0^2}.$$ 

We take $\epsilon = 1/\sqrt{y_0}$, to get

$$-(1 - \frac{1}{\sqrt{y_0}})\Delta^K_0 - C \frac{b(z_0)}{\sqrt{y_0}} \leq -\Delta^K_A \leq -(1 + \frac{1}{\sqrt{y_0}})\Delta^K_0 + C \frac{b(z_0)}{\sqrt{y_0}}.$$ 

As $b(z_0) \leq C \lambda$, the Lemma follows easily from the min-max principle and the well-known estimate for $N(\lambda; -\Delta^K_0)$.

**Proof of Proposition 3.2.**

It follows easily from Lemma 3.5 (for large $y$), using the same tricks as in [Mo-Tr].

## 4 Proof of the main Theorem 2.1

The proof comes easily from the three propositions 3.1 - - 3.3, following the method developped in [Mo-Tr].

## 5 Remark on the case of constant magnetic field

It is not always possible to have a constant magnetic field on $M$, (for topological reason), but for any $(b, \beta) \in \mathbb{R}^{J_1} \times \mathbb{R}^{J_2}$, there exists a one-form $A$, such that the corresponding magnetic field $dA$ satisfies

$$dA = \tilde{b}(z) dm \quad \left\{ \begin{array}{l} \tilde{b}(z) = b_j \forall z \in M_j \\ \tilde{b}(z) = \beta_k \forall z \in F_k \end{array} \right.$$  

(5.1)
Theorem 5.1 Assume (2.1) and (5.1).

If $J_1 = 0$ and $J_2 > 0$, then the essential spectrum of $-\Delta_A$ is

$$\text{sp}_{\text{ess}}(-\Delta_A) = \left[ \frac{1}{4} + \inf_{\beta_k} \beta_k^2, +\infty \right] \bigcup \left( \bigcup_{k=1}^{J_2} S(\beta_k) \right)$$

with $S(\beta_k) = \emptyset$ when $|\beta_k| \leq 1/2$ and when $|\beta_k| > 1/2$

$S(\beta_k) = \{(2j+1)|\beta_k| - j(j+1) ; j \in \mathbb{N}, j < |\beta_k| - 1/2\}$.

If $J_1$ and $J_2$ are $> 0$, then for any $j$, $1 \leq j \leq J_1$ and for any $z \in M_j$ there exists a unique closed curve through $z$, $C_{j,z}$ in $(M_j, g)$, not contractible and with zero $g-$curvature. The following limit exists and is finite:

$$[A]_{M_j} = \lim_{d(z) \to +\infty} \int_{C_{j,z}} A.$$  \hspace{1cm} (5.3)

If $J_1^A = \{ j \in \mathbb{N} , 1 \leq j \leq J_1 \ \text{s.t.} \ [A]_{M_j} \in 2\pi \mathbb{Z} \}$, then

$$\text{sp}_{\text{ess}}(-\Delta_A) = \left[ \frac{1}{4} + \min \{ \inf_{j \in J_1^A} \beta_j^2, \inf_{1 \leq k \leq J_2} \beta_k^2 \} , +\infty \right] \bigcup \left( \bigcup_{k=1}^{J_2} S(\beta_k) \right).$$  \hspace{1cm} (5.4)

If $J_2 = 0$ and $J_1^A = \emptyset$, then $\text{sp}_{\text{ess}}(-\Delta_A) = \emptyset$.

$-\Delta_A$ has purely discrete spectrum, (its resolvent is compact).

Remark 5.2 In Theorem 5.1, one can change $C_{j,z}$ into $S_{j,z}$, the unique closed curve through $z$, not contractible and with minimal $g-$length. $S_{j,z}$ is not smooth at $z$, $S_{j,z}$ is part of two geodesics through $z$, so there is an out-going tangent and an incoming tangent at $z$. It is easy to see that $C_{j,z} \cap S_{j,z} = \{ z \}$, so by Stokes formula

$$\int_{S_{j,z}} (A - A^0) = \int_{C_{j,z}} (A - A^0),$$

where $A^0$ is a one-form on $M$, such that

$$dA = dA^0 \quad \text{on} \quad M_j \quad \text{and} \quad [A^0]_{M_j} = 0 \quad \forall \ j.$$

The orientation in both cases $C_{j,z}$ and $S_{j,z}$, is chosen such that, if $u_z, v_z \in T_z M_j$, $g_z(u_z, v_z) = 0$, $dm(u_z, v_z) > 0$, and $u_z$ is tangent to the curve (in the positive direction), then $v_z$ points to boundary at infinity. (for $S_{j,z}$, one can take as $u_z$ the out-going tangent, or the incoming tangent).
Proof of Theorem 5.1. It is clear that

\[ \text{sp}_{\text{ess}}(-\Delta_A) = \left( \bigcup_{j=1}^{J_1} \text{sp}_{\text{ess}}(-\Delta_{A_j}) \right) \bigcup \left( \bigcup_{k=1}^{J_2} \text{sp}_{\text{ess}}(-\Delta_{F_k}) \right); \]  

(5.5)

so the proof will result on the two lemmas below.

Lemma 5.3

\[ \text{sp}_{\text{ess}}(-\Delta_{F_k}) = \left[ \frac{1}{4} + \beta_k^2, +\infty \right] \cup S(\beta_k). \]

Proof. We have 

\[ -\Delta_{F_k} = \tau_k^{-2} \cosh^{-2}(t)(D_\theta - A_1)^2 + \cosh^{-1}(t)(D_t - A_2) [\cosh(t)(D_t - A_2)]. \]

Since \( \beta_k = \tau_k^{-1} \cosh^{-1}(t)(\partial_\theta A_2 - \partial_t A_1) \), there exists a function \( \varphi \) such that \( A - \tilde{A} = d\varphi \) if \( \tilde{A} = (\xi - \beta_k \tau_k \sinh(t))d\theta \), (for some constant \( \xi \)). So we can assume that \( A = \tilde{A} \).

We change the density \( dm = \tau_k \cosh(t)d\theta dt \) for \( d\theta dt \), using the unitary operator \( Uf = (\tau_k \cosh(t))^{1/2}f \), so

\[ P = -U\Delta_{F_k}^* U = \tau_k^{-2} \cosh^{-2}(t)(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}(1 + \cosh^{-2}(t)). \]

We remind that \( \lambda \in \text{sp}_{\text{ess}}(-\Delta_{F_k}) \) iff there exists a sequence \((u_j)_j \in \text{Dom}(-\Delta_{F_k})\) converging weakly in \( L^2(F_k) \) to zero, \( \|u_j\|_{L^2(F_k)} = 1 \) and such that the sequence \((-\Delta_{F_k} u_k - \lambda u_k)_k\) converges strongly to zero.

It is clear that \( \text{sp}(-\Delta_{F_k}^*) = \text{sp}\left(\bigoplus_{\ell \in \mathbb{Z}} P_\ell\right) \),

\[ P_\ell = D_t^2 + \tau_k^{-2} \cosh^{-2}(t)(\ell + \beta_k \tau_k \sinh(t) - \xi)^2 + \frac{1}{4}(1 + \cosh^{-2}(t)), \]

for the Dirichlet condition on \( L^2(I; dt) ; I = [a_k^2, +\infty[. \)

So \( \text{sp}(-\Delta_{F_k}^*) = \bigcup_{\ell \in \mathbb{Z}} \text{sp}(P_\ell) \).

Writing that \( P_\ell = D_t^2 + \left( \frac{\ell - \xi}{\tau_k \cosh(t)} + \beta_k \tanh(t) \right)^2 + \frac{1}{4}(1 + \cosh^{-2}(t)), \)

we get easily that \( \text{sp}_{\text{ess}}(P_\ell) = \left[ \frac{1}{4} + \beta_k^2, +\infty \right], \) and that the number of eigenvalues < \( \frac{1}{4} + \beta_k^2 \) is finite for all \( \ell < \xi \) and equal to zero for all \( \ell \geq \xi \). Here
we assume $\beta_k > 0$ . So $[\frac{1}{4} + \beta_k^2, +\infty[ \subset \sp_{\text{ess}}(-\Delta^F_k) \text{ and the other part of } \sp_{\text{ess}}(-\Delta^F_k)$ is $S_\infty = \{\lambda; \lambda = \lim_{j \to +\infty} \lambda_{\ell(j)} \text{, } \lambda_{\ell(j)} \in \sp_d(P_{\ell(j)})\}$ ,

where $\ell(j), j$ denotes any decreasing sequence of negative integers.

Now we use again the formula,

$$P_\ell = D^2_\ell + \left(\frac{\ell - \xi}{\tau_k \cosh(t)} + \beta_k \tanh(t)\right)^2 + \frac{1}{4} (1 + \cosh^{-2}(t)) .$$

Assuming $\ell - \xi < 0 \text{, we set } \rho = |\ell - \xi|/\tau_k$ and we introduce the new variable $y = 2\rho e^{-t}$ . We get that $P_\ell$ is unitarily equivalent to $\tilde{P}_\rho$ defined as a Dirichlet type operator in $L^2([0, 2\rho e^{-\alpha}]; dy)$ , (zero boundary condition is only required on the right boundary):

$$\tilde{P}_\rho = D_\rho(y^2 D_y) + W_\rho(y) , \text{ with }$$

$$W_\rho(y) = \left(\beta_k \left(1 - y^2/(4\rho^2)\right) - y \frac{1}{1 + y^2/(4\rho^2)}\right)^2 + \left(\frac{y/(2\rho)}{1 + y^2/(4\rho^2)}\right)^2 .$$

So we have $\lim_{\rho \to +\infty} W_\rho(y) = W_\infty(y) = (\beta_k - y)^2$ , and the operator $\tilde{P}_\infty = D_\rho(y^2 D_y) + W_\infty(y)$ on $L^2([0, +\infty[; dy)$ satisfies, (see [Mo-Ti]), $\sp(\tilde{P}_\infty) = \sp_{\text{ess}}(\tilde{P}_\infty) \cup \sp_d(\tilde{P}_\infty)$ with $\sp_{\text{ess}}(\tilde{P}_\infty) = [\frac{1}{4} + \beta_k^2, +\infty[ ; \sp_d(\tilde{P}_\infty) = S(\beta_k) .$

We remind that the eigenfunctions associated to the eigenvalues in $S(\beta_k)$ of $\tilde{P}_\infty$ are exponentially decreasing, so if $\lambda_0(\rho) \leq \ldots \leq \lambda_j(\rho) \leq \lambda_{j+1}(\rho) \ldots$ are the eigenvalues of $\tilde{P}_\rho$ then for any $j$ ,

$$\lim_{\rho \to +\infty} \lambda_j(\rho) = \lambda_j(\infty) = (2j + 1)\beta_k - j(j + 1) , \text{ if } \beta_k > 1/2 \text{ and } j < \beta_k - 1/2 ,$$

otherwise $\lim_{\rho \to +\infty} \lambda_j(\rho) = \frac{1}{4} + \beta_k^2$ .

Therefore we get that $S_\infty = S(\beta_k)$ , or $S_\infty = S(\beta_k) \cup \{\frac{1}{4} + \beta_k^2\}$ : the formula of Lemma 5.4 follows.

**Lemma 5.4** If $1 \leq j \leq J_1 \text{ and } j \notin J_1^A$ , then

$$\sp_{\text{ess}}(-\Delta^M_A) = \emptyset .$$

If $j \in J_1^A$ , then

$$\sp_{\text{ess}}(-\Delta^M_A) = \left[\frac{1}{4} + b_j^2, +\infty[ .$$
Proof. Use the coordinate $t = \ln y$ instead of $y$, so

$$M_j = S \times [\alpha_j^2, +\infty[ \quad \text{and} \quad ds_j^2 = L_j^2 e^{-2t} d\theta^2 + dt^2 ; \quad (\alpha_j = e^{\alpha_j}) .$$

Then $-\Delta^{M_j}_A = L_j^{-2} e^{2t}(D_\theta - A_1)^2 + e'(D_t - A_2)(e^{-t}(D_t - A_2))$,

$$\tilde{b} = L_j^{-1} e^t(\partial_\theta A_2 - \partial_t A_1) \quad \text{and} \quad dm = L_j e^{-t} d\theta dt . \quad \text{As in Lemma 5.3, we have}$$

$$A - \tilde{A} = d\varphi \quad \text{if} \quad \tilde{A} = (\xi + L_j b_j e^{-t})d\theta , \quad \text{(for some constant} \ \xi) .$$

So we can also assume that $A = \tilde{A}$.

We replace the density $dm$ by $d\theta dt$, using the unitary operator

$$Uf = \sqrt{L_j} e^{-t/2} f , \quad \text{so} \quad P = -U \Delta^{M_j}_A U^* = L_j^{-2} e^{2t}(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4} .$$

Then we get also that

$$\text{sp}(-\Delta^{M_j}_A) = \text{sp}(P) = \bigcup_{\ell \in \mathbb{Z}} \text{sp}(P_\ell) ; \quad P_\ell = D_t^2 + \frac{1}{4} + \left( e^t(\ell + \xi) \right)^2 ,$$

for the Dirichlet condition on $L^2(I; dt) ; \quad I = [\alpha_j^2, +\infty[ .$

When $\ell + \xi \neq 0$, the spectrum of $P_\ell$ is discrete. More precisely

$$\text{sp}(P_\ell) = \text{sp}(P^\pm) , \quad \text{where} \quad P^\pm = D_t^2 + \frac{1}{4} + (\pm e^t + b_j)^2$$

for the Dirichlet condition on $L^2(I_{j,\ell}; dt) ; \quad I_{j,\ell} = [\alpha_j^2 + \ln(|\ell + \xi|/L_j), +\infty[ .$

and $\pm = \frac{\ell + \xi}{|\ell + \xi|} .$

So \( \lim_{|\ell| \to \infty} \inf \text{sp}(P_\ell) = +\infty \), and then we get easily that the spectrum of

$$-\Delta^{M_j}_A$$

is discrete, when $\xi = [A]_{M_j}/(2\pi) \notin \mathbb{Z} . \quad \text{If} \ \ell + \xi = 0 , \ \text{the spectrum of} \ P_\ell \ \text{is absolutely continuous} :$$

$$\text{sp}(P_{-\ell}) = \text{sp}_{ess}(P_{-\ell}) = \text{sp}_{ac}(P_{-\ell}) = \left[ \frac{1}{4} + b_j^2 , +\infty[ ;$$

and then, when $[A]_{M_j} \in 2\pi \mathbb{Z} , \ \text{sp}_{ess}(-\Delta^{M_j}_A) = \left[ \frac{1}{4} + b_j^2 , +\infty[ .$

This achieves the proof of Lemma 5.4.
References


