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Abstract

A procedure for determining two parameters to be used in Kautz approximation is presented. It is based on minimisation of an upper bound of the error energy.

Index terms: Orthonormal approximation, Signal representation, Modelling, Mathematical techniques.

Introduction:

Poorly damped systems are difficult to approximate with a reasonable number of Laguerre functions, so the so-called two-parameter Kautz functions which can approximate more efficiently signals with strong oscillatory behavior, have received much attention in the recent mathematical modelling and identification literature (see, e.g., [[1]] and the references therein). These functions can be defined by their Laplace transforms

\[
\hat{\varphi}_{2k} (s) = \frac{\sqrt{2bc}}{s^2 + bs + c} \left( \frac{s^2 - bs + c}{s^2 + bs + c} \right)^k \\
\hat{\varphi}_{2k+1} (s) = \frac{s\sqrt{2bc}}{s^2 + bs + c} \left( \frac{s^2 - bs + c}{s^2 + bs + c} \right)^k
\]

where the numbering of the functions as defined in [[1]] has been slightly modified for suitability. The time functions \(\varphi_n\) or as \(\varphi_n (t, b, c)\) whenever it is desirable to exhibit the parameters. The orthonormal set \(\{\varphi_n\}\) is complete in \(L^2 [0, \infty]\), thus any finite energy real causal signal \(f(t)\) can be approximated within any prescribed accuracy by truncating its infinite expansion \(f(t) = \sum_{n=0}^{\infty} a_n \varphi_n (t)\) where \(a_n = \langle f, \varphi_n \rangle\) is the \(n + 1\) th Fourier coefficient. The \(N\)-term truncated expansion yields the best approximation to \(f(t)\) of the form \(\tilde{f}(t) = \sum_{n=0}^{N-1} a_n \varphi_n (t)\) in the sense of minimising the integrated squared error (ISE)

\[
Q = \int_0^\infty [f(t) - \tilde{f}(t)]^2 dt = \|f\|^2 - \sum_{n=0}^{N-1} a_n^2 = \sum_{n=N}^{\infty} a_n^2 .
\]

Usually, since the \(a_n\) depend on \(b\) and \(c\), \(Q\) can be reduced further by a proper choice of these parameters. Nice optimality conditions for Kautz approximation, generalizing that of the Laguerre case [[2], [3]], have been derived by Oliveira e Silva [[4]] and den Brinker [[5]]. However, these conditions of great theoretical interest can result in complicated computations in practical cases. For Laguerre functions [[6], [7]] and other classical functions [[8], [9]] an alternative easy-to-use and efficient approach, based on minimisation of an upper bound of the error energy, has been proposed. It is the purpose of this Letter to derive a somewhat similar procedure for the specific set of non-classical two-parameter Kautz functions.

Key relationship:

Recently [[1]], it has been shown that the coefficients \(a_n\) can be found from power series calculations in the following manner. Denoting by \(\widehat{f}(s)\) the Laplace transform of \(f(t)\), assumed to be analytic outside an appropriate region in the \(s\)-plane, let \(F_i, i = 1, 2\), be defined by

\[
F_1 (s) = \left[ s^2 \widehat{f}(s) - c \widehat{f}(c/s) \right] / (s^2 - c) \\
F_2 (s) = \left[ \widehat{f}(c/s) - \widehat{f}(s) \right] s \sqrt{c} / (s^2 - c) .
\]

Since \(F_i (c/s) = F_i (s)\), both these functions are symmetric functions of \(c/s\) and \(s\) and so they can be represented as functions of \((c/s) + s\) and \((c/s) s = c\), whence

\[
F_i (s) = \hat{f}_i (s + c/s, c) , \quad i = 1, 2 .
\]

Notice that the transformation \( s \to s + c / s \) is familiar in filter design where it is used to design a band-pass filter from a low-pass filter. The trick to relate Kautz coefficients and power series is to observe the remarkable relationship

\[
\hat{f}_i(s, c) = \sum_{n=0}^{\infty} a_{2n+2-i} \hat{i}_n(s, b), \quad i = 1, 2
\]

(5)

where \( \hat{i}_n(s, b) = \sqrt{2b} (s - b)^n / (s + b)^{n+1} \) denotes the Laplace transform of the normalised Laguerre function \( l_n(bt) \). Thus, the Fourier coefficients associated with the expansion of \( f(t) \) with respect to the orthonormal set \( \{\varphi_n\} \) can be obtained via Laguerre expansions.

**Proposed procedure for pertinent parameters:**

Denoting by \( f_i(t, c) \) the inverse Laplace transform of \( \hat{f}_i(s, c) \), let us define moments \( M_j \) by

\[
M_0 = \int_0^{\infty} \left[ (f_1)^2 + (f_2)^2 \right] dt
\]

(6)

\[
M_1(c) = \int_0^{\infty} t \left[ (f_1)^2 + (f_2)^2 \right] dt
\]

(7)

\[
M_2(c) = \int_0^{\infty} t \left[ \left( \frac{df_1}{dt} \right)^2 + \left( \frac{df_2}{dt} \right)^2 \right] dt
\]

(8)

where we have used \( f_i \) as shorthand for \( f_i(t, c) \). Since the Laguerre functions are orthonormal and the Kautz functions are orthonormal also, both \( M_0 \) and \( \|f\|^2 \) are equal to \( \sum_{n=0}^{\infty} a_n^2 \), hence \( M_0 = \|f\|^2 \) is a constant. On the other hand, \( M_1 \) and \( M_2 \) depend on \( c \) but do not depend on \( b \).

**Theorem 1:**

Let \( q = Q / \|f\|^2 \), \( m_i(c) = M_i(c) / \|f\|^2 \), \( i = 1, 2 \). Then, the normalised ISE associated with a 2\( K \)-term Kautz approximation is bounded by

\[
q \leq B = \frac{1}{2K} \left[ \frac{m_2(c)}{b} + bm_1(c) - 1 \right].
\]

(9)

This bound attains its minimum when \( b = \sqrt{m_2(c) / m_1(c)} \). The minimum itself is \( B_{\min} = \left( 2\sqrt{m_1(c) m_2(c)} - 1 \right) / (2K) \).

**Proof:**

Let \( M_j = M_{j1} + M_{j2} \) where \( M_{jj} \) denotes the contribution of \( f_j \) \( (M_0 = \int_0^{\infty} (f_1)^2 dt, ... \) ). Then, the ISE \( Q_i = \sum_{n=K}^{\infty} a_{2n+1, j} \) associated with the \( K \)-term Laguerre approximation of \( f_i(t, c) \) (see eqn. 5) is known \([6]\) to be bounded by \( (M_{2i} / b + M_{1i} b - M_{0i}) / (2K) \), provided that \( (2K + 1) \geq (M_{2i} / b + M_{1i} b) / M_{0i} \), \( i = 1, 2 \), a condition which is assumed to hold in the following (\( K \) is sufficiently large). In view of eqn. 1 the ISE associated with the \( (N = 2K) \)-term Kautz approximation of \( f(t) \) is \( Q = Q_1 + Q_2 \) and can then be bounded as \( Q \leq (M_2 / b + M_1 b - M_0) / (2K) \). Dividing throughout by \( \|f\|^2 \) \( M_0 \) achieves the proof of eqn. 9. Writing \( dB / db = 0 \), the last part of the theorem follows readily.

For a fixed \( c > 0 \), let \( C = \text{C}(c; m_1, m_2) \) denote the class of signals \( f \in L^2[0, \infty] \) with given \( m_1 \) and \( m_2 \). There exist signals \( f \in C \) that achieve the bound in eqn. 9; as a simple example, consider \( C(5; 0.4, 1.6) \): it is a standard exercise to show that \( f(t) = 3\varphi_0(t, 2.5) + \varphi_0(t, 2.5) \) is in this class. Clearly, the 6-term Kautz approximation using \( \varphi_0(t, 2.5) \) is \( f(t) = 3\varphi_0(t, 2.5) \) with \( q = 0.1 \) and \( B = (1.6 / 2 \times 0.4 - 1) / 6 = 0.1 \), whence \( q = B \). Therefore, the bound in eqn. 9 is actually the maximum ISE for signals in \( C \) and theorem 1 gives the best \( b \), in the sense of minimising the maximum integrated squared error, that can be obtained with the knowledge of the signal limited to \( m_1 \) and \( m_2 \).

Now, suppose that \( m_1 \) and \( m_2 \) are known for more than one value of \( c \), say for \( c \in C \) where \( C \) represents a discrete or continuous set of positive numbers. Since the lowest \( m_1 \) and \( m_2 \) will result in the lowest \( B_{\min} \), we have the following theorem.

**Theorem 2:**

Let \( c_0 \) denote that value of \( c \in C \) at which the product \( m_1 \) and \( m_2 \) is minimum and let \( b_0 = \sqrt{m_2(c_0) / m_1(c_0)} \). Then, a pertinent choice for the pair of Kautz parameters is \( (b_0, c_0) \), which yields \( B_{\min} = \left( 2\sqrt{m_1(c_0) m_2(c_0)} - 1 \right) / (2K) \).

**Remark:**

Notice that \( b_0 \) and \( c_0 \) do not depend on the number \( N = 2K \) of functions to be used. Thus \( b_0 \) and \( c_0 \) can be computed in a first time and \( N \) can be chosen afterwards: for instance, one can choose \( N \) such that the upper bound \( B_{\min} \) is small enough or such that the exact \( q = 1 - \sum_{n=0}^{N-1} a_n^2 / \|f\|^2 \) is small enough.
**Illustrative example:**

Consider the Laplace transform

\[ \hat{f}(s) = \frac{s^3 + 4s^2 + 8s + 1}{s^4 + 5s^3 + 13s^2 + 19s + 18} \]

with a view to deriving a second-order approximation \((N = 2)\) Kautz functions. Letting for example \(c = 4\), eqns. 2-4 yield

\[ \hat{f}_1(s, 4) = \frac{9s^3 + 72s^2 + 182s + 127}{9s^4 + 83s^3 + 267s^2 + 349s + 164} \]

\[ \hat{f}_2(s, 4) = \frac{s^3 + 19s^2 + 81s + 87}{9s^4 + 83s^3 + 267s^2 + 349s + 164} \]

Using one of the available techniques (e.g. [10]), the required moments are computed as \(M_0 = \|f\|^2 = 0.5183\), \(M_1(4) = 0.2531\), \(M_2(4) = 0.3278\) and the error bound is minimised when \(b = \sqrt{M_2(4)/M_1(4)} = \sqrt{M_2(4)/M_1(4)} = 1.138\).

With \(b = 1.138\) and \(c = 4\), the first and second coefficients of the Kautz expansion are \(a_0 = 0.2965\) and \(a_1 = 0.6365\) from which the exact normalised ISE is obtained as \(q = 4.878 \times 10^{-2}\).

The normalised moments computed by repeating the procedure for \(c = 2\) and \(c = 3\) are shown in Table 1. For \(c \in \{2, 3, 4\}\), the product \(m_1(c) m_2(c)\) is minimum if \(c = 3\); therefore, in agreement with theorem 2, we select \(b_0 = \sqrt{0.4980/0.5138} = 0.9846\) and \(c_0 = 3\), improving the normalised ISE which becomes \(q = q_o = 2.505 \times 10^{-3}\). It is worth noting that \(q_o\) obtained using limited knowledge of the signal (Table 1) is, for this example, very close to the best possible value \(q_{opt} = 2.486 \times 10^{-3}\) that can be achieved with complete knowledge of the signal.

<table>
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<th>2</th>
<th>3</th>
<th>4</th>
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<td>(m_1(c))</td>
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<td>0.5138</td>
<td>0.4883</td>
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<tr>
<td>(m_2(c))</td>
<td>0.6747</td>
<td>0.4980</td>
<td>0.6325</td>
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<td>0.2559</td>
<td>0.3088</td>
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**Table 1:** Normalised moments for \(c = 2, 3, 4\)

**Conclusion:**

A procedure for improving a Kautz approximation, in the case of a limited number of expansion terms, by a proper choice of a pair of free parameters, has been presented. It possesses desirable features and can be readily adapted to the discrete time case. This work is underway.

**References**


