

Two proofs of the Bermond-Thomassen conjecture for tournaments with bounded minimum in-degree

Stéphane Bessy, Nicolas Lichiardopol, Jean-Sébastien Sereni

► **To cite this version:**

Stéphane Bessy, Nicolas Lichiardopol, Jean-Sébastien Sereni. Two proofs of the Bermond-Thomassen conjecture for tournaments with bounded minimum in-degree. *Discrete Mathematics*, Elsevier, 2010, 310 (3), pp.557–560. <10.1016/j.disc.2009.03.039>. <hal-00487977>

HAL Id: hal-00487977

<https://hal.archives-ouvertes.fr/hal-00487977>

Submitted on 31 May 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Two proofs of the Bermond-Thomassen conjecture for tournaments with bounded minimum in-degree

Stéphane Bessy* Nicolas Lichiardopol[†]
Jean-Sébastien Sereni[‡]

Abstract

The Bermond-Thomassen conjecture states that, for any positive integer r , a digraph of minimum out-degree at least $2r - 1$ contains at least r vertex-disjoint directed cycles. Thomassen proved that it is true when $r = 2$, and very recently the conjecture was proved for the case where $r = 3$. It is still open for larger values of r , even when restricted to (regular) tournaments. In this paper, we present two proofs of this conjecture for tournaments with minimum in-degree at least $2r - 1$. In particular, this shows that the conjecture is true for almost regular tournament. In the first proof, we prove auxiliary results about union of sets contained in other union of sets, that might be of independent interest. The second one uses a more graph-theoretical approach, by studying the properties of a maximum set of vertex-disjoint directed triangles.

1 Introduction

In 1981, Bermond and Thomassen [2] conjectured that for any positive integer r , any digraph of minimum out-degree at least $2r - 1$ contains at least r vertex-disjoint directed cycles. It is trivially true when r is one, and it was proved by Thomassen [7] when r is two in 1983. Very recently, the conjecture

*LIRMM, Université Montpellier 2r, Montpellier, France. E-mail: Stephane.Bessy@lirmm.fr.

[†]I3S-EPU, Sophia Antipolis, France. E-mail: lichiar@club-internet.fr.

[‡]MASCOTTE I3S (CNRS-UNSA) – INRIA, Sophia Antipolis, France. E-mail: sereni@sophia.inria.fr.

was also proved in the case where r is three [6]. It is still open for larger values of r . We prove, in two different ways, that the restriction of this conjecture to almost regular tournaments is true.

Chen, Gould and Li [3] proved that a k -strongly-connected tournament of order at least $5k - 3$, contains k vertex-disjoint directed cycles. Given a tournament T , let $q(T)$ be the maximum order of a transitive subtournament of T . Li and Shu [4] showed that any strong tournament T of order n with $q(T) \leq \frac{n-5k+8}{2}$ can be vertex-partitioned into k cycles. However, these results do not prove the Bermond-Thomassen conjecture for regular tournaments.

The following definitions are those of the monograph by Bang-Jensen and Gutin [1]. A *tournament* is a digraph T such that for any two distinct vertices x and y , exactly one of the couples (x, y) and (y, x) is an arc of T . The vertex set of T is $V(T)$, and its cardinality is the *order* of T . The set of arcs of T is $\mathcal{A}(T)$. A vertex y is a *successor* of a vertex x if (x, y) is an arc of T . A vertex y is a *predecessor* of a vertex x if x is a successor of y . The number of successors of x is the *out-degree* $\delta^+(x)$ of x , and the number of predecessors of x is the *in-degree* $\delta^-(x)$ of x . Let $\delta^+(T) := \min\{\delta^+(x) : x \in V(T)\}$, $\delta^-(T) := \min\{\delta^-(x) : x \in V(T)\}$ and $\delta(T) := \min\{\delta^+(T), \delta^-(T)\}$.

Given a tournament T , its *reversing tournament* is the tournament $T' = (V(T), \mathcal{A}')$, where $\mathcal{A}' := \{(x, y) : (y, x) \in \mathcal{A}(T)\}$. A tournament is *regular of degree d* if $\delta^+(x) = \delta^-(x) = d$ for every vertex x . Necessarily, the order of such a tournament is $2d + 1$. It is *almost regular* if $|\delta^+(x) - \delta^-(x)| \leq 1$ for every vertex x . An almost regular tournament of odd order is regular, and an almost regular tournament T of even order v is characterised by $\delta^+(T) = \delta^-(T) = \frac{v}{2} - 1$.

For any subset A of $V(T)$, we let $T(A)$ be the sub-tournament induced by the vertices of A . By a path or a cycle of a tournament T , we mean a directed path or a directed cycle of T , respectively. By disjoint cycles, we mean vertex-disjoint cycles. A cycle of length three is a *triangle*.

A tournament is *acyclic*, or *transitive*, if it does not contain cycles, i.e. if its vertices can be ranged into a unique Hamiltonian path x_1, \dots, x_n such that (x_i, x_j) is an arc if and only if $i < j$. As is well-known, and straightforward to prove, a non-acyclic tournament contains a triangle. In particular, note that if a tournament contains k disjoint cycles, then it contains k disjoint triangles.

2 Preliminary results

Let (x, y) be an arc of a tournament T . We set

$$\begin{aligned} A(x, y) &:= \{z \in V(T) : (z, x) \in \mathcal{A}(T) \text{ and } (z, y) \in \mathcal{A}(T)\}, \\ B(x, y) &:= \{z \in V(T) : (x, z) \in \mathcal{A}(T) \text{ and } (y, z) \in \mathcal{A}(T)\}, \\ E(x, y) &:= \{z \in V(T) : (z, x) \in \mathcal{A}(T) \text{ and } (y, z) \in \mathcal{A}(T)\}, \text{ and} \\ F(x, y) &:= \{z \in V(T) : (x, z) \in \mathcal{A}(T) \text{ and } (z, y) \in \mathcal{A}(T)\}. \end{aligned}$$

Note that $E(x, y)$ is the set of vertices z such that x, y and z form a triangle. We let $a(x, y), b(x, y), e(x, y)$ and $f(x, y)$ be the respective cardinalities of these four sets. The proof of the following proposition is straightforward, and can be found in [5], so we omit it.

Proposition 2.1. *If (x, y) is an arc of a tournament, then $e(x, y) = f(x, y) + \delta^+(y) - \delta^+(x) + 1$.*

A set of cardinality m is an m -set. We give now three new results, which may be of independent interest. The first one is essential in our first proof of the Bermond-Thomassen conjecture for almost regular tournaments.

Theorem 2.2. *Fix two integers $m \geq 3$ and $r \geq 1$. Let $n \in \{1, 2, \dots, r\}$ and $s = \lceil \frac{r+m-1}{2} \rceil$. For every $i \in \{1, 2, \dots, n\}$, let B_i be an m -set, and for every $j \in \{1, 2, \dots, s\}$, fix a set $A_j \subseteq \bigcup_{1 \leq i \leq n} B_i$ of cardinality at least $r + m + 1 - 2j$. Then, there exist $i \in \{1, 2, \dots, n\}$ and distinct elements j and k of $\{1, 2, \dots, s\}$ such that B_i has distinct elements x and y with $x \in A_j$ and $y \in A_k$.*

Proof. If $n < r$, then proving the result for the sets B'_1, B'_2, \dots, B'_r with $B'_i = B_i$ if $i \leq n$ and $B'_i = B_n$ if $i > n$ will yield the desired conclusion. So, we suppose now that $n = r$, and we use induction on r .

Observe that it is sufficient to prove that there exist $i \in \{1, 2, \dots, n\}$ and distinct integers $j, k \in \{1, 2, \dots, s\}$ such that $|A_j \cap B_i| \geq 1$ and $|A_k \cap B_i| \geq 2$.

The assertion is true when $r = 1$. Indeed, in this case, $s = \lceil \frac{1+m-1}{2} \rceil = \lceil \frac{m}{2} \rceil \geq 2$, $|A_1| \geq m \geq 3$, $|A_2| \geq m - 2 \geq 1$ and B_1 is an m -set such that $A_i \subseteq B_1$ for $i \in \{1, 2, \dots, s\}$. Therefore, $|A_1 \cap B_1| \geq 3$ and $|A_2 \cap B_1| \geq 1$, which yields the desired conclusion.

The assertion is true also for $r = 2$. Indeed, in this case, $s = \lceil \frac{2+m-1}{2} \rceil = \lceil \frac{m+1}{2} \rceil \geq 2$, $|A_1| \geq m + 1 \geq 4$, $|A_2| \geq m - 1 \geq 2$ and $A_1 \cup A_2 \subseteq B_1 \cup B_2$. Clearly, $A_1 \cap B_1 \neq \emptyset$ — otherwise B_2 would contain A_1 , which has at least $m + 1$ elements — and similarly, $A_1 \cap B_2 \neq \emptyset$. If $|A_1 \cap B_1| \geq 2$ and $|A_1 \cap B_2| \geq 2$, then the result holds. Otherwise, we have, say, $|A_1 \cap B_1| = 1$ and hence

$|A_1 \cap B_2| = m$. Now, either $|A_2 \cap B_1| \geq 2$ or $|A_2 \cap B_2| \geq 1$, so the result holds.

Suppose now that the assertion is true for every $k < r$, for some integer $r \geq 3$, and let us prove it for r . Then, $s = \lceil \frac{r+m-1}{2} \rceil \geq 3$, $|A_1| \geq r + m - 1$ and $|A_2| \geq r + m - 3 \geq r$. Without loss of generality, we assume that $|B_1 \cap A_1| \geq |B_2 \cap A_1| \geq \dots \geq |B_r \cap A_1|$.

Suppose first that $|B_2 \cap A_1| \leq 1$. Then, $B_2 \cup \dots \cup B_r$ contains at most $r - 1$ elements of A_1 and $B_1 \cup B_2 \cup \dots \cup B_r$ contains at least $r + m - 1$ elements of A_1 . So, we deduce that $|B_1 \cap A_1| = m$ and $|B_i \cap A_1| = 1$ for every $i \in \{2, 3, \dots, r\}$. The assertion of the theorem holds if $|B_1 \cap A_2| \geq 1$. If $|B_1 \cap A_2| = 0$, then there exists $i \in \{2, 3, \dots, r\}$, such that $|B_i \cap A_2| \geq 2$ — otherwise we would have $|(B_1 \cup B_2 \cup \dots \cup B_r) \cap A_2| \leq r - 1$, a contradiction. Clearly, B_i contains distinct elements x and y with $x \in A_1$ and $y \in A_2$.

Suppose now that $|B_2 \cap A_1| \geq 2$. In this case, $|B_1 \cap A_1| \geq 2$, $|B_2 \cap A_1| \geq 2$ and the desired conclusion holds if $B_1 \cup B_2$ contains an element of $A_2 \cup \dots \cup A_s$. If $B_1 \cup B_2$ does not contain an element of $A_2 \cup \dots \cup A_s$, let $A'_i := A_{i+1}$ for $i \in \{1, 2, \dots, s-1\}$. We have $s-1 = \lceil \frac{r-2+m-1}{2} \rceil$, $|A'_i| \geq r-2+m+1-2i$ and $A'_i \subseteq \bigcup_{3 \leq j \leq r} B_j$ for $i \in \{1, 2, \dots, s-1\}$. Therefore, by the induction hypothesis

there exist $i \in \{3, \dots, r\}$ and distinct elements j and k of $\{2, \dots, s\}$ such that B_i contains distinct elements x and y with $x \in A_j$ and $y \in A_k$, which concludes the proof. \square

The second and third results can be proved analogously, and we omit their proofs.

Theorem 2.3. *Fix two integers $m \geq 3$ and $r \geq 2$. Let $n \in \{1, 2, \dots, r\}$, and for every $i \in \{1, 2, \dots, n\}$, denote by B_i an m -set. For every $j \in \{1, 2, \dots, r\}$, let $A_j \subseteq \bigcup_{1 \leq i \leq n} B_i$ with $|A_j| \geq r+m+1-2j$. Then, there exist $i \in \{1, \dots, n\}$ and distinct elements j and k of $\{1, \dots, r\}$ such that B_i has distinct elements x and y with $x \in A_j$ and $y \in A_k$.*

The best result is a combination of the first two.

Theorem 2.4. *Fix two integers $m \geq 3$ and $r \geq 2$. Let $n \in \{1, 2, \dots, r\}$ and set $s = \min(\lceil \frac{r+m-1}{2} \rceil, r)$. For $i \in \{1, 2, \dots, n\}$, denote by B_i an m -set, and for every $j \in \{1, 2, \dots, s\}$, let $A_j \subseteq \bigcup_{1 \leq i \leq n} B_i$ with $|A_j| \geq r+m+1-2j$. Then, there exist $i \in \{1, \dots, n\}$ and distinct elements j and k of $\{1, \dots, s\}$ such that B_i has distinct elements x and y with $x \in A_j$ and $y \in A_k$.*

3 Disjoint cycles in tournaments T with $\delta(T) \geq 2r - 1$

In this section, we give two different proofs of the following result.

Theorem 3.1. *For any $r \geq 1$, every tournament T with $\delta(T) \geq 2r - 1$ contains r disjoint cycles.*

Proof. The case $r = 1$ being a simple observation, we assume that $r \geq 2$. Let v be the order of T , and let n be the maximum number of disjoint cycles of T . Thus, n is also the maximum number of disjoint triangles: let T_i , $i \in \{1, 2, \dots, n\}$ be n disjoint triangles. Let $V' := V(T) \setminus \bigcup_{1 \leq j \leq n} V(T_j)$ and $p := |V'|$. Suppose that $n \leq r - 1$. Thus, $p \geq v - 3(r - 1)$, that is $p \geq r + 2$, since $v \geq 4r - 1$. The subtournament $T(V')$ is acyclic — otherwise, we would have an extra cycle — and, consequently, its vertices can be ranged into a Hamiltonian path x_1, \dots, x_p such that (x_i, x_j) is an arc of $T(V')$ if and only if $i < j$, see Figure 1.

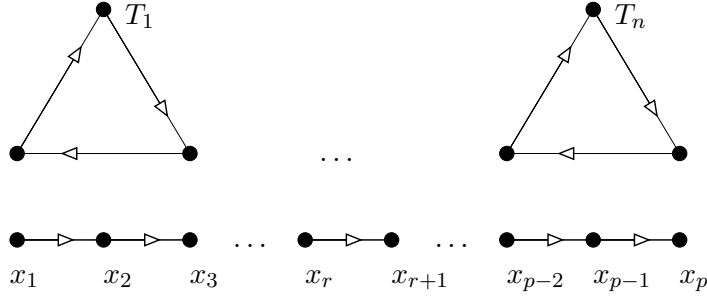


Figure 1: Disjoint triangles and Hamiltonian path of $T(V')$

For $i \in \{1, 2, \dots, \lceil \frac{r+1}{2} \rceil\}$, consider the arc (x_i, x_{p+1-i}) : each vertex x_j with $j \in \{i + 1, i + 2, \dots, r + 2 - i\}$ belongs to $F(x_i, x_{p+1-i})$. Therefore,

$$f(x_i, x_{p+1-i}) \geq p - 2i \geq v - 3n - 2i.$$

By Proposition 2.1,

$$e(x_i, x_{p+1-i}) \geq p - 2i + \delta^+(x_{p+1-i}) - \delta^+(x_i) + 1.$$

Since $2r - 1 \leq \delta^+(x) \leq v - 2r$ for every vertex x , we deduce that

$$e(x_i, x_{p+1-i}) \geq v - 3n - 2i + 2r - 1 - (v - 2r) + 1 \geq (r - 1) + 3 + 1 - 2i,$$

as $n \leq r - 1$.

Observe now that every vertex of $E(x_i, x_{p+1-i})$ forms a triangle with the vertices x_i and x_{p+1-i} . Moreover, as $T(V')$ is acyclic, we have $E(x_i, x_{p+1-i}) \subseteq \bigcup_{1 \leq j \leq n} V(T_j)$ for $i \in \{1, 2, \dots, \lceil \frac{r+1}{2} \rceil\}$. Hence, the conditions of Theorem 2.2 are fulfilled — the r of the theorem being $r - 1$, m being three, $s = \lceil \frac{r+1}{2} \rceil$, $A_i = E(x_i, x_{p+1-i})$ and $B_j = V(T_j)$. Consequently, with $s = \lceil \frac{r+1}{2} \rceil$, there exist $i \in \{1, \dots, n\}$ and distinct elements j and k of $\{1, \dots, s\}$ such that $V(T_i)$ contains distinct vertices x and y with $x \in E(x_j, x_{p+1-j})$ and $y \in E(x_k, x_{p+1-k})$. Each T_q , for $q \in \{1, 2, \dots, n\} \setminus \{i\}$, and the tournaments induced by x_j, x_{p+1-j}, x and by x_k, x_{p+1-k}, y are $n+1$ disjoint triangles, which contradicts the definition of n . Therefore, T contains at least r disjoint cycles, as desired. \square

Second proof of Theorem 3.1. As mentioned in the Introduction, Thomassen [7] proved the conjecture in the general case for $r \leq 2$, and the general case for $r = 3$ was recently proved [6]. Thus, we assume in this proof that $r \geq 4$.

Suppose that V' is a subset of at least 6 vertices such that $T(V')$ is acyclic. Let $\{x_1, x_2, \dots, x_p\}$ be the vertices of V' , indexed such that (x_i, x_j) is an arc if and only if $i < j$. We set $A_{V'} := \{x_1, x_2, x_3\}$ and $B_{V'} := \{x_{p-2}, x_{p-1}, x_p\}$. For a vertex x , let $s_{V'}^-(x)$ be the *in-score of x with respect to V'* , that is the number of predecessors of x in $B_{V'}$. Analogously, $s_{V'}^+(x)$ is the *out-score of x with respect to V'* , that is the number of successors of x in $A_{V'}$. Given a subgraph H of T , the *in-score of H with respect to V'* is

$$s_{V'}^-(H) := \sum_{x \in V(H)} s_{V'}^-(x).$$

We define $s_{V'}^+(H)$, the *out-score of H with respect to V'* , analogously regarding the out-scores of the vertices of H . Last, the *score of H with respect to V'* is $s_{V'}(H) = s_{V'}^-(H) + s_{V'}^+(H)$. In all these notations, we may omit the subscript if the context is clear.

As in the first proof, let n be the maximum number of disjoint triangles, and consider a family T_i , $i \in \{1, 2, \dots, n\}$, of n disjoint triangles. We set $V' := V(T) \setminus \bigcup_{1 \leq j \leq n} V(T_j)$ and $p := |V'|$. Again, we consider the Hamiltonian path x_1, \dots, x_p of the acyclic tournament $T(V')$ such that (x_i, x_j) is an arc of $T(V')$ if and only if $i < j$.

Suppose that $n \leq r - 1$. Then, we obtain that $p \geq 4r - 1 - 3(r - 1)$, that is $p \geq r + 2$, and hence $p \geq 6$ since $r \geq 4$.

For each triangle T_i , we have $s^-(T_i) \leq 9$ and $s^+(T_i) \leq 9$. If $s^-(T_i) \geq 7$ and $s^+(T_i) \geq 4$, then there exists a matching of size three from $B_{V'}$ to T_i , and a matching of size two from T_i to $A_{V'}$. Therefore, $T(A_{V'} \cup B_{V'} \cup V(T_i))$

contains two disjoint triangles, which contradicts the maximality of n . Thus, either $s^-(T_i) \leq 6$ or $s^+(T_i) \leq 3$. Similarly, either $s^+(T_i) \leq 6$ or $s^-(T_i) \leq 3$.

We assert that $s(T_i) \leq 12$ for each triangle T_i : indeed, if $s^-(T_i) > 6$, then $s^+(T_i) \leq 3$, and since $s^-(T_i) \leq 9$, we infer that $s(T_i) \leq 12$. In the same way, if $s^+(T_i) > 6$, one can deduce that $s(T_i) \leq 12$. Finally, if $s^-(T_i) \leq 6$ and $s^+(T_i) \leq 6$, we also have $s(T_i) \leq 12$. Hence, the sum s of the scores of the n triangles is at most $12n$.

Observe that the vertices x_p, x_{p-1} and x_{p-2} have $\delta^+(x_p), \delta^+(x_{p-1}) - 1$ and $\delta^+(x_{p-2}) - 2$ successors in $\bigcup_{1 \leq j \leq n} V(T_j)$, respectively. Moreover, the vertices x_1, x_2 and x_3 have respectively $\delta^-(x_1), \delta^-(x_2) - 1$ and $\delta^-(x_3) - 2$ predecessors in $\bigcup_{1 \leq j \leq n} V(T_j)$. It follows that

$$s = \delta^+(x_p) + \delta^+(x_{p-1}) + \delta^+(x_{p-2}) + \delta^-(x_1) + \delta^-(x_2) + \delta^-(x_3) - 6.$$

Therefore, it holds that

$$\delta^+(x_p) + \delta^+(x_{p-1}) + \delta^+(x_{p-2}) + \delta^-(x_1) + \delta^-(x_2) + \delta^-(x_3) - 6 \leq 12n \leq 12r - 12.$$

Recall that $\delta^+(x) \geq 2r - 1$ and $\delta^-(x) \geq 2r - 1$ for every vertex x . Thus, we infer that $\delta^+(x_p) = \delta^+(x_{p-1}) = \delta^+(x_{p-2}) = \delta^-(x_1) = \delta^-(x_2) = \delta^-(x_3) = 2r - 1$, $n = r - 1$ and $s(T_i) = 12$ for every triangle T_i . Note that this assertion holds for any set on n disjoint triangles — their score being with respect to the remaining vertices.

For each integer $i \in \{4, 5, \dots, p-3\}$, the vertex x_i belongs to $F(x_3, x_{p-2})$, and hence $f(x_3, x_{p-2}) \geq p - 6$. Therefore, by Proposition 2.1,

$$\begin{aligned} e(x_3, x_{p-2}) &\geq p - 6 + \delta^+(x_{p-2}) - \delta^+(x_3) + 1 \\ &\geq v - 3(r - 1) - 6 + (2r - 1) - (v - 1 - 2r + 1) + 1 \\ &\geq r - 3 \geq 1. \end{aligned}$$

Consequently, there exists a vertex x of some triangle T_j such that the vertices x_3, x_{p-2}, x induce a triangle T' . Let y and z be the vertices of T_j different from x . The triangles T' and T_i for $i \neq j$ form a new collection of n disjoint triangles, and $V'' := (V' \setminus \{x_3, x_{p-2}\}) \cup \{y, z\}$ is the set of the remaining vertices. Consider now the set $A_{V''}$: observe that x_3 has at most two successors in $A_{V''}$, and it can have two only if both y and z belong to $A_{V''}$. Furthermore, the predecessors of x_3 in $B_{V''}$ can only be y and z . Therefore, it follows that $s_{V''}^-(x_3) + s_{V''}^+(x_3) \leq 3$ with equality only if both y and z belong to $B_{V''}$. Similarly, $s_{V''}^-(x_{p-2}) + s_{V''}^+(x_{p-2}) \leq 3$ with equality only if both y and z belong to $A_{V''}$. Thus, the score of the triangle T' with respect to V'' is at most 11, a contradiction. This concludes the proof. \square

References

- [1] J. Bang-Jensen and G. Gutin. *Digraphs*. Springer-Verlag London Ltd., London, 2001. Theory, algorithms and applications.
- [2] J.-C. Bermond and C. Thomassen. Cycles in digraphs—a survey. *J. Graph Theory*, 5(1):1–43, 1981.
- [3] G. Chen, R. J. Gould, and H. Li. Partitioning vertices of a tournament into independent cycles. *J. Combin. Theory Ser. B*, 83(2):213–220, 2001.
- [4] H. Li and J. Shu. The partition of a strong tournament. *Discrete Math.*, 290(2-3):211–220, 2005.
- [5] N. Lichiardopol. Dominated and dominating arcs in tournaments. Vertex strong connectivity of doubly regular tournaments. Submitted for publication.
- [6] N. Lichiardopol, A. Pór, and J.-S. Sereni. A step towards the Bermond-Thomassen conjecture about disjoint cycles in digraphs. Submitted for publication. ITI Series 2007-339, 2007.
- [7] C. Thomassen. Disjoint cycles in digraphs. *Combinatorica*, 3(3-4):393–396, 1983.