A note on antisymmetric flows in graphs

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Abstract

We prove that any orientation of a graph without bridges and directed 2-edge-cuts admits a $\mathbb{Z}_2^3 \times \mathbb{Z}_3^9$-antisymmetric flow, which improves the bounds obtained by DeVos, Johnson and Seymour, and DeVos, Nešetřil and Raspaud.

1 Introduction

Nowhere-zero flows form an extensively studied topic in graph theory. They were introduced by Tutte [14] in 1954 as the dual notion for graph colorings, and they have spawned many deep and important conjectures such as the Cycle Double Cover Conjecture of Seymour [10] and Szekeres [13], and Tutte's 3-Flow, 4-Flow and 5-Flow Conjectures [14, 15, 16, 17]. Several modifications of classical nowhere-zero flows were later developed, e.g., bidirected flows by Bouchet [1] as notion dual to colorings of graphs embedded on non-orientable surfaces. Nešetřil and Raspaud [9], motivated by the oriented chromatic

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number and their passion for dualities, introduced *antisymmetric flows*, the notion dual to strong oriented colorings of graphs.

Let us now introduce some basic notation related to flows. For an Abelian group M and a directed graph $G = (V, A)$, an M-flow is a mapping $f : A \to M$ that satisfies Kirchhoff's law at every vertex. The flow f is *nowhere-zero* if $0 \notin f(A)$, and it is *antisymmetric* if $f(A) \cap -f(A) = \emptyset$. So an antisymmetric flow is in particular nowhere-zero, and antisymmetric flows can be viewed as "nowhere-inverse" flows.

Note that if a directed graph G admits a nowhere-zero M-flow (M-NZF), then so does any directed graph obtained by re-orienting some of the edges of G. Moreover, Tutte [14] proved that G then admits an M'-NZF for any Abelian group M' of order at least $|M|$. In other words, having a nowhere-zero flow depends neither on the orientation of the graph, nor on the structure of the considered Abelian group. On the other hand, the existence of an antisymmetric M-flow (M-ASF) depends on the structure of the Abelian group M, and not only on its order. As an example, consider a directed circuit: assigning 1 to every arc yields a $\mathbb{Z}_4$-ASF, while no graph admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$-ASF since every element of this latter group is self-inverse. The existence of an M-ASF also depends on the chosen orientation: if the edges of a circuit are not directed cyclically, then any two arcs outgoing from a same vertex must be assigned inverse elements of M. Hence, a directed circuit has an M-ASF if and only if it is cyclically directed and M is not $\mathbb{Z}_2^k$ for some positive integer k.

Let us now look for some obstructions for the existence of nowhere-zero flows and antisymmetric flows in graphs. Bridges are the only obstacle for the existence of a nowhere-zero flow (since the value assigned to a bridge by any flow must be 0). For antisymmetric flows, another obstacle is a directed 2-edge-cut, i.e., a cut formed by two edges directed in the same direction. Any flow must assign inverse values to two such arcs. These are the only two obstacles for the existence of an antisymmetric flow: Nešetřil and Raspaud [9] proved that any bridgeless directed graph G without a directed 2-edge-cut admits a $\mathbb{Z}_3^t$-ASF, where t is the dimension of the cycle space of G.

A natural question is whether there exists a fixed Abelian group M such that every bridgeless directed graph with no directed 2-edge-cut has an M-ASF, and if so, what is the smallest order of such a group M. For nowhere-zero flows, the answer to the first part of this question was given by Kilpatrick [7] and Jaeger [5] who independently showed that every bridgeless graph has a $\mathbb{Z}_8$-NZF. Seymour [11] later improved this result by establishing that every bridgeless graph has a $\mathbb{Z}_6$-NZF. Note that the Tutte 5-Flow Conjecture [14] from 1954 asserts that every bridgeless graph has a $\mathbb{Z}_5$-NZF and is still open.

For antisymmetric flows, Nešetřil and Raspaud [9] proved that every planar directed 3-edge-connected graph has a $\mathbb{Z}_6^5$-ASF, and asked whether any orientation of a 3-edge-connected (not necessarily planar) graph has an M-ASF for a fixed Abelian group M. This was answered positively by Devos, Johnson and Seymour [2].

Theorem 1 (DeVos, Johson and Seymour [2]). *Every directed bridgeless graph without a directed* 2*-edge-cut admits a* $\mathbb{Z}_2^8 \times \mathbb{Z}_3^{17}$*-*ASF*.*

As noted by DeVos, Johnson and Seymour, the order of the group offered by the theorem does not seem to be the smallest possible. Indeed, DeVos, Nešetřil and Raspaud [3] managed to improve this result.

Theorem 2 (DeVos, Nešetřil and Raspaud [3]). *Every directed bridgeless graph without a directed* 2*-edge-cut admits a* $\mathbb{Z}_2^6 \times \mathbb{Z}_3^9$*-*ASF*;*

We improve Theorem 2 by a multiplicative factor of 8.

Theorem 3. *Every directed bridgeless graph without a directed* 2*-edge-cut admits a* $\mathbb{Z}_2^3 \times \mathbb{Z}_3^9$*-*ASF*.*

Sopena [12] proved the existence of directed 3-edge-connected (planar) graphs with no M-ASF for any Abelian group M of order less than 16, and Marshall [8] improved this bound to 17. It follows from this result and Theorem 3 that the smallest order of an Abelian group M such that every orientation of a 3-edge-connected graph has an M-ASF is between 17 and $2^3 \cdot 3^9 = 157\,464$.

2 Preliminaries

We now introduce basic notation and results used in this paper. The reader is referred to the monograph of Zhang [18], or the book by Diestel [4, Chapter 6] for further exposition on flows.

Let $G = (V, A)$ be a directed graph (the graphs considered in this paper may contain parallel edges and loops). For each $X \subseteq V$, we define $\delta^+(X)$ to be the set of arcs directed from X to $V \setminus X$, and we set $\delta^-(X) := \delta^+(V \setminus X)$. Given an Abelian group M, an M*-flow* of G is a mapping $f : A \to M$ such that for every set $X \subseteq V$,

$$\sum_{e\in\delta^-(X)} f(e) \;=\; \sum_{e\in\delta^+(X)} f(e).$$

As already mentioned, the flow f is *nowhere-zero* if $0 \notin f(A)$, and *antisymmetric* if $f(A) \cap -f(A) = \emptyset$.

Let $G = (V, A)$ be a directed graph and let $e \in A$. The directed graph G/e is obtained by *contracting the arc* e, i.e. identifying its endvertices (and keeping parallel edges and loops if they appear). If H is a subgraph of G, the graph G/H is obtained by successively contracting all the arcs within H. The definition readily extends to undirected graphs, for which we keep the same notations. In our considerations, we will need the following property of flows which directly follows from the definition: if G is a connected graph and f is an M-flow of G/H for some subgraph H of G, then G admits an M-flow whose restriction to G/H is equal to f.

In the proof of Theorem 3, we utilize some of the results of DeVos, Nešetřil and Raspaud [3] on antisymmetric flows in highly edge-connected graphs. In particular, the following is of use to us [3, Theorems 4 and 7].

Theorem 4 (DeVos, Nešetřil and Raspaud [3])**.**

(i) Every directed 4*-edge-connected graph admits a* $\mathbb{Z}_2^2 \times \mathbb{Z}_3^4$-ASF*; and*

(ii) every directed 5*-edge-connected graph admits a* $\mathbb{Z}_3^5$-ASF.

Another ingredient for our proof is a structural result of Kaiser and Škrekovski [6, Corollary 1.3] on cycles in graphs. By *cycle*, we mean a subgraph all of whose vertices have even degree.

Theorem 5. *Every* 3*-edge-connected graph G contains a cycle C such that the graph G/C (obtained by contracting each component of C to a vertex) is* 5*-edge-connected.*

Suppose that G is an orientation of a 2-edge-connected graph, and let v be a vertex of degree $d > 3$. By *expanding v to a circuit* we mean the following (see Figure 1 for an example). Let $e_1, e_2, \ldots, e_d$ be the arcs of G incident with v, and say that the arc e_i joins v and u_i (note that the vertices u_i need not be all distinct). First, we replace v by a circuit $C(v) := v_1 v_2 \ldots v_d$ of length d, and we orient its edges cyclically. Let $C_1, C_2, \ldots, C_k$ be the connected components of $G - v$. Since G is 2-edge-connected, v is joined to each component C_i by at least 2 edges (and hence $d \geq 2k$). So we can choose the ordering of the arcs e_i such that u_i and u_{k+i} both belong to the component C_i for $i \in \{1, 2, \ldots, k\}$. For each $i \in \{1, \ldots, d\}$, we add an arc between v_i and u_i, directed in the same way as the arc e_i. A consequence of our choice of the correspondence between the vertices of the circuit $C(v)$ and the edges incident with v is that no matter how the vertices of $C(v)$ are split into two connected parts, there exists a component C_i having a neighbor in both the parts. We let $H(G)$ be any directed graph obtained from G by recursively expanding vertices of degree more than 3 to circuits.

Finally, a graph G is *d-regular* if every vertex has degree d, and a *d-factor* of G is a spanning d-regular subgraph of G.

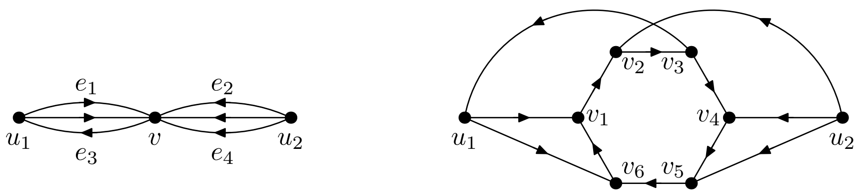

Figure 1: The vertex v is expanded to a circuit of length 6. When removing v from the original graph (on the left), we obtain two components C_1 (composed of the vertex u_1) and C_2. Thus, we order the arcs incident with v so that e_1 and e_3 join v to C_1, while e_2 and e_4 join v to C_2 (the remaining arcs being ordered arbitrarily).

3 The Result

Let us start this section with the following observation on the expansion graph $H(G)$ defined in Section 2.

Lemma 6. *If G is a 3-edge-connected graph, then so is $H(G)$.*

Proof. Let v be a vertex of $G = (V, A)$ of degree $d > 3$. It suffices to show that the graph G_1 obtained from G by expanding v to a circuit $C(v)$ is 3-edge-connected. Let e and e' be any two edges of G_1, set $G' := G_1 - \{e, e'\}$ and suppose that G' is not connected. Then both e and e' are edges of $C(v)$, since otherwise $\{e, e'\} \cap A$ would be an edge-cut of G, a contradiction. Thus, the vertices of $C(v)$ are split into two connected parts P_1 and P_2, according to the connected component of G' to which they belong. Consequently, as we noted earlier, there exists a connected component of $G - v$ having a neighbor in each of P_1 and P_2, a contradiction. $\square$

We are now ready to prove Theorem 3.

Proof of Theorem 3. The proof proceeds by contradiction. Set $M := \mathbb{Z}_2^3 \times \mathbb{Z}_3^9$, and let $G = (V, A)$ be a minimal counter-example to Theorem 3. The graph G is a directed 2-edge-connected graph without a directed 2-edge-cut. Moreover, we may assume that G is 3-edge-connected (and thus has minimum degree at least 3). Indeed, suppose on the contrary that $\{e_1, e_2\}$ is a 2-edge-cut of G. Let G' be the graph obtained from G by contracting the arc e_1 (see Figure 2). Thus, G' is a directed 2-edge-connected graph without a directed 2-edge-cut. Hence, by the minimality of G, there exists an M-ASF f of G'. Since $\{e_1, e_2\}$ is not a directed 2-edge-cut of G, we can extend f to an M-ASF of G by setting $f(e_1) := f(e_2)$, a contradiction.

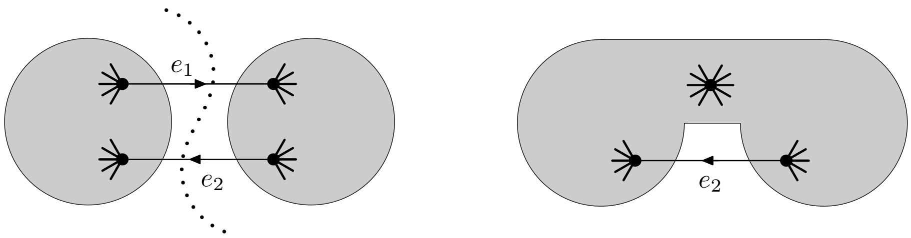

Figure 2: The graph on the right, obtained by contracting the arc e_1, has an M-ASF f by the minimality of G. We obtain an M-ASF of G by setting $f(e_1) := f(e_2)$.

Let $H := H(G)$. The graph H is 3-edge-connected by Lemma 6. Moreover, if H admits an M-ASF, then so does G. So it only remains to prove that H has an M-ASF to obtain a contradiction.

By Theorem 5, there exists a cycle C of the underlying undirected graph of H such that H/C is 5-edge-connected. Since H is 3-regular, the cycle C is a 2-factor of H. By Theorem 4(ii), the graph H/C admits a $\mathbb{Z}_3^5$-ASF h. As noted in Section 2, we may extend h to a $\mathbb{Z}_3^5$-flow of H (not necessarily nowhere-zero on the edges of C).

Let F be the 1-factor of H induced by the edges of H that are not in C. The graph H/F is 4-regular. Hence, it has no k-edge-cut with $k \in \{1, 3\}$, for otherwise one of the components would be a graph with an odd number of vertices of odd degree, a contradiction. Moreover, observe that contracting an edge in a graph cannot decrease the edge-connectivity. Therefore, H/F has no 2-edge-cut since H is 3-edge-connected. Consequently, H/F is 4-edge-connected. By Theorem 4(i), it admits a $\mathbb{Z}_2^2 \times \mathbb{Z}_3^4$-ASF g. We extend g to a $\mathbb{Z}_2^2 \times \mathbb{Z}_3^4$-flow of H.

Set $f(e) := (\varepsilon(e), h(e), g(e))$ for every arc e of H, where $\varepsilon(e) := 1$ if e is in C and $\varepsilon(e) := 0$ otherwise. First, note that no two arcs e and e' of H are assigned by f opposite values of M. Indeed, if e and e' both belong to F or to C, then $f(e) \neq -f(e')$ since both h (restricted to H/C) and g (restricted to H/F) are antisymmetric flows. Moreover, if e belongs to C and e' to F, then $\varepsilon(e) \neq -\varepsilon(e')$.

Finally, f is a flow of H since both f and g are, and C is a cycle. □

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