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To cite this version:

Charles de Clercq. A going-down theorem for Chow-Grothendieck motives. 2009. hal-00443691v1

HAL Id: hal-00443691

https://hal.archives-ouvertes.fr/hal-00443691v1

Submitted on 3 Jan 2010 (v1), last revised 13 Sep 2012 (v4)

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A going-down theorem for Chow-Grothendieck motives

Charles De Clercq

January 5, 2010

Abstract

Let \((M(X), p)\) be a direct summand of the motive associated with a geometrically split, geometrically irreducible variety over a field \(F\) satisfying the nilpotence principle. We show that under some conditions, if \((M(X_E), p_E)\) is a direct summand of another motive \(M_E\) over a field extension \(E\), then \((M(X), p)\) is a direct summand of \(M\) over \(F\).

I Introduction

Throughout this note, \(\Lambda\) will always be a finite field. Given a field \(F\), an \(F\)-variety will be understood as a separated scheme of finite type over \(F\). Given such \(\Lambda\) and an \(F\)-variety \(X\), we can consider \(\text{CH}_i(X; \Lambda)\), the Chow group of \(i\)-dimensional cycles on \(X\) with coefficients in \(\Lambda\), defined by \(\text{CH}_i(X) \otimes \mathbb{Z}/\Lambda\). These groups are the first step in the construction of the category \(\text{CM}(F; \Lambda)\) of Chow-Grothendieck motives with coefficients in \(\Lambda\). This category is constructed as the pseudo-abelian envelope of the category \(\text{CR}(F; \Lambda)\) of correspondences with coefficients in \(\Lambda\). Our main reference for the construction and the main properties of these categories is [2].

Definition I.1. Let \(X\) be an \(F\)-variety. A field extension \(E/F\) is a splitting field of \(X\) if the \(E\)-motive \(X_E\) is isomorphic to a finite direct sum of twisted Tate motives.

Following [2], we will write \(\text{CH}(X; \Lambda)\) for the colimit of all \(\text{CH}(X_E; \Lambda)\), where \(E\) runs through all field extensions \(E/F\). We will say that an \(F\)-variety \(X\) is geometrically split if \(X\) splits over the algebraic closure of \(F\).

For any field extension \(E/F\), we will say that any element of \(\text{CH}(X; \Lambda)\) lying in the image of the restriction morphism \(\text{CH}(X_E; \Lambda) \to \text{CH}(X; \Lambda)\) is \(E\)-rational.

The purpose of this note is to generalize the following result, proved by N. Karpenko ([3, Proposition 4.6]).

Theorem I.2. Let \(X\) be a geometrically irreducible, geometrically split variety satisfying the nilpotence principle and \(M\) be a motive. Assume that there exists a field extension \(E/F\) such that

1. the field extension \(E(X)/F(X)\) is purely transcendental;
2. the upper indecomposable summand of \(M(X)_E\) is also lower and is a summand of \(M_E\).

Then the upper indecomposable summand of \(M(X)\) is a summand of \(M\).

The class of geometrically split, geometrically irreducible \(F\)-varieties satisfying the nilpotence principle is quite large (see [2], [1], [4]). N. Karpenko uses this result for instance to prove several motivic decompositions in \(\text{CM}(F; \Lambda)\) ([3, lemma 7.1]). Here we generalize theorem I.2 by replacing the motive \(M(X)\) by the direct summand \((M(X), p)\) associated with a projector \(p \in \text{CH}_{\dim(X)}(X \times X; \Lambda)\).

I would like to thank Nikita Karpenko, my Ph.D. thesis adviser, for raising this question and guiding me during this work.

II The Chow group of a geometrically split \(F\)-variety

Let \(X\) be a geometrically split \(F\)-variety. Let us consider the motivic decomposition

\[
M(X_E) \simeq \bigoplus_{i=1}^m \Lambda(a_i)
\]
where \(m\) and \(a_i\) are integers, over a splitting field \(E\) of \(X\).

By definition of morphisms in \(CM(E; \Lambda)\), \(\text{Hom}_{CM(E; \Lambda)}(\Lambda(i), \Lambda(j)) \simeq \delta_{ij}\Lambda\), where \(\delta_{ij}\) stands for the Kronecker symbol. Therefore there is an integer \(n\) called the rank of \(X\) such that

\[
CH(X; \Lambda) \simeq \bigoplus_{i=1}^{n} \Lambda.
\]

Moreover, the isomorphism between \(M(X_E)\) and the previous direct sum of twisted Tate motives defines an homogeneous basis \((x_k)_{k=1}^{n}\) of the \(\Lambda\)-module \(CH(X_E; \Lambda)\).

**Proposition II.1.** Let \(X\) be a geometrically split \(F\)-variety. then the pairing

\[
\Psi : CH(\overline{X}; \Lambda) \times CH(\overline{X}; \Lambda) \longrightarrow \Lambda
\]

\[
(\alpha, \beta) \longmapsto \deg(\alpha \cdot \beta)
\]

is bilinear, symmetric and non-degenerate.

**Proof.** c.f. [5, Remark 5.6] \(\square\)

Since the bilinear form \(\Psi\) is non-degenerate, it gives rise to an isomorphism of \(\Lambda\)-modules between \(CH(\overline{X}; \Lambda)\) and its dual space \(\text{Hom}_\Lambda(CH(\overline{X}; \Lambda), \Lambda)\) given by

\[
f : CH(\overline{X}; \Lambda) \longrightarrow \text{Hom}_\Lambda(CH(\overline{X}; \Lambda), \Lambda)
\]

\[
f(u) \longmapsto \Psi(u, \cdot)
\]

Now consider the dual basis \((x'_i)_{i=0}^{n}\) of the basis \((x_i)_{i=0}^{n}\). We define the antedual basis of \((x_i)_{i=0}^{n}\) by \((x'_i)_{i=0}^{n}\), where \(x'_i : = f^{-1}(x_i)\). By definition of the antedual basis, we have \(\Psi(x_i, x'_j) = \delta_{ij}\), where \(\delta_{ij}\) is the Kronecker symbol.

**Proposition II.2.** Let \(M\) and \(N\) be two motives in \(CM(F; \Lambda)\), with \(M\) split. Then there is an isomorphism of \(\Lambda\)-modules

\[
CH^*(M; \Lambda) \otimes CH^*(N; \Lambda) \longrightarrow CH^*(M \otimes N; \Lambda)
\]

**Proof.** c.f. [2, Proposition 64.3]. \(\square\)

Computations become much simpler on split varieties.

**Lemma II.3.** Let \(X\) be a split variety of rank \(n\) and two smooth complete varieties \(Y\) and \(Y'\). Consider the homogeneous basis \((x_k)_{k=1}^{n}\) defined previously and its antedual basis \((x'_k)_{k=1}^{n}\). Then for all cycles \(y \in CH(Y; \Lambda)\), \(y' \in CH(Y'; \Lambda)\) and for all \(1 \leq i, j \leq n\),

\[
(x_i \times y) \circ (y' \times x'_j) = \delta_{ij}(y' \times y).
\]

**Proof.** We set 1 for the identity class, either in \(CH(Y; \Lambda)\) or in \(CH(Y'; \Lambda)\).

\[
(x_i \times y) \circ (y' \times x'_j) = (Y' \times Y')_*(y' \times x'_j \cdot (p'_Y \times Y')^*(x_i \times y))
\]

\[
= (Y' \times Y')_*(y' \times x'_j \cdot (1 \times x_i \times y))
\]

\[
= (Y' \times Y')_*(y' \times (x'_j \cdot x_i) \times y)
\]

\[
= \deg(x'_j \cdot x_i)(y' \times y)
\]

\[
= \delta_{ij}(y' \times y)
\]

\(\square\)
III Direct summands of geometrically split $F$-varieties

Let $X$ be an $F$-variety and $p \in \text{CH}_{\dim(X)}(X \times X; \Lambda)$ be the projector defining the direct summand $(X, p)$. A direct summand of the motive $(X, p)$ is given by a projector in $\text{End}_{CM(F; \Lambda)}((X, p), (X, p))$, by definition of the category $CM(F; \Lambda)$.

**Lemma III.1.** Let $X$ be an $F$-variety, and $p \in \text{CH}_{\dim(X)}(X \times X; \Lambda)$ the projector defining the direct summand $(X, p)$. Then every direct summand of $(X, p)$ is a summand $(X, q)$ associated with a projector $q \in \text{CH}_{\dim(X)}(X \times X; \Lambda)$ satisfying $p \circ q \circ p = q$.

**Proof.** Indeed we have $\text{End}_{CM(F; \Lambda)}((X, p), (X, p)) = p \circ \text{CH}_{\dim(X)}(X \times X; \Lambda) \circ p$, thus for any projector $q \in \text{End}_{CM(F; \Lambda)}((X, p), (X, p))$, there is a cycle $s \in \text{CH}_{\dim(X)}(X \times X; \Lambda)$ such that $q = p \circ s \circ p$. But $p$ is itself a projector, therefore $p \circ q \circ p = p^2 \circ s \circ p^2 = p \circ s \circ p = q$. \hfill \Box

We now study the notion of upper and lower direct summands of a direct summand $(M, p)$. From now on, we consider a geometrically split $F$-variety $X$ of rank $n$. We keep the notation $(x_k)_{k=1}^n$ for the homogeneous basis of $\text{CH}(X; \Lambda)$ and $(x_k^* n_{k=1}^n$ for the antidual basis of $(x_k)_{k=1}^n$ associated with the pairing $\Psi$.

**Definition III.2.** Let $X$ a geometrically split $F$-variety and $p \in \text{CH}_{\dim(X)}(X \times X; \Lambda)$ be a non-zero projector. We denote by $\overline{p}$ the image of $p$ under the restriction map to a splitting field $E$ of $X$. Considering the bases $(x_i)_{i=1}^n$ and $(x_i^*)_{i=1}^n$ of $CH(X_E; \Lambda)$, we have

$$\overline{p} = \sum_{i=1}^n \sum_{j=1}^n p_{ij} (x_i \times x_j^*).$$

We define the lowest codimension of $p$ by

$$\text{cdmin}(p) := \min_{p_{ij} \neq 0} (\text{codim}(x_i))$$

where $\text{codim}(x_i)$ stands for the codimension in $CH(X; \Lambda)$.

As $X$ is geometrically split, the summand $(X, p)$ associated with the projector $p \in \text{CH}_{\dim(X)}(X \times X; \Lambda)$ splits over a field extension $E/F$:

$$(X_E, p_E) \simeq \bigoplus_{i=1}^m (\Lambda(a_i))^{m_i} \quad (\text{III.1})$$

where $a_i$, $m_i$ and $m$ are integers, and we set $i < j \Rightarrow a_i < a_j$.

**Definition III.3.** Let $X$ be a geometrically split $F$-variety, and $(X, p)$ be a summand of $X$. Let also $(X, q)$ be a direct summand of the motive $(X, p)$. Consider the decomposition (**III.1**) of $(X, p)$ in a splitting field $E$ of $X$. The summand $(X, q)$ is :

1. upper in $(X, p)$ if $\text{CH}^{a_1}((X_E, q_E)) \neq 0$.
2. lower in $(X, p)$ if $\text{CH}^{a_m}((X_E, q_E)) \neq 0$.

By definition, a direct summand $(X, q)$ is upper in $(X, p)$ if and only if $(X(-\text{dim}(X)), q)$ is lower in the dual motive $(X(-\text{dim}(X)), p)$. We can now study the link between the lowest codimension of a summand and the position of its motivic decomposition.

**Proposition III.4.** Let $X$ be a geometrically split $F$-variety, and $p \in \text{CH}_{\dim(X)}(X \times X; \Lambda)$ the projector defining the summand $(X, p)$. Consider the motivic decomposition (**III.1**) given above of the motive $(X_E, p_E)$, where $E$ is a splitting field of $X$. Then

$$a_1 = \text{cdmin}(p)$$

**Proof.** Let us show first that $\text{cdmin}(p) \geq a_1$. Indeed we consider the motivic decomposition (**III.1**), thus
Consequently $CH^a(X_\mathcal{E};\Lambda)\circ p_\mathcal{E} \neq 0$ by definition of Chow groups in $CM(\mathcal{E};\Lambda)$.

Let $x \in CH^a(X_\mathcal{E};\Lambda)$ such that $x \circ p_\mathcal{E} \neq 0$. Since $\{x_i\}^n_{i=1}$ is homogeneous, we can consider the base $(x_k)_{k \in K}$ of the $\Lambda$-module $CH^a(X_\mathcal{E};\Lambda)$ and write $x = \sum_{k \in K} \lambda_k x_k$. Then

$$x \circ p_\mathcal{E} \neq 0 \Rightarrow \left(\sum_{k \in K} \lambda_k x_k\right) \circ \left(\sum_{i=1}^n p_{ij} (x_i \times x_j^*)\right) \neq 0$$

$$\Rightarrow \sum_{i=1}^n \sum_{k \in K} \lambda_k p_{ik} x_i \neq 0$$

$$\Rightarrow \exists (i_0, k_0) \in \{1,..,n\} \times K, \ p_{i_0 k_0} \neq 0$$

Then since $\text{codim}(x_{i_0}) = \dim(X_\mathcal{E}) - \text{codim}(x_{i_0}^*) = \text{codim}(x_{i_0}) = a_1$ we have proven that $\text{cdmin}(p) \leq a_1$.

Suppose on the other hand that $\text{cdmin}(p) = s < a_1$. Then there are two indices $i_0, j_0$ in $\{1,..,n\}$ such that $p_{i_0 j_0} \neq 0$, and $\text{codim}(x_{i_0}) = s$. Then $x_{j_0} \circ p_\mathcal{E} = \sum_{i=0}^n p_{j_0 x_i} \neq 0$, therefore $CH^s(X_\mathcal{E};\Lambda) \circ p_\mathcal{E} \neq 0$, as $\text{codim}(x_{i_0}) = \text{codim}(x_{j_0})$. But this contradicts the fact that

$$CH^s(X_\mathcal{E};\Lambda) \circ p_\mathcal{E} = Hom_{CM(\mathcal{E};\Lambda)}((X_\mathcal{E}, p_\mathcal{E}); \Lambda(s))$$

$$= \bigoplus_{i=1}^m Hom_{CM(\mathcal{E};\Lambda)}(\Lambda(a_i); \Lambda(s))^{m_i}$$

$$= 0$$

Therefore $\text{cdmin}(p) = a_1$.

\[\square\]

**Proposition III.5.** Let $X$ be a geometrically split $F$-variety, and $p$ the projector defining the summand $(M(X), p)$. Let also be $(M(X), q)$ a direct summand of $(M(X), p)$. Then

$$\text{cdmin}(p) = \text{cdmin}(q) \iff (X, q) \text{ is upper in } (X, p).$$

**Proof.** We keep the same notations as in proposition III.4.

As $q$ defines a direct summand of $(X, p)$ we have $q = p \circ q \circ p$ by lemma III.1, thus for any $k$

$$CH^k((X_\mathcal{E}, q_\mathcal{E}); \Lambda) = CH^k((X_\mathcal{E}, \Lambda) \circ q_\mathcal{E}$$

$$= CH^k((X_\mathcal{E}, \Lambda) \circ (p_\mathcal{E} \circ q_\mathcal{E} \circ p_\mathcal{E})$$

But $q_\mathcal{E} \circ p_\mathcal{E} = (p_\mathcal{E} \circ q_\mathcal{E} \circ p_\mathcal{E}) \circ p_\mathcal{E} = p_\mathcal{E} \circ q_\mathcal{E} \circ p_\mathcal{E} = q_\mathcal{E}$, thus

$$CH^k((X_\mathcal{E}, q_\mathcal{E}); \Lambda) = (CH^k((X_\mathcal{E}, \Lambda) \circ p_\mathcal{E})) \circ q_\mathcal{E} = CH^k((X_\mathcal{E}, p_\mathcal{E}); \Lambda) \circ q_\mathcal{E}(\ast)$$

Therefore for any $k$ such that $CH^k((X_\mathcal{E}, p_\mathcal{E}); \Lambda) = 0$, we have shown that $CH^k((X_\mathcal{E}, q_\mathcal{E}); \Lambda) = 0$.

Now write the decomposition of $(X_\mathcal{E}, q_\mathcal{E})$

$$(X_\mathcal{E}, q_\mathcal{E}) \simeq \bigoplus_{j=1}^s (\Lambda(b_j))^{\ast_i}$$

with $i < j \Rightarrow b_i < b_j$. Equality $(\ast)$ implies that for all $j \in \{1,..,s\}$, $b_j \geq a_1$. But proposition III.4 implies that $b_1 = \text{cdmin}(q)$, thus

$$\text{cdmin}(p) = \text{cdmin}(q) \iff b_1 = a_1$$

$$\iff CH^{a_1}((X_\mathcal{E}, q_\mathcal{E}); \Lambda) \neq 0$$

\[\square\]
IV Proof of the main theorem

Following [3], we recall some results on the category generated by geometrically split $F$-varieties satisfying the nilpotence principle in $CM(F; \Lambda)$. We will say that a pseudo-abelian category satisfies the Krull-Schmidt principle if every object of this category decomposes in a unique way as a direct sum of indecomposable objects (up to permutation of these indecomposable summands).

**Lemma IV.1.** ([3, Corollary 3.2]) Let $X$ be a geometrically split $F$-variety satisfying the nilpotence principle. Then an appropriate power of any endomorphism of the motive $X$ is a projector.

**Proposition IV.2.** ([3, Corollary 3.3]) The Krull-Schmidt principle holds for the pseudo-abelian Tate subcategory of $CM(F; \Lambda)$ generated by the motives of geometrically split $F$-varieties satisfying the nilpotence principle.

**Remark IV.3.** More generally, proposition IV.2 is true when $\Lambda$ is any finite commutative ring (see [3, Corollary 3.2]). The Krull-Schmidt principle also holds for the category generated by motives of quadrics when the ring of coefficients $\Lambda$ is $\mathbb{Z}$. Nevertheless, the Krull-Schmidt principle does not hold in the category generated by the motives of projective homogeneous varieties when $\Lambda = \mathbb{Z}$ (see [6]) and this is one of the reasons why we suppose that $\Lambda$ is a finite field.

Let $X$ be a geometrically split $F$-variety. Let $p \in CH_{dim(X)}(X \times X; \Lambda)$ be the projector defining the direct summand $(X, p)$ of $X$. As seen before, the homogeneous base $(x_k)_{k=1}^n$ of $X$ and the antidual $x_k^* \in CH_{dim(X)}(X \times X; \Lambda)$ relatively to the pairing $\Psi$ allow us to decompose $p$ in a splitting field $E$ of the motive $X$, that is to say $p_E = \sum_{i=1}^n \sum_{j=1}^n p_{ij} (x_i \times x_j^*)$.

**Definition IV.4.** Let $X$ be a geometrically split $F$-variety and $E$ a splitting field of the motive $X$. Let $p$ be the projector defining the direct summand $(X, p)$. We say that couple $(i, j) \in \{1, \ldots, n\}^2$ is contained in $p_E$ if $p_{ij} \neq 0$. We will also say that an element $x_i \times x_j^*$ of the base of $CH_{dim(X_E)}(X_E \times X_E; \Lambda)$ is contained in $p_E$ if the couple $(i, j)$ is contained in $p_E$.

**Notation IV.5.** The set of all elements $x_i \times x_j^*$ of $CH_{dim(X_E)}(X_E \times X_E; \Lambda)$ which are contained in the projector $p_E$ will be called the base of $p_E$, and denoted $B(p_E)$.

**Definition IV.6.** Let $X$ be a geometrically split $F$-variety and two projectors $p$ and $q$ defining the two direct summand $(X, p)$ and $(X, q)$. Let $E$ be a splitting field of $X$. We say that $p$ contains $q$ if $B(q_E) \subset B(p_E)$. We will say that $q$ intersects $p$ if $B(q_E) \cap B(p_E) \neq \emptyset$.

**Remark IV.7.** Given a geometrically split $F$-variety $X$ and two direct summands $(X, p)$ and $(X, q)$ of the motive $X$ defined by the two projectors $p$ and $q$, we will also say that the summand $(X, p)$ contains the summand $(X, q)$ if the projector $p$ contains the projector $q$.

Since $\Lambda$ is a finite field, we have seen that the category generated by motives of geometrically split $F$-varieties satisfying the nilpotence principle satisfies the Krull-Schmidt principle (proposition IV.2). By definition of the category $CM(F; \Lambda)$, direct summands of the motive of an $F$-variety correspond to projectors in $End_{CM(F; \Lambda)}(X)$, thus in a splitting field $E$ of the motive $X$, every $F$-rational cycle in $CH_{dim(X)}(X \times X; \Lambda)$ is a linear combination of minimal $E$-rational cycles, corresponding to indecomposable direct summands of $X$. We now prove the key-lemma.

**Lemma IV.8.** Let $X$ be a geometrically split, geometrically irreducible $F$-variety satisfying the nilpotence principle. Let $p$ be the projector defining the direct summand $N := (X, p)$ of $X$. Let also $Y$ be a smooth complete $F$-variety, and $M := (Y, q)$ the direct summand of $Y$ induced by the projector $q$. Assume that there is a field extension $E/F$ and two $E$-rational correspondences $h : X_E \to M_E$ and $k : M_E \to X_E$ such that

1. every $E(X)$-rational cycle in $CH(X \times Y; \Lambda)$ is also $F(X)$-rational;
2. $k \circ h \in CH_{dim(X_E)}(X_E \times X_E; \Lambda)$ defines an upper direct summand $(X_E, k \circ h)$ of $N_E$. 


Then there exists an $F$-rational cycle $h'$ and a $E$-rational cycle $k'$ such that

- $k' \circ h'_E$ is a non-zero projector in $CH_{\dim(X_E)}(X_E \times X_E; \Lambda)$;
- the $E$-motive $(X_E, k' \circ h'_E)$ contains every upper indecomposable summand of $(X_E, k \circ h)$;
- the cycle $k'$ is $F$-rational if $k$ is $F$-rational.

**Proof.** We denote by $s$ the composite $k \circ h$. By assumption, $s$ is a projector in $CH_{\dim(X_E)}(X_E \times X_E; \Lambda)$ satisfying $s = p_E \circ s \circ p_E$ by lemma III.1. Moreover we suppose that $(M(X), s)$ is upper in $(M(X), p_E)$, that is to say $p \circ \text{ad}(s) = \text{ad}(p)$ by proposition III.5.

Let us decompose the images of the cycles $h$, $k$, $s$ and $p$ in a splitting field of the $F$-variety $X$ of rank $n$:

1. $\overline{h} = \sum_{i=1}^{n} h_i(x_i \times y_i)$
2. $\overline{k} = \sum_{j=1}^{n} k_j(y_j \times x_j)$
3. $\overline{s} = \sum_{i=1}^{n} \sum_{j=1}^{n} s_{ij}(x_i \times x_j)$
4. $\overline{p} = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}(x_i \times x_j) $

with $s_{ij} = h_i k_j \deg(y_j \cdot y_i)$ by definition of $s$.

Since $E(X)$ is a field extension of $E$, the cycle $\overline{h}$ is $E(X)$-rational, hence $F(X)$-rational by assumption 1).

Now consider the morphism $Spec(F(X)) \longrightarrow \overline{X}$ induced by the generic point of the geometrically irreducible variety $X$. It induces a morphism

$$\epsilon : (\overline{X} \times \overline{Y})_{F(X)} \longrightarrow \overline{X} \times \overline{Y} \times \overline{X}$$

whose pull-back $\epsilon^* : CH_{\dim(X)}(\overline{X} \times \overline{Y} \times \overline{X}; \Lambda) \longrightarrow CH_{\dim(X)}(\overline{X} \times \overline{Y}; \Lambda)$ maps every cycles of the form $\sum_i \lambda_i(\alpha_i \times \beta_i \times 1)$ on $\sum_i \lambda_i(\alpha_i \times \beta_i)$ and vanishes on elements of the form $\sum_i \lambda_i(\alpha_i \times \beta_i \times \gamma_i)$, where $\text{codim}(\gamma_i) > 0$.

Moreover, the pull-back $\epsilon^*$ induces a surjection of $F$-rational cycles onto $F(X)$-rational cycles ([2, corollary 57.11]). Therefore we can choose an $F$-rational cycle $\overline{h}_1 \in CH_{\dim(X)}(\overline{X} \times \overline{Y} \times \overline{X}; \Lambda)$ such that $\epsilon^*(\overline{h}_1) = \overline{h}$.

By definition of $\epsilon^*$ we have

$$\overline{h}_1 = \sum_{i=1}^{n} h_i(x_i \times y_i \times 1) + \sum \alpha \times \beta \times \gamma$$

where cycles $\gamma$ are of strictly positive codimension.

Now let us look at the cycle $\overline{h}_1$ as a correspondence $\overline{h}_1 : \overline{X} \longrightarrow \overline{Y} \times \overline{X}$. Setting $\overline{h}_2 := \overline{h}_1 \circ \overline{p}$, $\overline{h}_2$ is $F$-rational and its image in a splitting field of $X$ is

$$\overline{h}_2 = \left( \sum_{i=1}^{n} h_i(x_i \times y_i \times 1) \right) \circ p + \left( \sum \alpha \times \beta \times \gamma \right) \circ p$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} p_{ij}(x_i \times y_j \times 1) + \sum \tilde{\alpha} \times \tilde{\beta} \times \tilde{\gamma}$$

where cycles $\tilde{\gamma}$ are of strictly positive codimension and cycles $\tilde{\alpha}$ satisfies $\text{codim}(\tilde{\alpha}) \geq \text{cdmin}(p)$.

Consider now the diagonal embedding

$$\Delta : (\overline{X} \times \overline{Y}) \hookrightarrow (\overline{X} \times \overline{Y} \times \overline{X})$$

The morphism $\Delta$ induces a pull-back $\Delta^* : CH_{\dim(X)}(\overline{X} \times \overline{Y} \times \overline{X}; \Lambda) \longrightarrow CH_{\dim(X)}(\overline{X} \times \overline{Y}; \Lambda)$.

Setting $\overline{h}_3 := \Delta^*(\overline{h}_2)$, $\overline{h}_3$ is an $F$-rational cycle and

$$\overline{h}_3 = \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} p_{ij}(x_i \times y_j) + \sum (\tilde{\alpha} \cdot \tilde{\gamma}) \times \tilde{\beta}$$
where \( \text{codim}(\tilde{\alpha} \cdot \tilde{\gamma}) > \text{cdmin}(p) \) since \( \text{codim}(\tilde{\alpha}) \geq \text{cdmin}(p) \) and \( \text{codim}(\tilde{\gamma}) > 0 \).

The next step consists of computing the composite \( \overline{k} \circ \overline{h_3} \):

\[
\overline{k} \circ \overline{h_3} = \left( \sum_{i=1}^{n} k_l(y_l \times x_i^*) \right) \circ \left( \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} p_{ij}(x_i \times y_j) \right) + \left( \sum_{l=1}^{n} k_l(y_l^* \times x_i^*) \right) \circ \left( \sum (\tilde{\alpha} \cdot \tilde{\gamma}) \times \tilde{\beta} \right)
\]

\[
= \sum_{i=1}^{n} \sum_{l=1}^{n} \left( \sum_{j=1}^{n} k_l h_{ij} p_{ij}(y_l^* \times x_i^*) \right) \circ (x_i \times y_j) + \sum \tilde{\alpha} \times \tilde{\beta}
\]

\[
= \sum_{i=1}^{n} \sum_{l=1}^{n} \left( \sum_{j=1}^{n} k_l h_{ij} p_{ij} \deg(y_l^* \cdot y_j) \right) (x_i \times x_i^*) + \sum \tilde{\alpha} \times \tilde{\beta}
\]

Where cycles \( \overline{\pi} \) satisfy \( \text{codim}(\overline{\pi}) > \text{cdmin}(p) = \text{cdmin}(s) \).

Let us show that for any \((i, l) \in \{1, \ldots, n\}^2\), \(\sum_{j=1}^{n} k_l h_{ij} p_{ij} \deg(y_l^* \cdot y_j) = s_{il}\). First \(s\) defines a direct summand of \((X, p_E)\), thus by lemma III.1 \(s \circ p_E = p_E \circ s \circ p_E \circ p_E = p_E \circ s \circ p_E = s\). Now compute \(\overline{\pi} \circ \overline{p} \):

\[
\overline{\pi} \circ \overline{p} = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} s_{ij}(x_i \times x_j^*) \right) \circ \left( \sum_{l=1}^{n} \sum_{t=1}^{n} p_{lt}(x_l \times x_t^*) \right)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} s_{ij} p_{lt}(x_i \times x_j^*) \circ (x_l \times x_t^*)
\]

\[
= \sum_{i=1}^{n} \sum_{l=1}^{n} \left( \sum_{j=1}^{n} s_{lj} p_{lt} \right) (x_l \times x_t^*)
\]

Therefore for any \((i, l)\) with \(\text{codim}(x_i) = \text{cdmin}(s)\), the component of \(\overline{k} \circ \overline{h_3}\) over \(x_i \times x_i^*\) is equal to \(\sum_{j=1}^{n} k_l h_{ij} p_{ij} \deg(y_l^* \cdot y_j)\) and we have

\[
\sum_{j=1}^{n} k_l h_{ij} p_{ij} \deg(y_l^* \cdot y_j) = \sum_{j=1}^{n} s_{lj} p_{ij} = s_{il}.
\]

We thus have obtained

\[
(\overline{k} \circ \overline{h_3}) = \sum_{\text{codim}(x_i) = \text{cdmin}(p)} \sum_{j=1}^{n} s_{ij}(x_i \times x_j^*) + \sum \overline{\pi} \times \overline{\beta}
\]

where cycles \(\overline{\pi}\) satisfy \(\text{codim}(\overline{\pi}) > \text{cdmin}(p) = \text{cdmin}(s)\).

As \(\Lambda\) is finite, there exists an integer \(n_0\) such that \((g \circ (f_3)_{E \circ E})^{n_0}\) is a projector by lemma IV.1. Setting \(h' = \overline{h_3}\) and \(k' = (\overline{k} \circ \overline{h_3})^{n_0-1} \circ \overline{k}\), we see that

\[
(\overline{k'} \circ \overline{h'}) = \sum_{\text{codim}(x_i) = \text{cdmin}(p)} \sum_{j=1}^{n} s_{ij}(x_i \times x_j^*) + \sum \tilde{\alpha} \times \tilde{\beta}
\]

where \(\text{codim}(\tilde{\alpha}) > \text{cdmin}(p)\), therefore \(k' \circ h'_3\) is non-zero.

If \(k\) is \(F\)-rational, then \(k' = (\overline{k} \circ \overline{h_3})^{n_0-1} \circ k\) is also \(F\)-rational. Moreover, the cycle \(k' \circ h'\) intersects every projector \(w \in CH(X \times X; \Lambda)\) satisfying \(\text{cdmin}(w) = \text{cdmin}(k \circ h)\), and \(\mathcal{B}(w) \subset \mathcal{B}(k \circ h)\). In particular \(k' \circ h'\) intersects every upper direct summand of the motive \((X_E, k \circ h)\), and therefore contains every indecomposable direct summand of the motive \((X_E, k \circ h)\).

The main theorem is a direct consequence of lemma IV.8.

**Theorem IV.9.** Let \(X\) be a geometrically irreducible, geometrically split \(F\)-variety satisfying the nilpotence principle. Consider the projector \(p\) defining the indecomposable direct summand \(N := (X, p)\) of the motive \(X\). Let also \(Y\) be a smooth complete \(F\)-variety and \(M := (Y, q)\) the direct summand induced by a projector \(q\). Assume the existence of a field extension \(E/F\) such that
1. every $E(X)$-rational cycle in $\text{CH}(\overline{X} \times \overline{Y}; \Lambda)$ is also $F(X)$-rational;

2. the $E$-motive $N_E$ contains an indecomposable direct summand $(X_E, s)$ both lower and upper and is a direct summand of $M_E$.

Then the motive $N$ is a direct summand of the motive $M$.

**Proof.** By assumption, the $E$-motive $N_E$ is a direct summand of $M_E$, that is to say there are two correspondences $f \in \text{CH}_{\dim(X_E)}(X_E \times Y_E; \Lambda)$ and $g \in \text{CH}_{\dim(Y_E)}(Y_E \times X_E; \Lambda)$ such that $g \circ f = p_E$. Let us apply lemma IV.8.

Lemma IV.8 implies that there are two correspondences $g' \in \text{CH}_{\dim(Y_E)}(Y_E \times X_E; \Lambda)$ and $f' \in \text{CH}_{\dim(X_E)}(X_E \times Y_E; \Lambda)$, with $f'$ $F$-rational, such that $g' \circ f'_E$ is a non-zero projector and the motive $(X_E, g' \circ f'_E)$ contains every upper indecomposable summand of $(X_E, p_E)$.

Now consider the cycle $p_E \circ g' \circ f'_E \circ p_E$. Lemma IV.1 justifies the existence of an integer $n_0$ such that $(p_E \circ g' \circ f'_E \circ p_E)^{n_0}$ is a projector. Moreover we have the decomposition

$$p_E \circ f'_E = \sum_{\text{codim}(x_i) = \text{cdim}(p)} n \sum_{j=1}^{n} p_{ij}(x_i \times x_j^*) + \sum \alpha \times \beta$$

with $\text{codim}(\alpha) > \text{cdim}(p)$.

Set $\tilde{g} := (p_E \circ g' \circ f'_E \circ p_E)^{n_0-1} \circ p_E \circ g'$ and $\tilde{f} := f' \circ p$. The cycle $\tilde{g} \circ \tilde{f}_E$ is a projector satisfying $p_E \circ \tilde{g} \circ \tilde{f}_E \circ p_E = \tilde{g} \circ \tilde{f}_E$. Besides in a splitting field of $X$ $p_E \circ \tilde{g} \circ \tilde{f}_E = \sum_{\text{codim}(x_i) = \text{cdim}(p)} n \sum_{j=1}^{n} p_{ij}(x_i \times x_j^*) + \sum \tilde{\alpha} \times \tilde{\beta}$

with $\text{codim}(\tilde{\alpha}) > \text{cdim}(p)$, therefore the motive $(X_E, \tilde{g} \circ \tilde{f}_E)$ is an upper direct summand of the motive $N_E$. Besides $(X_E, \tilde{g} \circ \tilde{f}_E)$ intersects any upper direct summand of $N_E$, thus contains the indecomposable direct summand $(X_E, s)$. It follows that the summand $(X_E, \tilde{g} \circ \tilde{f}_E)$ is both upper and lower in $N_E$.

Transposing, the summand $(X_E(-\dim(X_E)), f'_E \circ \tilde{g})$ is a direct summand of the dual motive $N^*_E$ which is upper, as $(X_E, \tilde{g} \circ \tilde{f}_E)$ is lower in $N_E$, therefore we can apply lemma IV.8 again.

Lemma IV.8 justifies the existence of an $F$-rational cycle $\tilde{g}$ and another cycle $\tilde{f}$ which is also $F$-rational, such that $f'_E \circ \tilde{f} \circ \tilde{g}_E$ is a non-zero projector in $\text{End}_{\text{CM}(E, \Lambda)}(N^*_E)$. Transposing again, the cycle $\tilde{g}_E \circ \tilde{f}_E$ is a non-zero projector in $\text{CH}_{\dim(X_E)}(X_E \times X_E; \Lambda)$.

By nilpotence principle, an appropriate power $(\tilde{g} \circ \tilde{f})^n$ is a non-zero projector in $\text{CH}_{\dim(X)}(X \times X; \Lambda)$. Setting $g := (\tilde{g} \circ \tilde{f})^{-1} \circ \tilde{g}$ and $f := \tilde{f}$, the cycle $g \circ f$ is a non-zero projector. But the motive $N$ is indecomposable, therefore we can conclude that $g \circ f = p$, that is to say $N$ is a direct summand of $M$. \qed
References


